A Mathematician’s apology
(with a foreword by C. P. SNOW)

Godfrey Harold HARDY
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66 TEACHING

It is one of the first duties of a professor, for example, in any subject, to exaggerate a little both the importance of his subject and his own importance in it. A man who is always asking ‘Is what I do worth while?’ and ‘Am I the right person to do it?’ will always be ineffective himself and a discouragement to others. He must shut his eyes a little and think a little more of his subject and himself than they deserve.

81/153 ‘IMMORTALITY’ OF MATHEMATICS

Archimedes will be remembered when Aeschylus is forgotten, because languages die and mathematical ideas do not. ‘Immortality’ may be a silly word, but probably a mathematician has the best chance of whatever it may mean.

[…]

Dr Snow has also made an interesting minor point about § 8. Even if we grant that ‘Archimedes will be remembered when Aeschylus is forgotten’, is not mathematical fame a little too ‘anonymous’ to be wholly satisfying? We could form a fairly coherent picture of the personality of Aeschylus (still more, of course, of Shakespeare or Tolstoi) from their works alone, while Archimedes and Eudoxus would remain mere names.

Mr J. M. Lomas put this point more picturesquely when we were passing the Nelson column in Trafalgar Square. If I had a statue on a column in London, would I prefer the column to be so high that the statue was invisible, or low enough for the features to be recognizable? I would choose the first alternative, Dr Snow, presumably, the second.

Commentaire. La première phrase mérite quelque développement pour éviter d’y voir une provocation facile et gratuite de la part de la communauté mathématique.

Même si les langues devaient mourir au fil des âges, il y a quelque chose qui perdure : ce qui a animé les langues mortes, ce qui anime les langues d’aujourd’hui et qui animera celles à venir. Peut-être Achille tombera dans l’oubli, mais ce qui l’animait s’exprimera d’une autre façon, chez une autre personne, dans une autre époque. En ce sens, peu importe Achille ou Racine ou Shakespeare ou Goethe ou Beethoven : leur témoignage sont certes d’inestimables joyaux de l’humanité, mais l’âme de l’orfèvre perdure à travers cette dernière et engendrera çà et là d’autres bijoux tout aussi remarquables. Voilà pour nuancer le mépris qui peut être ressenti envers Achille à la lecture de Hardy.

D’où viendrait maintenant l’immortalité des idées mathématiques ? Partons d’une constatation empirique : face à une même question, les mathématiciens dégagent souvent les mêmes ressorts conceptuels fondamentaux, sous une forme plus ou moins déguisée, les plus brillants épurant à l’extrême les concepts ainsi mis à jour. En ce sens, les idées mathématiques ont un caractère *convergeant*, on aimerait dire *invariable* – mais certainement pas immortel. Vient alors la fossilisation des idées mathématiques ainsi dégagées sous forme de théorèmes : une fois établis, ces derniers marquent à jamais le paysage mathématique et c’est là que réside pour Hardy, il nous semble, le caractère immortel des idées mathématiques. Mais c’est là une immortalité bien tautologique, sauf à confondre le vrai et le prouvable et à invoquer une « réalité » mathématique permettant d’entretenir la confusion (tout est permis sur le terrain de la métamathématique dont nous nous gardons toute intrusion) et surtout une immortalité bien relative : viendraient à disparaître les écrits d’Euclide et à se défigurer la géométrie de notre univers, est-il seulement raisonnable d’imaginer pouvoir concevoir ne serait-ce qu’une bribe de géométrie euclidienne ?

Afin d’en finir avec le caractère méprisant de cette citation, demandons-nous : ce qui a animé un Achille, un Hugo, un Monet, le mathématicien peut-il prétendre s’en passer pour faire revivre ses fossiles ? Nous nous contenterons humblement d’une mathématique vivante, laissant l’immortalité aux éclaireurs de la métamathématique.

84-5/87 BEAUTY OF MATHEMATICS

A mathematician, like a painter or a poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with *ideas*. A painter makes patterns with shapes and colours, a poet with words. A painting may embody an ‘idea’, but the idea is usually commonplace and unimportant. In poetry, ideas count for a good deal more; but, as Housman insisted, the importance of ideas in poetry is exaggerated: ‘I cannot satisfy myself that there are any such things as poetical ideas…. Poetry is not the thing said but a way of saying it.'

Not all the water in the rough rude sea
Can wash the balm from an anointed King.

Could lines be better, and could ideas be at once be more trite and false? The poverty of the ideas seems hardly to affect the beauty of the verbal pattern. A mathematician, on the other hand, has no material to work with but ideas, and so his patterns are likely to last longer, since ideas wear less with time than words.

The mathematician's patterns, like the painter's or the poet's, must be *beautiful*; the ideas, like the colours or the words, must fit together in a harmonious way. Beauty is the first test: there is no permanent place in the world for ugly mathematics. And here I must deal with a misconception […], what Whitehead has called the 'literary superstition' that love of and aesthetic appreciation of mathematics is 'a monomania confined to a few eccentrics in each generation'.

[…]

A very little reflection is enough to expose the absurdity of the 'literary superstition'. There are masses of chess-players in every civilized country–in Russia, almost the whole educated population; and every chess-player can recognize and appreciate a 'beautiful' game of problem. Yet a chess problem is *simply* an exercise in pure mathematics (a game not entirely, since psychology also plays a part), and everyone who calls a problem 'beautiful' is applauding mathematical beauty, even if it is the beauty of a comparatively lowly kind. Chess problems are the hymn-tuned of the mathematics.

89-90 Seriousness of mathematics

The best mathematics is *serious* as well as beautiful–'important' if you like, but the word is very ambiguous, and 'serious' expresses what I mean better.

[…] The 'seriousness' of a mathematical theorem lies, not in its practical consequences, which are usually negligible, but in the *significance* of the mathematical ideas which it connects. We may say, roughly, that a mathematical idea is 'significant' if it can be connected, in a natural and illuminating way, with a large complex of other mathematical ideas. Thus a serious mathematical theorem, a theorem which connects significant ideas, is likely to lead to important advances in mathematics itself and even in other sciences. […]

The seriousness of a theorem, of course, does not *lie in* its consequences, which are merely the *evidence* for its seriousness. […]

103/108-10 Seriousness: generality and depth

A 'serious theorem is a theorem which contains 'significant' ideas, and I suppose that I ought to try to analyse a little more closely the qualities which make a mathematical idea significant. […] There are two things at any rate which seem essential, a certain *generality* and a certain *depth*; but neither quality is easy to define at all precisely.

[…] It is not mere 'piling of subtlety of generalization' which is the outstanding achievement of modern mathematics. Some measure of generality must be present in any high-class theorem, but *too much* tends inevitably to insipidity. 'Everything is what it is, and not another thing', and the differences between things are quite as interesting as their resemblances. We do not choose our friends because they embody all the pleasant qualities of humanity, but because they are the people that they are. And so in mathematics; a property common to too many objects can hardly be very exciting, and mathematical ideas also become dim unless they have plenty of individuality. Here at any rate I can quote Whitehead on my side: 'it is the large generalization, limited by a happy particularity, which is the fruitful conception.'

[…]

It seems that mathematical ideas are arranged somehow in strata, the ideas in each stratum being linked by a complex of relations both among themselves and with those above and below. The lower the stratum, the deeper (and in general the more difficult) the idea. Thus the idea of an 'irrational' is deeper than that of an integer; and Pythagoras's theorem is, for that reason, deeper that Euclid's.

113 Beauty of proofs

In both [Euclid's and Pythagoras's] theorems […] there is a very high degree of *unexpectedness*, combined with *inevitability* and *economy*. The arguments take so odd and surprising a form; the weapons used seem so childishly simple when compared with the far-reaching results; but there is no escape from the conclusions. There are no complications of detail–one line of attack is enough in each case; and this is true of the proofs of many more difficult theorems, the full appreciation of which demands quite a high degree of technical proficiency. We do not want many 'variations' in the proof of a mathematical theorem: 'enumeration of cases', indeed, is one of the duller forms of mathematical arguments. A mathematical proof should resemble a simple and clear-cut constellation, not a scattered cluster in the Milky Way.

Commentaire. Même si, au sein d'une preuve, l'on se voit examiner fastidieusement de nombreux cas n'ayant apparemment aucun lien entre eux, nous disposons en fin de compte bel et bien d'une *preuve*, aussi laide soit-elle, sur laquelle nous pouvons nous appuyer pour avancer. Certes, l'on *doit* se sentir insatisfait d'une preuve recelant de tels arguments et l'on *doit* en rechercher une explication plus élégante, mais la modestie nous suggère également qu'une telle « simple and clear-cut constellation » pourrait bien être hors de portée au stade où nous en sommes. À chacun d'avoir les idées claires sur ses priorités (efficaces, esthétiques, littéraires…) selon le moment du discours et selon la pièce d'architecture concernée.

125-6 Geometry is no reality

the point which is important to us now is this, that there is one thing at any rate of which pure geometries are *not* pictures, and that is the spatio-temporal reality of the physical world. It is obvious, surely, that they cannot be, since earthquakes and eclipses are not mathematical concepts.

This may sound a little paradoxical to an outsider, but it is a truism to a geometer; and I may perhaps be able to make clearer by an illustration. Let us suppose that I am giving a lecture on some system of geometry, such as ordinary Euclidean geometry, and that I draw figures on the blackboard to stimulate the imagination of my audience, rough drawings of straight lines or circles or ellipses. It is plain, first, that the truth of the theorems which I prove is in no way affected by the quality of my drawings. Their function is merely to bring home may meaning to my hearers, and, if I can do that, there would be no gain in having them redrawn by the most skilful draughtsman. They are pedagogical illustrations, not part of the real subject-matter of the lecture.

Now let us go a stage further. The room in which I am lecturing is part of the physical world, and has itself a certain pattern. The study of that pattern, and of the general pattern of physical reality, is a science in itself, which we may call 'physical geometry'. Suppose now that a violent dynamo, or a massive gravitating body, is introduced into the room. Then the physicists tell us that the geometry of the room is changed, its whole physical pattern slightly but definitively distorted. Do the theorems which I have proved become false? Surely it would be nonsense to suppose that the proofs of them which I have given are affected in any way. It would be like supposing that a play of Shakespeare is changed when a reader spills his tea over a page. The play is independent of the pages on which it is printed, and 'pure geometries' are independent of lecture rooms, or of any other detail of the physical world.

Commentaire.. On entend souvent que la géométrie est l'art de raisonner juste sur des figures fausses ; mais la géométrie ne raisonne *pas* sur des *figures*, ses objets sont de tout autre nature ; elle utilise certes ces dernières comme guides intuitifs, only as « pedagogical illustrations ». En cela, il vaut mieux lors d'une preuve faire des dessins grossiers plutôt que des figures propres prétendant coller à une « réalité » mathématique (dont la soi-disant écrasante évidence en a effectivement écrasé plus d'un). Toutefois, lorsque l'on n'est pas encore au stade de la preuve, mais seulement à celui d'observation, le soin apporté au tracé de figures ne peut que servir notre regard, vu en tant qu'instrument de notre intuition, sans que nous n'ayons aucunement à rentrer dans de vaines querelles métamathématiques – combien de fois en effet avons-nous intuité une vérité en suivant une intuition qui s'est révélée infondée *a posteriori* ?

Nous renvoyons à Stella Baruk pour ce mythe de la réalité mathématique ainsi qu'à Pierre Duhem pour la distinction entre les phénomènes physiques et les énoncés mathématiques supposés les régir (nécessairement au nom de quelque métaphysique). Vu que Duhem ne semble pas conscient de ce mythe, reprendre ses propos en remplaçant le duo « physique – mathématique » par « mathématique – logique » serait certainement des plus fructueux.

129 From the physical facts to physical reality

I went on to say that neither physicists nor philosophers have ever given any convincing account of what 'physical reality' is, or of how the physicist passes, from the confuses mass of fact or sensation with which he starts, to the construction of the objects which he calls 'real'. Thus we cannot be said to know what the subject-matter of physics is; but this need not prevent us from understanding roughly what a physicist is trying to do. It is plain that he is trying to correlate the incoherent body of crude fact confronting him with some definite and orderly scheme of abstract relations, the kind of which he can borrow only from mathematics.

A mathematician, on the other hand, is working with his own mathematical reality.

Commentaire.. Comment ne pas entendre Duhem à travers ces lignes ? Comment pourrions-nous, à la lecture de la dernière phrase, lever la critique que nous lui adressons concernant la « réalité » mathématique ?

133-4 'Useful' mathematics

We must […] remember that a reserve of knowledge is always an advantage, and that the most practical of mathematicians may be seriously handicapped if his knowledge is the minimum which is essential to him; and for that reason, we must add a little under every heading.