Asymptotics of Hurwitz numbers with any number of partitions

Marc SAGE

2011 December

Contents

Abstract

We derive in this paper the asymptotics of several-partition Hurwitz numbers, relying on a theorem of Maxim Kazarian for the one-partition case. Essentially, the asymptotics for several partitions is the same as the one-partition asymptotics obtained by concatenating the partitions. The genus-depending constant appearing in the asymptotics is trivially linked to Bender-Gao-Richmond's map universal constant t_g.

1 Introduction

Hurwitz wondered at the end of the XIX$^{\text{th}}$ century in how many ways a permutation in $\mathfrak{S}_n$ of given type λ could be factorised into a product of a minimal number of transpositions whose generated group acts transitively. If one denotes by $h_n^0(\lambda)$ that number divided par $n!$, Hurwitz proved in [3] the following elegant formula (write $\lambda = (d_1, ..., d_p)$)

$$\frac{h_n^0(\lambda)}{(n+p-2)!} = \frac{1}{\#\mathrm{Aut}\lambda}\left(\prod_{i=1}^{p}\frac{d_i^{d_i}}{d_i!}\right)n^{p-3}. \tag{1}$$

A fruitful generalisation of the original wonderings of Hurwitz is to look for such factorisation numbers $h_n^g(\lambda_1, ..., \lambda_k)$ with prescribed genus g (not only 0) and by remplacing the single permutation σ by a product of an arbitrary number of permutations of given types $\lambda_1, ..., \lambda_k$.

It is remarkable that three factors always appear in all known formulas of Hurwitz numbers. The first is the decremented genus

$$g' := g - 1. \tag{2}$$

The second can be seen on the denominator of the fraction above on the left: if one sets $r(\lambda) := \sum(d_i - 1)$ for any partition $\lambda = (d_1, ..., d_p)$, this factor is equal to the factorial of the number

$$T_n := 2n + 2g' - r(\lambda_1) - r(\lambda_2) - \cdots - r(\lambda_k). \tag{3}$$

We will consequently encode Hurwitz numbers by the generating series

$$H^g(\lambda_1, ..., \lambda_k) := \sum_{n\geq 1}\frac{h_n^g(\lambda_1, ..., \lambda_k)}{T_n!}t^n. \tag{4}$$

The third factor depends only on the partitions λ_i: if one sets $\boxed{\lambda} := \frac{1}{\#\mathrm{Aut}\lambda}\prod_{i=1}^{p}\frac{d_i^{d_i}}{d_i!}$, then this third factor equals the product $\boxed{\lambda_1}\boxed{\lambda_2}\cdots\boxed{\lambda_k}$. It will therefore be convenient to normalize Hurwitz numbers and series by this third factor.

Explicit formulas are known in spherical and toric genus ($g = 0$ or $g = 1$) for one-partition Hurwitz numbers $h_n^0(\lambda)$ and $h_n^1(\lambda)$, thanks to the Ekedahl-Lando-Shapiro-Vainshtein formula (see [2]). Moreover, there exist some algorithms to compute $h_n^g(\lambda)$ with g-free complexity $O(C^n)$ (for some constant $C > 0$) as well as explicit formulas to express the series $H^g(\lambda_1, ..., \lambda_k)$ as polynomials in the series $H^g(\lambda)$ – both are described in [4]. However, except for these algorithms, computing Hurwitz numbers $h_n^g(\lambda_1, ..., \lambda_k)$ remains to our knowledge an open problem.

Nevertheless, the asymptotics of all Hurwitz numbers $h_n^g(\lambda_1, ..., \lambda_k)$ when n becomes infinitely large is now completely understood.

The central tool is the algebra $\mathcal{A} := \mathbb{Q}[Y, Z]$ spanned by the exponential generating functions of one- and two-rooted Cayley trees $Y := \sum_{n\geq 1}\frac{n^{n-1}}{n!}t^n$ and $Z := \sum_{n\geq 1}\frac{n^n}{n!}t^n$. This algebra was introduced by Zvonkine in [8] (and, to his knowledge, earlier by D. Zagier).

The algebra $\mathcal{A}$ is very convenient to study asymptotics. Indeed, the linear freedom of Y and Z and the linearisation $YZ = Z - Y$ allows one to assign to every series in $\mathcal{A} = \mathbb{Q}[Y] + \mathbb{Q}[Z]$ two polynomials up to their constant coefficients; once these polynomials are known, the asymptotics of the coefficient in t^n of the given series is straightforward.

On the other hand, Zvonkine proved in [7] that all series $H^g(\lambda_1, ..., \lambda_k)$ but $H^1(\emptyset)$ lay in the algebra $\mathcal{A}$ by induction on the number k of partitions. The induction relies on two items: an unexplicited induction formula to decrease k, and – in the more recent proof [Zvon06] – a theorem of Kazarian (see also [4]) which gives an explicit formula for one-partition series $H^g(\lambda)$ as a polynomial in Y and Z.

The main idea of our paper is to make explicit the induction formula of Zvonkine, to combine it with the theorem of Kazarian, and to use the asymptotics proprieties of the algebra $\mathcal{A}$ to derive the asymptotics of the coefficient $h_n^g(\lambda_1, ..., \lambda_k)$ in the series $H^g(\lambda_1, ..., \lambda_k)$. Our result states that the asymptotic for several partitions $\lambda_1, ..., \lambda_k$ is the same as the asymptotics for the concatenation $\lambda_1 \sqcup \cdots \sqcup \lambda_k$: for any genus $g \geq 0$, one has the following asymptotics for some constant c_g:

$$\frac{h_n^g(\lambda_1, ..., \lambda_k)}{T_n!\,\boxed{\lambda_1}\boxed{\lambda_2}\cdots\boxed{\lambda_k}} \overset{n\to\infty}{\sim} c_g\frac{e^n}{n}n^{\frac{5}{2}g'}n^{l(\lambda_1)+\cdots+l(\lambda_k)}. \tag{5}$$

The constants c_g do not appear in the induction formula and come only from the theorem of Kazarian. In genera 0 and 1, the explicit formulas for one-partition Hurwitz numbers can be seen through their generating series, whence the values

$$c_0 = \frac{1}{\sqrt{2\pi}} \qquad \text{and} \qquad c_1 = \frac{1}{48}. \tag{6}$$

In higher genera, the theorem of Kazarian links the constants $c_{g\geq 2}$ with some integrals on the Deligne-Mumford compactification $\overline{\mathcal{M}}_n^g$ of the moduli space $\mathcal{M}_n^g$ of g-genused n-marked curves. A theorem of Kontsevitch (formerly a conjecture of Witten) yields recursion formulas to compute some integrals on $\overline{\mathcal{M}}_n^g$ and can be used to show that the rational numbers $\alpha_{g'} := c_g 2^{\frac{5g-1}{2}} \Gamma\left(\frac{5g-1}{2}\right)$ satisfy the recursion formula

$$\alpha_{-1} = -1 \text{ and } \alpha_g = \frac{25g^2-1}{12} + \sum_{p+q=g'}^{p,q\geq 0} \alpha_p \alpha_q. \tag{7}$$

From this recursion, it is easy to derive the equality

$$c_g = \sqrt{2}^{g-3} t_g \tag{8}$$

where t_g is the Bender-Gao-Richmond universal constant appearing in map asymptotics (see [1]).

Another way to describe the above recursion is to say that the series

$$u(t) := \sum_{g\geq 0} c_g \frac{\Gamma\left(\frac{5g-1}{2}\right)}{t^{\frac{5g-1}{2}}} = \sum_{g\geq 0} \frac{\alpha_{g'}}{(2t)^{\frac{5g-1}{2}}} \tag{9}$$

satisfies the Painlevé I equation

$$u(t)^2 + \frac{1}{6}\frac{d^2 u(t)}{dt^2} = 2t. \tag{10}$$

Therefore, until someone clears out the combinatorial meaning of the above recursion (which amounts to yielding a more profound understanding on the Painlevé I equation), we do not believe that much else can be said about the constants c_g.

This paper is about clearing the asymptotic of all Hurwitz numbers. Our new results are the general asymptotics (5) of Hurwitz numbers $h_n^g(\lambda_1, ..., \lambda_k)$ (Corollary 1) and the relation (8) between the asymptotics constants c_g and t_g (Theorem 7). However, we do not believe our explicited induction formula (Theorem 8) is computationally efficient (the simple case $H^0((a),(3))$, not included in this paper, is already very cumbersome).

Plan of the paper.

We first recall in Section 2 some properties of the algebra $\mathcal{A} := \mathbb{Q}[Y, Z]$ of formal power series in the indeterminate variable t introduced by Zvonkine in [8]. The asymptotics of the coefficient in t^n of a series lying in $\mathcal{A}$ only requires to know the leading coefficient in Z.

We then define in Section 3 Hurwitz numbers $h_n^g(\lambda_1, ..., \lambda_k)$ and their corresponding generating function $H^g(\lambda_1, ..., \lambda_k)$ for any degree $n \geq 0$, genus $g \geq 0$ and partitions $\lambda_1, ..., \lambda_k$, as well as convenient renormalisations $\mathbb{h}_n^g(\lambda_1, ..., \lambda_k)$ and $\mathbb{H}_n^g(\lambda_1, ..., \lambda_k)$. We are then ready in Section 3.3 to state our main theorem on the asymptotics of all Hurwitz numbers (Theorem 1 and Corollary 1). We recall in Section 3.4 explicit formulas in spherical and toric genera for the numbers $h_n^g(\lambda)$ which come from the ELSV formula (see [2]) and then carry on with the asymptotics by expliciting the corresponding series $H^g(\lambda)$.

So as to be able, in Section 4.3, to prove and use the theorem of Kazarian (see [4] and [Zvon06]), which gives for genera $g \geq 2$ an explicit formula for one-partition series $H^g(\lambda)$ as a polynomial in Y and Z, we recall in Section 4.1 and Section 4.2 some definitions and facts about the Deligne-Mumford compactification $\overline{\mathcal{M}}_n^g$ of the moduli space $\mathcal{M}_n^g$ of g-genused n-marked curves. In Section 4.4, we derive as an immediate corollary of Kazarian formulas the asymptotics of one-partition Hurwitz numbers in any genus.

The latter asymptotics involves some constants c_g, only depending on the genus g, which can be encoded in a function satisfying the Painlevé I equation (see Section 4.5), a fact already proved in [9] in the "physical" part. One can show the same behaviour for the universal constants t_g defined in [1] appearing in the asymptotics of map enumeration (see Section 4.6), a fact that boils down to the equality $c_g = \sqrt{2}^{g-3} t_g$ (Theorem 7).

Zvonkine proved in [7] that, with the only exception of empty partitions in genus 1, all series $H^g(\lambda_1, ..., \lambda_k)$ lay in the algebra $\mathcal{A}$ by induction on the number k of partitions, the case $k = 1$ being an immediate corollary from the theorem of Kazarian. In Section 5, which contains the most technical part of our paper, we explicit the (unexplicited) induction formula used by Zvonkine so as to control the leading coefficients in Z of the series $H^g(\lambda_1, ..., \lambda_k)$ and derive the asymptotics of the numbers $h_n^g(\lambda_1, ..., \lambda_k)$.

Notations.

For sake of conciseness, we will use throughout the paper the genus-notation

$$g' := g - 1. \tag{11}$$

We will also use the de/increasing power notations

$$\begin{cases} a^{\uparrow k} = a(a+1)(a+2)\cdots(a+k-1) \\ a^{\downarrow k} = a(a-1)(a-2)\cdots(a-k+1) \end{cases} \quad \text{where both have comprise } k \text{ factors.} \tag{12}$$

One has the identity $a^{\uparrow(p+q)} = a^{\uparrow p} (a+p)^{\uparrow q}$ for any integers $p, q \geq 0$. So that the latter remains valid for negative integers, one has to define

$$a^{\uparrow(-k)} := \frac{1}{(a-k)^{\uparrow k}} \text{ for any integer } k \geq 0. \tag{13}$$

The (***mean***) ***elementary symmetric functions*** will be written with the letter e:

$$e_k (x_1, ..., x_n) \quad := \quad \sum_{k_1 < k_2 < \cdots < k_n} x_{i_1} x_{i_2} \cdots x_{i_n} \tag{14}$$

$$\widetilde{e}_k (x_1, ..., x_n) \quad := \quad \frac{1}{\binom{n}{k}} e_k (x_1, ..., x_n). \tag{15}$$

We define a ***symmetry*** (or ***automorphism***) of a family $(a_i)_{i\in I}$ as a permutation $\sigma \in \mathfrak{S}_I$ of the set I leaving the family $(a_i) = (a_{\sigma(i)})$ invariant. Their set will be denoted $\mathrm{Sym}(a_i)$ and is clearly in bijection with the product of the symmetric groups of the multiplicities of the values of the family:

$$\mathrm{Sym}(a_i) \cong \prod_{\alpha \in \{a_i\}_{i\in I}} \mathfrak{S}_{\#\{i\in I \ ; \ a_i = \alpha\}}. \tag{16}$$

2 The algebra $\mathbb{Q}\left[\sum_{n\geq1} \frac{n^{n-1}}{n!} t^n, \sum_{n\geq1} \frac{n^n}{n!} t^n\right]$ used to study asymptotics

Details for the following claims can be found in [8] and [Zvon06].

Let us define an algebra $\mathcal{A} := \mathbb{Q}[Y, Z] \subset \mathbb{Q}[[t]]$ where

$$Y := \sum_{n\geq1} \frac{n^{n-1}}{n!} t^n \text{ and } Z := \sum_{n\geq1} \frac{n^n}{n!} t^n. \tag{17}$$

Cayley trees are enumerated by the exponenial generating function $\sum \frac{n^{n-2}}{n!} t^n$, so that Y and Z enumerate Cayley trees with one or two marked vertices. From that description of Y, one can derive the equality

$$Y = te^Y \tag{18}$$

(erasing the root of a rooted tree yields a forest of rooted trees). Notice that leading coefficients in Z are obtained from those in Y by mulplication by n, *i.e.*, by applying the differential operator

$$D := t\frac{\partial}{\partial t} : (a_n) \mapsto (na_n), \text{ so that } Z = DY. \tag{19}$$

One can therefore linearise the product

$$YZ = Z - Y, \ i.e., \ (1-Y)(1+Z) = 1,$$

whence many identites

$$\begin{array}{c} DZ = Z(1+Z)^2 \\ \forall P \in \mathbb{Q}[X], \ DP(Z) = Z(1+Z)^2 P'(Z) \\ \forall k \in \mathbb{N}, \ Y^k Z = Z - Y - Y^2 - Y^3 - \cdots - Y^k \\ \forall k \in \mathbb{N}, \ YZ^k = Z^k - Z^{k-1} + Z^{k-2} - \cdots + (-1)^{k-1} Z + (-1)^k Y \end{array}. \tag{20}$$

This linearisation allows one to dispose of "cross-terms" in a series lying in $\mathcal{A}$; in others words, one has the description

$$\mathcal{A} = \mathbb{Q}[Y] + \mathbb{Q}[Z]. \tag{21}$$

As an immediate consequence, one can see by setting $\{X, X^{-1}\} := \{1+Z, 1-Y\}$ that $\mathcal{A}$ is formally isomorphic to an algebra $\mathbb{Q}[X, X^{-1}]$, but we won't make use of that description.

The linear freedom of Y and Z allows one to assign to every series in $\mathcal{A}$ a Z-polynomial (*i.e.*, an element of $\mathbb{Q}[Z]$) up to the constant coefficient. Then, a series lies in

$$\mathcal{A}^Z := \mathcal{A} \backslash \mathbb{Q}[Y] \tag{22}$$

if and only if its corresponding Z-polynomial is non-constant.

The above description shows that, in order to carry out calculations in $\mathcal{A}$, one only has to describe powers of Y and Z. When dealing with the coefficient in t^n, it will be more adequate to study the action of the operator D.

For sake of convenience, we will use a pseudo-inverse of the operator D, defined by

$$D^{-1}: \sum_{n\geq 0} a_n t^n \mapsto 0 + \sum_{n\geq 1} \frac{a_n}{n} t^n. \tag{23}$$

(Recall that an element i is a ***pseudo-inverse*** of an element a in a monoid if $iai = i$. Obviously, an inverse is a pseudo-inverse.)

Claim 1 (powers of Y**).**

1. *Powers of* Y *are given by the formula (for any* $k \geq 1$*)*

$$\frac{Y^k}{k} = \sum_{n\geq 1} \frac{n^{\downarrow k}}{n^{k+1}} \frac{n^n}{n!} t^n = \sum_{n\geq k} \frac{n^{n-k-1}}{(n-k)!} t^n. \tag{24}$$

2. *The algebra* $\mathbb{Q}[Y]$ *is stable by* D^{-1}*: one has for any* $a \geq 1$

$$D^{-1}\frac{Y^a}{a} = \sum_{n\geq a} \frac{n^{\downarrow a}}{n^{a+2}} \frac{n^n}{n!} t^n = \frac{1}{a}\left(\frac{Y^a}{a} - \frac{Y^{a+1}}{a+1}\right). \tag{25}$$

3. *The algebra spanned by* Y *is that of* D^{-1} *applied on* Y*:*

$$\mathbb{Q}[Y] = \mathbb{Q}\left[D^{-1}\right](Y). \tag{26}$$

To study powers of Z, let us first describe its square $Z^2 = \sum \frac{A_n}{n!} t^n$ where we set

$$\frac{A_n}{n!} := \sum_{p+q=n}^{p,q\geq 1} \frac{p^p q^q}{p! q!} = \sum_{i=0}^{n-2} \frac{n^i}{i!} \overset{n\to\infty}{\sim} \frac{e^n}{2}. \tag{27}$$

The above equivalent is a standard exercise, the second equality amounts to the following identity

$$Y + \ln(1-Y) = -D^{-1}Z^2 = -\sum_{n\geq 2} \frac{Y^n}{n}. \tag{28}$$

Claim 2 (powers of Z**).** *Let* $k \geq 0$ *an integer.*

1. *The series* $D^k Z$ *and* $D^k Z^2$ *are polynomials in* Z *whose coefficients are non-negative integers. Their degrees and leading coefficients are given as follows:*

$$\begin{array}{rclcl} D^k Z & = & \sum_{n\geq 1} \frac{n^{n+k}}{n!} t^n & \in & (2k-1)!!\ Z^{2k+1} + \mathbb{N}_{2k}[Z], \\ D^k Z^2 & = & \sum_{n\geq 1} \frac{n^k A_n}{n!} t^n & \in & (2k)!!\ Z^{2k+2} + \mathbb{N}_{2k+1}[Z]. \end{array} \tag{29}$$

2. *The series* Z^k *is a linear combinaison of the* k *firsts terms of the list* $(Z, Z^2, DZ, DZ^2, D^2Z, D^2Z^2, ...)$, *with weight* $\frac{1}{(k-2)!!}$ *for the* k*-th term (and weight* 1 *for* $k = 1, 2$*).*
3. *The subalgebra spanned by* Z *is described by*

$$\mathbb{Q}[Z] = \mathbb{Q}[D]\ (Z) + \mathbb{Q}[D]\ (Z^2). \tag{30}$$

From the previous claims, one can give the following description of the algebra $\mathcal{A}$ which proves it to be stable by D:

$$\mathcal{A} = \mathbb{Q}\left[D, D^{-1}\right](Z) + \mathbb{Q}[D]\ (Z^2). \tag{31}$$

In other words, each series in $\mathcal{A}$ equals $const + \sum_{n\geq 1} \frac{L(n)n^n + P(n)A_n}{n!} t^n$ for some Laurent polynomial L and polynomial P.

Claim 3 (asymptotics in $\mathcal{A}$**).** *For any integer* $i \geq 1$*, the sequence of the coefficient in* t^n *of the sequence of series* $\left(\frac{Y^i}{i}, Z, Z^2, Z^3, Z^4, ...\right)$ *form a comparison scale following the powers* ≥ -1 *of* $\sqrt{n}$*, up to a factor* $\frac{e^n}{n}$*:*

$$\forall i, k \geq 1, \quad \begin{array}{l} \left(\text{coefficient of } t^n \text{ in } \frac{Y^i}{i}\right) \overset{n\to\infty}{\sim} C_{-1} \frac{e^n}{n} \sqrt{n}^{-1} \\ \left(\text{coefficient of } t^n \text{ in } Z^k\right) \overset{n\to\infty}{\sim} C_k \frac{e^n}{n} \sqrt{n}^k \end{array} \quad \text{where} \quad \begin{array}{l} \frac{1}{C_{-1}} = \sqrt{2\pi} \\ \frac{1}{C_k} = \Gamma\left(\frac{k}{2}\right) 2^{\frac{k}{2}} \end{array}. \tag{32}$$

Remarks. Notices that there are no i's appearing in the asymptotics of the series $\frac{Y^i}{i}$. The formula for C_k actually holds for $k=-1$ if one remplaces Γ by $|\Gamma|$. For $k=1$, the formula merely states Stirling's equivalent of $n!$. For $k=0$, one will remind $\lim_0 \Gamma = \infty$.

The previous claim shows that two series in $\mathcal{A}^Z$ have the same asymptotics if and only if their Z-leading terms are equal. For sake of convenience, we will introduce the following notation.

Definition 1 (Z-equality and notation $\overset{Z}{=}$). *Two series in $\mathcal{A}^Z = \mathcal{A}\backslash\mathbb{Q}[Y]$ will be called **Z-equal** if their Z-leading terms are equal. Such an equivalence will be denoted as*

$$S \overset{Z}{=} T \overset{def.}{\iff} S \text{ and } T \text{ have same } Z\text{-leading terms.} \tag{33}$$

For instance, one can write

$$\forall k \geq 0,\ D^k Z \overset{Z}{=} (2k-1)!!\ Z^{2k+1} \tag{34}$$
$$\forall k \geq 0,\ D^k Z^2 \overset{Z}{=} (2k)!! Z^{2k+2} \tag{35}$$
$$\forall p, q \geq 1,\ Z^p Y^q \overset{Z}{=} Z^p \tag{36}$$
$$\forall (P, q) \in \mathbb{Q}[X] \times \mathbb{N}^*,\ P(Z) Y^q \overset{Z}{=} P(Z) \tag{37}$$
$$\forall P, Q \in \mathbb{Q}[X],\ P(Z) Q(Y) \overset{Z}{=} P(Z) \iff Q(1) \neq 0. \tag{38}$$

3 Hurwitz numbers

3.1 Reminders on partitions

Recall that a ***partition*** is any finite non-increasing sequence λ of positive integers. The integers appearing in the sequence are the ***parts*** of the partition. Its ***length*** $l(\lambda)$ is the number of parts, its ***size*** $|\lambda|$ is the sum of the parts. The ***multiplicity*** $m_k(\lambda)$ of any integer k is the number of parts in λ equal to k, the ***ramification*** $r(\lambda)$ is $|\lambda| - l(\lambda)$ (for a topological interpretation, see [5]), its ***symmetries*** are the permutation of $\mathfrak{S}_{l(\lambda)}$ leaving the sequence λ unchanged after acting on the indexes $1, ..., l(\lambda)$. A partition is called ***reduced*** if it contains no ones, *i.e.*, if $m_1 = 0$. The ***reduction*** $\mathring{\lambda}$ of a partition λ is the partition obtained by removing all ones from λ. When an integer $n \geq |\lambda|$ is contextly given, we defined the ***completion*** $\overline{\lambda}$ of λ by the n-sized partition obtained from λ by additing as many ones as necessary. The ***concatenation*** $\lambda \sqcup \mu$ of two partitions λ and μ is the partition whose parts are those of λ union those of μ. The length, size and ramification are morphisms from the concatenation to the addition and can therefore be extended to a tuple of partitions by concatenating the latter.

Let us summarise all notations used for a partition $\lambda = (d_1 \geq d_2 \geq \cdots \geq d_p) = 1^{m_1} 2^{m_2} \cdots$:

length	$l(\lambda) := \sum_{k\geq 1} m_k$ (also denoted above by p)
size	$\lvert\lambda\rvert := \sum d_i = \sum_{k\geq 1} k m_k$ (also denoted above by a)
ramification	$r(\lambda) := \sum (d_i - 1) = \lvert\lambda\rvert - l(\lambda)$
number of symmetries	$\#\mathrm{Sym}\lambda := \prod_{k\geq 1} m_k!$
reduction	$\mathring{\lambda} := \lambda \backslash 1^{m_1}$
(n-)completion	$\overline{\lambda} := \lambda \sqcup 1^{n-m_1}$
concatenation	$\lambda \sqcup \mu$

. (39)

Let $n \geq 1$ an integer and σ a permutation in $\mathfrak{S}_n$. Recall that its ***support*** is the complement $S\sigma = \mathrm{Supp}\sigma$ in $[1, n]$ of all σ-fixed points and its ***type*** is the partition $\mathrm{type}(\sigma)$ whose parts are the lengths of the cycles of σ (including fixed cycles). For instance, the type of the disjoint product of two permutations is the concatenation of their types and the cardinality of the support of a permutation is equal to the size of the reduction of its type.

Let us recall that conjugacy classes in $\mathfrak{S}_n$ are indexed by partitions of size n.

3.2 Constellations and Hurwitz numbers

Set n and k positive integers. Define a k-***constellation*** of degree n to be a k-tuple $\overrightarrow{\sigma} = (\sigma_1, ..., \sigma_k) \in \mathfrak{S}_n^k$ such that $\sigma_1 \cdots \sigma_k = \mathrm{Id}$ and that the subgroup $\langle \sigma_1, ..., \sigma_k \rangle$ acts transitively on $[1, n]$. The ***type*** of a k-constellation $\overrightarrow{\sigma}$ is the k-tuple of the types of the σ_i. Its ***ramification*** is the sum of those of the σ_i's. Its ***genus*** g is defined by the ***Riemann-Hurwitz formula***:

$$r = 2n + 2g' \qquad \text{or} \qquad g := \frac{r}{2} - (n-1)\,. \tag{40}$$

Although the genus has a topological interpretation that we won't need, its existence can proved by showing r is even, which comes from

$$1 = \varepsilon(\mathrm{Id}) = \varepsilon\left(\prod \sigma_i\right) = \prod (-1)^{r(\sigma_i)} = (-1)^r\,. \tag{41}$$

The term ***constellation*** has a topological interpretation (lifting up a star graph *via* a ramified covering yields some interlaced stars, *i.e.*, a constellation) that can be found in [5]. One could instead speak of ***transitive factorisations***. When translated in topological terms, the transitivity amounts to a connectedness condition.

With the above langage, the original question raised by Hurwitz in [3] was to enumerate the constellations of degree n, genus 0, and type $(\lambda, 2, 2, ..., 2)$. Their number involves a factor $\frac{1}{\#\mathrm{Sym}\lambda}\prod_{i=1}^{p}\frac{d_i^{d_i}}{d_i!}$ which happens to be rampant throughout formulas on Hurwitz numbers $h_n^g\left(\overrightarrow{\lambda}\right)$ we are about to define. It is therefore more than reasonable to introduce the following notations:

$$\boxed{\lambda} := \frac{1}{\#\mathrm{Sym}\lambda}\frac{d_1^{d_1}\cdots d_p^{d_p}}{d_1!\cdots d_p!} \qquad \boxed{\overrightarrow{\lambda}} = \boxed{\lambda_1}\boxed{\lambda_2}\cdots\boxed{\lambda_k} \qquad \boxed{h}_n^g\left(\overrightarrow{\lambda}\right) = \frac{h_n^g\left(\overrightarrow{\lambda}\right)}{\boxed{\overrightarrow{\lambda}}}. \tag{42}$$

A fruitful generalisation of the original wonderings of Hurwitz is to enumerate constellations with a prescribed genus (not only 0) and with prescribed types of an arbitrary number of the first factors (not only the very fisrt one).

Definition 2 (the Hurwitz numbers $h_n^g\left(\overrightarrow{\lambda}\right)$ and the Hurwitz series $H^g\left(\overrightarrow{\lambda}\right)$). *Set g and n two non-negative integers and set partitions λ_1, ..., λ_k of non-negative integers.*

Define $T = T_n^g(\lambda_1, ..., \lambda_k)$ by the Riemann-Hurwitz formula:

$$2n + 2g' = T + \sum r(\lambda_i)\,, \text{ namely } \quad T := 2n + 2g' - r. \tag{43}$$

Define $h_n^g(\lambda_1, ..., \lambda_k)$ by $\frac{1}{n!}$ times the number of ordered pairs(C, F) where C is a constellation $(\overrightarrow{\sigma}, \overrightarrow{\tau}) \in \mathfrak{S}_n^k \times \mathfrak{S}_n^T$ of type $\begin{cases} \forall i, \ \mathrm{type}(\sigma_i) = \overline{\lambda_i} \\ \forall j, \ \mathrm{type}(\tau_j) = \overline{2} \end{cases}$ (in particular, all τ_i's are transpositions) *and where F is a k-tuple of parts in $[1, n]$ such that* $\begin{cases} \forall i, & F_i \subset \mathrm{Fix}\sigma_i \\ & |F_i| = m_1(\lambda_i) \end{cases}$ (choosing F_i amounts to choosing a subset of $[1, n]$ where σ_i can be seen as a permutation of type λ_i).

Define Hurwitz series by the following generating functions:

$$H^g\left(\overrightarrow{\lambda}\right) := \sum_{n\geq 1}\frac{h_n^g\left(\overrightarrow{\lambda}\right)}{T_n!}t^n, \tag{44}$$

$$\boxed{H}^g\left(\overrightarrow{\lambda}\right) := \sum_{n\geq 1}\frac{\boxed{h}_n^g\left(\overrightarrow{\lambda}\right)}{T_n!}t^n = \frac{H^g\left(\overrightarrow{\lambda}\right)}{\boxed{\overrightarrow{\lambda}}}, \tag{45}$$

$$H^g\left(\overline{\lambda_1}, ..., \overline{\lambda_k}\right) := \sum_{n\geq 1}\frac{h_n^g\left(\overline{\lambda_1}, ..., \overline{\lambda_k}\right)}{T_n!}t^n. \tag{46}$$

Some remarks.

By choosing first the constellation then the fixed parts, one has

$$h_n^g\left(\overrightarrow{\lambda}\right) := \mathrm{Card}\left\{\text{constellations} \begin{pmatrix}\overrightarrow{\sigma} \\ \overrightarrow{\tau}\end{pmatrix} \in \mathfrak{S}_n^k \times \mathfrak{S}_n^T\right\}_{\forall j,\ \mathrm{type}(\tau_j) = \overline{2}}^{\forall i,\ \mathrm{type}(\sigma_i) = \overline{\lambda_i}} \tag{47}$$

$$\times\frac{1}{n!}\prod_{i=1}^{k}\binom{n - (|\lambda_i| - m_1(\lambda_i))}{m_1(\lambda_i)}, \text{ namely}$$

$$h_n^g(\lambda_1, ..., \lambda_k) = h_n^g\left(\overline{\lambda_1}, ..., \overline{\lambda_k}\right) \times \prod_{i=1}^{k}\binom{n - (|\lambda_i| - m_1(\lambda_i))}{m_1(\lambda_i)}. \tag{48}$$

When seen through Hurwitz series, the latter identity becomes

$$H^g(\lambda_1,...,\lambda_k)=\left[\prod_{i=1}^{k}\binom{D-(|\lambda_i|-m_1(\lambda_i))}{m_1(\lambda_i)}\right]H^g\left(\overline{\lambda_1},...,\overline{\lambda_k}\right). \tag{49}$$

When adding a partition which contains either nothing, either some ones, or one two, one can easily check the following:

$$\begin{cases} h_n^g\left(\overrightarrow{\lambda},\emptyset\right)=h_n^g\left(\overrightarrow{\lambda}\right) \\ h_n^g\left(\overrightarrow{\lambda},(1^p)\right)=\binom{n}{p}h_n^g\left(\overrightarrow{\lambda}\right) \\ h_n^g\left(\overrightarrow{\lambda},(2)\right)=h_n^g\left(\overrightarrow{\lambda}\right)\end{cases} \text{ and } \begin{cases} H^g\left(\overrightarrow{\lambda},\emptyset\right)=H^g\left(\overrightarrow{\lambda}\right) \\ H^g\left(\overrightarrow{\lambda},(1^p)\right)=\binom{D}{p}H^g\left(\overrightarrow{\lambda}\right) \\ H^g\left(\overrightarrow{\lambda},(2)\right)=\left(2D+2g'-r\left(\overrightarrow{\lambda}\right)\right)H^g\left(\overrightarrow{\lambda}\right) \\ \mathbb{H}^g\left(\overrightarrow{\lambda},(2)\right)=\left(D+g'-\frac{r\left(\overrightarrow{\lambda}\right)}{2}\right)\mathbb{H}^g\left(\overrightarrow{\lambda}\right)\end{cases}. \tag{50}$$

As the series H^g suggest, we always consider the number h_n^g divided by $T_n!$. Nonetheless, if ever needed, one can derive the asymptotics of $T_n!$ by Stirling (in the following, α, β and γ are constants with $\gamma>1$):

$$T_n^g\left(\overrightarrow{\lambda}\right)!\overset{n\to\infty}{\sim}\sqrt{\pi}2^{2g'-r+1}\left(\frac{4}{e^2}\right)^n n^{2g'-r+\frac{1}{2}}n^{2n}=\alpha\frac{n^\beta}{\gamma^n}n^{2n}. \tag{51}$$

3.3 The main theorem and the general asymptotics of Hurwitz numbers

We can now state the main theorem of this paper, which reduces the understanding of the asymptotics of several-partition Hurwitz numbers to that of single-partition Hurwitz numbers. Since the latter is a straightforward corollary of Corollary 3 (see Section 4.3), we will be able to derive the following corollary (see Section 4.4).

Theorem 1. *For any partitions $\lambda_1,...,\lambda_k$ and any genus $g\geq 0$, one has the following Z-equality in the algebra $\mathcal{A}^Z$:*

$$D^3\mathbb{H}^g(\lambda_1,...,\lambda_k)\overset{Z}{=}D^{3+m_1(\lambda_i)+\cdots+m_1(\lambda_k)}\mathbb{H}^g\left(\mathring{\lambda}_1\sqcup\mathring{\lambda}_2\sqcup\cdots\sqcup\mathring{\lambda}_k\right). \tag{52}$$

Corollary 1 (general asymptotics of Hurwitz numbers). *For any partitions $\lambda_1,...,\lambda_k$ and any genus $g\geq 0$, one has the following asymptotics for some constant c_g:*

$$\frac{\mathbb{h}_n^g(\lambda_1,...,\lambda_k)}{T_n!}\overset{n\to\infty}{\sim}c_g\frac{e^n}{n}n^{\frac{5}{2}g'}n^{l(\lambda_1)+\cdots+l(\lambda_k)}. \tag{53}$$

More precisely, $c_0=\frac{1}{\sqrt{2\pi}}$, $c_1=\frac{1}{48}$ and $c_{g\geq 2}=\frac{\left\langle\tau_2^{3g'}\right\rangle}{(3g')!}\frac{1}{\Gamma\left(\frac{5}{2}g'\right)2^{\frac{5}{2}g'}}$ for some rationnal numbers $\left\langle\tau_2^{3g'}\right\rangle$ defined in Section 4.2.

It is shown in Section 4.5 that the constants c_g can be encoded in a function satisfying the Painlevé I equation (a fact already known in [9]), which enables one to compute them recursively. More precisely, *if one sets*

$$\forall g\geq 0,\alpha_{g'}:=c_g\Gamma\left(\frac{5g-1}{2}\right)2^{\frac{5g-1}{2}}, \tag{54}$$

then one has $\alpha_{-1}=-1$ and

$$\forall g\geq 0,\ \alpha_g=\frac{25g^2-1}{12}\alpha_{g'}+\frac{1}{2}\sum_{p+q=g'}^{p,q\geq 0}\alpha_p\alpha_q.$$

Sanity-checks.

When $k=1$, Theorem 5 yields the above asymptotic.

When adding a empty partition $\emptyset$, the relation $H^g\left(\overrightarrow{\lambda},\emptyset\right)=H^g\left(\overrightarrow{\lambda}\right)$ show that the asymptotics is unchanged, which is consistant with the above equivalent (the empty partition has length 0).

When adding a partition (1), the series $\mathbb{H}^g$ is multiplied by the operator D, hence the asymptotics is multiplied by $n=n^{l((1))}$.

When adding a partition (2), the series $\mathbb{H}^g$ is multiplied by the operator $D+g'-\frac{r\left(\overrightarrow{\lambda}\right)}{2}$, hence the asymptotics is multiplied by $n+g'-\frac{r\left(\overrightarrow{\lambda}\right)}{2}\overset{n\to\infty}{\sim}n^{l((2))}$.

3.4 One-partition Hurwitz numbers and series in genera 0 and 1

In spherical ($g = 0$) or toric ($g = 1$) genus, considerations from algebraic geometry (more specifically the ELSV formula, see [2] or [5] and Section 4 below) allow one to compute all one-partition Hurwitz numbers.

Claim 4 (Hurwitz numbers in genera 0 and 1).
For $n \geq 1$ and λ partition of an integer $a \leq n$, one has:

$$\frac{\mathbb{h}_n^0(\lambda)}{T_n!} = \frac{\mathbb{h}_n^0(\lambda)}{(2n-2-r)!} = \frac{n^{n-r-3}}{(n-a)!} = \frac{n^{\downarrow a}}{n^{r+3}}\frac{n^n}{n!} \overset{n\to\infty}{\sim} \frac{1}{\sqrt{2\pi}} e^n n^{p-\frac{7}{2}}. \tag{55}$$

For λ (non empty) partitionning a (positive) integer n, one has

$$\frac{\mathbb{h}_n^1(\lambda)}{T_n!} = \frac{\mathbb{h}_n^1(\lambda)}{(n+p)!} = \frac{1}{24}\left(n^p - n^{p-1} - \sum_{i=2}^{p}(i-2)!e_i(\lambda)\, n^{p-i}\right). \tag{56}$$

For λ empty and $n \geq 1$, one has

$$\frac{h_n^1(\emptyset)}{(2n)!} = \frac{\mathbb{h}_n^1(\emptyset)}{(2n)!} = \frac{1}{24}\frac{1}{n!}\frac{A_n}{n} \overset{n\to\infty}{\sim} \frac{1}{48}\frac{e^n}{n}. \tag{57}$$

When seen through the generating functions, these identities become

$$\mathbb{H}^0(\lambda) = \sum_{n\geq 1}\frac{n^{\downarrow a}}{n^{r+3}}\frac{n^n}{n!}t^n \tag{58}$$

$$\mathbb{H}^1(\lambda) = \frac{1}{24}\sum_{n\geq 1}\frac{n^{n-r}}{(n-a)!}t^n\left(1 - \frac{1}{n} - \sum_{i=2}^{n-r}(i-2)!\frac{e_i(\lambda)}{n^i}\right) \tag{59}$$

$$\mathbb{H}^1(\emptyset) = \frac{1}{24}\sum_{n\geq 1}\frac{A_n}{n}\frac{n^n}{n!}t^n. \tag{60}$$

We now carry on with the description of the series $H^0(\lambda)$ and $H^1(\lambda)$ as polynomials in Y and Z in the algebra $\mathcal{A}$. The following theorem is a reformulation of results already known by Kazarian in [4] but yet apparently unpublished – the reason why we produce all detailed computations.

Theorem 2 (Hurwitz series in genera 0 and 1). *Set a partition λ of an integer $a \geq 0$ in $p \geq 0$ parts. Then, one has the identities*

$$\mathbb{H}^0(\lambda) = D^{p-3}\left(Y^{a-1}Z\right) \qquad \textit{and} \tag{61}$$

$$24\mathbb{H}^1(\lambda) = D^{p-1}\left(Y^{a-1}Z^2\right) + (a-1)\,D^{p-1}\left(Y^{a-1}Z\right) - \sum_{x=2}^{p}(x-2)!e_x(\lambda)\,D^{p-x}\left(Y^{a-x}Z^x\right). \tag{62}$$

Examples.
In spherical genus, one has the identites

$$\mathbb{H}^0_{p=2} = \frac{Y^a}{a} \qquad \mathbb{H}^0_{p=1} = \frac{1}{a}\left(\frac{Y^a}{a} - \frac{Y^{a+1}}{a+1}\right) \qquad \mathbb{H}^0_{p=1}(1) = Y - \frac{Y^2}{2} \tag{63}$$

$$\mathbb{H}^0_{p=0} = Y - \frac{3}{2}\left(\frac{Y^2}{2}\right) + \frac{1}{2}\left(\frac{Y^3}{3}\right) = \left(Y - \frac{Y^2}{2}\right) - \frac{1}{2}\left(\frac{Y^2}{2} - \frac{Y^3}{3}\right) = D^{-1}\mathbb{H}^0_{p=1}(1). \tag{64}$$

In toric genus, one has the equalities

$$24\mathbb{H}^1(\emptyset) = D^{-1}Z^2 \qquad 24\mathbb{H}^1((1)) = Z^2 \qquad 24\mathbb{H}^1((2)) = Z^2 \tag{65}$$

and for any $d \geq 0$:

$$24\mathbb{H}^1((d+1)) = Y^dZ\,(Z+d) = Z^2 - Y^2 - 2Y^3 - 3Y^4 - \cdots - (d-1)\,Y^d. \tag{66}$$

The reader will notice that $\mathbb{H}^1(\emptyset)$ is the only series $\mathbb{H}^g(\lambda)$ for $g \in \{0,1\}$ not belonging to the algebra $\mathcal{A}$.

Corollary 2 (asymptotics of Hurwitz number in genera 0 **and** 1**).** *For any partition* λ *and any genus* $g \in \{0,1\}$*, one has the following asymptotics:*

$$\frac{\boxed{h}_n^g(\lambda)}{T_n!} \overset{n\to\infty}{\sim} c_g e^n n^{\frac{5}{2}g'+p-1} \text{ where } (c_0, c_1) = \left(\frac{1}{\sqrt{2\pi}}, \frac{1}{48}\right). \tag{67}$$

Proof of Corollary 2. The equivalent of $\frac{\boxed{h}_n^0(\lambda)}{T_n!} = \frac{n^{\downarrow a}}{n^{r+3}}\frac{n^n}{n!}$ is straightforward by Stirling formula but we prefer using the Z-techniques coming from the algebra $\mathcal{A}$ so as to get acquainted to them.

In null genus, one has for any partition λ the relation

$$\boxed{H}^0(\lambda) = D^{p-3}\left(Y^{a-1}Z\right) \overset{Z}{\equiv} D^{p-3}Z.$$

Its coefficient in t^n is therefore equivalent to $C_1\frac{e^n}{n}\sqrt{n}^1$ (Claim 3) multiplied by n^{p-3} (Claim 2):

$$\text{coefficient in } t^n \text{ of } \boxed{H}^0(\lambda) \overset{n\to\infty}{\sim} C_1\frac{e^n}{\sqrt{n}}n^{p-3} = \frac{1}{\sqrt{2\pi}}e^n n^{p-\frac{7}{2}}.$$

In toric genus, the first term $D^{p-1}\left(Y^{a-1}Z^2\right) \overset{Z}{\equiv} D^{p-1}Z^2$ has degree $2+2(p-1) = 2p$ whereas the following terms $D^{p-x}\left(Y^{a-x}Z^x\right) \overset{Z}{\equiv} D^{p-x}Z^x$ for $x \geq 1$ have Z-degrees $x+2(p-x) = 2p-x < 2p$. One has therefore $24\boxed{H}^1(\lambda) \overset{Z}{\equiv} D^{p-1}Z^2$, hence the asymptotics

$$\text{coefficient in } t^n \text{ of } \boxed{H}^1(\lambda) \overset{n\to\infty}{\sim} n^{p-1} \times C_2\frac{e^n}{n}\sqrt{n}^2 = \frac{1}{2}e^n n^{p-1}.$$

Proof of Theorem 2 for genus 0**.**

Using Formulas (55) from Claim 4, one can derive

$$\begin{aligned}\boxed{H}^0(\lambda) &= \sum_{n\geq 1}\frac{\boxed{h}_n^0(\lambda)}{T_n!}t^n = \sum_{n\geq a}\frac{n^{n-r-3}}{(n-a)!}t^n = \sum_{n\geq 1}\frac{n^{\downarrow a}}{n^{r+3}}\frac{n^n}{n!}t^n \\ &= D^{a-r-2}\sum_{n\geq 1}\frac{n^{\downarrow a}}{n^{a+1}}\frac{n^n}{n!}t^n = D^{p-2}\frac{Y^a}{a},\end{aligned}$$

hence the results for $p \geq 2$.

For $p = 1$, one can check that

$$\begin{aligned}\frac{Y^a}{a} - \frac{Y^{a+1}}{a+1} &= \sum_{n\geq a}\left(\frac{n^{\downarrow a}}{n^{a+1}} - \frac{n^{\downarrow a+1}}{n^{a+21}}\right)\frac{n^n}{n!}t^n = \sum_{n\geq a}\frac{n^{\downarrow a}}{n^{a+2}}(n-(n-a))\frac{n^n}{n!}t^n \\ &= a\sum_{n\geq a}\frac{n^{\downarrow a}}{n^{a+2}}\frac{n^n}{n!}t^n = a\boxed{H}^0(\lambda).\end{aligned}$$

At last, for $p = 0$, recall that $\boxed{H}^0(\emptyset) = \sum_{n\geq 1}\frac{n^{n-3}}{n!}t^n$. By expanding

$$\frac{Y^2}{2} = \sum_{n\geq 1}n(n-1)\frac{n^{n-3}}{n!}t^n = Y - \sum_{n\geq 1}\frac{n^{n-2}}{n!}t^n,$$

one derives $\sum_{n\geq 1}\frac{n^{n-2}}{n!}t^n = Y - \frac{Y^2}{2}$, then by expanding

$$\frac{Y^3}{3} = \sum_{n\geq 1}\underbrace{n(n-1)(n-2)}_{=n^3-3n^2+2n}\frac{n^{n-4}}{n!}t^n = Y - 3\left(Y - \frac{Y^2}{2}\right) + 2\sum_{n\geq 1}\frac{n^{n-3}}{n!}t^n,$$

one obtains the announced expression for $H^0(\emptyset) = \boxed{H}^0(\emptyset)$.

Proof of Theorem 2 for genus 1**.**

Noticing that the equality $\boxed{\lambda} m_1(\lambda)! = \boxed{\overline{\lambda}} m_1(\overline{\lambda})!$ can be rewritten

$$\frac{\boxed{\lambda}}{\boxed{\overline{\lambda}}} = \frac{(m_1(\lambda)+n-|\lambda|)!}{m_1(\lambda)!} = \binom{n-(|\lambda|-m_1(\lambda))}{m_1(\lambda)}(n-|\lambda|)!, \tag{68}$$

and using Formula (56) for $h_n^1\left(\overline{\lambda}\right)$, one can derive the equalities

$$\begin{aligned}
\frac{\boxed{h}_n^1(\lambda)}{T_n!} &= \binom{n-(a-m_1(\lambda))}{m_1(\lambda)} \frac{\boxed{h}_n^1\left(\overline{\lambda}\right)}{T_n!} = \frac{\boxed{\overline{\lambda}}}{\boxed{\lambda}} \binom{n-(a-m_1(\lambda))}{m_1(\lambda)} \frac{\boxed{h}_n^1\left(\overline{\lambda}\right)}{T_n!} \\
&= \frac{1}{24} \frac{n^{n-r}}{(n-a)!} \left(1 - \frac{1}{n} - \sum_{i=2}^{n-r} (i-2)! \frac{e_i\left(\overline{\lambda}\right)}{n^i}\right).
\end{aligned}$$

When $\lambda = \emptyset$, one can work out further the sum on the right (multiplied by $\frac{n^{n-r}}{(n-a)!} = \frac{n^n}{n!}$):

$$\begin{aligned}
\sum_{i=2}^{n} (i-2)! \binom{n}{i} \frac{n^{n-i}}{n!} &= \sum_{i=2}^{n} \frac{(i-2)!n!}{i!\,(n-i)!} \frac{n^{n-i}}{n!} \\
&= \sum_{i=2}^{n} \left(\frac{1}{i-1} - \frac{1}{i}\right) \frac{n^{n-i}}{(n-i)!} \\
&= \sum_{i=1}^{n-1} \frac{1}{i} \frac{n^{n-i-1}}{(n-i-1)!} - \sum_{i=2}^{n} \frac{1}{i} \frac{n^{n-i}}{(n-i)!} \\
&= \frac{n^{n-2}}{(n-2)!} - \frac{1}{n} + \sum_{i=2}^{n-1} \frac{n^{n-i-1}}{(n-i)!} \frac{1}{i} \left((n-i) - n\right) \\
&= \frac{n(n-1)\,n^{n-2}}{n!} - \frac{1}{n} \left(1 + \sum_{i=2}^{n-1} \frac{n^{n-i}}{(n-i)!}\right) \\
&= \frac{n^n}{n!} \left(1 - \frac{1}{n}\right) - \frac{1}{n} \sum_{i=0}^{n-2} \frac{n^i}{i!}.
\end{aligned}$$

Recalling that $\sum_{i=0}^{n-2} \frac{n^i}{i!} = \frac{A_n}{n!}$, one obtains the formula for $p = 0$.

The case $p > 0$ is far longer to carry out.
The generating series of the first two terms are easy to compute:

$$\begin{aligned}
\sum_{n \geq 1} \frac{n^{n-r}}{(n-a)!} t^n &= D^p \sum_{n \geq 1} \frac{n^{n-a}}{(n-a)!} t^n = D^p \left(Y^{a-1} Z\right) \text{ and} \\
\sum_{n \geq 1} \frac{n^{n-r}}{(n-a)!} \frac{1}{n} t^n &= D^p \sum_{n \geq 1} \frac{n^{n-a-1}}{(n-a)!} t^n = D^p \left(\frac{Y^a}{a}\right) = D^{p-1} \left(Y^{a-1} Z\right).
\end{aligned}$$

Let us then look at each term of the x-sum. Since $e_i\left(\overline{\lambda}\right) = \sum_{x=0}^{p} e_x \binom{n-a}{i-x}$ (some terms being of course likely to cancel out according to the position of i with respect to p and $n-a$), one has

$$\begin{aligned}
24 \frac{\boxed{h}_n^1(\lambda)}{T_n!} &= \frac{n^{n-r}}{(n-a)!} \left(1 - \frac{1}{n} - \sum_{x=0}^{p} e_x \sum_{\substack{i=2 \\ x \leq i \leq x+n-a}}^{n-r} \frac{(i-2)!}{n^i} \binom{n-a}{i-x}\right) \\
&= \frac{n^{n-r}}{n'!} \left(1 - \frac{1}{n} - \sum_{x=0}^{p} e_x \sum_{i \geq 2-x, 0}^{n\prime} \binom{n'}{i} \frac{(i+x')!}{n^{i+x}}\right) \quad \text{where} \quad \begin{array}{l} n' := n-a \\ x' = x-2 \end{array}.
\end{aligned}$$

Let us look at the terms for $x \geq 2$:

$$\frac{n^{n-r}}{n'!} \sum_{i=0}^{n'} \frac{n'!}{i!\,(n'-i)!} \frac{(i+x')!}{n^{i+x}} = n^{p-x} \sum_{i=0}^{n'} x'! \binom{i+x'}{i} \frac{n^{n'-i}}{(n'-i)!}.$$

Taking the series of this term in t^n yields $x'!D^{p-x}$ applied on

$$\begin{aligned}
\sum_{n\geq a}\sum_{i=0}^{n'}\binom{i+x'}{i}\frac{n^{n'-i}}{(n'-i)!}t^n &= \sum_{i\geq 0}\binom{i+x'}{i}\sum_{n\geq i+a}\frac{n^{n-(a+i)}}{(n-(a+i))!}t^n \\
&= \sum_{i\geq 0}\binom{i+x'}{i}Y^{a+i-1}Z \\
&= Y^{a-1}Z\sum_{i\geq 0}\binom{i+x'}{i}Y^i \\
&= Y^{a-1}Z\left(\frac{1}{1-Y}\right)^{x-1} \\
&= Y^{a-1}Z\,(1+Z)^{x-1} \\
&= Y^{a-x}Z^x.
\end{aligned}$$

For $x=1$, we carry out the same thing by replacing $x'!\binom{i+x'}{i}$ by $\frac{(i-1)!}{i!}=\frac{1}{i}$ and by starting the sum at $i=1$: the generating series becomes D^{p-1} applied on $Y^{a-1}Z\sum_{i\geq 1}\frac{Y^i}{i}$. Notice by the way that

$$D\left(\sum_{i\geq 1}\frac{Y^i}{i}\right)=\sum_{i\geq 0}Y^iZ=\frac{Z}{1-Y}=Z\,(1+Z)\,.$$

For $x=0$, idem by replacing $x'!\binom{i+x'}{i}$ by $\frac{1}{i(i-1)}$: the series becomes D^p applied on

$$Y^{a-1}Z\sum_{i\geq 2}\left(\frac{Y^i}{i-1}-\frac{Y^i}{i}\right)=Y^{a-1}Z\left[\frac{Y^1}{1}+(Y-1)\sum_{i\geq 1}\frac{Y^i}{i}\right]=Y^aZ-Y^a\sum_{i\geq 1}\frac{Y^i}{i}.$$

The second term is equal to $-D^{p-1}$ applied on $aY^{a-1}Z\sum_{i\geq 1}\frac{Y^i}{i}+Y^aZ\,(1+Z)$: in this sum, the first terms kills the contribution of $x=1$ (the latter is weighted by $-e_1=-a$) and the second becomes $Y^{a-1}Z^2$.

Therefore, the four first series $(1-\frac{1}{n}-e_0?-e_1?)$ bring a contribution

$$\begin{aligned}
& D^p\left(Y^{a-1}Z\right)-D^{p-1}\left(Y^{a-1}Z\right)-\left[D^p\left(Y^aZ\right)-D^{p-1}\left(Y^{a-1}Z^2\right)\right] \\
=\ & D^p\left[Y^{a-1}Z\,(1-Y)\right]+D^{p-1}\left[Y^{a-1}\left(Z^2-Z\right)\right] \\
=\ & D^{p-1}DY^a+D^{p-1}\left(Y^{a-1}Z^2\right)-D^{p-1}\left(Y^{a-1}Z\right) \\
=\ & D^{p-1}\left(Y^{a-1}Z^2\right)+(a-1)\,D^{p-1}\left(Y^{a-1}Z\right).
\end{aligned}$$

Adding everything yields the result.

4 A present from algebraic geometry

4.1 The Ekedahl-Lando-Shapiro-Vainshtein formula

Let us give some definitions and intuitions (taken from [10]) of the space and of the integration theory involved in the ELSV formula proved in [2].

Define a ***n-marked Riemann surface*** as a compact connected one-dimensional complex manifold S together with a family of n pairwise distinct points on S. An ***isomorphism*** between two n-marked Riemann surfaces $(S,a_1,...,a_n)$ and $(T,b_1,...,b_n)$ is an isomorphism $f:S\widetilde{\longrightarrow}T$ of Riemann surfaces such that $f(a_i)=b_i$ for any $i=1,...,n$.

One can prove that a g-genused n-marked Riemann surface has a finite group of automorphisms if and only if

$$[g\geq 2]\ \text{ or }\ [g=1\text{ and }n\geq 1]\text{ or }[g=0\text{ and }n\geq 3]\,, \tag{69}$$

which can be summarised as

$$n+2g'>0. \tag{70}$$

Such a pair (g,n) will be called ***stable***.

For a stable pair (g, n), define the ***moduli space*** $\mathcal{M}_n^g$ as the set of isomorphism classes of g-genused n-marked Riemann surfaces. For instance, the classical fact that homographies act 3-transitively on the Riemann sphere $\mathbb{S}$ shows that $\mathcal{M}_3^0$ is a point and that $\mathcal{M}_4^0 \simeq \mathbb{S}\backslash\{0, 1, \infty\}$. This fact also explains why the pairs $(0, n)$ are unstable for $n \leq 2$.

The example of $\mathcal{M}_4^0$ shows that the moduli spaces $\mathcal{M}_n^g$ are not always compact. One can build a convenient compactification $\overline{\mathcal{M}}_n^g$, called the ***Deligne-Mumford compactification*** of $\mathcal{M}_n^g$, which is a manifold for g fixed and n large enough. It can be shown that, for any stable pair (g, n), the space $\overline{\mathcal{M}}_n^g$ is a compact complex ***orbifold*** of dimension $3g' + n$. Intuitively, a d-dimensional complex orbifold is locally isomorphic to an open ball in $\mathbb{C}^d$ *modulo* the orbits of a finite group; the finiteness explains why we consider the spaces $\overline{\mathcal{M}}_n^g$ only when the pair (g, n) is stable.

On a compact orbifold O, one can define a ***$\mathbb{Q}$-algebra of cohomology*** $H^*(O) = \bigoplus_{k\geq 0} H^k(O)$: it is an anti-commutative algebra over the field $\mathbb{Q}$, graded by the $H^k(O)$'s, whose elements are called ***cohomology classes***. A cohomology class lying in a homogeneous component $H^k(O)$ is said ***to have pure degree*** k.

One can then define an ***integration theory*** of cohomology classes on a compact orbifold O. By construction, the integral of a cohomology class lying in $H^k(O)$ is zero unless $k = \dim O$. Therefore, if α_i are (commutating) homogeneous cohomology classes, the integral $\int_O \prod \alpha_i$ will cancel out unless $\sum \deg \alpha_i = \dim O$. This allows to write expressions like $\int_O \frac{1}{1-\alpha}\frac{1}{1-\beta}$ by formally expanding the fractions into power series and by keeping only the terms of degree $\dim O$.

The reason for introducing integration of cohomology classes on $\overline{\mathcal{M}}_n^g$ is the ELSV formula which expresses $\frac{h_n^g(\lambda)}{T_n!}$ as an integral involving on the one hand the ***first Chern classes*** $\psi_k \in H^2\left(\overline{\mathcal{M}}_n^g\right)$ of the cotangent bundles at the k-th marked point (for any $k = 1, ..., n$) and on the other hand the Chern classes $(-1)^i \lambda^i \in H^{2i}\left(\overline{\mathcal{M}}_n^g\right)$ of the ***dual Hodge bundle*** Λ_n^{g*}.

We will not need the precise definition of the Chern classes ψ_k and λ^i but only the knowing that they have pure even degree (and therefore commute)

$$\begin{array}{l} \forall k = 1, ..., n,\ \deg \psi_k = 2 \\ \forall i = 1, ..., g,\ \deg \lambda^i = 2i \end{array}. \tag{71}$$

Theorem 3 (ELSV formula). *Let $g, n \geq 0$ be integers and $\lambda = (a_1, ..., a_n)$ a partition of an integer $\leq n$. The following formula holds as long as $n + 2g' > r$:*

$$\frac{h_n^g(\lambda)}{T_n!} = \frac{1}{(n-|\lambda|)!}\int_{\overline{\mathcal{M}}_{n-r(\lambda)}^g} \frac{1-\lambda^1+\lambda^2-\cdots+(-1)^g\lambda^g}{(1-a_1\psi_1)\cdots(1-a_p\psi_p)(1-\psi_{p+1})\cdots\left(1-\psi_{n-r(\lambda)}\right)}. \tag{72}$$

When the size $|\lambda|$ is equal to n, the formula becomes

$$\frac{h_n^g(\lambda)}{T_n!} \overset{|\lambda|=n}{=} \int_{\overline{\mathcal{M}}_p^g} \frac{1-\lambda^1+\lambda^2-\cdots+(-1)^g\lambda^g}{(1-a_1\psi_1)\cdots(1-a_p\psi_p)}. \tag{73}$$

When expanding the fraction of the integrand, one obtains a finite linear combinaison of integrals whose generic form is $\int_{\overline{\mathcal{M}}_p^g} \lambda^j \psi_1^{k_1} \cdots \psi_p^{k_p}$ for some integers $j, k_1, ..., k_p$.

The other reason for speaking about integration on $\overline{\mathcal{M}}_n^g$ is a formula from Kazarian (Theorem 4 in Section 4.3) which expresses the series $H^g(\lambda)$ as a polynomial in Y and Z, whose Z-leading term can be expressed as an integral of the classes ψ_i. We are actually only interested in the Z-leading term, which we give as a straightforward corollary (Corollary 3 in Section 4.3) of the Kazarian formula .

Before stating the Kazarian formula, let us recall some classical notations and facts about the classes ψ_i. (For details and proofs see [10] or [5].)

4.2 The Witten brackets $\langle\tau_{d_1}, ..., \tau_{d_n}\rangle$

For integers $d_1, ..., d_n \geq 0$, one denotes the integral of $\prod \psi_i^{d_i}$ by the ***Witten (pointed-) bracket***:

$$\langle\tau_{d_1}, ..., \tau_{d_n}\rangle := \int_{\overline{\mathcal{M}}_n^g} \psi_1^{d_1} \cdots \psi_n^{d_n}. \tag{74}$$

Notice that the n in the integrand is the same as the n of the $\overline{\mathcal{M}}_n^g$ on which the integration is carried out and that the genus is defined by the relation $3g' + n = \sum d_i$. Moreover, since the n marked points in $\mathcal{M}_n^g$ play symmetric roles, the bracket is invariant under permutations of the d_i's.

For instance, when considering the bracket $\langle\tau_0^5\tau_3\tau_7^2\rangle$, one must have $n=5+1+2$, then $n+3g'=5\cdot 0+3+2\cdot 7$, hence $(g,n)=(4,8)$. One therefore integrates over $\overline{\mathcal{M}}_8^4$ the class $\psi_1^0\psi_2^0\psi_3^0\psi_4^0\psi_5^0\psi_6^3\psi_7^7\psi_8^7$, which amounts to integrating the class $\alpha^7\beta^7\gamma^3$ for any α,β,γ distinct ψ_i's.

The brackets $\langle\tau_{d_1},...,\tau_{d_n}\rangle$ satisfy two recurrence formulas on the number of d_i's whenever one of the d_i's is zero or one (see [10] or [5] for details).

Claim 5 (string and dilaton equations). *For any stable pair (g,n), one has*

$$\langle\tau_{d_1},...,\tau_{d_n},\tau_{\underline{0}}\rangle \underset{\text{equation}}{\overset{\text{string}}{=}} \sum_{i=1}^{n}\left\langle\tau_{d_1},...,\tau_{d_{i-1}},\tau_{\underline{d_i-1}},\tau_{d_{i+1}},...,\tau_{d_n}\right\rangle, \tag{75}$$

$$\langle\tau_{d_1},...,\tau_{d_n},\tau_{\underline{1}}\rangle \underset{\text{equation}}{\overset{\text{dilaton}}{=}} (2g'+n)\langle\tau_{d_1},...,\tau_{d_n}\rangle \qquad (\text{set } \tau_{-1}=0). \tag{76}$$

In spherical genus, the dimension condition $\sum d_i = n-3$ shows that all brackets obtained in the ELSV formula have at least three null indexes, which allows their computation with the knowledge of $\langle\tau_0^3\rangle=1$. In toric genus, the dimensional identity $\sum d_i = n$ show that at least one the index is 0 or 1, hence the 1-genused Witten brackets once known $\langle\tau_1\rangle=\frac{1}{24}$. Combining these formulas (Claim 6) with the ELSV formula (Theorem 3) yields all spherical and toric one-partition Hurwitz numbers (Claim 4).

Claim 6 (Witten brackets in genera 0 and 1). *For any integers $d_1,...,d_n\geq 0$ and genus $g\in\{0,1\}$ satisfying the dimensional condition $n+3g'=\sum d_i$, one has the identities*

$$\langle\tau_{d_1},...,\tau_{d_n}\rangle_{g=0} = \binom{n-3}{d_1,...,d_n} \text{ and} \tag{77}$$

$$\langle\tau_{d_1},...,\tau_{d_n}\rangle_{g=1} = \frac{1}{24}\binom{n}{d_1,...,d_n}\left(1-\sum_{i=2}^{n}\frac{\widetilde{e}_i(d_1,...,d_n)}{i(i-1)}\right). \tag{78}$$

4.3 The Kazarian formulas

The following theorem and remark are entirely borrowed from [4]. The given proof was however not complete (and unpublished), the reason why we carry out all the details.

Theorem 4 (Kazarian formula). *Give $\lambda=(a_1,...,a_p)$ a partition of an integer $a\geq 0$ and set a genus $g\geq 0$. Then one has the following equality whenever the pair (g,p) is stable:*

$$\mathbb{H}^g(\lambda) = Y^a(Z+1)^{2g'+p}\;{}^{\lambda}P^g(Z) \tag{79}$$

where ${}^{\lambda}P^g(Z)$ is the following series in Z:

$${}^{\lambda}P^g(Z) := \sum_{k\geq 0}\frac{Z^k}{k!}\int_{\overline{\mathcal{M}}_{p+k}^g}\frac{1-\lambda_1+\lambda_2-\cdots+(-1)^g\lambda_g}{(1-a_1\psi_1)\cdots(1-a_p\psi_p)}\frac{\psi_{p+1}^2\cdots\psi_{p+k}^2}{(1-\psi_{p+1})\cdots(1-\psi_{p+k})}. \tag{80}$$

Remark. Some combinatorial explanation for the factor $Y^a(Z+1)^{2g'+p}$ can be found in [8]. Essentially, once a certain graph has been constructed, the presence of powers of Y corresponds to planting a Cayley tree on each vertex and the presence of the powers of $1+Z$ to planting a Cayley tree (or nothing) on each edge.

The series ${}^{\lambda}P^g$ in Theorem 4 is actually a *polynomial* for dimensional reasons: the integrand has degree $\geq 2+2+\cdots+2=2k$, but that very degree must equal the dimension $3g'+p+k$, hence $3g'+p+k\geq 2k$ and $k\leq 3g'+p$. The equality $k=3g'+p$ yields the leading term of ${}^{\lambda}P^g(Z)$: $\frac{Z^d}{d!}\langle\tau_2^d\tau_0^p\rangle$. Multiplying by $Y^a(Z+1)^{2g'+p}\overset{Z}{\cong}Z^{2g'+p}$ yields Corollary 3. Knowing that ${}^{\lambda}P^g$ is a polynomial reduces the computation of the infinite list $(h_n^g(\lambda))_{n\geq 1}$ to that of one of its *finite* sublists (see remark below).

Examples. Let us check, thanks to Theorem 2, that $\frac{\mathbb{H}^g(\lambda)}{Y^a(Z+1)^{2g'+p}}$ belongs to $\mathbb{Q}[Z]$ in genera 0 and 1 with a partition λ having minimal length.

On spherical gender, when λ has length 3, one has

$$\frac{\mathbb{H}^g(\lambda)}{Y^a(Z+1)^{2g'+p}} = \frac{\mathbb{H}^0_{p=3}}{Y^a(Z+1)} = \frac{Y^{a-1}Z}{Y^a(Z+1)} = \frac{Z}{Y(1+Z)} = 1. \tag{81}$$

In toric gender, when $\lambda = (d+1)$ has length 1, one has

$$\frac{\mathbb{H}^g(\lambda)}{Y^a(Z+1)^{2g'+p}} = \frac{\mathbb{H}^1((d+1))}{Y^{d+1}(Z+1)} = \frac{1}{24}\frac{Y^d Z(Z+d)}{Y^{d+1}(Z+1)} = \frac{Z+d}{24}\frac{Z}{Y(1+Z)} = \frac{Z+d}{24}. \tag{82}$$

Unstable pairs. When (g,p) is one of the four unstable pairs, the first terms of the polynomial ${}^{\lambda}P^g$ fail to exist. One can nevertheless wonder if the series ${}^{\lambda}P^g := \frac{\mathbb{H}^g(\lambda)}{Y^a(Z+1)^{2g'+p}}$ is still a Z-polynomial. The following identities (which are straightforward computations from Theorem 2 left to the reader) show that ${}^{\lambda}P^g \in \mathbb{Q}[Z]$ *if and only if* λ *is non-empty*:

$$\begin{array}{lcl} (g,p)=(0,2) & {}^{(x,a-x)}P^0 & = & \frac{1}{a} \\ (g,p)=(0,1) & {}^{(a)}P^0 & = & \frac{1}{a^2}\left(1+\frac{Z}{a+1}\right) \\ (g,p)=(0,0) & {}^{\emptyset}P^0 & = & Z+\frac{5Z^2}{12}-\frac{Z}{6}+\frac{Y}{6} \\ (g,p)=(1,0) & {}^{\emptyset}P^1 & = & \frac{D^{-1}Z^2}{24} \end{array}.$$

Corollary 3 (Kazarian Z-formula). *For any partition λ and genus $g \geq 0$ such that (g,p) is stable, one has the following Z-equality:*

$$\mathbb{H}^g(\lambda) \overset{Z}{=} \frac{\left\langle \tau_0^p \tau_2^{3g'+p}\right\rangle}{(3g'+p)!} Z^{5g'+2p}. \tag{83}$$

Examples. Let us cross-check Corollary 3 in spherical and toric genus with the explicit formulas of Theorem 2.

In null genus, Claim 2 and Theorem 2 allow us to write

$$\mathbb{H}^0(\lambda) = D^{p-3}\left(Y^{a-1}Z\right) \overset{Z}{=} D^{p-3}Z \overset{Z}{=} (2p-7)!!Z^{2p-5}.$$

According to Kazarian, one should have $\mathbb{H}^0(\lambda) \overset{Z}{=} \frac{\langle \tau_0^p \tau_2^{p-3}\rangle}{(p-3)!} Z^{2p-5}$. But Claim 6 for spherical Witten brackets yields

$$\frac{\langle \tau_0^p \tau_2^{p-3}\rangle}{(p-3)!} = \frac{1}{(p-3)!}\binom{(2p-3)-3}{0,...,0,2,...,2} = \frac{(2p-6)!}{(p-3)!}\frac{1}{2^{p-3}} = (2p-7)!!.$$

In genus 1, Claim 2 and Theorem 2 allow us to write

$$24\mathbb{H}^1(\lambda) \overset{Z}{=} D^{p-1}\left(Y^{a-1}Z^2\right) \overset{Z}{=} D^{p-1}Z^2 \overset{Z}{=} (2p-2)!!Z^{2p}.$$

Compared to Kazarian, one should expect $\mathbb{H}^1(\lambda) \overset{Z}{=} \frac{\langle \tau_0^p \tau_2^p\rangle}{p!} Z^{2p}$. But the string and dilaton equations show that

$$\begin{array}{rcl} \frac{\langle \tau_0^p \tau_2^p\rangle}{p!} & \overset{\text{string}}{=} & \frac{p\langle \tau_0^{p-1}\tau_1\tau_2^{p-1}\rangle}{p!} \overset{\text{dilaton}}{=} (2p-2)\frac{\langle \tau_0^{p-1}\tau_2^{p-1}\rangle}{(p-1)!} = \cdots \\ & = & (2p-2)!!\langle \tau_0\tau_2\rangle \overset{\text{string}}{=} (2p-2)!!\langle \tau_1\rangle = \frac{(2p-2)!!}{24}. \end{array}$$

Compacity remark. So as to recall the variable t in $H^g(\lambda)$, the latter will also be written ${}^{\lambda}H^g$ or ${}^{\lambda}H^g(t)$. Recalling computations in the algebra $\mathcal{A}$, one can write

$$t = Ye^{-Y} = \frac{Z}{1+Z}e^{-\frac{Z}{1+Z}} \in Z\mathbb{Q}[[Z]]. \tag{84}$$

One the other hand, since $h_n^g(\lambda) = 0$ when $n < |\lambda| = a$, one can also write

$${}^{\lambda}H^g(t) \in t^a\mathbb{Q}[[t]] \subset Z^a\mathbb{Q}[[Z]]. \tag{85}$$

As a result, the series $\frac{H^g(\lambda)}{Z^a}\underbrace{(1+Z)^{a-l-2g'}}_{\in 1+Z\mathbb{Q}[[Z]]}$ is a Z-polynomial of degree $3g'+l$ (it equals the polynomial ${}^{\lambda}P^g$ in Theorem 4) and is therefore determined by its first coefficients. Theorem 4 then yields Corollary 4.

Corollary 4. *For any fixed genus $g \geq 0$,*

one can compute the Hurwitz numbers $h_n^g(\lambda)$ for any n
once known the numbers $h_n^g(\lambda)$ for $n \leq a + 3g' + l$.

More precisely, for any function f, define $\mathrm{Trunc}_{\deg_x = d} f := \sum_{k=0}^{d} \frac{x^k}{k!}\left[\partial_x^k f\right](0)$ *as its Taylor-Mac Laurin truncation of degree d. Then one has*

$$H^g(\lambda) = Y^a (1+Z)^{2g'+l} \times \mathrm{Trunc}_{\deg_Z = 3g'+l}\left[\frac{{}^{\lambda}H^g(t(Z))}{Z^a}(1+Z)^{a-l-2g'}\right]. \quad (86)$$

Proof of Theorem 4. Let us abbreviate $\check{\lambda} := 1 - \lambda_1 + \lambda_2 - \cdots \pm \lambda_g$ and

$$\frac{1}{1-a\psi} := \frac{1}{(1-a_1\psi_1)\dots(1-a_p\psi_p)} \text{ and } a^n := a_1^{n_1}\cdots a_p^{n_p} \text{ for any } n \in \mathbb{N}^p. \quad (87)$$

When one encodes the ELSV formula (Theorem 3) in the series

$$\begin{aligned} \mathbb{H}^g(\lambda) &= \sum_{n\geq a} \frac{t^n}{(n-a)!}\int_{\overline{\mathcal{M}}^g_{n-r}} \frac{1-\lambda^1+\lambda^2-\cdots+(-1)^g\lambda^g}{(1-a_1\psi_1)\cdots(1-a_p\psi_p)(1-\psi_{p+1})\cdots(1-\psi_{n-r})} && (88)\\ &\overset{n\leftarrow a+k}{=} \sum_{k\geq 0}\frac{t^{a+k}}{k!}\int_{\overline{\mathcal{M}}^g_{p+k}} \frac{\check{\lambda}}{1-a\psi}\frac{1}{(1-\psi_{p+1})\cdots(1-\psi_{p+k})} \\ &= t^a \sum_{k,d_1,\dots,d_k\geq 0}\frac{t^k}{k!}\int_{\overline{\mathcal{M}}^g_{p+k}}\frac{\check{\lambda}}{1-a\psi}\psi_{p+1}^{d_1}\cdots\psi_{p+k}^{d_k}, && (89)\end{aligned}$$

one can make the classes ψ_i in the Hodge integral to be of degree $d_i \geq 2$ thanks to the string and dilaton equations, hence the pattern of the polynomial ${}^{\lambda}P^g$ in Theorem 4. Let us be more precise.

Define a series F in $\mathbb{Q}[[t_0, t_1, \dots]]$ by

$$\begin{aligned} F(t_0,t_1,\dots) &:= \sum_{k\geq 0}\frac{1}{k!}\int_{\overline{\mathcal{M}}^g_{p+k}}\left[\frac{\check{\lambda}}{1-a\psi}\sum_{d_1,\dots,d_k\geq 0}\left(t_{d_1}\psi_{p+1}^{d_1}\right)\cdots\left(t_{d_k}\psi_{p+k}^{d_k}\right)\right] && (90)\\ &= \sum_{k,d_1,\dots,d_k\geq 0}\frac{t_{d_1}\cdots t_{d_k}}{k!}\int_{\overline{\mathcal{M}}^g_{p+k}}\check{\lambda}\left(\sum_{n_1,\dots,n_p\geq 0} a_1^{n_1}\cdots a_p^{n_p}\psi_1^{n_1}\cdots\psi_p^{n_p}\right)\psi_{p+1}^{d_1}\cdots\psi_{p+k}^{d_k} \\ &= \sum_{k,d_1,\dots,d_k,n_1,\dots,n_p\geq 0} a^n\frac{t_{d_1}\cdots t_{d_k}}{k!}\left\langle\check{\lambda},\tau_{n_1},\tau_{n_2},\dots,\tau_{n_p},\tau_{d_1},\dots,\tau_{d_k}\right\rangle. && (91)\end{aligned}$$

The theorem is a trivial consequence of the two following lemmas (mind the switch of the starting index of the series F in Lemma B).

Lemma A. *One has the following identities and differential equations:*

$$\begin{cases} {}^{\lambda}H^g(t) = t^a F(t,t,\dots) \\ {}^{\lambda}P^g(t) = F(0,0,t\dots) \end{cases} \text{ and } \begin{cases} \partial_{t_0}F = aF + \sum_{i\geq 0} t_{i+1}\,\partial_{t_i}F \\ \partial_{t_1}F = (2g'+p)F + \sum_{i\geq 0} t_i\,\partial_{t_i}F \end{cases}. \quad (92)$$

Lemma B. *Let K be a field and F be a series in $K[[t_1, t_2, \dots]]$ which satisifes both equations*

$$\begin{cases} \partial_{t_1}F = \lambda F + \sum_{i\geq 1} t_{i+1}\,\partial_{t_i}F \\ \partial_{t_2}F = \mu F + \sum_{i\geq 1} t_i\,\partial_{t_i}F \end{cases} \text{ for some scalars } \lambda \text{ and } \mu. \quad (93)$$

Then, one has the following equality in the algebra $K[[Y, Z]]$:

$$t^{\lambda}F(t,t,t,\dots) = Y^{\lambda}(1+Z)^{\mu}F(0,0,Z,Z,Z,\dots). \quad (94)$$

Proof of Lemma A. When one cancels $t_i \leftarrow 0$ for any $i < N$ (where N is a given integer), then the powers appearing in the product $\psi_{p+1}^{d_1}\dots\psi_{p+k}^{d_k}$ range from N to ∞, whence the equality (recall the generic identity $\sum_{d_1,\dots,d_k\geq N} x_1^{d_1}\dots x_k^{d_k} = \frac{x_1^N\dots x_k^N}{(1-x_1)\cdots(1-x_k)}$)

$$F\Big(\underbrace{0,\dots,0}_{N\text{ zeros}},t,t,t\dots\Big) = \sum_{k\geq 0}\frac{t^k}{k!}\int_{\overline{\mathcal{M}}_{g,p+k}}\frac{\check{\lambda}}{1-a\psi}\frac{\psi_{p+1}^N\dots\psi_{p+k}^N}{(1-\psi_{p+1})\cdots(1-\psi_{p+k})}. \quad (95)$$

For instance, $N = 0$ and $N = 2$ yields the first two wanted identities

$${}^{\lambda}H^g(t) = t^a F(t,t,\dots) \text{ and } {}^{\lambda}P^g(t) = F(0,0,t,\dots).$$

To handle the differentiation of F, let us cluster all factors t_{d_i} with same d_i's.

Since the summand is symmetric in the indexes d_i, one can cluster the summand having same multiset $\left[\binom{0}{m_0},\binom{1}{m_1},\binom{2}{m_2},...\right]$ where $m_i := \#\{n\ ;\ d_n = i\}$ for any $i \geq 0$. Once chosen the multiplicities $(m_0, m_1, m_2, ...) \in \mathbb{N}^{(\mathbb{N})}$, choosing a family $(d_1, ..., d_k) \in \bigcup_{p\geq 0} \mathbb{N}^p$ such that $k = \sum_{i\geq 0} m_i$ amounts to choosing a set partition of $\{1, ..., k\}$ into disjoint parts of cardinalities m_i, hence a multinomial $\binom{k}{\overrightarrow{m}}$. The series F can therefore be rewritten

$$\begin{aligned} F &= \sum_{\substack{\overrightarrow{m}\in\mathbb{N}^{(\mathbb{N})} \\ n_1,\dots,n_p\geq 0}} a^n \frac{t_0^{m_0}}{m_0!}\frac{t_1^{m_1}}{m_1!}\frac{t_2^{m_2}}{m_2!}\cdots\left\langle\widehat{\lambda},\tau_{n_1},...,\tau_{n_p},\tau_0^{m_0},\tau_1^{m_1},\tau_2^{m_2},...\right\rangle \\ &= \sum_{\substack{\overrightarrow{m}\in\mathbb{N}^{(\mathbb{N})} \\ n_1,\dots,n_p\geq 0}} a^n t^m \left\langle\widehat{\lambda},\tau_n,\tau^m\right\rangle \text{ with the obvious abbreviations.} \end{aligned}$$

Let us compute

$$\begin{aligned} \sum_{i\geq 0} t_{i+1}\ \partial_{t_i} F &= \sum_{i\geq 0}\sum_{\substack{\overrightarrow{m}\in\mathbb{N}^{(\mathbb{N})},\ m_i\geq 1 \\ n_1,\dots,n_p\geq 0}} a^n \frac{t_0^{m_0}}{m_0!}\cdots\frac{t_i^{m_i-1}}{(m_i-1)!}\frac{t_{i+1}^{m_{i+1}+1}}{m_{i+1}!}\cdots\left\langle\widehat{\lambda},\tau_n,\tau_0^{m_0},...,\tau_i^{m_i},...\right\rangle \\ &\overset{m_i\leftarrow m_i+1}{=} \sum_{i\geq 0}\sum_{\substack{\overrightarrow{m}\in\mathbb{N}^{(\mathbb{N})} \\ n_1,\dots,n_p\geq 0}} a^n \frac{t_0^{m_0}}{m_0!}\cdots\frac{t_i^{m_i}}{m_i!}\frac{t_{i+1}^{m_{i+1}+1}}{m_{i+1}!}\cdots\left\langle\widehat{\lambda},\tau_n,\tau_0^{m_0},...,\tau_i^{m_i+1},...\right\rangle. \end{aligned}$$

On the other hand, the partial derivative $\partial_{t_0} F$ equals

$$\begin{aligned} \partial_{t_0} F &= \sum_{\substack{\overrightarrow{m}\in\mathbb{N}^{(\mathbb{N})},\ m_0\geq 1 \\ n_1,\dots,n_p\geq 0}} a^n \frac{t_0^{m_0-1}}{(m_0-1)!}\frac{t_1^{m_1}}{m_1!}\frac{t_2^{m_2}}{m_2!}\cdots\left\langle\widehat{\lambda},\tau_n,\tau^m\right\rangle \\ &\overset{m_0\leftarrow m_0+1}{=} \sum_{\substack{\overrightarrow{m}\in\mathbb{N}^{(\mathbb{N})} \\ n_1,\dots,n_p\geq 0}} a^n t^m \left\langle\widehat{\lambda},\tau_n,\tau^m,\boxed{\tau_0}\right\rangle \\ &\overset{\substack{\text{string} \\ \text{equation}}}{=} \sum_{\substack{\overrightarrow{m}\in\mathbb{N}^{(\mathbb{N})} \\ n_1,\dots,n_p\geq 0}} a^n t^m \left(\begin{array}{c} \sum_{j=1,\dots,p}\left\langle\widehat{\lambda},\tau_{n_1},...,\tau_{n_j-1},...,\tau_{n_p},\tau^m\right\rangle \\ +\sum_{i\geq 0}\left\langle\widehat{\lambda},\tau_n,\tau_0^{m_0},...,\tau_i^{m_i+1},\tau_{i+1}^{m_{i+1}-1},...\right\rangle \end{array}\right). \end{aligned}$$

The sum $\sum_{j=1,\dots,p}$ equals (after renaming $n_j \leftarrow n_j + 1$) the series F multiplied by a factor $\sum_{j=1,\dots,p} a_j = a$, the second sum $\sum_{i\geq 0}$ equals (after renaming $m_{i+1} \leftarrow m_{i+1} + 1$) the above computation of $\sum_{i\geq 0} t_{i+1}\ \partial_{t_i} F$, hence the first differential equation.

On the other hand, the partial derivative $\partial_{t_1} F$ equals

$$\begin{aligned} \partial_{t_1} F &= \sum_{\substack{\overrightarrow{m}\in\mathbb{N}^{(\mathbb{N})},\ m_1\geq 1 \\ n_1,\dots,n_p\geq 0}} a^n \frac{t_0^{m_0}}{m_0!}\frac{t_1^{m_1-1}}{(m_1-1)!}\frac{t_2^{m_2}}{m_2!}\cdots\left\langle\widehat{\lambda},\tau_n,\tau^m\right\rangle \\ &\overset{m_1\leftarrow m_1+1}{=} \sum_{\substack{\overrightarrow{m}\in\mathbb{N}^{(\mathbb{N})} \\ n_1,\dots,n_p\geq 0}} a^n t^m \left\langle\widehat{\lambda},\tau_n,\tau^m,\tau_1\right\rangle \\ &\overset{\substack{\text{dilaton} \\ \text{equation}}}{=} \sum_{\substack{\overrightarrow{m}\in\mathbb{N}^{(\mathbb{N})} \\ n_1,\dots,n_p\geq 0}} a^n t^m \left((2g'+(p+k))\left\langle\widehat{\lambda},\tau_n,\tau^m,...\right\rangle\right) \text{ where } k := \sum_{i\geq 0} m_i \\ &= (2g'+p)\,F + \sum_{i\geq 0}\sum_{\substack{\overrightarrow{m}\in\mathbb{N}^{(\mathbb{N})} \\ n_1,\dots,n_p\geq 0}} a^n \frac{t_0^{m_0}}{m_0!}\cdots\frac{t_i^{m_i}}{(m_i-1)!}\cdots\left\langle\widehat{\lambda},\tau_n,\tau^m\right\rangle \\ &= (2g'+p)\,F + \sum_{i\geq 0} t_i\ \partial_{t_i} F. \end{aligned}$$

Proof of Lemma B. We first restrict ourselves to only three variables by setting $f := F(x, y, z, z, z...)$ in the algebra $K[[x, y, z]]$. By differentiating the latter definition, one gets the equalities

$$\left\{\begin{array}{c} \partial_x f = \frac{\partial\ F(x,y,z,z,z...)}{\partial x} = [\partial_1 F](x, y, z, z, ...) \\ \partial_y f = \frac{\partial\ F(x,y,z,z,z...)}{\partial y} = [\partial_2 F](x, y, z, z, ...) \\ \partial_z f = \frac{\partial\ F(x,y,z,z,z...)}{\partial z} = \sum_{i\geq 3}[\partial_i F](x, y, z, z, ...) \end{array}\right..$$

Therefore, evaluating the two equations along $(t_1, t_2, t_{i\geq 3}) = (x, y, z)$ yields

$$\begin{cases} \partial_x f = \lambda f + y\ \partial_x f + z\ \partial_y f + z\ \partial_z f \\ \partial_y f = \mu f + x\ \partial_x f + y\ \partial_y f + z\ \partial_z f \end{cases},$$

which we rewrite as

$$\begin{cases} (1-y)\ \partial_x f - z\ \partial_y f - z\ \partial_z f = \lambda f \\ -x\ \partial_x f + (1-y)\ \partial_y f - z\ \partial_z f = \mu f \end{cases}.$$

Set $g := f(0, y, z)$ in $K[[y, z]]$. Differentiating the latter definition and evaluating $x \leftarrow 0$ in the second equation (with μ) yields

$$(1-y)\ \partial_y g - z\ \partial_z g = \mu g.$$

Let us seek two series $\binom{\widetilde{y}}{\widetilde{z}}$ in $K[[s, y, z]]$ satisfying $\begin{cases} \partial_s \widetilde{y} = 1 - \widetilde{y} \\ \partial_s \widetilde{z} = -\widetilde{z} \end{cases}$, so that the evaluation of the above equation along $\binom{y}{z} \leftarrow \binom{\widetilde{y}}{\widetilde{z}}$ yields

$$\mu\ g\begin{pmatrix} \widetilde{y} \\ \widetilde{z} \end{pmatrix} = \partial_s \widetilde{y} \times [\partial_y g]\begin{pmatrix} \widetilde{y} \\ \widetilde{z} \end{pmatrix} + \partial_s \widetilde{z} \times [\partial_z g]\begin{pmatrix} \widetilde{y} \\ \widetilde{z} \end{pmatrix} = \partial_s\ g\begin{pmatrix} \widetilde{y} \\ \widetilde{z} \end{pmatrix},$$

hence $g\binom{\widetilde{y}}{\widetilde{z}}$ is exponential in s. A solution is $\binom{\widetilde{y}}{\widetilde{z}} := \binom{1-e^{-s}}{\frac{z}{1-y}e^{-s}}$ in $K\left[\left[s, \frac{z}{1-y}\right]\right]$. Therefore, the series $g\binom{\widetilde{y}}{\widetilde{z}}$ in $K\left[\left[s, \frac{z}{1-y}\right]\right]$ equals $G\left(\frac{z}{1-y}\right)e^{\mu s}$ for some series G. Evaluating $s \leftarrow 0$ yields $g\binom{0}{\frac{z}{1-y}} = G\left(\frac{z}{1-y}\right)$. Then, evaluating $s \leftarrow -\ln(1-y)$ yields $\binom{\widetilde{y}}{\widetilde{z}} = \binom{y}{z}$, whence

$$g = g\begin{pmatrix} y \\ z \end{pmatrix} = g\begin{pmatrix} \widetilde{y} \\ \widetilde{z} \end{pmatrix} = G\left(\frac{z}{1-y}\right)e^{\mu(-\ln(1-y))} = g\begin{pmatrix} 0 \\ \frac{z}{1-y} \end{pmatrix}\frac{1}{(1-y)^{\mu}}.$$

We have therefore derived the equality

$$f(0, y, z) = f\left(0, 0, \frac{z}{1-y}\right)\frac{1}{(1-y)^{\mu}}.$$

Set $h := f(x, z, z)$ in $K[[x, z]]$. Differentiating the definition of h and evaluating $y \leftarrow z$ in the first equation (with λ) yields

$$(1-z)\ \partial_x h - z\ \partial_z h = \lambda h.$$

Like before, set $\binom{\widetilde{x}}{\widetilde{z}} := \binom{s+u(e^{-s}-1)}{ue^{-s}}$ in $K[[u, s]]$ so that $\begin{cases} \partial_s \widetilde{x} = 1 - \widetilde{z} \\ \partial_s \widetilde{z} = -\widetilde{z} \end{cases}$: from the equation $\partial_s\ h\binom{\widetilde{x}}{\widetilde{z}} = \lambda h\binom{\widetilde{x}}{\widetilde{z}}$, one can deduce $h\binom{\widetilde{x}}{\widetilde{z}} = H(u)e^{\lambda s}$ for some series H. Evaluating $s \leftarrow 0$ yields $h\binom{0}{u} = H(u)$, then evaluating $u \leftarrow s$ yields $h\binom{se^{-s}}{se^{-s}} = h\binom{0}{s}e^{\lambda s}$. At last, evaluating $s \leftarrow Y$ yields $\binom{\widetilde{x}}{\widetilde{z}} = \binom{t}{t}$, whence

$$\begin{aligned} h\begin{pmatrix} t \\ t \end{pmatrix} &= h\begin{pmatrix} \widetilde{x} \\ \widetilde{z} \end{pmatrix} = e^{\lambda Y}h\begin{pmatrix} 0 \\ Y \end{pmatrix} = \left(\frac{Y}{t}\right)^{\lambda} f(0, Y, Y) = \left(\frac{Y}{t}\right)^{\lambda}\frac{1}{(1-Y)^{\mu}}f\left(0, 0, \frac{Y}{1-Y}\right), \\ \text{i.e., } F(t, t, t, \ldots) &= \frac{Y^{\lambda}}{t^{\lambda}}(1+Z)^{\mu}F(0, 0, Z, Z, Z, \ldots),\ Q.E.D.. \end{aligned}$$

4.4 Asymptotics of one-partition Hurwitz numbers

Theorem 5 (asymptotics of one-partition Hurwitz numbers in any genus). *For any partition λ and any integers $g, n \geq 0$, one has the asymptotics*

$$\frac{h_n^g(\lambda)}{T_n!} \overset{n\to\infty}{\sim} c_g e^n n^{\frac{5}{2}g'+p-1} \text{ where } \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2\pi}} \\ \frac{1}{48} \end{pmatrix} \text{ and } c_{g\geq 2} = \frac{\left\langle \tau_2^{3g'} \right\rangle}{(3g')!}\frac{1}{\Gamma\left(\frac{5}{2}g'\right)2^{\frac{5}{2}g'}}. \tag{96}$$

Proof. We already know from Corollory 2 the asymptotics for $g \in \{0, 1\}$. When $g \geq 2$, combining Corollary 3 with Claim 3 immediatly yields the asymptotics of all $h_n^g(\lambda)$'s:

$$\frac{h_n^g(\lambda)}{T_n!} \overset{n\to\infty}{\sim} \frac{\left\langle \tau_2^{3g'+p}\tau_0^p \right\rangle}{(3g'+p)!}C_{5g'+2p}\frac{e^n}{n}\sqrt{n}^{5g'+2p} = Cst_g(\lambda) \times e^n n^{\frac{5}{2}g'+p-1}.$$

Let us proove that the constant $Cst_g(\lambda) := \frac{\left\langle \tau_2^{3g'+p}\tau_0^p \right\rangle}{(3g'+p)!} C_{5g'+2p}$ is actually λ-free, as we already know in genus 0 and 1 (recall $Cst_0 = \frac{1}{\sqrt{2\pi}}$ and $Cst_1 = \frac{1}{48}$). Just like in the previous Examples, we use the string and dilaton equations (Claim 5):

$$\begin{aligned}\frac{\left\langle \tau_2^{3g'+p}\tau_0^p \right\rangle}{(3g'+p)!} &\overset{\text{string}}{=} \frac{(3g'+p)}{(3g'+p)!}\left\langle \tau_2^{3g'+p-1}\tau_1\tau_0^{p-1}\right\rangle \\ &\overset{\text{dilaton}}{=} \frac{(5g'+2(p-1))}{(3g'+p-1)!}\left\langle \tau_2^{3g'+p-1}\tau_0^{p-1}\right\rangle \\ &= \cdots = \frac{(5g')^{\uparrow\uparrow p}}{(3g')!}\left\langle \tau_2^{3g'}\right\rangle.\end{aligned}$$

Besides, recalling the definition of the constants C_k (Claim 3) and the identity $\Gamma(x+p) = x^{\uparrow p}\Gamma(x)$ yields (set $G := \frac{5}{2}g'$)

$$\frac{1}{C_{2G+2p}} = \Gamma(G+p)\,2^{G+p} = 2^p G^{\uparrow p}\Gamma(G)\,2^G = \frac{(2G)^{\uparrow\uparrow p}}{C_{2G}}.$$

Multiplying both equalites leads to the conclusion.

We restate here Corollary 1, since it follows from the conjunction of Theorem 5 and of Theorem 1.

Corollary 1 (general asymptotics of Hurwiz numbers). *For any partitions $\lambda_1, ..., \lambda_k$ and any genus $g \geq 0$, one has the following asymptotics for some constant c_g:*

$$\frac{\mathbb{h}_n^g(\lambda_1, ..., \lambda_k)}{T_n!} \overset{n\to\infty}{\sim} c_g \frac{e^n}{n} n^{\frac{5}{2}g'} n^{l(\lambda_1)+\cdots+l(\lambda_k)}. \tag{97}$$

Proof. Set $m_1 := \sum m_1(\lambda_i)$ and $p := \sum l(\lambda_i)$. Theorem 1 gives us the Z-equality

$$D^3\mathbb{H}^g(\lambda_1, ..., \lambda_k) \overset{Z}{=} D^{3+m_1}\mathbb{H}^g\left(\mathring{\lambda}_1 \sqcup \mathring{\lambda}_2 \sqcup \cdots \sqcup \mathring{\lambda}_k\right),$$

whence the asymptotics (notice $l\left(\mathring{\lambda}\right) + m_1(\lambda) = l(\lambda)$ for any partition λ)

$$\frac{\mathbb{h}_n^g(\lambda_1, ..., \lambda_k)}{T_n!} \overset{n\to\infty}{\sim} n^{m_1}\frac{\mathbb{h}_n^g\left(\mathring{\lambda}_1 \sqcup \cdots \sqcup \mathring{\lambda}_k\right)}{T_n^g\left(\mathring{\lambda}_1 \sqcup \cdots \sqcup \mathring{\lambda}_k\right)!} \overset{\substack{\text{Theorem 5}\\ n\to\infty}}{\sim} n^{m_1} c_g e^n n^{\frac{5}{2}g'+(p-m_1)-1} = c_g e^n n^{\frac{5}{2}g'+p-1}.$$

4.5 Computing the brackets $\left\langle \tau_2^{3k}\right\rangle$ and the Hurwitz constants c_g

All Witten brackets can be encoded in a series

$$F(t_0, t_1, ...) := \sum_{\substack{g\geq 0, n\geq 1\\ (g,n)\text{ stable}}} \sum_{\substack{d_1+\cdots+d_n=3g'+n\\ d_1,\ldots,d_n\geq 0}} \langle \tau_{d_1}, ..., \tau_{d_n}\rangle \frac{t_{d_1}\cdots t_{d_n}}{n!}. \tag{98}$$

A conjecture of Witten (see [5] and [6]), now proved by Kontsevitch, gives a differential equation satisfied by F which allows, together with the string and dilaton equations, to recursively compute *all* Witten brakets. As an exemple, one can retrieve the value of $\langle \tau_1 \rangle = \frac{1}{24}$. Our interest in the theorem of Kontsevitch is deriving a recursion formula for the numbers $\left\langle \tau_2^{3g'}\right\rangle$ appearing in the asymptotics of Hurwitz numbers, which amounts to satisfying a Painevé I equation. Our proof is, in its ouline, similar to that in [9].

Theorem 6 (Kontsevitch). *If one denotes $\frac{\partial}{\partial t_d}$ by a subscript d, then one has*

$$F_{0,1} = \frac{1}{2}F_{0,0}^2 + \frac{1}{12}F_{0,0,0,0}. \tag{99}$$

Corollary 5 (computing the brackets $\left\langle \tau_2^{3k}\right\rangle$ and the constants c_g).

1. *Define* $(\alpha_{-1},\alpha_0) := \left(-1,\frac{1}{12}\right)$ *and* $\frac{\alpha_k}{(5k)(5k+2)} := \frac{\left\langle\tau_2^{3k}\right\rangle}{(3k)!}$ *for any* $k\geq 1$. *Then, the numbers* α_k *satisfy the recursion formula for any* $g\geq 0$:

$$\alpha_g = \frac{25g^2-1}{12}\alpha_{g'} + \frac{1}{2}\sum_{p+q=g'}^{p,q\geq 0}\alpha_p\alpha_q. \tag{100}$$

2. *The function* $u(t) := \sum_{g\geq 0} c_g \frac{\Gamma\left(\frac{5g-1}{2}\right)}{t^{\frac{5g-1}{2}}}$ *satisfies the Painlevé I equation*

$$u(t)^2 + \frac{1}{6}\frac{d^2u(t)}{dt^2} = 2t. \tag{101}$$

Proof of Corollary 5. Let us first derive the Painlevé equation from the recursion formula. Once noticed that, for any $g\geq 2$, one has

$$c_g\Gamma\left(\frac{5g-1}{2}\right)2^{\frac{5g-1}{2}} = \frac{\left\langle\tau_2^{3g'}\right\rangle}{(3g')!}\frac{\frac{5g-1}{2}\frac{5g-3}{2}\Gamma\left(\frac{5g-5}{2}\right)2^{\frac{5g-1}{2}}}{\Gamma\left(\frac{5g-5}{2}\right)2^{\frac{5g-5}{2}}} = \alpha_{g'}, \tag{102}$$

which also holds when $g\in\{0,1\}$, it then remains to show that the function $u(t) = \sum_{g\geq 0}\frac{\alpha_{g'}}{(2t)^{\frac{5g-1}{2}}}$ satisifies the above equation:

$$\begin{aligned}
\frac{1}{6}u'' + u^2 &= \frac{1}{6}\sum_{g\geq 0}\frac{\alpha_{g'}}{2^{\frac{5g-1}{2}}t^{\frac{5g-5}{2}}}\left(-\frac{5g-1}{2}\right)\left(-\frac{5g+1}{2}\right) + \sum_{k\geq 0}\sum_{i+j=k}^{i,j\geq 0}\frac{\alpha_{i'}\alpha_{j'}}{(2t)^{\frac{5k-2}{2}}}\\
&= \left(\sum_{g\geq 0}\frac{1}{(2t)^{\frac{5g+3}{2}}}\frac{25g^2-1}{6}\alpha_{g'}\right) + 2t + \sum_{k\geq 1}\frac{1}{(2t)^{\frac{5k-2}{2}}}\sum_{i+j=k}^{i,j\geq 0}\alpha_{i'}\alpha_{j'}\\
&= 2t + \sum_{g\geq 0}\frac{2}{(2t)^{\frac{5k+3}{2}}}\left(\frac{25g^2-1}{12}\alpha_{g'} + \frac{1}{2}\sum_{i+j=g+1}^{i,j\geq 0}\alpha_{i'}\alpha_{j'}\right).
\end{aligned}$$

To obtain the nullity of the brackets, use the recursion formula and rewrite the sum $\sum_{i+j=g+1}^{i,j\geq 0}\alpha_{i'}\alpha_{j'}$ as $-2\alpha_g + \sum_{p+q=g-1}^{p,q\geq 0}\alpha_p\alpha_q$.

Let us now prove the recursion formula.

One wants the coefficients $\left\langle\tau_2^k\right\rangle$ in $F(t_0,t_1,...)$ of the powers of t_2. One therefore cancels all t_i's with $i\neq 2$ in Theorem 6:

$$F_{0,1}(0,0,t,0,0,...) = \frac{1}{2}F_{0,0}^2(0,0,t,0,0,...) + \frac{1}{12}F_{0,0,0,0}(0,0,t,0,0,...).$$

When applying to F a differential operator $\prod_{i\neq 2}\frac{\partial^{d_i}}{(\partial t_i)^{d_i}}$ before cancelling all $t_{i\neq 2}$'s, the terms in F which yield a non-zero contribution to $\prod_{i\neq 2}\frac{\partial^{d_i}}{(\partial t_i)^{d_i}}F$ are necessarily like $\left(\prod_{i\neq 2}t_i^{d_i}\right)t_2^*$ for some power $*$: after differentiating, the $t_{i\neq 2}$'s disappear and leave a factor $\prod_{i\neq 2}d_i!$ behind. Therefore, when looking at $\frac{1}{2}F_{0,0}^2(0,0,t,0,...)$, the n-tuples of d_i's which have a non-zero contribution are necessarily such that exactly two d_i's equal 0 and all others equal $n-2$ (this implies $n\geq 2$). Similarly, when looking at $F_{0,0,0,0}(0,0,t,0,...)$, the n-tuples involved have four d_i's equal to 0 and all others equal to 2 (hence $n\geq 4$). At last, the d_i's involved in $F_{0,1}(0,0,t,0,0,...)$ have one d_i equal to 0, one d_i equal to 1 and all other d_i's equal to 2 (hence $n\geq 2$). One can therefore begin to explicit the three series above. We shall use the values of some brackets that can be computed thanks to the string and dilaton equation (see Lemma C below).

1. To start with, one has

$$F_{0,0}(0,0,t,0,...) = \sum_{n\geq 1,\ g\geq 0}^{(g,n)\text{ stable}}\frac{1}{n!}\left\langle\tau_0^2\tau_2^{n-2}\right\rangle\ 2!t^{n-2}\sum_{d_1+\cdots+d_n=3g'+n}^{\substack{\text{two } d_i\text{'s equal 0}\\ \text{all other equal 2}}}1.$$

The factor 2! comes from the differentiation of t_0^2. The choice of the d_i's yield a binomial $\binom{n}{2}$, which kills all terms where $n=1$. The dimension condition states $2n-2=3g'+n$, hence $n=3g+1$ (and

$g \geq 1$ since $n \geq 2$). One therefore has

$$\begin{aligned} F_{0,0}(0,0,t,0,...) &= \sum_{n=3g'+4,\ g\geq 1} t^{n-2} \frac{\langle \tau_0^2 \tau_2^{n-2} \rangle}{(n-2)!} = \sum_{g\geq 1} \frac{\langle \tau_0^2 \tau_2^{n-2} \rangle}{(n-2)!} t^{3g'+2} \\ &\overset{\text{Lemma C}}{=} t^2 \frac{(2\cdot 2-2)!!}{24} + \sum_{g\geq 2} \frac{\left\langle \tau_2^{3g'} \right\rangle}{(3g')!} (5g')^{\uparrow\uparrow 2} t^{3g'+2} \\ &= t^2 \left(\frac{1}{12} + \sum_{g\geq 2} (5g')^{\uparrow\uparrow 2} \frac{\left\langle \tau_2^{3g'} \right\rangle}{(3g')!} t^{3g'} \right) \\ &= t^2 \sum_{g\geq 1} \alpha_{g'} t^{3g'}. \end{aligned}$$

2. Similarly, when looking at $F_{0,0,0,0}(0,0,t,0,...)$, choosing the d_i's yields a binomial $\binom{n}{4}$ (killing all terms where $n \leq 3$), the dimension condition becomes $2(n-4) = 3g' + n$, hence $n = 3g + 5 = 3g' + 8$ and

$$F_{0,0,0,0}(0,0,t,0,...) = \sum_{g\geq 0}^{n=3g'+8} \frac{1}{n!} \left\langle \tau_0^4 \tau_2^{n-4} \right\rangle 4! t^{n-2} \binom{n}{4} = \sum_{g\geq 0}^{n=3g'+8} \frac{\left\langle \tau_0^4 \tau_2^{n-4} \right\rangle}{(n-4)!} t^{3g'+4}.$$

Lemma C yields the first two terms

$$t \times (4-3)!\,(2(4-3)-1)!! + t^4 \times \frac{(2\cdot 4-2)!!}{24} = t + 2t^4$$

as well as the coefficients when $g \geq 2$

$$\frac{\left\langle \tau_0^4 \tau_2^{n-4} \right\rangle}{(n-4)!} = \frac{\left\langle \tau_2^{3g'} \right\rangle}{(3g')!} (5g')^{\uparrow\uparrow 4} = \alpha_{g'} (5g'+4)(5g'+6) = \left(25g^2 - 1\right) \alpha_{g'},$$

(notice that, when $g = 1$, one obtains $24\alpha_0 = 2$), whence

$$F_{0,0,0,0}(0,0,t,0,...) = t + t^4 \sum_{g\geq 1} \left(25g^2 - 1\right) \alpha_{g'} t^{3g'}.$$

3. Finally, the series $F_{0,1}(0,0,t,0,...)$ involves a factor $n(n-1)$ (disposing of all terms where $n = 1$) while the dimension condition $2(n-2) + 1 = 3g' + n$ yields $n = 3g$ (and $g \geq 1$), whence the equality

$$F_{0,1}(0,0,t,0,...) = \sum_{g\geq 1}^{n=3g} \frac{1}{n!} \left\langle \tau_0 \tau_1 \tau_2^{n-2} \right\rangle t^{n-2} n(n-1) = \sum_{g\geq 1}^{n=3g} \frac{\left\langle \tau_0 \tau_1 \tau_2^{n-2} \right\rangle}{(n-2)!} t^{3g'+1}.$$

Lemma C provides the first term $\frac{\frac{1}{12}}{(3-2)!} t$ and the following $\frac{(5g')^{\uparrow\uparrow 2} \left\langle \tau_2^{3g'} \right\rangle}{(3g')!} t^{3g'+1} = \alpha_{g'} t^{3g'+1}$, hence

$$F_{0,1}(0,0,t,0,...) = t \sum_{g\geq 1} \alpha_{g'} t^{3g'}.$$

Now, one can write down the equality binding the three series above:

$$\frac{t}{12} + t \sum_{g\geq 2} \alpha_{g'} t^{3g'} = \frac{1}{2} \left(t^2 \sum_{g\geq 1} \alpha_{g'} t^{3g'} \right)^2 + \frac{t}{12} + t^4 \sum_{g\geq 1} \frac{25g^2 - 1}{12} \alpha_{g'} t^{3g'}.$$

Killing the $\frac{t}{12}$, simplifying by t^4, setting $T := t^3$ and $k := g'$ yields

$$\sum_{k\geq 0} \alpha_{k+1} T^k = \frac{1}{2} \left(\sum_{k\geq 0} \alpha_k T^k \right)^2 + \sum_{k\geq 0} \frac{25g^2 - 1}{12} \alpha_k T^k = \frac{1}{2} \sum_{k\geq 0} \left(\sum_{p+q=k} \alpha_p \alpha_q \right) T^k + \sum_{k\geq 0} \frac{25g^2 - 1}{12} \alpha_k T^k,$$

whence the announced recursion formula.

Lemma C. *For any stable (g, n) one has*

$$\frac{\left\langle \tau_0 \tau_1 \tau_2^{n-2} \right\rangle}{(n-2)!} = \begin{cases} \frac{(5g')^{\uparrow\uparrow 2} \left\langle \tau_2^{3g'} \right\rangle}{(3g')!} & \text{if} \quad g \geq 2 \\ \frac{1}{12} & \text{if} \quad g = 1 \end{cases} \tag{103}$$

and, for any positive integer k,

$$\frac{\left\langle \tau_0^k \tau_2^{n-k} \right\rangle}{(3g'+k)!} = \begin{cases} \frac{\left\langle \tau_2^{3g'} \right\rangle}{(3g')!} (5g')^{\uparrow\uparrow k} & \text{if} \quad g \geq 2 \\ \frac{(2k-2)!!}{24} & \text{if} \quad g = 1 \quad (\text{and } k \geq 1) \\ (2k-7)!! & \text{if} \quad g = 0 \quad (\text{and } k \geq 3) \end{cases} . \tag{104}$$

Proof of Lemma C. All relations can be proved by using the string and dilaton equations (Claim 5).

1. Let us first look at $\langle \tau_0 \tau_1 \tau_2^{n-2} \rangle$. The dimension condition says $(n-2)\,2+1 = 3g'+n$, hence $n = 3g'+3$. When $g \geq 2$, one has

$$\begin{aligned} \left\langle \tau_0 \tau_1 \tau_2^{n-2} \right\rangle &\overset{\text{dilaton}}{=} (2g'+n-1) \left\langle \tau_0 \tau_2^{n-2} \right\rangle \\ &\overset{\text{string}}{=} (5g'+2)(n-2) \left\langle \tau_1 \tau_2^{n-3} \right\rangle \\ &\overset{\text{dilaton}}{=} (5g'+2)(n-2)(5g') \left\langle \tau_1 \tau_2^{n-3} \right\rangle . \end{aligned}$$

When $g = 1$, do the same as above except the last use of the dilaton equation (which would be meaningless) and recall $\langle \tau_1 \rangle = \frac{1}{24}$:

$$\left\langle \tau_0 \tau_1 \tau_2^{n-2} \right\rangle = (5g'+2)(n-2) \left\langle \tau_1 \tau_2^{n-3} \right\rangle \overset{n=3}{=} 2 \cdot 1 \cdot \langle \tau_1 \rangle .$$

2. Let us now look at $\langle \tau_0^k \tau_2^{n-k} \rangle$. The dimension condition says $(n-k)\,2 = 3g'+n$, hence $n = 3g'+2k$. When $g \geq 2$, one has

$$\begin{aligned} \left\langle \tau_0^k \tau_2^{n-k} \right\rangle &\overset{\text{string}}{=} (n-k) \left\langle \tau_0^{k-1} \tau_1 \tau_2^{n-k-1} \right\rangle \overset{\text{dilaton}}{=} (n-k)(2g'+n-2) \left\langle \tau_0^{k-1} \tau_2^{n-k-1} \right\rangle \\ &\overset{\substack{\text{induction} \\ \text{on } k}}{=} (3g'+k)^{\downarrow k} (5g'+2k-2)^{\downarrow\downarrow k} \left\langle \tau_2^{n-2k} \right\rangle = (3g'+k)! \, (5g')^{\uparrow\uparrow k} \left\langle \tau_2^{3g'} \right\rangle . \end{aligned}$$

When $g < 2$, do the same as above until just before using the first meaningless equation. When $g = 1$, this happens just before the last dilaton equation and yields (recall $\langle \tau_1 \rangle = \frac{1}{24}$):

$$\left\langle \tau_0^k \tau_2^{n-k} \right\rangle_{g=1} = (0+k)^{\downarrow k} (0+2k-2)^{\downarrow\downarrow (k-1)} \left\langle \tau_1 \tau_2^0 \right\rangle = k! \, (2k-2)!! \, \langle \tau_1 \rangle .$$

When $g = 0$, one has to stop when one gets a τ_0 to the power 3 (use both string and dilaton equations $k-3$ times), which concludes (recall $\langle \tau_0^3 \rangle = 1$):

$$\left\langle \tau_0^k \tau_2^{n-k} \right\rangle_{g=0} = (-3+k)^{\downarrow (k-3)} (-5+2k-2)^{\downarrow\downarrow (k-3)} \left\langle \tau_0^3 \tau_2^{n-2k+2} \right\rangle = (k-3)! \, (2k-7)!! \, \langle \tau_0^3 \rangle .$$

4.6 Linking the constants c_g to the Bender-Gao-Richmond constants t_g

In [1], it is recalled that the number of g-genused rooted maps with n edges is, when n grows to ∞, asymptotically equivalent to $t_g 12^n n^{\frac{5}{2}g'}$ for some constants t_g and that the asymptotics of many other interesting families of maps behave like $\alpha t_g (\beta n)^{\frac{5}{2}g'} \gamma^n$ for some constants α, β, γ depending on the considered family. It is remarkable to find the same universal exponent $\frac{5}{2}g'$ as in the asymptotics of Hurwitz numbers. It is also worth noticing that the constants t_g can be very easily related to the contants c_g thanks to the following theorem.

Theorem 7. *One has for any genus $g \geq 0$ the equality*

$$c_g = \sqrt{2}^{g-3} t_g . \tag{105}$$

Proof. We shall prove that a sequence closely related to t_g satisfies the same recursion and has the same initial value as the sequence $(\alpha_{g'})$ of the Corollary 5.

Recall from [1] the definition of t_g in Theorem 1: *define $u_0 := \frac{1}{10}$ and for any $g \geq 1$*

$$u_g := u_{g'} + \sum_{i=1}^{g'} \frac{\left(\frac{1}{5}\right)^{\uparrow i} \left(\frac{1}{5}\right)^{\uparrow (g-i)}}{\left(\frac{1}{5}\right)^{\uparrow g}} \frac{\left(\frac{4}{5}\right)^{\uparrow (i-1)} \left(\frac{4}{5}\right)^{\uparrow (g-i-1)}}{\left(\frac{4}{5}\right)^{\uparrow g'}} u_i u_{g-i} ; \tag{106}$$

then, one has $t_g 8^{g'} \Gamma\left(\frac{5g-1}{2}\right) = \left(\frac{25}{12}\right)^g \left(\frac{1}{5}\right)^{\uparrow g} \left(\frac{4}{5}\right)^{\uparrow g'} u_g$ for any $g \geq 0$.

If one defines $v_g := \left(\frac{1}{5}\right)^{\uparrow g}\left(\frac{4}{5}\right)^{\uparrow g'} u_g$ for any $g \geq 0$ and notices that

$$\frac{\left(\frac{1}{5}\right)^{\uparrow g}\left(\frac{4}{5}\right)^{\uparrow g'}}{\left(\frac{1}{5}\right)^{\uparrow (g-1)}\left(\frac{4}{5}\right)^{\uparrow (g'-1)}} = \left(\frac{1}{5}+g-1\right)\left(\frac{4}{5}+g'-1\right) = \frac{5g-4}{5}\frac{5g-6}{5} = \frac{25g'^2-1}{25},$$

then the above recursion becomes $v_g = \frac{25g'^2-1}{25}v_{g'} + \sum_{i=1}^{g'} v_i v_{g-i}$. Setting $w_g := \left(\frac{25}{12}\right)^g v_g$ for any $g \geq 0$ leads to $w_g = \frac{25g'^2-1}{12}w_{g'} + \sum_{i+j=g}^{i,j>0} w_i w_j$. Setting $x_g := 2w_{g+1}$ for any $g \geq -1$ leads to $x_g = \frac{25g^2-1}{12}x_{g'} + \frac{1}{2}\sum_{i+j=g+1}^{i,j>0} x_{i-1}x_{j-1}$; since the previous sum can be rewritten $\sum_{p+q=g'}^{p,q\geq 0} x_p x_q$, one obtains the same recursion for the sequence $(x_{g'})$ as that defining $(\alpha_{g'})$. Once checked the initial values

$$x_{-1} = 2w_0 = 2v_0 = 2u_0 1\frac{1}{\left(\frac{4}{5}-1\right)^{\uparrow 1}} = -10u_0 = -1 = \alpha_{-1},$$

one may identify the sequences $x_{g'} = 2\left(\frac{25}{12}\right)^g\left(\frac{1}{5}\right)^{\uparrow g}\left(\frac{4}{5}\right)^{\uparrow g'} u_g = 2t_g 8^{g'}\Gamma\left(\frac{5g-1}{2}\right)$ and $\alpha_{g'} = c_g\Gamma\left(\frac{5g-1}{2}\right)2^{\frac{5g-1}{2}}$, hence $t_g 2^{3g'+1} = c_g 2^{\frac{5g-1}{2}}$ and the announced equality.

Examples. Recall in the values $t_0 = \frac{2}{\sqrt{\pi}}$, $t_1 = \frac{1}{24}$, $t_2 = \frac{7}{2^5 3^3 5\sqrt{\pi}}$, to be compared to those given in [9] page 19:

$$c_0 = \frac{1}{\sqrt{2\pi}},\ c_1 = \frac{1}{48},\ c_2 = \frac{1}{\sqrt{2\pi}}\frac{7}{2^5 3^3 5}. \tag{107}$$

5 Reduction formula

We now carry out with the most technical part of the paper. We explicit the induction formula used in [7] and derive Theorem 1.

5.1 The reduction formula

We first carry out an analysis of what becomes a constellation after merging its first two permutations. We reproduce mostly what is explained in [7].

Let $(\sigma, \rho, \sigma_3, \sigma_4, ..., \sigma_k)$ be a constellation and define $\pi := \sigma\rho$. One gets $k-1$ permutations $\pi, \sigma_3, ..., \sigma_k$ whose product is the identity, but one generally loses the transitivity condition. Set N for the number of orbits of our new group $\langle \pi, \sigma_3, ..., \sigma_k \rangle$, denote Ω any of the orbits and write σ_i^Ω for the permutation σ_i induced on Ω. One thus obtains N constellations $\left(\pi^\Omega, \sigma_3^\Omega, ..., \sigma_k^\Omega\right)$ whose degrees are the $|\Omega|$'s.

We haven't labelled the orbits because they are *a priori* indistinguishable, which will account for the appearing of a symmetry factor $\frac{1}{N!}$ in the reduction formula.

Notice than *the number N of orbits is bounded whatever the chosen constellation.* This is trivial when $S\sigma \cup S\rho$ is empty (since one has then $\sigma = \mathrm{Id}$ and $N = 1$) and let us explain why, when $S\sigma \cup S\rho$ is non empty, every orbit must intersect it (hence $N \leq \left|\mathring{\lambda}\right| + |\mathring{\mu}|$): if the group $\langle \sigma\rho, \sigma_3, ..., \sigma_k \rangle$ stabilised an orbit disjoint from $S\sigma \cup S\rho$, then so would the group $\langle \sigma, \rho, \sigma_3, ..., \sigma_k \rangle$ since σ and ρ acts trivially out of $S\sigma \cup S\rho$, but the latter group is by assumption transitive, so the mentionned orbit must equal all $[1, n]$, consequently intersecting $S\sigma \cup S\rho$, which is a contradiction. As a result, one always has $N \leq \left|\mathring{\lambda}\right| + |\mathring{\mu}| + 1$

The genera g^Ω's satisfy the Riemann-Hurwitz relation $2n^\Omega + 2g^{\Omega\prime} = r\left(\pi^\Omega\right) + \sum_{i=3}^k r\left(\sigma_i^\Omega\right)$. By summing up these relations and recalling that of our first constellation, one gets

$$\begin{aligned} 2n + 2\sum g^{\Omega\prime} &= \sum r\left(\pi^\Omega\right) + \sum_{i=3}^k \sum_\Omega r\left(\sigma_i^\Omega\right) \\ \sum_{i\geq 2} r(\sigma_i) - 2g' + 2\sum g^{\Omega\prime} &= r(\pi) + \sum_{i=3}^k r(\sigma_i) \\ \sum g^{\Omega\prime} &= g' - \frac{r(\lambda) + r(\mu) - r(\pi)}{2}. \end{aligned} \tag{108}$$

As a consequence (the fraction is non-negative), *the familly $\left(g^\Omega\right)$ of such genera is bounded whatever the chosen constellation*: $g^{\Omega\prime} \leq \sum_\omega g^{\omega\prime} \leq g'$.

Furthermore, since $S\pi \subset S\sigma \cup S\rho$, one can consider the type ν of $\pi_{|S\sigma\cup S\rho}$ as a partition of an integer smaller than $|S\sigma \cup S\rho| \leq |\lambda|+|\mu|$. Let us be more precise and set ν^{Ω} for the type of the permutation $\pi_{|S\sigma\cup S\rho}$ induced of Ω: the ν^{Ω}'s are all non empty (unless $S\sigma \cup S\rho = \emptyset$, namely unless $\mathring{\lambda} = \mathring{\mu} = \emptyset$) and their sizes always sum up to that of $|\nu|$. Then $\pi_{|S\sigma\cup S\rho}$ has $m_1\left(\nu^{\Omega}\right)$ fixed points in Ω and the knowledge of these fixed points for all Ω's allows one to rebuild $S\sigma \cup S\rho$ (add for any Ω these $m_1\left(\nu^{\Omega}\right)$ points to the support of π^{Ω}).

One can therefore assign to a constellation $(\sigma, \rho, \sigma_3, \sigma_4, ..., \sigma_k)$ the following data:

1. an integer $N \leq \left|\mathring{\lambda}\right| + |\mathring{\mu}| + 1$;
2. a set partition $\mathcal{O} = \{\Omega\}_{\Omega\in\mathcal{O}}$ of the underlying set into N orbits;
3. a family $\left(g^{\Omega}\right)_{\Omega\in\mathcal{O}}$ of genera (all $\leq g$) satisfying the relation $\sum_{\Omega} g^{\Omega\prime} = g' - \frac{r(\sigma)+r(\rho)-r(\sigma\rho)}{2}$;
4. a family $\left(\nu^{\Omega}\right)$ of partitions whose size is $|\nu| \leq |\lambda| + |\mu|$;
5. some families $\left(\lambda_3^{\Omega}\right), ..., \left(\lambda_k^{\Omega}\right)$ of partitions such that $\forall i,\ \text{type}\,(\sigma_i) = \bigsqcup^{\Omega} \lambda_i^{\Omega}$.

By collecting constellations according to these datas, one obtains the formula used by Zvonkine in [7] to prove that, with the only exception of empty partitions in genus 1, all series $H^g\left(\overrightarrow{\lambda}\right)$ lie in the algebra $\mathcal{A}$. However, this formula was not given explicitely: since we want to precisely compute the Z-degree of $H^g\left(\overrightarrow{\lambda}\right)$, we will carry out its making explicit. The reduction formula thus obtained relies only on a family $\left(f^{\overrightarrow{\nu}}_{\lambda,\mu}\right)$ of integers that we will define just after stating the reduction formula.

Theorem 8 (reduction formula). *Let $g \geq 0$ be a genus and $\overrightarrow{\lambda} = (\lambda, \mu, \lambda_3, \lambda_4, ..., \lambda_k)$ be k partitions. One has the following (k-)induction formula*

$$H^g\left(\overline{\lambda}, \overline{\mu}, \lambda_3, ..., \lambda_k\right) = \sum_{\overrightarrow{\nu}, \overrightarrow{g}} \frac{f^{\overrightarrow{\nu}}_{\lambda,\mu}}{N!} \sum_{\overrightarrow{\lambda_3}, ..., \overrightarrow{\lambda_k}} \prod_{j=1}^{N} H^{g^j}\left(\nu^j, \lambda_3^j, ..., \lambda_k^j\right) \tag{109}$$

where one sums overs

1. *integers $N \geq 1$ smaller or equal to $\left|\mathring{\lambda}\right| + |\mathring{\mu}| + 1$;*
2. *N-tuples $\begin{pmatrix}\overrightarrow{\nu}\\ \overrightarrow{g}\end{pmatrix}$ such that $g' = \frac{r(\lambda)+r(\mu)-r\left(\overrightarrow{\nu}\right)}{2} + \sum g^{j\prime}$ (all ν^j's being non-empty unless $\mathring{\lambda} = \mathring{\mu} = \emptyset$);*
3. *for any $i = 3, ..., k$ families of partitions $\left(\lambda_i^1, ..., \lambda_i^N\right)$ whose concatenation $\lambda_i^1 \sqcup \cdots \sqcup \lambda_i^N$ is λ_i (when $k = 2$, one sums (not over nothing but) over the empty list).*

It is easy to adapt the oncoming proof to derive the following reduction formula:

$$H^g\left(\overline{\lambda}, \overline{\mu}, \overline{\lambda_3}, ..., \overline{\lambda_k}\right) = \sum_{\overrightarrow{\nu}, \overrightarrow{g}} \frac{f^{\overrightarrow{\nu}}_{\lambda,\mu}}{N!} \sum_{\overrightarrow{\lambda_3}, ..., \overrightarrow{\lambda_k}} \prod_{j=1}^{N} H^{g^j}\left(\nu^j, \overline{\lambda_3^j}, ..., \overline{\lambda_k^j}\right). \tag{110}$$

Definition 3 (the numbers $f^{\overrightarrow{\nu}}_{\lambda,\mu}$). *Let N be a positive integer and set $N+2$ partitions $\lambda, \mu, \overrightarrow{\nu}$. For any j, consider Ω^j a $\left|\nu^j\right|$-sized set and π^j a ν^j-typed permutation in $\mathfrak{S}_{\Omega^j}$. Set $\Omega := \bigsqcup \Omega^j$ and $\pi := \prod \pi^j$. Define $f^{\overrightarrow{\nu}}_{\lambda,\mu}$ as the number of factorisations in $\mathfrak{S}_{\Omega}$ of the permutation π in a product $\sigma\rho$ satisfying the three conditions:*

1. *the types of σ and ρ are respectively $\overline{\lambda}$ and $\overline{\mu}$;*
2. *the supports of σ and ρ cover all Ω, namely $\text{Fix}\sigma \cap \text{Fix}\rho = \emptyset$;*
3. ***(junction condition)** for any $j \neq j'$, there is a finite sequence $j = j_0, ..., j_L = j'$ such that, for any $p = 1, ..., L$, there is a cycle of σ or ρ which intersects both orbits $\Omega^{j_{p-1}}$ and Ω^{j_p}.*

Remarks on the numbers $f^{\overrightarrow{\nu}}_{\lambda,\mu}$.
The first condition shows that $f^{\overrightarrow{\nu}}_{\lambda,\mu}$ does not depend of the parts equal to 1 in λ or μ:

$$f^{\overrightarrow{\nu}}_{\lambda,\mu} = f^{\overrightarrow{\nu}}_{\mathring{\lambda},\mathring{\mu}}.$$

The second condition shows that, whenever there exists such a factorisation, then $|\overrightarrow{\nu}| = |\Omega| = |S\sigma \cup S\rho|$ is smaller than $|S\sigma| + |S\rho| = \left|\mathring{\lambda}\right| + |\mathring{\mu}|$. In other words, one has the implication

$$f^{\overrightarrow{\nu}}_{\lambda,\mu} > 0 \implies |\overrightarrow{\nu}| \leq \left|\mathring{\lambda}\right| + |\mathring{\mu}|.$$

When $\overrightarrow{\nu}$ is made only with one partition, the junction condition vanishes.

When, moreover ν is the concatenation of λ and μ, the above inequality $|\overrightarrow{\nu}| \leq \left|\mathring{\lambda}\right| + |\mathring{\mu}|$ implies both λ and μ to be reduced and the supports to be disjoint. Then, choosing a factorisation amounts to choosing for any $k \geq 2$ which k-lengthed cycles of π will appear in σ. Therefore, one has $f_{\lambda,\mu}^{\mathring{\lambda} \sqcup \mathring{\mu}} = \prod_{k \geq 2} \binom{m_k(\lambda) + m_k(\mu)}{m_k(\lambda)}$, which can be rewritten in a more convient way (for future application) as

$$\frac{f_{\lambda,\mu}^{\mathring{\lambda} \sqcup \mathring{\mu}}}{m_1(\lambda)!\, m_1(\mu)!} \frac{\boxed{\mathring{\lambda} \sqcup \mathring{\mu}}}{\boxed{\lambda}\boxed{\mu}} = 1. \tag{111}$$

Examples.

Add a empty partition at the beginning of the list of partitions. The corresponding σ is the identity, hence $\sigma\rho = \rho$ and $\nu = \mathring{\mu}$. The junction condition then implies all cycles to be in the same orbit, hence $N = 1$. The genus relation then becomes $\overrightarrow{g} = (g)$. Since the above formula becomes $f_{\emptyset,\mu}^{\mathring{\mu}} = 1$, the reduction formula states

$$H^g(\emptyset, \overline{\mu}, \lambda_3, ..., \lambda_k) = H^g(\mathring{\mu}, \lambda_3, ..., \lambda_k)$$

which we know to be true since $H^g\left(\overline{\mu}, \overrightarrow{\lambda}\right) = H^g\left(\overline{\mathring{\mu}}, \overrightarrow{\lambda}\right)$ and since removing the bar above $\mathring{\mu}$ amounts to appplying $\binom{D - (|\mathring{\mu}| - m_1(\mathring{\mu}))}{m_1(\mathring{\mu})} = 1$.

Add now the partition (1^p). Again, the corresponding σ equals Id, hence $\nu = \mathring{\mu}$, $N = 1$, $\overrightarrow{g} = (g)$ and $f_{(1^p),\mu}^{\mathring{\mu}} = 1$, so that the reduction formula says

$$H^g\left(\overline{(1^p)}, \overline{\mu}, \lambda_3, ..., \lambda_k\right) = H^g(\mathring{\mu}, \lambda_3, ..., \lambda_k).$$

This was obvious anyway since (1^p) and $\emptyset$ have the same reduction.

Let us now compute $H^0((2),(2))$. This example is detailed in topological terms in [7] pages 34-35. Merging two transpositions yields either a commuting product of transpositions (hence $N = 1$, $\overrightarrow{\nu} = ((2,2))$ and $g^1 = g = 0$), either a 3-cycle (hence $N = 1$, $\overrightarrow{\nu} = ((3))$ and $g^1 = g - \frac{1+1-2}{2} = 0$), or the identity permutation (hence two cases: $N = 2$ and $\overrightarrow{\nu} = ((1),(1))$, or $N = 1$ and $\overrightarrow{\nu} = ((1,1))$). But the genus condition yields in the last case $-2 \leq g^{1\prime}\left(+g^{2\prime}\right) = g' - \frac{1+1-0}{2} = -2$, which forces $N = 2$ and $\left(g^1, g^2\right) = (0,0)$. The sum in Theorem 8 will therefore have three terms. It is moreover easy to compute the numbers $f_{(2),(2)}^{((2,2))} = 2$, $f_{(2),(2)}^{((3))} = 3$ and $f_{(2),(2)}^{((1),(1))} = 1$. Since the partition (2) has no fixed point, one can remove the top bars in Theorem 8:

$$\begin{aligned}
H^0((2),(2)) &= H^0\left(\overline{(2)}, (2)\right) = 2H^0(2,2) + 3H^0(3) + \frac{1}{2!}H^0(1)^2 \\
&= 2\boxed{(2,2)}H^0_{\substack{p=2\\a=4}} + 3\boxed{(3)}H^0_{\substack{p=1\\a=3}} + \frac{1}{2}\left(\boxed{(1)}H^0_{\substack{p=1\\a=1}}\right)^2 \\
&= 2\left(\frac{1}{2!}\frac{2^2 2^2}{2!2!}\right)\frac{Y^4}{4} + 3\left(\frac{3^3}{3!}\right)\frac{1}{3}\left(\frac{Y^3}{3} - \frac{Y^4}{4}\right) + \frac{1}{2}\left(\frac{1^2}{1!}\left(Y - \frac{Y^2}{2}\right)\right)^2 \\
&= Y^4 + \frac{9}{2}\left(\frac{Y^3}{3} - \frac{Y^4}{4}\right) + \left(\frac{Y^2}{2} - \frac{Y^3}{2} + \frac{Y^4}{8}\right) \\
&= Y^3 + \frac{Y^2}{2}.
\end{aligned}$$

But we already know from formula (51) that

$$\begin{aligned}
H^g((2),(2)) &= \left[2D + 2g' - r(2)\right] H^0((2)) = [2D - 3]\boxed{(2)}H^0_{\substack{p=1\\a=2}} = \frac{2^2}{2!}\left(2H^0_{\substack{p=2\\a=2}} - 3H^0_{\substack{p=1\\a=2}}\right) \\
&= 2\left(Y^2 - \frac{3}{2}\left(\frac{Y^2}{2} - \frac{Y^3}{3}\right)\right) = Y^3 + \frac{Y^2}{2}.
\end{aligned}$$

Proof of Theorem 8. Two lemmas will naturally be needed at some moments. They are given at the end of the proofs.

The number $h_n^g\left(\overline{\lambda}, \overline{\mu}, \lambda_3, ..., \lambda_k\right)$ enumerates (up to a factor $\frac{1}{n!}$ and some binomials) the constellations $(\sigma, \rho, \sigma_3, ..., \sigma_k, \tau_1, ..., \tau_T)$ of degree n, genus g and type $\left(\overline{\lambda}, \overline{\mu}, \overline{\lambda_3}, ..., \overline{\lambda_k}, \overline{2}, ..., \overline{2}\right)$. We will rely on the analysis presented at the beginning of this section. After setting $\pi := \sigma\rho$, one obtains N constellations $\left(\pi^\Omega, \sigma_3^\Omega, ..., \sigma_k^\Omega, \tau_1^\Omega, ..., \tau_{T^\Omega}^\Omega\right)$ on the orbits Ω, where the τ_i^Ω's are transpositions with $\sum^\Omega T^\Omega = T$. Notice that two transpositions τ_i^Ω and $\tau_{i'}^{\Omega'}$ commute if they lie in differents orbits $\Omega \neq \Omega'$. Therefore, when going backwards to reform the constellation , one will be allowed to interlace the N blocks of transpositions without changing the product $\sigma\rho$, hence $\binom{T}{(T^\Omega)}$ possible choices.

We now evaluate $h_n^g\left(\overline{\lambda},\overline{\mu},\lambda_3,...,\lambda_k\right)$ by enumerating (up to a factor $\frac{1}{n!}$) the ordered pairs formed on the one hand of a constellation $(\sigma,\rho,\sigma_3,...,\sigma_k,\tau_1,...,\tau_T)$ of degree n, genus g and type $\left(\overline{\lambda},\overline{\mu},\overline{\lambda_3},...,\overline{\lambda_k},\overline{2},...,\overline{2}\right)$, on the other hand for any $i\geq 3$ of a part of $m_1(\lambda_i)$ points in $[1,n]$.

Start by choosing the constellation according to the data given in the analysis above.

For this, choose a number of orbits $N\geq 1$, then the orbits Ω, then the types $\left(\nu^\Omega\right)$, then the types $\left(\lambda_i^\Omega\right)$ (for all $i=3,...,k$), then the genera $\left(g^\Omega\right)$. One obtains so far the following operator (which is a *finite* sum)

$$\sum_{N\geq 1}\ \sum_{\substack{\text{set partition } \mathcal{O} \text{ of}\\ \{1,...,n\} \text{ in } N \text{ orbits}}}\ \sum_{\substack{(\nu^\Omega)_{\Omega\in\mathcal{O}}\\ \sum|\nu^\Omega|\leq|\lambda|+|\mu|}}\ \sum_{\substack{\forall i=3,...,k,\\ \sum|\lambda_i^\Omega|=|\lambda_i|}}\ \sum_{\substack{\sum g^{\Omega\prime}\leq g'\\ \forall\Omega,\ g^\Omega\geq 0}}.$$

Choose then for any Ω a constellation $\left(\pi^\Omega,\sigma_3^\Omega,...,\sigma_k^\Omega,\tau_1^\Omega,...,\tau_{T^\Omega}^\Omega\right)$ on the set Ω of genus g^Ω and type $\left(\overline{\nu^\Omega},\overline{\lambda_3^\Omega},...,\overline{\lambda_k^\Omega},\overline{2},...,\overline{2}\right)$ as well as $m_1\left(\nu^\Omega\right)$ points fixed by π^Ω, which adds an operator

$$\prod_\Omega |\Omega|!\ h_{|\Omega|}^{g^\Omega}\left(\overline{\nu^\Omega},\overline{\lambda_3^\Omega},...,\overline{\lambda_k^\Omega},\overline{2},...,\overline{2}\right)\binom{|\Omega|-\left(\left|\nu^\Omega\right|-m_1\left(\nu^\Omega\right)\right)}{m_1\left(\nu^\Omega\right)}=\prod_\Omega |\Omega|!\ h_{|\Omega|}^{g^\Omega}\left(\nu^\Omega,\overline{\lambda_3^\Omega},...,\overline{\lambda_k^\Omega}\right)$$

It remains to choose a factorisation of $\pi:=\prod\pi^\Omega$ in a product $\sigma\rho$ satisfying the three following conditions:

1. σ and ρ have respective types $\overline{\lambda}$ and $\overline{\mu}$;
2. the union $S\sigma\cup S\rho$ equals $S\pi$ union the preceedingly-chosen points;
3. the group $\langle\sigma,\rho,\sigma_3,...,\sigma_k,\tau_1,...,\tau_T\rangle$ acts transitively.

Lemma D shows that the third condition amounts to a ***junction condition*** on the orbits Ω by the cycles of σ or ρ; Lemma E then shows that the number $f_{\lambda,\mu}^{\overrightarrow{\nu}}$ of such factorisations does not depend of neither n, g, the orbits Ω, the permutations π^Ω, nor the chosen points.

One will eventually not forget the choices of the transpositions.

Once chosen the constellation, the choice (with fixed i) of the $m_1(\lambda_i)=\sum_\Omega m_1\left(\lambda_i^\Omega\right)$ points in $[1,n]=\bigsqcup\Omega$ amounts to choosing for any Ω some points in Ω in number $m_1\left(\lambda_i^\Omega\right)$, hence for all Ω a factor

$$\prod_{i=3}^k\binom{n^\Omega-\left(\left|\lambda_i^\Omega\right|-m_1\left(\lambda_i^\Omega\right)\right)}{m_1\left(\lambda_i^\Omega\right)}.$$

Finally, one deduces from everything above that $h_n^g\left(\overline{\lambda},\overline{\mu},\lambda_3,...,\lambda_k\right)$ equals the following sum (for sake of lightness, the conditions on the "indexes" have been removed)

$$\frac{1}{n!}\sum_{\substack{\text{sets } \{\Omega\} \text{ of orbits, families}\\ (g^\Omega),\,(v^\Omega),\,(\lambda_3^\Omega),\,...,\,(\lambda_k^\Omega)}}\left[\prod_\Omega|\Omega|!h_{|\Omega|}^{g^\Omega}\left(\nu^\Omega,\overline{\lambda_3^\Omega},...,\overline{\lambda_k^\Omega}\right)\right]f_{\lambda,\mu}^{\overrightarrow{\nu}}\binom{T}{(T^\Omega)}\prod_\Omega^{i=3,...,k}\binom{|\Omega|-\left(\left|\lambda_i^\Omega\right|-m_1\left(\lambda_i^\Omega\right)\right)}{m_1\left(\lambda_i^\Omega\right)}.$$

The first sum can be rewritten as a sum over (unordered) *sets* of -tuples $\left(\Omega,g^\Omega,\nu^\Omega,\lambda_3^\Omega,...,\lambda_k^\Omega\right)$ or over *sets* of pairs $\left(\Omega,u^\Omega\right)$ where u^Ω stands short for the -tuple $\left(g^\Omega,\nu^\Omega,\lambda_3^\Omega,...,\lambda_k^\Omega\right)$. Since the conditions on the Ω's and on the u^Ω's are invariant under permutation and because they ensure the $\left(\Omega,u^\Omega\right)$ to be *distinct*, summing on sets $\left\{\left(\Omega,u^\Omega\right)\right\}_\Omega$ satisfying theses conditions amounts to $\frac{1}{N!}$ times summing on *families* $\left(\left(\Omega^j,u^j\right)\right)_{j=1,...,N}$ satisfying the same conditions. Clustering then the orbits according to their cardinalities allows one to describe the operator

$$\sum_{\substack{\text{sets } \{\Omega\} \text{ of orbits, families}\\ (g^\Omega),\,(v^\Omega),\,(\lambda_3^\Omega),\,...,\,(\lambda_k^\Omega)}}=\frac{1}{N!}\sum_{\overrightarrow{\lambda_3},\,...,\,\overrightarrow{\lambda_k}}^{\overrightarrow{g},\,\overrightarrow{v}}\ \sum_{\text{families } \overrightarrow{\Omega}}=\frac{1}{N!}\sum_{\overrightarrow{\lambda_3},\,...,\,\overrightarrow{\lambda_k}}^{\overrightarrow{g},\,\overrightarrow{v}}\ \sum_{n^1+\cdots n^N=n}\binom{n}{n^1,...,n^N}\sum_{\substack{\text{families } \overrightarrow{\Omega} \text{ such that}\\ \forall j=1,...,N,\ |\Omega^j|=n^j}}.$$

After integrating the binomials in the $h_{|\Omega|}^{g^\Omega}$'s and clustering the orbits according to the family $\left(n^1,...,n^N\right)$ of their cardinalities, the number $h_n^g\left(\overline{\lambda},\overline{\mu},\lambda_3,...,\lambda_k\right)$ becomes

$$\frac{1}{n!}\sum_{\overrightarrow{\lambda_3},\,...,\,\overrightarrow{\lambda_k}}^{\overrightarrow{g},\,\overrightarrow{v}}\frac{1}{N!}\sum_{n^1+\cdots n^N=n}\binom{n}{n^1,...,n^N}\left[\prod_{j=1}^N n^j!\ h_{n^j}^{g^j}\left(\nu^j,\lambda_3^j,...,\lambda_k^j\right)\right]f_{\lambda,\mu}^{\overrightarrow{\nu}}\binom{T}{T^1,...,T^N}.$$

After multiplying by $\frac{t^n}{T_n!}$, simplifying both multinomials and sharing out the powers of $t^n = \prod t^{n^j}$, one gets the equality (write T_{n^j} for T^j)

$$h_n^g\left(\overline{\lambda}, \overline{\mu}, \lambda_3, ..., \lambda_k\right) \frac{t^n}{T_n!} = \sum_{\overrightarrow{\lambda_3},\ ...,\ \overrightarrow{\lambda_k}}^{\overrightarrow{g},\ \overrightarrow{\nu}} \frac{f^{\nu}_{\lambda,\mu}}{N!} \sum_{n^1+\cdots+n^N=n} \left[\prod_j h^{g^j}_{n^j}\left(\nu^j, \lambda_3^j, ..., \lambda_k^j\right) \frac{t^{n^j}}{T_{n^j}!}\right].$$

Summing on the $n \geq 1$, one obtains by multidistributing

$$H^g\left(\overline{\lambda}, \overline{\mu}, \lambda_3, ..., \lambda_k\right) = \sum_{\overrightarrow{\lambda_3},\ ...,\ \overrightarrow{\lambda_k}}^{\overrightarrow{g},\ \overrightarrow{\nu}} \frac{f^{\nu}_{\lambda,\mu}}{N!} \prod_j H^{g^j}\left(\nu^j, \lambda_3, ..., \lambda_k\right),\ Q.\ E.\ D..$$

We now state and prove the two lemmas that were needed in our proof.

Lemma D. *Let $N \geq 1$ be an integer and $\Omega^1, ..., \Omega^N$ mutually disjoint sets. Set $\Omega := \bigsqcup \Omega^j$. For any $j \in [1, N]$, let π^j a permutation of Ω^j and G^j a transitive subgroup of $\mathfrak{S}_{\Omega^j}$. Assume the (disjoint) product $\prod \pi^j = \sigma\rho$ being factorised in $\mathfrak{S}_\Omega$. Then the group $\langle \sigma, \rho, G^1, ..., G^N \rangle$ acts transitively on Ω if and only if, for any $j \neq j'$ in $[1, N]$, there is a finite sequence $j = j_0, ..., j_L = j'$ such that, for any $p = 1, ..., L$, there is a cycle of σ or ρ which intersects both orbits $\Omega^{j_{p-1}}$ and Ω^{j_p}.*

Lemma E. *Set $N \geq 1$ an integer and let $\lambda, \mu, \overrightarrow{\nu}$ be $N+2$ partitions. For any choice of*

1. *an integer $n \geq |\overrightarrow{\nu}|$;*
2. *a composition $\left(n^1, ..., n^N\right)$ of n such as $n^j \geq \left|\nu^j\right|$ for any j;*
3. *a n^j-sized part $\Omega^j \subset \{1, ..., n\}$ for any j such as all Ω^j's are mutually disjoint;*
4. *a $\overline{\nu^j}$-typed permutation $\pi^j \in \mathfrak{S}_{\Omega^j}$ for any j;*
5. *a π^j-fixed part $F^j \subset \Omega^j$ of size $m_1\left(\nu^j\right)$ for any j,*

the number of factorisations in $\mathfrak{S}_n$ of $\pi := \prod \pi^j$ in a product $\sigma\rho$ satisfying the junction condition and both equalities $\operatorname{type}(\sigma, \rho) = \left(\overline{\lambda}, \overline{\mu}\right)$ *and $S\sigma \cup S\rho = S\pi \sqcup \bigsqcup F^j$ always equals $f^{\overrightarrow{\nu}}_{\lambda,\mu}$.*

Proof of lemma D.

$\boxed{\Longleftarrow}$ Let x, x' two points in Ω. Denote by ω, ω' and G, G' the corresponding orbits and subgroups. If x and x' lie in the same orbit, the transitivity of G concludes. If there is a cycle or σ or ρ which intersects both ω and ω', say in two points $(y, y') \in \omega \times \omega'$, then we have the following action $x \overset{G}{\longmapsto} x' \overset{\sigma \text{ or } \rho}{\longmapsto} y' \overset{G'}{\longmapsto} x'$ of the subgroup $\left\langle \sigma, \rho, G^1, ..., G^N \right\rangle$, hence the result by following the cycles of the sequence $j_0, ..., j_L$.

$\boxed{\Longrightarrow}$ Set any $j \neq j'$ and take a pair (x, x') of points in $\Omega^j \times \Omega^{j'}$. We have a permutation of $\left\langle \sigma, \rho, G^1, ..., G^N \right\rangle$ which sends x to x'. Write that permutation $*a*b*c*\cdots z*$ where all $*$'s are in $\left\langle G^1, ..., G^N \right\rangle$ and where letters $a, b, c, ..., z$ each denote a element of $\langle\sigma\rangle$ or $\langle\rho\rangle$. Since the $*$ can't make points get out of a given orbit Ω^j, only permutations $a, b, c, ..., z$ can, whence the sought cycle sequence.

Proof of lemma E.

Let us first fix n and the n^j's and set $\overrightarrow{\Omega}, \overrightarrow{\pi}, \overrightarrow{F}$ and $\overrightarrow{\Omega'}, \overrightarrow{\pi'}, \overrightarrow{F'}$ as above. Since π^j and $\pi^{j\prime}$ have same type, there is a bijection φ exchanging their *non-fixed cycles* for any j. Since F^j and $F^{j\prime}$ have same size and are π^j-fixes, one can complete φ so as to exchange them for any j. Finally, since the orbits Ω^j and Ω'^j have same size, one can complete φ in a permutation of all $\{1, ..., n\}$, which yields, by conjugation, a bijection between the factorisations associated with $\left(\overrightarrow{\Omega}, \overrightarrow{\pi}, \overrightarrow{F}\right)$ and those associated with $\left(\overrightarrow{\Omega'}, \overrightarrow{\pi'}, \overrightarrow{F'}\right)$.

Let us fix now any $\left(\overrightarrow{\Omega}, \overrightarrow{\pi}, \overrightarrow{F}\right)$ as above. Set $F := \bigsqcup F^j$ and $\Omega := \bigsqcup \Omega^j$. The condition $S\sigma \cup S\rho = S\pi \sqcup F \subset \Omega$ ensures that σ and ρ fix every point outside of Ω, so that all three permutations σ, ρ, π lie in $\mathfrak{S}_\Omega$ (and therefore all conditions happen in Ω). Therefore, one has a trivial bijection between: on the one hand, the factorisations associated with $\left(\overrightarrow{\Omega}, \overrightarrow{\pi}, \overrightarrow{F}\right)$ and the given composition $\overrightarrow{n}$; on the other hand, the factorisations associated with $\left(\overrightarrow{\Omega}, \overrightarrow{\pi}, \overrightarrow{F}\right)$ and the same but *reduced* composition $\overrightarrow{n} = \left(\left|\nu^j\right|\right)$. One could also see everything in $\mathfrak{S}_\infty$ with the corresponding trivial bijections.

5.2 Proof of the main theorem

We restate Theorem 1: *for any partitions* $\lambda_1, ..., \lambda_k$ *and any genus* $g \geq 0$*, one has the following* Z*-equality in the algebra* $\mathcal{A}^Z$ *for* M *large enough:*

$$D^M \mathbb{H}^g(\lambda_1, ..., \lambda_k) \overset{Z}{=} D^{M+m_1(\lambda_i)+\cdots+m_1(\lambda_k)} \mathbb{H}^g\left(\mathring{\lambda}_1 \sqcup \mathring{\lambda}_2 \sqcup \cdots \sqcup \mathring{\lambda}_k\right). \tag{112}$$

For the wondering reader, the exponent M is a trick to get rid of the exceptional cases. It is indeed a useful trick we frequently make use of and that therefore requires some details.

The D-trick.

As a first example, the information about the Z-degree in Corollary 3 can be stated without the stability condition by the simple equality $\deg_Z D^3 H^g(\lambda) = 5g + 2p + 1$ (one just has to check the four exceptional cases).

Let us prove the following generalisation *for any* $M \geq 0$: *if the series* $D^M H^g(\lambda)$ *lies in* $\mathcal{A}^Z$*, then it has degree*

$$\deg_Z D^M H^g(\lambda) = 2M + 5g' + 2p. \tag{113}$$

Indeed, setting $S := H^g(\lambda)$, one can write on the one hand $D^3\left(D^M S\right) = 2 \cdot 3 + \deg_Z D^M S$ and on the other hand $D^M\left(D^3 S\right) = 2M + 5g' + 2p + 6$; equalling both members leads to the conclusion.

Let us now prove the following equivalences *for any series* S *lying in* $\mathcal{A}$*:*

$$S \in \mathcal{A}^Z \iff \forall M \geq 0,\ \deg_Z D^M S \geq 2M \iff \exists M \geq 0,\ \deg_Z D^M S \geq 2M. \tag{114}$$

The arrows $\Longrightarrow$ are trivial (see claim on the powers of Z) and one even has a strict inequality. Conversely, if S is a polynomial $P(Y)$, then $DS = P'(Y)Z$ has Z-degree ≤ 1 and hence $D^M S = D^{M-1} DS$ has degree $\leq 1 + 2(M-1)$.

Finally, let us prove the following corollary of Theorem 1.

Corollary 6 (which series H^g lie in $\mathcal{A}^Z$). *For any non-empty partitions* $\lambda_1, ..., \lambda_k, \lambda, \mu$*:*

1. $H^g(\lambda_1, ..., \lambda_k)$ *always lies in* $\mathcal{A}^Z$ *when* $k \geq 3$.
2. $H^g(\lambda, \mu)$ *does **not** lie in* $\mathcal{A}^Z$ *if and only if* $g = 0$ *and if both* λ *and* μ *have one part.*
3. $H^g(\lambda)$ *does **not** lie in* $\mathcal{A}^Z$ *if and only if* $\begin{pmatrix} g \\ l(\lambda) \end{pmatrix} \in \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$.

Proof. Take the Z-degree in the given Z-equality and use Theorem 4:

$$\begin{aligned} \deg D^M \mathbb{H}^g(\lambda_1, ..., \lambda_k) &= 2M + 2\sum m_1(\lambda_i) + \left(5g' + 2\sum l\left(\mathring{\lambda}_i\right)\right) \\ &= 2M + 5g' + 2\sum l(\lambda_i). \end{aligned}$$

Since all lengths are ≥ 1, the above degree is $\geq 2M$ when $k \geq 3$. When $k = 2$, the above degree is $< 2M$ if and only if $g = 0$ and $l(\lambda_i) = 1$ for $i = 1, 2$. When $k = 1$, one retrieves the already-known exceptional cases of Theorem 4.

We now proceed with the proof of Theorem 1.

To derive the wanted Z-equality from Theorem 8, one has to analyse the contribution in Z of each product $\prod H^{g^j}$, which invites us to carry out an induction on the number k of partitions. We will require an inequality (Lemma F) which we state (and prove) at the end of the proof.

We will thus prove the following reduction Z-formula:

$$H^g\left(\overline{\lambda}, \overline{\mu}, \lambda_3, ..., \lambda_k\right) \overset{Z}{=} f_{\lambda,\mu}^{\mathring{\lambda} \sqcup \mathring{\mu}} H^g\left(\mathring{\lambda} \sqcup \mathring{\mu}, \lambda_3, ..., \lambda_k\right). \tag{115}$$

Notice it already stands as a plain equality when $\mathring{\lambda} = \mathring{\mu} = \emptyset$, the reason for which we will leave that case aside below. From then on, it will be easy to conclude. To remove the bars on top of λ and μ, one has to multiply by the binomials $\begin{pmatrix} D - (|\lambda| - m_1(\lambda)) \\ m_1(\lambda) \end{pmatrix} \begin{pmatrix} D - (|\lambda| - m_1(\lambda)) \\ m_1(\lambda) \end{pmatrix}$; since D strictly increases $\deg_Z$, one can multiply instead by $\frac{D^{m_1(\lambda)+m_1(\mu)}}{m_1(\lambda)!\ m_1(\mu)!}$ and still gets a Z-equality:

$$H^g(\lambda, \mu, \lambda_3, ..., \lambda_k) \overset{Z}{=} \frac{D^{m_1(\lambda)+m_1(\mu)}}{m_1(\lambda)!\ m_1(\mu)!} f_{\lambda,\mu}^{\mathring{\lambda} \sqcup \mathring{\mu}} H^g\left(\mathring{\lambda} \sqcup \mathring{\mu}, \lambda_3, ..., \lambda_k\right).$$

To get from H to $\boxed{H}$, divide both sides by $\boxed{\lambda}\boxed{\mu}\boxed{\lambda_3}\cdots\boxed{\lambda_k}$. Then remember Formula 111 and use the induction hypothesis:

$$\begin{aligned} \boxed{H}^g(\lambda,\mu,\lambda_3,...,\lambda_k) &\overset{Z}{=} D^{m_1(\lambda)+m_1(\mu)}\boxed{H}^g\left(\mathring{\lambda}\sqcup\mathring{\mu},\lambda_3,...,\lambda_k\right) \\ &\overset{Z}{=} D^{m_1(\lambda)+m_1(\mu)}D^{0+m_1(\lambda_3)+\cdots+m_1(\lambda_k)}\boxed{H}^g\left(\mathring{\lambda}\sqcup\mathring{\mu}\sqcup\mathring{\lambda}_3\sqcup\cdots\sqcup\mathring{\lambda}_k\right), \quad Q.E.D.. \end{aligned}$$

The case $k=1$ is an immediate corollary of Corollary 3. Because of the number of exceptional cases, the case $k=2$ will be the longest to deal with, the case $k=3$ much similar and much easier, and greater k's will be straightforward. We start with $k\geq 4$ to get used to the idea, then $k=3$ and finally $k=2$, the induction hypothesis allowing to use the corresponding parts of Corollary 6 above.

$\boxed{k\geq 4}$. Theorem 8 implies

$$H^g\left(\overline{\lambda},\overline{\mu},\lambda_3,...,\lambda_k\right)=\sum_{\overrightarrow{\nu},\overrightarrow{g}} f^{\overrightarrow{\nu}}_{\lambda,\mu}\sum_{\overrightarrow{\lambda_3},...,\overrightarrow{\lambda_k}}\prod_j H^{g^j}\left(\nu^j,\lambda_3^j,...,\lambda_k^j\right)$$

where every factor $H^{g^j}\left(\nu^j,\lambda_3^j,...,\lambda_k^j\right)$ lies in $\mathcal{A}^Z$ by the Corollary 6 (remind all ν^j's to be non empty since we left aside the case $\mathring{\lambda}=\mathring{\mu}=\emptyset$). The product $\prod H^{g^j}\left(\nu^j,\lambda_3^j,...,\lambda_k^j\right)$ has therefore Z-degree

$$\begin{aligned} &\sum_j 5g^{j\prime}+2\left(l\left(\nu^j\right)+\sum_{i\geq 3} l\left(\lambda_i^j\right)\right) \\ =\;& 5g'-5\frac{r_\lambda+r_\mu}{2}+\frac{5}{2}\left(\left|\overrightarrow{\nu}\right|-l\left(\overrightarrow{\nu}\right)\right)+2l\left(\overrightarrow{\nu}\right)+2\sum_{i\geq 3} l\left(\lambda_i\right) \\ =\;& 2\sum_{i\geq 3} l\left(\lambda_i\right)+5g'-\frac{5}{2}\left(r_\lambda+r_\mu\right)+\frac{5\left|\nu\right|-l\left(\nu\right)}{2}. \end{aligned}$$

Everything is constant except $\frac{5|\nu|-l(\nu)}{2}$. Lemma F then shows that the above quantity is maximal if and only if $\nu=\mathring{\lambda}\sqcup\mathring{\mu}$; since this implies $N=1$ and $\overrightarrow{g}=(g)$, we can deduce that the term $\prod_j H^{g^j}\left(\nu^j,\lambda_3^j,...,\lambda_k^j\right)$ of maximal Z-degree in the sum $H^g\left(\overline{\lambda},\overline{\mu},\lambda_3,...,\lambda_k\right)$ is precisely $H^g\left(\mathring{\lambda}\sqcup\mathring{\mu},\lambda_3,...,\lambda_k\right)$, which yields the announced Z-reduction formula.

$\boxed{k=3}$. We go along the same idea. Fix a genus $g\geq 0$ and three partitions λ,μ,ξ. Let $p:=l(\nu)+l(\xi)$ and p^j defined alike for all j. Theorem 8 then implies for any integer $M\geq 0$

$$D^M H^g\left(\overline{\lambda},\overline{\mu},\xi\right)=\sum_{\overrightarrow{\nu},\overrightarrow{g},\overrightarrow{\xi},\overrightarrow{M}} f^{\overrightarrow{\nu}}_{\lambda,\mu}\binom{M}{\overrightarrow{M}}\prod_j D^{M^j}H^{g^j}\left(\nu^j,\xi^j\right)$$

where the sum over $\overrightarrow{M}$ is taken over the N-tuples of non-negative integers M^j which sum up to M. By the induction hypothesis for $k=2$, the term $D^M H^g\left(\mathring{\lambda}\sqcup\mathring{\mu},\xi\right)$ lies in $\mathcal{A}^Z$ for M large enough. Fix such an M. We then show that all other terms have Z-degree smaller than the latter.

By Corollary 6 for two partitions, a factor $D^{M^j}H^{g^j}\left(\nu^j,\xi^j\right)$ will belong to $\mathbb{Q}[Y]$ if and only if $\left(g^j,p^j,M^j\right)=(0,2,0)$; multiplying by such an element will decrease the Z-degree (strictly if and only if its (Y-)coefficients sum up to zero). As for the other factors, the D-trick combined with Corollary 6 for two partitions shows that their Z-degree is $5g^{j\prime}+2p^j+2M^j$. The product $\prod_j D^{M^j}H^{g^j}\left(\nu^j\right)$ has therefore Z-degree $\leq\sum_Z 5g^{j\prime}+2p^j+2M^j$ where the index Z means that $D^{M^j}H^{g^j}\left(\nu^j\right)$ lies in $\mathcal{A}^Z$ (we will index "no Z" otherwise).

Set $e:=\#\left\{j;\ \left(g^j,p^j,M^j\right)=(0,2,0)\right\}$ for the number of (exceptional) factors with no Z. The three previous Z-sums can be linked to the same sums without restriction:

$$\begin{aligned} \sum_Z g^{j\prime} &= \sum_{\text{no } Z}1+\sum_j g^{j\prime}=e+g'-\frac{r_\lambda+r_\mu-r_{\overrightarrow{\nu}}}{2}, \\ \sum_Z p^j &= \left(l\left(\overrightarrow{\nu}\right)-e\right)+\left(l\left(\xi\right)-e\right)=l\left(\overrightarrow{\nu}\right)+l\left(\xi\right)-2e, \\ \sum_Z M^j &= M. \end{aligned}$$

One can thus derive the majoration

$$\begin{aligned}\deg_Z \prod_j D^{M^j} H^{g^j}\left(\nu^j,\xi^j\right) &\le 5\sum_Z g^{j\prime} + 2\sum_Z p^j + 2\sum_M M^j \\ &= 5e + 5g' - \frac{5}{2}\left(r_\lambda + r_\mu\right) + \frac{5\left|\overrightarrow{\nu}\right| - 5l\left(\overrightarrow{\nu}\right)}{2} \\ &\quad + 2l\left(\overrightarrow{\nu}\right) + 2l\left(\xi\right) - 4e + 2M \\ &= 2M + 5g' - \frac{5}{2}\left(r_\lambda + r_\mu\right) + l\left(\xi\right) + \frac{5\left|\nu\right| - l\left(\nu\right)}{2} + e.\end{aligned}$$

As above, everything is constant except $\frac{5|\nu|-l(\nu)}{2} + e$; since there is (thanks to the trick of applying D) at least one $M^j \ge 1$, one has $e \le N-1 \le 3(N-1)$ and Lemma F still holds: the maximal-Z-degreed term $\prod_j D^{M^j} H^{g^j}\left(\nu^j,\xi^j\right)$ in the sum $D^M H^g\left(\overline{\lambda},\overline{\mu},\xi\right)$ is precisely $D^M H^g\left(\mathring{\lambda} \sqcup \mathring{\mu},\xi\right)$. One then concludes exactly the same way as in the case $k=4$.

$\boxed{k=2}$.

The proof goes as above. Fix $g \ge 0$ any genus and λ,μ two partitions. Theorem 8 implies for any $M \ge 0$

$$D^M H^g\left(\overline{\lambda},\overline{\mu}\right) = \sum_{\overrightarrow{\nu},\overrightarrow{g},\overrightarrow{M}} f^{\overrightarrow{\nu}}_{\lambda,\mu}\binom{M}{\overrightarrow{M}}\prod_j D^{M^j} H^{g^j}\left(\nu^j\right).$$

By Corollary 6, a factor $D^{M^j} H^{g^j}\left(\nu^j\right)$ will belong to $\mathbb{Q}[Y]$ if and only if $\begin{pmatrix} g^j \\ p^j \\ M^j\end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1\end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0\end{pmatrix}$ or $\begin{pmatrix} 0 \\ 2 \\ 0\end{pmatrix}$. For the other factors, we have already stated their degrees were $5g^{j\prime} + 2p^j + 2M^j$. The product $\prod_j D^{M^j} H^{g^j}\left(\nu^j\right)$ has therefore Z-degree $\le \sum_Z 5g^{j\prime} + 2p^j + 2M^j$. Now link the Z-sums to the (no Z)-sums:

$$\begin{aligned}\sum_Z g^{j\prime} &= \sum_{\text{no } Z} 1 + \sum g^{j\prime} = \#\left\{j;\ \begin{pmatrix} g^j \\ p^j \\ M^j\end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1\end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0\end{pmatrix} \text{ or } \begin{pmatrix} 0 \\ 2 \\ 0\end{pmatrix}\right\} + g' - \frac{r_\lambda + r_\mu - r_\nu}{2}, \\ \sum_Z p^j &= l(\nu) - \#\left\{j;\ \begin{matrix} g^j = 0 \\ p^j = 1 \\ M^j = 0 \text{ or } 1\end{matrix}\right\} - 2\#\left\{j;\ \begin{matrix} g^j = 0 \\ p^j = 2 \\ M^j = 0\end{matrix}\right\}, \\ \sum_Z M^j &= M - \#\left\{j;\ \begin{matrix} g^j = 0 \\ p^j = 1 \\ M^j = 1\end{matrix}\right\}.\end{aligned}$$

One can thus derive the majoration

$$\begin{aligned}\deg_Z \prod_j D^{M^j} H^{g^j}\left(\nu^j\right) &\le 5\sum_Z g^{j\prime} + 2\sum_Z p^j + 2\sum_M M^j \\ &= 2M + 5g' - \frac{5}{2}\left(r_\lambda + r_\mu\right) + \frac{5}{2}\left(|\nu| - l(\nu)\right) + 2l(\nu) \\ &\quad + 5\#\left\{j;\ \begin{pmatrix} g^j \\ p^j \\ M^j\end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1\end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0\end{pmatrix} \text{ or } \begin{pmatrix} 0 \\ 2 \\ 0\end{pmatrix}\right\} - 4\#\left\{j;\ \begin{matrix} g^j = 0 \\ p^j = 2 \\ M^j = 0\end{matrix}\right\} \\ &\quad - 2\#\left\{j;\ \begin{matrix} g^j = 0 \\ p^j = 1 \\ M^j = 0 \text{ or } 1\end{matrix}\right\} - 2\#\left\{j;\ \begin{matrix} g^j = 0 \\ p^j = 1 \\ M^j = 1\end{matrix}\right\} \\ &= 2M + 5g' - \frac{5}{2}\left(r_\lambda + r_\mu\right) + \frac{5|\nu| - l(\nu)}{2} \\ &\quad + \#\left\{j;\ \begin{matrix} g^j = 0 \\ p^j = 1 \\ M^j = 1\end{matrix}\right\} + 3\#\left\{j;\ \begin{matrix} g^j = 0 \\ p^j = 1 \\ M^j = 0\end{matrix}\right\} + \left\{j;\ \begin{matrix} g^j = 0 \\ p^j = 2 \\ M^j = 0\end{matrix}\right\}.\end{aligned}$$

The three sets whose cardinalities are involved being mutually disjoint, the corresponding sum is $\le N$ and one can even remplace N by $N-1$ if there is at least one $M^j \ge 2$, which can be realised by choosing $M \ge 2\left(\left|\mathring{\lambda}\right| + |\mathring{\mu}| + 1\right) \ge 2N$. Therefore, one can still apply Lemma F and conclude, which finishes the proof of the Theorem 1.

Lemma F. *Let λ, μ be two partitions and σ, ρ two permutations in $\mathfrak{S}_\infty$ of type $\left(\overline{\lambda}, \overline{\mu}\right)$. Denote by ν the partition of $\sigma\rho$ induced on $S\sigma \cup S\rho$. Cluster the cycles of ν into N orbits such that the junction condition is satisfied. Then the quantity $\frac{5|\nu|-l(\nu)}{2} + 3(N-1)$ is maximal if and only if σ and ρ have disjoint supports. (And, in that case, one has $N = 1$.)*

Remarks. To get the intuition of it, make the intersection of the supports $S\sigma$ and $S\rho$ growing one by one from the empty set (the announced equality case): when a point is added to $S\sigma \cap S\rho$, the size decreases quicker than the length thanks to the factor 5. (Of course, that intution does not explain why the factor 5 suffices to kill the annoying term $3(N-1)$.)

Proof. Call a cycle of σ or ρ to be ***interlaced*** if it intersects another cycle of σ or ρ (and two such cycles will be called ***interlaced with*** each other). Set c for the number of interlaced cycles and c' for the number of cycles (included fixed cycles) of the product $\sigma\rho$ induced on the interlaced cycles (of σ and ρ).

A crucial remark is the following: for the junction condition to be satisfied, every cycle of ν must lie in the same orbit as an interlaced cycle, hence the inequality $N \leq c'$.

If one sets $k := |S\sigma \cap S\rho|$ for the number of contact points of the supports, one can write

$$\begin{cases} |\nu| = \left|\mathring{\lambda}\right| + |\mathring{\mu}| - k \\ l(\nu) = l\left(\mathring{\lambda}\right) + l(\mathring{\mu}) - c + c' \end{cases},$$

hence the quantity to be majorised

$$Q := \frac{5(-k) - (c' - c)}{2} + 3(N-1). \tag{116}$$

When $S\sigma \cap S\rho = \emptyset$, all variables $c, c', k, N-1$ equal 0 and so does Q. One has therefore to show $Q < 0$, namely $-2Q \geq 1$, for any other ν than $\mathring{\lambda} \sqcup \mathring{\mu}$. By the crucial remark, it suffices to show the same inequality $5k + c' - c - 2(3N-3) \overset{?}{\geq} 1$ with some N's been replaced by the same number of c''s: so as to kill the c' in the inequality, we remplace one N out of six, which lead us to muse over the inequality $5(k+1-N) \overset{?}{\geq} c$. We are going to show by induction on $|S\sigma| + |S\rho|$ the stronger inequality

$$2(k - N + 1) \overset{?}{\geq} c. \tag{117}$$

When $\sigma = \rho = \mathrm{Id}$, then all three quantitites $c, k, N-1$ equal 0, hence the inequality (even though ν equals $\mathring{\lambda} \sqcup \mathring{\mu}$).

Assume now $\left|\mathring{\lambda}\right| + |\mathring{\mu}| > 0$. Because on the assumption $\nu \neq \mathring{\lambda} \sqcup \mathring{\mu}$ one has $k \geq 1$: take one contact point x in $S\sigma \cap S\rho$, set $y := \sigma(x)$ and $\tau := (x, y)$ the transposition exchanging these points. Finally, write $\sigma = \tau\sigma_*$ where $\sigma_* := \tau\sigma$ fixes x and therefore satifies $|\sigma^*| < |\sigma|$. Thus, one obtains the cycle decomposition of $\sigma\rho$ by multiplying that of $\sigma_*\rho$ by the transposition τ on the left (and conversely). Denote by a $*$-subscript the quantities c_*, k_*, N_* associated to the product of σ_* and ρ; notice that N_* is not well-defined and can be chosen arbitrarily as long as the junction condition is satisfied. For such an N_*, one has the induction hypothesis

$$c_* \leq 2(k_* - N_* + 1).$$

What we want is to dispose of the $*$'s.

Since x is fixed by σ_*, it disappears from the contact points, hence $k_* < k$. Besides, σ_* loses at most one interlaced cycle (it can only be the σ-orbit of x) and ρ loses at most two interlaced cycles (those maybe interlaced with τ), hence $c_* \geq c - 3$. But the case $c_* = c - 3$ implies the σ-orbit of x to be a transposition interlaced with two ρ-cycles, each of which not being interlaced with another σ-cycle; since σ and ρ play symmetric roles (set $y := \rho(x)$ instead of $\sigma(x)$), one can avoid this case and hence assume $c_* \geq c - 2$.

Let us look at what happens to the cycles of $\sigma\rho$ when composing (on the left) by τ. If a ($\sigma\rho$-)cycle γ is split in two cycles, cluster them in the same orbit as that of γ (hence $N_* = N$). If two cycles are joined, either both cycles were in the same orbit (then, do not change the orbits, hence $N_* = N$) or they were in distinct orbits (then, merge these orbits and do not change the others, hence $N_* = N - 1$). Whenever $N_* = N$, one can conclude by writing

$$c \leq c_* + 2 \leq 2(k_* - N_* + 1) + 2 \leq 2((k-1) - N + 1) + 2 = 2(k - N + 1), \ Q.\ E.\ D..$$

We can consequently assume $N_* = N - 1$ and hence τ *joining* two $\sigma\rho$-cycles, which goes the same as saying x and y not to lie in the same $\sigma\rho$-orbit. But that implies both σ- and ρ-orbits of x to remain interlaced for σ_* and ρ (if not, iterate $\sigma\rho$ in a not-interlaced orbit to join x and y), hence $c_* = c$ and the induction hypothesis yields

$$c = c_* \leq 2(k_* - N_* + 1) \leq 2((k-1) - (N-1) + 1) = 2(k - N + 1), \ Q.\ E.\ D..$$

6 References

References

[1] Edward A. Bender, Zhicheng Gao, and L. Bruce Richmond. The map asymptotics constant t_g. *The Electronic Journal of Combinatorics*, 15(1)(R51), Mar 27 2008.

[2] T. Ekedahl, S. K. Lando, M. Shapiro, and A. Vainshtein. Hurwitz numbers and intersections on moduli spaces of curves. *Inventiones Mathematicae*, 106:297–327, 2001. arxiv:math.AG/0004096.

[3] A. Hurwitz. Uber die anzahl der riemann'schen flächen mit gegebenen verzweigungspunkten. *Math. Ann. 55 (1902), 53-66.*, 55:53–66, 1902.

[4] M. Kazarian. On computations of hurwitz numbers. Unpublished (in Russian).

[5] S. Lando and A. Zvonkin. *Graph on surfaces and their applications.* Springer, 2004.

[6] A. Okounkov and R. Pandharipande. Gromov-witten theory, hurwitz numbers, and matrix models, I. 2001. arXiv:math/0101147v2 [math.AG].

[7] D. Zvonkine. Counting ramified coverings and intersection theory on hurwitz spaces II. 2003. arxiv:math.AG/0304251.

[8] D. Zvonkine. An algebra of power series arising in the intersection theory of the moduli spaces of curves and in the enumeration of ramified coverings of the sphere. 2004. arxiv:math.AG/0403092.

[9] D. Zvonkine. Enumeration of ramified coverings of the sphere and 2-dimensional gravity. 2005. arxiv:math.AG/0506248.

[10] D. Zvonkine. Intersection theory of moduli spaces of stable curves, Mar 1 2008. Lecture given during the *Journées mathématiques de Glanon.*