### Computing isogenies of small degrees on Abelian Varieties

#### Jean-Charles Faugère<sup>1</sup>, David Lubicz<sup>2,3</sup>, Damien Robert<sup>4</sup>

<sup>1</sup>INRIA, Centre Paris-Rocquencourt, SALSA Project

#### <sup>2</sup>CÉLAR

<sup>3</sup>IRMAR, Université de Rennes 1

<sup>4</sup>Nancy Université, CNRS, Inria Nancy Grand Est

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#### Definition

#### An Abelian variety is a complete connected group variety over a base field *k*.

- Abelian variety = points on a projective space (locus of homogeneous polynomials) + an algebraic group law.
- Abelian varieties are projective, smooth, irreducible with an Abelian group law
   ⇒ can be used for public key cryptography (Discrete Logarithm Problem).
- *Example:* Elliptic curves, Jacobians of genus *g* curves...

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A (separable) isogeny is a finite surjective (separable) morphism between two Abelian varieties.

- Isogenies = Rational map + group morphism + finite kernel.
- Isogenies ⇔ Finite subgroups.

$$(f : A \to B) \mapsto \operatorname{Ker} f$$
  
 $(A \to A/H) \leftrightarrow H$ 

• *Example:* Multiplication by  $\ell \implies \ell$ -torsion), Frobenius (non separable).

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- Compute the class field polynomials.
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## Vélu's formula

#### Theorem

Let  $E: y^2 = f(x)$  be an elliptic curve. Let  $G \subset E(k)$  be a finite subgroup. Then E/G is given by  $Y^2 = g(X)$  where

$$X(P) = x(P) + \sum_{Q \in G \setminus \{0_E\}} x(P+Q) - x(Q)$$
  
$$Y(P) = y(P) + \sum_{Q \in G \setminus \{0_E\}} y(P+Q) - y(Q)$$

• Uses the fact that x and y are characterised in k(E) by

$$v_{0_E}(x) = -3 \qquad v_P(x) \ge 0 \quad \text{if } P \neq 0_E$$
  

$$v_{0_E}(y) = -2 \qquad v_P(y) \ge 0 \quad \text{if } P \neq 0_E$$
  

$$y^2/x^3(O_E) = 1$$

• No such characterisation in genus  $g \ge 2$ .

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- The modular polynomial is a polynomial  $\varphi_n(x, y) \in \mathbb{Z}[x, y]$  such that  $\varphi_n(x, y) = 0$  iff x = j(E) and y = j(E') with *E* and *E' n*-isogeneous.
- If  $E: y^2 = x^3 + ax + b$  is an elliptic curve, the *j*-invariant is

$$j(E) = 1728 \frac{4a^3}{4a^3 + 27b^2}$$

- Roots of  $\varphi_n(j(E), .) \Leftrightarrow$  elliptic curves *n*-isogeneous to *E*.
- In genus 2, modular polynomials use Igusa invariants. The height explodes:  $\varphi_2 = 50MB$ .
- $\Rightarrow$  Use the moduli space given by theta functions.
- $\Rightarrow~$  Fix the form of the isogeny and look for coordinates compatible with the isogeny.

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## Complex abelian varieties

- Abelian variety over  $\mathbb{C}$ :  $A = \mathbb{C}^g / (\mathbb{Z}^g + \Omega \mathbb{Z}^g)$ , where  $\Omega \in \mathcal{H}_g(\mathbb{C})$  the Siegel upper half space.
- The theta functions with characteristic give a lot of analytic (quasi periodic) functions on  $\mathbb{C}^{g}$ .

$$\vartheta(z,\Omega) = \sum_{n \in \mathbb{Z}^g} e^{\pi i n' \Omega n + 2\pi i n' z}$$
$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (z,\Omega) = e^{\pi i a' \Omega a + 2\pi i a' (z+b)} \vartheta(z+\Omega a+b,\Omega) \ a, b \in \mathbb{Q}^g$$

• The quasi-periodicity is given by

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (z + m + \Omega n, \Omega) = e^{2\pi i (a'm - b'n) - \pi i n'\Omega n - 2\pi i n'z} \vartheta \begin{bmatrix} a \\ b \end{bmatrix} (z, \Omega)$$

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## Projective embeddings given by theta functions

#### Theorem

• Let  $\mathcal{L}_{\ell}$  be the space of analytic functions f satisfying:

$$f(z+n) = f(z)$$
  
$$f(z+n\Omega) = \exp(-\ell \cdot \pi i n' \Omega n - \ell \cdot 2\pi i n' z) f(z)$$

• A basis of  $\mathcal{L}_{\ell}$  is given by

$$\left\{\vartheta \begin{bmatrix} 0\\b \end{bmatrix} (z,\Omega/\ell)\right\}_{b \in \frac{1}{\ell} \mathbb{Z}^g / \mathbb{Z}^g}$$

• Let  $Z_{\ell} = \mathbb{Z}^{g}/\ell\mathbb{Z}^{g}$ . If  $i \in Z_{\ell}$  we define  $\vartheta_{i} = \vartheta \begin{bmatrix} 0\\ i/\ell \end{bmatrix} (., \Omega/\ell)$ . If  $l \ge 3$  then  $z \mapsto (\vartheta_{i}(z))_{i \in Z_{\ell}}$ 

*is a projective embedding*  $A \to \mathbb{P}_{\mathbb{C}}^{\ell^g - 1}$ *.* 

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*is a projective embedding*  $A \to \mathbb{P}_{\mathbb{C}}^{\ell^g - 1}$ .

- The point  $(a_i)_{i \in \mathbb{Z}_{\ell}} := (\vartheta_i(0))_{i \in \mathbb{Z}_{\ell}}$  is called the theta null point of level  $\ell$  of the Abelian Variety  $A := \mathbb{C}^g / (\mathbb{Z}^g + \Omega \mathbb{Z}^g)$ .
- $(a_i)_{i \in \mathbb{Z}_{\ell}}$  determines the equations of the projective embedding of A of level  $\ell$ .
- The symplectic basis Z<sup>g</sup> ⊕ ΩZ<sup>g</sup> induce a decomposition into isotropic subgroups for the commutator pairing:

$$A[\ell] = A[\ell]_1 \oplus A[\ell]_2$$
$$= \frac{1}{\ell} \mathbb{Z}^g / \mathbb{Z}^g \oplus \frac{1}{\ell} \Omega \mathbb{Z}^g / \Omega \mathbb{Z}^g$$

This decomposition can be recovered by  $(a_i)_{i \in \mathbb{Z}_{\ell}}$ .

• The action by translation is given by

$$\vartheta_k\left(z-\frac{i}{\ell}-\Omega\frac{j}{\ell}\right)=e_{\mathcal{L}_\ell}(i+k,j)\vartheta_{i+k}$$

where  $e_{\mathcal{L}_{\ell}}(x, y) = e^{2\pi i/\ell \cdot x' y}$  is the commutator pairing.

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$$A[\ell] = A[\ell]_1 \oplus A[\ell]_2$$
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### The isogeny theorem

#### Theorem

- Let  $\ell = n.m$ , and  $\varphi : Z_n \to Z_\ell, x \mapsto m.x$  be the canonical embedding. Let  $K = A[m]_2 \subset A[\ell]_2$ .
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- We have:

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$$\vartheta_i^B(z) = \vartheta \begin{bmatrix} 0\\ i/n \end{bmatrix} \left( z, \frac{\Omega}{m}/n \right) = \vartheta \begin{bmatrix} 0\\ mi/\ell \end{bmatrix} \left( z, \Omega/\ell \right) = \vartheta_{m \cdot i}^A(z)$$

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### Theorem (car $k \neq \ell$ )

• The theta null point of level  $\ell(a_i)_{i \in \mathbb{Z}_{\ell}}$  satisfy the Riemann Relations:

$$\sum_{e \in \mathbb{Z}_2} a_{x+t} a_{y+t} \sum_{t \in \mathbb{Z}_2} a_{u+t} a_{v+t} = \sum_{t \in \mathbb{Z}_2} a_{z-u+t} a_{z-y+t} \sum_{t \in \mathbb{Z}_2} a_{z-x+t} a_{z-v+t}$$
(1)

We note  $\mathcal{M}_{\ell}$  the moduli space given by these relations together with the relations of symmetry:

$$a_x = a_{-x}$$

*M*<sub>ℓ</sub>(k) is the modular space of k-Abelian variety with a theta structure of level ℓ. The locus of theta null points of level ℓ is an open subset *M*<sup>0</sup><sub>ℓ</sub>(k) of *M*<sub>ℓ</sub>(k).

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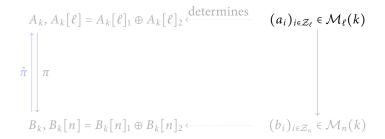
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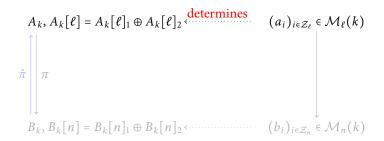




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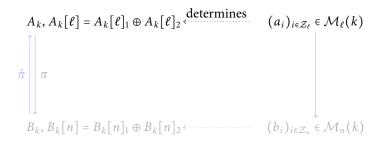
### Summary



- The kernel of  $\pi$  is  $A_k[m]_2 \subset A_k[\ell]_2$
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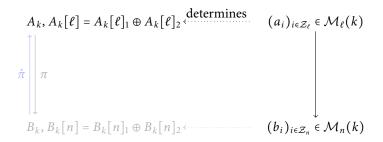




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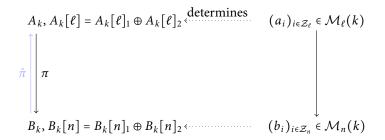




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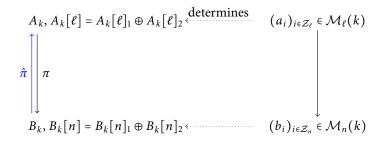




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We will show an example with g = 1, n = 4 and  $\ell = 12$ .

- Let *B* be the elliptic curve  $y^2 = x^3 + 11.x + 47$  over  $k = \mathbb{F}_{79}$ . The corresponding theta null point  $(b_0, b_1, b_2, b_3)$  of level 4 is  $(1:1:12:1) \in \mathcal{M}_4(\mathbb{F}_{79})$ .
- We note  $V_B(k)$  the subvariety of  $\mathcal{M}_{12}(k)$  defined by

$$a_0 = b_0, a_3 = b_1, a_6 = b_2, a_9 = b_3$$

• By the isogeny theorem, to every valid theta null point  $(a_i)_{i \in \mathbb{Z}_{\ell n}} \in V_B^0(k)$  corresponds a 3-isogeny  $\pi : A \to B$ :

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## The kernel of the dual isogeny

Let (a<sub>i</sub>)<sub>i∈Z<sub>ℓ</sub></sub> be a valid theta null point solution. Let ζ be a primitive 3-th root of unity.
 The kernel K of π is

 $\{(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}), \\ (a_0, \zeta a_1, \zeta^2 a_2, a_3, \zeta a_4, \zeta^2 a_5, a_6, \zeta a_7, \zeta^2 a_8, a_9, \zeta a_{10}, \zeta^2 a_{11}), \\ (a_0, \zeta^2 a_1, \zeta a_2, a_3, \zeta^2 a_4, \zeta a_5, a_6, \zeta^2 a_7, \zeta a_8, a_9, \zeta^2 a_{10}, \zeta a_{11})\}$ 

• The kernel  $\tilde{K}$  of the dual isogeny is given by the projection of the dual of K:  $\tilde{K} = \{(a_0, a_3, a_6, a_9), (a_4, a_7, a_{10}, a_1), (a_8, a_{11}, a_2, a_5)\}$ 

#### Theorem

Let  $(a_i)_{i \in \mathbb{Z}_{12}}$  be any solution. Then  $(a_i)_{i \in \mathbb{Z}_{12}}$  is valid if and only if

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## The automorphisms of the theta group

• If  $(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11})$  is a valid solution corresponding to an Abelian variety *A*, the solutions isomorphic to *A* are given by

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• In general, for each *m*-isogeny, there will be  $\simeq m^{g^2+g(g+1)/2}$  solutions.

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### *The solutions*

### Solutions of the system

• We have the following valid solutions (*v* is a primitive root of degree 3):

$$\begin{aligned} & (v^{490931}:1:46:v^{490931}:37:54:v^{54782}:54:37:v^{490931}:46:1) \\ & (v^{476182}:1:68:v^{476182}:67:10:v^{40033}:10:67:v^{476182}:68:1) \\ & (v^{465647}:1:3:v^{465647}:40:16:v^{29498}:16:40:v^{465647}:3:1) \\ & (v^{450898}:1:33:v^{450898}:69:24:v^{14749}:24:69:v^{450898}:33:1) \end{aligned}$$

• And the following degenerate solutions:

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## *The solutions*

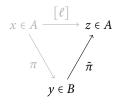
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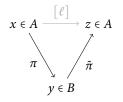


Let  $\pi : A \rightarrow B$  be the isogeny associated to  $\pi \qquad \tilde{\pi} \qquad \tilde{\pi} \qquad (a_0, \cdots, a_{11}). \text{ Let } y = (y_0, y_1, y_2, y_3) \in B. \text{ Let} \\ x = (x_0, \cdots, x_{11}) \text{ be one of the 3 antecedents.}$ 

$$y = (x_0, x_3, x_6, x_9)$$
  

$$y + P_1 = (x_4, x_7, x_{10}, x_1)$$
  

$$y + 2P_1 = (x_8, x_{11}, x_2, x_5)$$

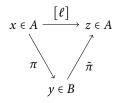


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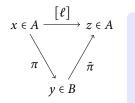


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Let  $\pi : A \to B$  be the isogeny associated to  $(a_0, \dots, a_{11})$ . Let  $y = (y_0, y_1, y_2, y_3) \in B$ . Let  $x = (x_0, \dots, x_{11})$  be one of the 3 antecedents. Then  $\tilde{\pi}(y) = 3x$ 

• Let  $P_1 = (a_4, a_7, a_{10}, a_1) \in \tilde{K}$ ,  $P_1$  is a point of 3-torsion in *B*. We have:

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So x can be recovered from y,  $y + P_1$ ,  $y + 2P_1$  up to three projective factors  $\lambda_0, \lambda_{P_1}, \lambda_{2P_1}$ .

$$x \in A \xrightarrow{[\ell]} z \in A$$

$$\pi \xrightarrow{\tilde{\mu}} \tilde{\pi}$$

$$y \in B$$
Let  $\pi : A \to B$  be the isogeny associated to
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# The addition formula

### Theorem (Addition formula)

$$2^{g} \vartheta \begin{bmatrix} a'\\ e' \end{bmatrix} (x+y) \vartheta \begin{bmatrix} b'\\ f' \end{bmatrix} (x-y) \vartheta \begin{bmatrix} c'\\ g' \end{bmatrix} (0) \vartheta \begin{bmatrix} d'\\ h' \end{bmatrix} (0) = \sum_{\alpha, \beta \in \frac{1}{2}\mathbb{Z}^{g}/\mathbb{Z}^{g}} e^{2\pi i \beta' (a+b+c+d)} \vartheta \begin{bmatrix} a+\alpha\\ e+\beta \end{bmatrix} (x) \vartheta \begin{bmatrix} b+\alpha\\ f+\beta \end{bmatrix} (x) \vartheta \begin{bmatrix} c+\alpha\\ g+\beta \end{bmatrix} (y) \vartheta \begin{bmatrix} d+\alpha\\ h+\beta \end{bmatrix} (y)$$

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- Using the addition formulas, we have  $\lambda_{2P_1} = \lambda_{P_1}^2$ .
- Since  $y + 3P_1 = y$ , we obtain a formula

$$\lambda_{P_1}^3 = \alpha$$

hence we can find the three antecedents.

- In fact when computing  $3 \cdot x$ , the projective factors become  $\lambda_0^3$ ,  $\lambda_{P_1}^3$ ,  $\lambda_{2P_1}^3$  so we don't need to extract roots.
- *Vélu's like formulas:* If we know the kernel  $\tilde{K}$  of the isogeny, we can use the same methods to compute the valid theta null points in  $\mathcal{M}_{\ell n}(k)$ , by determining the g(g+1)/2 indeterminates  $\lambda_{ij}$ .

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