Synthetic probability theory: an axiomatic approach to nondeterminism

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  Fritz's seminal paper also deals with conditional independence, sufficient statistics, almost sure equality...

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- Today: we will focus on a core first-order theory.

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  Let $\mathcal{F}$ denote the set of all such Markov kernels. We may write $f : X \to Y$ for deterministic functions, viewed as kernels.

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- Basic equations involving copy, del, ...

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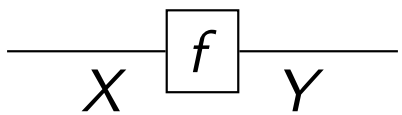

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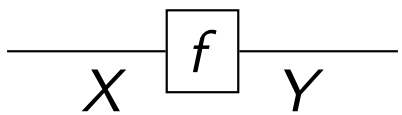
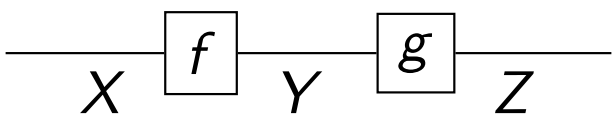


Sequential composites $f ; g$ shall be drawn as



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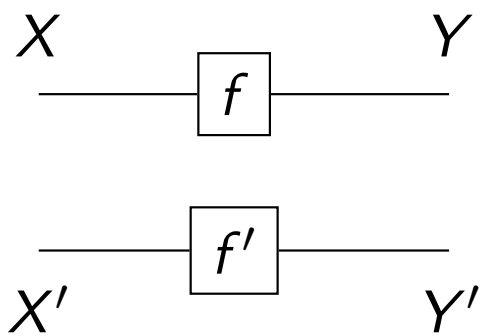

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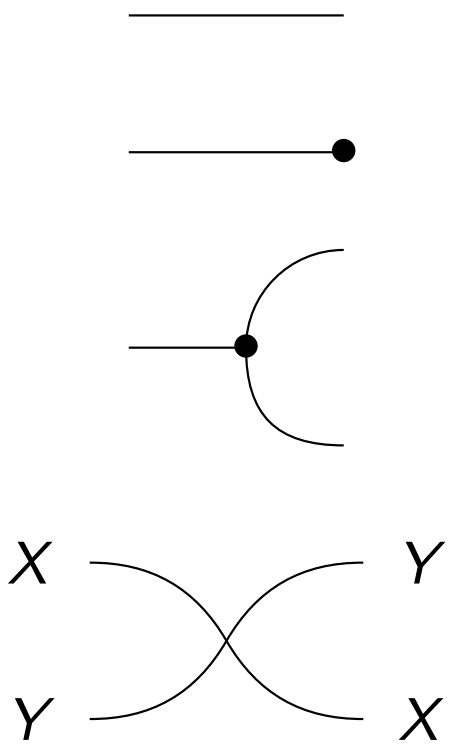
Identities, deleting, copying and symmetry kernels will be drawn as follows:



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  Note: these axioms are "baked in" the graphical representation.

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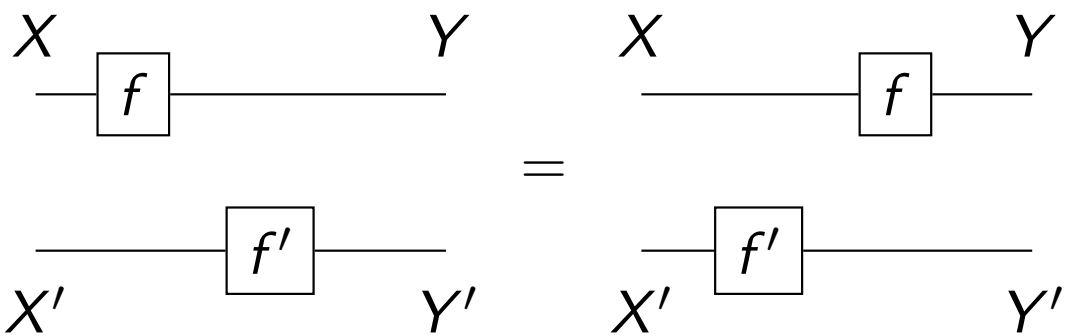

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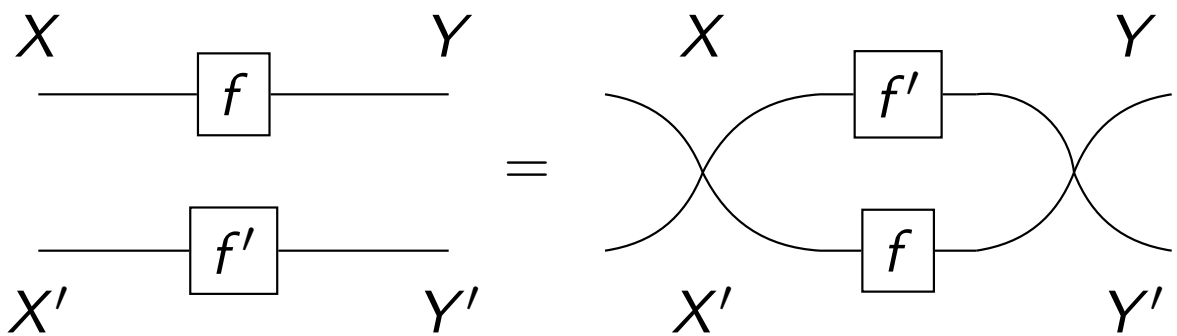

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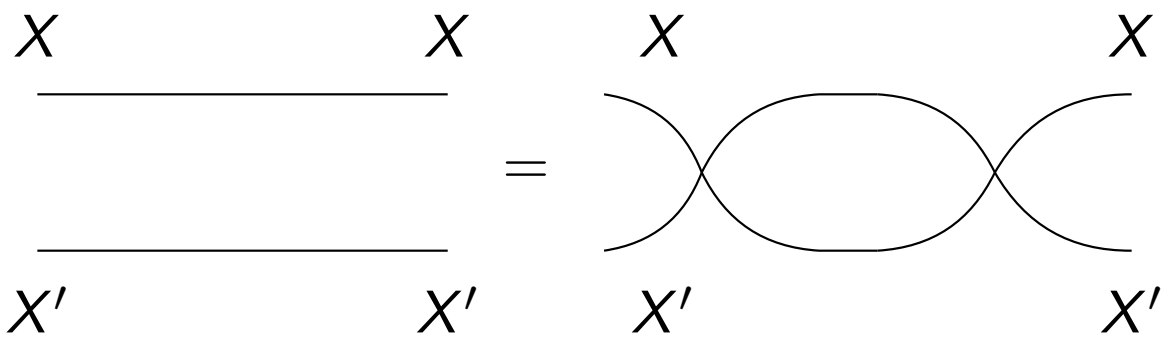

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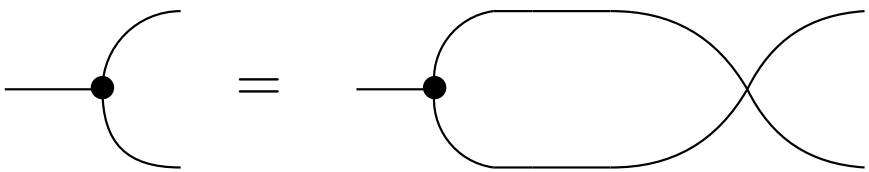

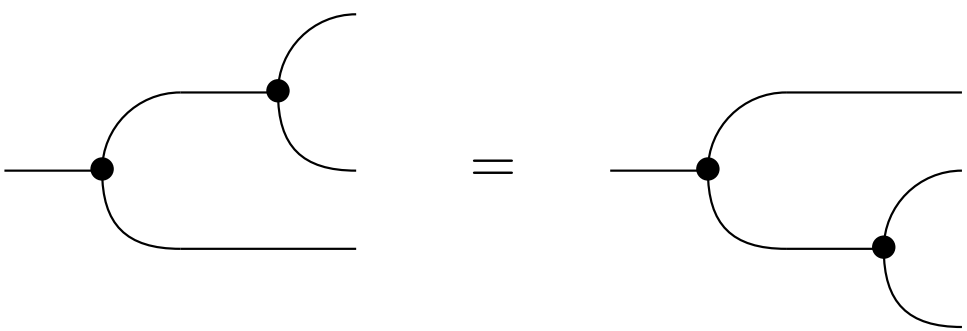

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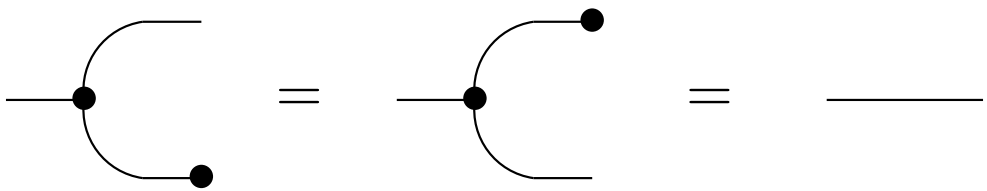

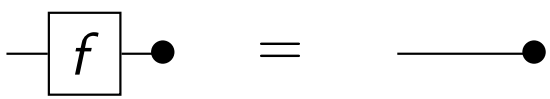

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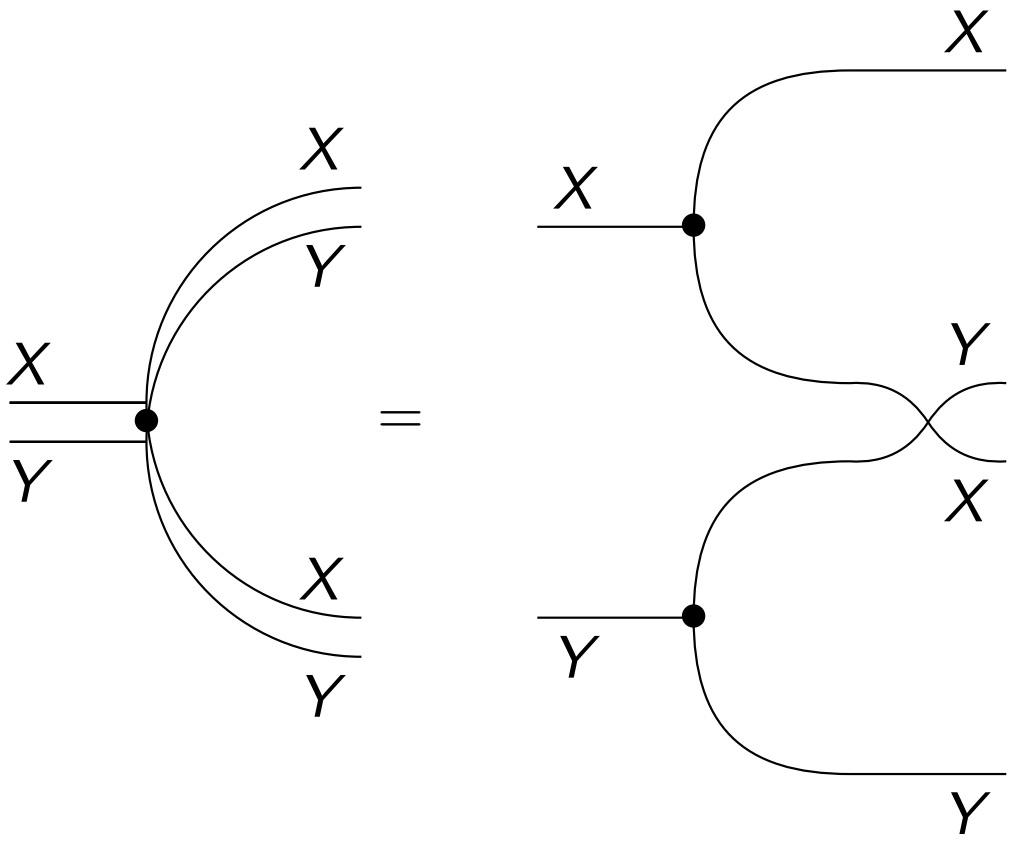

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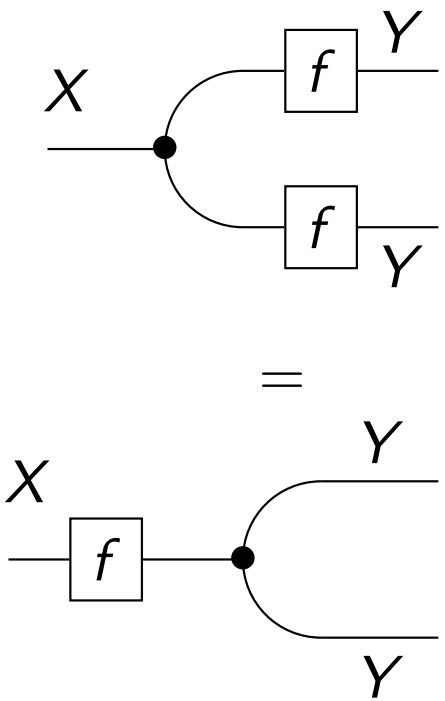

Outline

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- There is a special sort Bool, and a term constructor "If Then Else".

Syntax

$$FV(x \leftarrow t; u) = \begin{cases} FV(u) & \text{if } x \text{ not free in } u \\ FV(t) \cup (FV(u) \setminus \{x\}) & \text{otherwise} \end{cases}$$

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$$FV(\text{If } b \text{ Then } u \text{ Else } v) = FV(b) \cup FV(u) \cup FV(v).$$

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- For instance, one might add a sort I (for $[0, 1]$), and a function symbol $\mathrm{bernoulli} : I \to \mathrm{Bool}$.

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Proposition

Under this assumption, there exists an (inductive) interpretation for terms of the programming language, that satisfies the desiderata above.

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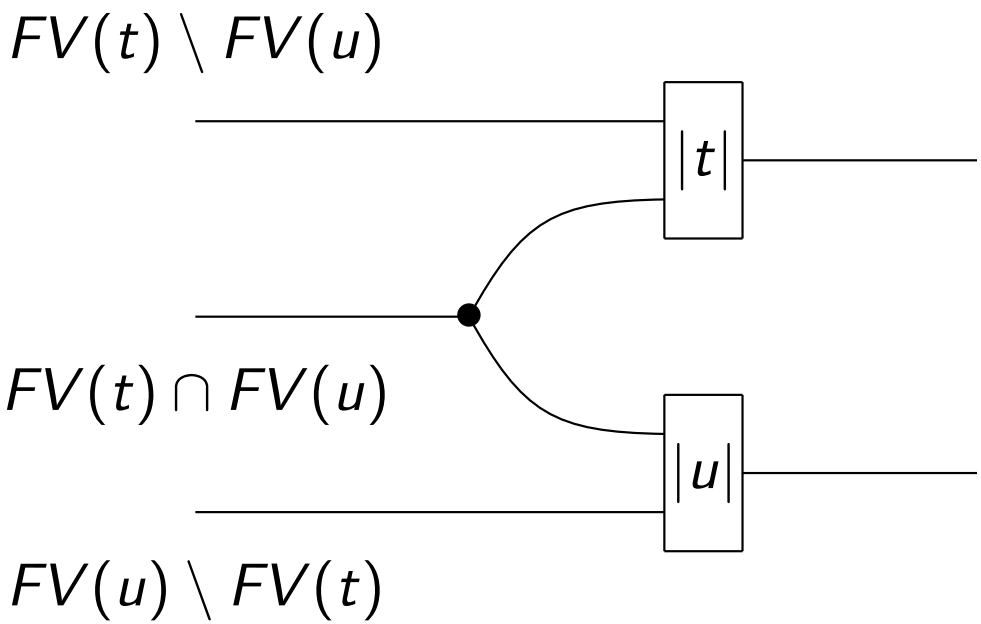

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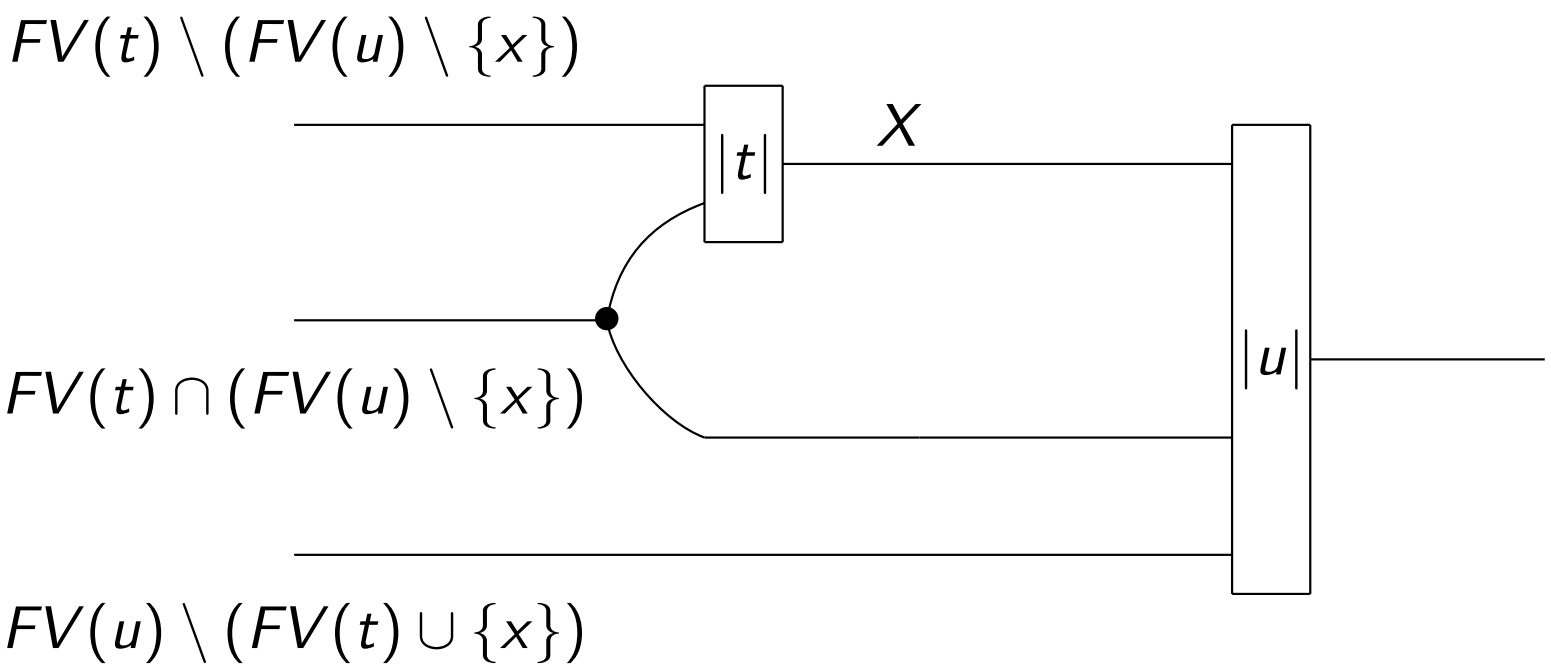

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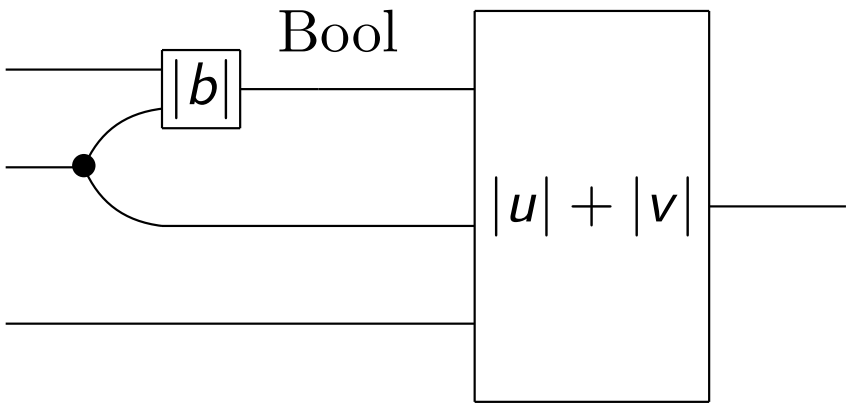

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References


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