Divers

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http ://en.wikipedia.org/wiki/Lubell-Yamamoto-Meshalkin_inequality
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1 Sur les permutations

Soit S un ensemble de cardinal n et S_k l'ensemble des parties de S à k éléments. k étant fixé, peut-on injecter pour n assez grad $S_k \hookrightarrow S_{k+1}$ de sorte que $A \subset \iota(A)$ pour tout $A \in S_k$?

Clairement, il faut $n \leq 2k+1$ par des considérations sur les cardinaux. Est-ce suffisant ?

1.1 1

En fait, by a theorem of de Bruijn, Tenbergen and Kruyswijk, les parties de S peuvent s'écrire comme l'union disjointe de chaînes symétriques, où une chaîne symétrique consiste en un emboitement de parties

$$A_1 \subset A_2 \subset ... \subset A_m$$

avec $|A_{i+1}| = |A_i| + 1$ et $|A_1| + |A_m| = n$.

See e.g. chapter 3 of I. Anderson, "Combinatorics of Finite Sets",
Oxford 1987, ISBN 0-19-853367-5 and 0-19-853379-9.

1.2 2

This is done with the marriage lemma. Viz. : if $R \subset X \times Y$ is a binary relation between finite sets X and Y, such that for every subset E of X the cardinality of the projection in Y of $R \cap (E \times Y)$ (i.e., the image of E in Y) is at least equal to the cardinality of E, then there exists an injetion of X in Y whose graph is included in R).

With the notation above, let X be S_k and Y be S_{k+1}, and let R be the relation "$A \subset B$" (for A in X and B in Y). The marriage lemma shows that to obtain an f such as you ask, it is sufficient that, for any set E of m subsets of S of size k, the set E' obtained by adding one element in every possible way to every element of E is of cardinality at least m. But the relation R restricted to $E \times E'$ matches each element of E with exactly $n-k$ elements of E' and each element of E' with at most $k+1$ elements of E ; so as long as $n-k \geq k+1$, i.e. $n \geq 2k+1$, we have indeed $|E'| \geq |E|$ (by double counting).

1.3 3

It is sufficient indeed. Let S have $2k+1$ elements and consider the poset $(P(S),\subset)$ of subsets of S, ordered by set-theoretic inclusion. A chain on $P(S)$ is a collection $\mathcal{C}$ of subsets of S, such that every two elements of $\mathcal{C}$ are comparable by $\subset$. An antichain is a collection $\mathcal{A}$ of such subsets, such that *no* two elements of $\mathcal{A}$ are so comparable.

A well-known theorem of Sperner's states that the largest antichain in $P(S)$ has size $N = \binom{2k+1}{k}$

http ://en.wikipedia.org/wiki/Sperner_family

Next, one invokes Dilworth's theorem, which says that, if a poset's largest antichain has size k, then that poset is the union of k chains.

http ://en.wikipedia.org/wiki/Dilworth's_theorem

Combining these, we may decompose $P(S)$ into the union of N chains. Let $\mathcal{D}$ be the set of chains in this decomposition.

It is clear that any chain in $P(S)$ has at most one element of each cardinality. Further, both S_k and S_{k+1} have N elements, so each chain in $\mathcal{D}$ must contain exactly one element from each of these. Now build your function from that : given $A \in S_k$, there is a unique chain $C(A)$ in $\mathcal{D}$ such that $A \in C(A)$; define $f(A)$ to be the unique element of $S_{k+1} \cap C(A)$.

If you're interested, I have a couple posts on my weblog about Dilworth's theorem, with a visual proof :
http ://pietrokc.wordpress.com/2008/03/06/dilworths-theorem-i/
http ://pietrokc.wordpress.com/2008/03/17/dilworths-theorem-ii/

2 Petit pb de combi

Soient k et m deux entiers fixes, avec $k \geq 2$, $m \geq 5$ et $m \geq k+2$.

Soient $x_1, ..., x_m$ des reels (ou des complexes).

Pour I inclus dans $\{1, ..., m\}$ de cardinal k, on considere la somme $s(I)$ des x_i, pour i appartenant à I.

Il est facile de voir que si $s(I)$ est nul pour tout I (de cardinal k fixe), alors les x_i sont tous nuls.

Ma question est la suivante : quel est le plus petit entier u tel que l'existence de u parties I (de cardinal k) telles que $s(I) = 0$ implique que les x_i sont tous nuls ?

On a necessairement $u \geq 1 + \binom{m-1}{k}$ en prenant x_1 non nul et les autres x_i nuls ; on a en effet dans ce cas $s(I) = 0$ des que I est contenu dans $\{2, .., m\}$.

Hé bien $u = 1 + \binom{m-1}{k}$ justement, non ?

En effet tu regardes des matrices à m colonnes et u lignes distinctes telles qu'on ait k '1' et $m-k$ '0' dans chaque ligne, et tu cherches un u minimal tel que toute matrice de ce type ait un rang m.

On peut donc regarder un u maximal compatible avec un rang $m-1$. On a des lignes génératrices $L_1, ..., L_{m-1}$.

Est-il possible d'obtenir une ligne valide avec une combinaison linéaire de $k+1$ L_i ? Si ce n'est pas le cas, le théorème est démontré.

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