Markov Categories of Definable Kernels and Relations

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We construct two Markov categories whose morphisms are definable in o-minimal structures. The probabilistic category DefStoch($\mathbb{R}_{\text{an}}$), over the structure of globally subanalytic sets, has morphisms given by Markov kernels with constructible densities on definable latent spaces; composition corresponds to a fibre-product construction on latent spaces, and the Cluckers–Miller stability theorem ensures that integration against presented kernels preserves the function class. This Markov category does not have conditionals, but is positive and causal, as a sub-Markov-category of BorelStoch. The possibilistic category DefRel$_+(\mathcal{M})$, over an arbitrary o-minimal structure $\mathcal{M}$, has morphisms given by definable total relations; it has conditionals, thus positivity and causality. Both constructions exploit model-theoretic tameness — cell decomposition and closure under integration (probabilistic) or projection (possibilistic) — and we isolate four hypotheses on an abstract function class from which the categorical structure follows.

1 Introduction

Markov categories, introduced by Fritz [Fri20], provide a synthetic framework for probability theory in which the basic operations — composing stochastic processes, copying random variables, and marginalising — are expressed as categorical structure. The key examples are BorelStoch (Markov kernels on standard Borel spaces) and FinStoch (stochastic matrices on finite sets). These categories are large: their morphisms range over all measurable or finite-probability structures, with no tameness constraints.

In this paper we construct two Markov categories whose morphisms are *definable* in an o-minimal structure $\mathcal{M}$ — a setting in which definable sets decompose into finitely many cells and are closed under projections — ensuring that all probability distributions involved have tame, finitely described behaviour. The possibilistic construction works over any such $\mathcal{M}$; the probabilistic one is carried out over the globally subanalytic structure $\mathbb{R}_{\text{an}}$. Crucially, associated classes of functions on definable sets are closed under parametric integration, making composition of stochastic kernels well-defined. A key finding is that this tameness comes at a cost: the probabilistic category does not have conditionals (Proposition 3.20), revealing a structural tension between definability and Bayesian inversion.

Why definable probability? Three motivations guide the constructions. First, *tameness*: definability excludes the measure-theoretic pathologies available over $\mathbb{R}$ — non-measurable sets, fractal supports, oscillatory densities — so that every distribution in sight has a finite combinatorial description, and the category is closed under the basic operations of probabilistic inference. Second, a link to *statistics*: a wide range of continuous laws are definable — the uniform, triangular, Beta (rational parameters), and Cauchy families, piecewise-polynomial densities, and the semialgebraic models of algebraic statistics [Sul18] — and the constructible function class arises unavoidably, because marginalising even a subanalytic density introduces logarithms (the Lion–Rolin phenomenon [LR98]). The boundary is equally informative: laws whose densities require the unrestricted exponential, such as the Gaussian and Gamma, fall outside the setting (Remark 3.22). The setting thus captures a large, algebraically-and-logarithmically tame fragment of continuous probability, with a sharp and explicit edge. Third, a source of *controlled counterexamples*:

To appear in EPTCS.

a category small enough to be governed by cell decomposition, yet large enough to contain genuinely continuous kernels, is a natural test-bed for conjectures about conditionals and dilations. As we show, it already separates positivity and causality (which hold) from the existence of conditionals (which fails).

The probabilistic case (§3). We define $\mathrm{DefStoch}(\mathbb{R}_{\mathrm{an}})$, a Markov category whose morphisms $X \to Y$ are Markov kernels admitting a *latent-variable presentation*: a finite index set I, definable latent spaces $U_i \subseteq X \times \mathbb{R}^{m_i}$, nonnegative constructible density functions $g_i \in \mathcal{C}(U_i)$, and definable output maps $h_i : U_i \to Y$ ($i \in I$), such that the kernel is given by

$$\kappa(x,B) = \sum_{i \in I} \int_{(U_i)_x} \mathbf{1}_B(h_i(x,u))\, g_i(x,u)\, du.$$

Here the latent coordinate $u \in \mathbb{R}^{m_i}$ is an auxiliary random draw: to sample $\kappa(x,-)$ one selects a component i, draws u according to the density $g_i(x,-)$, and returns the output $h_i(x,u)$, the latent draw being discarded (marginalised) so that only its effect on Y survives. The densities g_i are *constructible functions* in the sense of Cluckers–Miller [CM11]: sums of products of globally subanalytic functions and their logarithms. This class is closed under parametric integration [CM11, Thm. 1.3], which is the key property making composition work. We prove that composition corresponds to a fibre-product construction on latent spaces (Theorem 3.7), and that $\mathrm{DefStoch}(\mathbb{R}_{\mathrm{an}})$ is a Markov category (Proposition 3.10).

The construction is axiomatic: we isolate four hypotheses (H1)–(H4) on an abstract function class $\mathcal{C}_{\mathcal{M}}$ (Definition 3.1) and derive the Markov category structure from these, then verify them for the Cluckers–Miller class. This separates the categorical argument from the model-theoretic input.

The possibilistic case (§4). We construct $\mathrm{DefRel}_{+}(\mathcal{M})$, a possibilistic Markov category whose morphisms are definable total relations with nonempty fibres. This is a sub-Markov-category of the Kleisli category of the nonempty powerset monad P_{+} on Set [Fri20, Cor. 3.2], restricted to definable objects and morphisms. Unlike the probabilistic case, $\mathrm{DefRel}_{+}(\mathcal{M})$ has conditionals (Proposition 4.4) and hence is positive and causal.

The definable setting. Both constructions rely on the same model-theoretic ingredients: closure of definable sets under projections (for composition) and cell decomposition (for tameness). In the possibilistic case, the fibre construction gives conditionals directly (Proposition 4.4); in the probabilistic case, conditioning requires dividing by a marginal density, which can leave the constructible function class. The parallel is summarised in Remark 4.5. The relationship between the two categories — in particular, whether a support functor $\mathrm{DefStoch}(\mathbb{R}_{\mathrm{an}}) \to \mathrm{DefRel}_{+}(\mathcal{M})$ preserves the Markov structure — remains open.

Related work. The connection between o-minimal definability and measure theory is developed by Kaiser [Kai12], who introduces $\mathcal{M}$*-tame* measures and proves that the Lebesgue measure is strongly $\mathbb{R}_{\mathrm{an}}$-tame. Our latent-variable presentation can be seen as a syntactic counterpart to Kaiser's semantic notion; the relationship is discussed in Remark 3.23. Categorical probability in the Markov categories framework is surveyed by Fritz [Fri20], with extensions by Fritz–Rischel [FR20], Moss–Perrone [MP23], and Perrone [Per24]. In profile, $\mathrm{DefStoch}(\mathbb{R}_{\mathrm{an}})$ sits alongside the classical category Stoch of measurable-space Markov kernels, which is likewise positive and causal yet lacks conditionals [Fri20, Examples 11.3, 11.25, 11.35]; the obstruction, however, differs in kind — measure-theoretic for Stoch, definability-theoretic here. Quasi-Borel spaces [HKSY17] take a complementary approach: they *enlarge* the category of standard Borel spaces, adding objects such as function spaces to obtain cartesian closure, and BorelStoch embeds faithfully into their Markov category. Di Lavore and Román's *partial Markov categories* [DLR23] extend the framework in another direction, adjoining partial morphisms to give a synthetic account of observations, constraints, and Bayesian updates. Both directions *enlarge* the category of kernels; ours instead *restricts* it, carving out sub-Markov-categories by definability, with the resulting failure of conditionals (Proposition 3.20) as a structural consequence.

2 Preliminaries

2.1 O-minimal structures

This subsection collects the model-theoretic background; we refer to van den Dries [Dri98] for a comprehensive treatment. An *expansion* of the real ordered field $(\mathbb{R},<,+,\cdot)$ is a structure on the same underlying set $\mathbb{R}$ obtained by adjoining further distinguished functions and relations (for example, the exponential function). A set $X \subseteq \mathbb{R}^n$ is *definable* in such a structure $\mathcal{M}$ if it is the solution set of a first-order formula in the language of $\mathcal{M}$ — built from the distinguished symbols using the Boolean connectives and the quantifiers $\exists,\forall$ ranging over $\mathbb{R}$ — with parameters from $\mathbb{R}$; a map or family is definable if its graph is. For readers unfamiliar with formal logic, the same class admits a purely set-theoretic description, due to van den Dries, that never mentions formulas.

Definition 2.1 (Definable sets [Dri98, Ch. 1, (2.1)])**.** A *structure* on $\mathbb{R}$ is a sequence $\mathcal{S} = (\mathcal{S}_n)_{n\geq 1}$, with each $\mathcal{S}_n$ a collection of subsets of $\mathbb{R}^n$, such that for all $n \geq 1$:

(S1) $\mathcal{S}_n$ is a boolean algebra of subsets of $\mathbb{R}^n$ (closed under finite unions, intersections, and complements);

(S2) if $A \in \mathcal{S}_n$, then $\mathbb{R}\times A$ and $A\times\mathbb{R}$ belong to $\mathcal{S}_{n+1}$;

(S3) $\{(x_1,\dots,x_n) \in \mathbb{R}^n : x_1 = x_n\} \in \mathcal{S}_n$;

(S4) if $A \in \mathcal{S}_{n+1}$, then $\pi(A) \in \mathcal{S}_n$, where $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the projection onto the first n coordinates.

The *definable sets* of $\mathcal{M}$ are the members of the smallest structure on $\mathbb{R}$ that contains every singleton $\{r\}$ ($r \in \mathbb{R}$) and the graphs of $<$, $+$, $\cdot$, and of every distinguished function of $\mathcal{M}$. A map or family is *definable* if its graph is.

From these axioms one derives closure under Cartesian products, permutations, and identifications of variables [Dri98, (2.2)]; projection (S4) realises the existential quantifier, the boolean operations (S1) the connectives $\wedge,\vee,\neg$, and the singletons supply the parameters, so the definable sets are exactly those cut out by the first-order formulas above.

Definition 2.2 ([Dri98, Ch. 1, (3.2)])**.** An *o-minimal structure* is an expansion $\mathcal{M}$ of $(\mathbb{R},<,+,\cdot)$ such that every definable subset of $\mathbb{R}$ is a finite union of points and open intervals.

The defining condition restricts only the definable subsets of the line, but through cell decomposition (below) it propagates to every dimension: definable sets are constrained to have finite combinatorial complexity — no fractal boundaries, no space-filling curves, no pathological measure-theoretic behaviour. O-minimal structures thus realise Grothendieck's programme of *géométrie modérée* (tame topology) [Gro97].

Example 2.3. Three structures recur below, in increasing strength.

- $(\mathbb{R},<,+,\cdot)$ itself: the definable sets are the *semialgebraic sets* (finite Boolean combinations of polynomial inequalities), and o-minimality is Tarski's quantifier elimination for real closed fields.
- $\mathbb{R}_{\text{an}}$: the expansion by every *restricted analytic function* $f : [-1,1]^n \to \mathbb{R}$ (a function analytic on a neighbourhood of the cube, set to 0 outside it). Its definable sets are exactly the *globally subanalytic sets* [DD88]. This is the primary structure of the paper.

- $\mathbb{R}_{\text{an,exp}} = (\mathbb{R}_{\text{an}}, \exp)$: the further expansion by the unrestricted real exponential, equivalently the unrestricted logarithm. Its o-minimality is due to van den Dries–Miller [DM94].

Two structural theorems are used throughout. *Cell decomposition* [Dri98, Ch. 3]: every definable set partitions into finitely many *cells*, defined inductively — in $\mathbb{R}$, a point $\{a\}$ or an open interval (a,b) (allowing $a=-\infty$, $b=+\infty$); in $\mathbb{R}^{n+1}$, either the *graph* $\{(\bar{x}, f(\bar{x})) : \bar{x} \in C\}$ or the *band* $\{(\bar{x},y) : \bar{x} \in C,\ f(\bar{x}) < y < g(\bar{x})\}$ over a cell $C \subseteq \mathbb{R}^n$, for continuous definable functions $f < g$ on C (allowing $f \equiv -\infty$ and/or $g \equiv +\infty$). Moreover, every definable function is continuous on each cell of a suitable decomposition, and every definable set has finitely many connected components. *Definable choice* [Dri98, Ch. 6]: if $S \subseteq X \times Y$ is definable with nonempty fibres S_x for all $x \in \pi_X(S)$, there is a definable function $f : \pi_X(S) \to Y$ with $f(x) \in S_x$. Finally, the projection closure $\pi : \mathbb{R}^{m+n} \to \mathbb{R}^m$ of Definition 2.1 is the key operation behind both constructions below — relational composition in §4 and latent-variable marginalisation in §3.

All definable subsets of $\mathbb{R}^n$ are equipped with their Borel σ-algebra; globally subanalytic sets are standard Borel spaces and so form objects of BorelStoch. Section 3 develops an axiomatic framework valid over any o-minimal $\mathcal{M}$ equipped with a suitable function class (Definition 3.1), whose motivating instance is $\mathcal{M} = \mathbb{R}_{\text{an}}$ with the Cluckers–Miller constructible functions; Section 4 uses only the o-minimality of $\mathcal{M}$.

2.2 Markov categories

A *Markov category* [Fri20, Def. 2.1] is a symmetric monoidal category $(\mathbf{C}, \otimes, I)$ in which every object X carries a commutative comonoid structure $\text{copy}_X : X \to X \otimes X$ and $\text{del}_X : X \to I$, compatible with the monoidal product, such that del is natural: $\text{del}_Y \circ f = \text{del}_X$ for every morphism $f : X \to Y$. Intuitively, copy_X duplicates a random variable and del_X discards it; naturality of del expresses normalisation.

A morphism f is *deterministic* if $\text{copy}_Y \circ f = (f \otimes f) \circ \text{copy}_X$ [Fri20, Def. 10.1]. The deterministic morphisms form a cartesian monoidal subcategory.

Example 2.4. BorelStoch, the Kleisli category of the Giry monad [Gir82] on standard Borel spaces, is a Markov category whose morphisms are Markov kernels and whose deterministic morphisms are measurable functions. FinStoch (finite sets, stochastic matrices) is another standard example.

By [Fri20, Cor. 3.2], the Kleisli category of any symmetric monoidal affine monad on a Markov category is again a Markov category. The constructions in §3 and §4 are modelled on this pattern, though neither is literally a Kleisli category (see Remark 4.2).

2.3 Constructible functions

Globally subanalytic functions are not closed under parametric integration: the integral $\int_0^x f(t)\,dt$ of a subanalytic integrand may involve logarithms [LR98]. This motivates the following class.

Definition 2.5 ([CM11, Def. 1.2])**.** For each globally subanalytic set X, the *constructible functions* $\mathcal{C}(X)$ form the $\mathbb{R}$-algebra generated by all globally subanalytic functions on X and the functions $x \mapsto \log f(x)$ for $f : X \to (0,\infty)$ globally subanalytic. Every $g \in \mathcal{C}(X)$ has the form $g(x) = \sum_i f_i(x) \prod_j \log f_{ij}(x)$ with f_i subanalytic and f_{ij} positive subanalytic.

Recall 2.6 ([CM11, Thm. 1.3])**.** *For $g \in \mathcal{C}(X \times \mathbb{R}^m)$, the function $x \mapsto \int_{\mathbb{R}^m} g(x,u)\,du$ — taken to be 0 at the parameters x for which $g(x,\cdot)$ is not Lebesgue integrable — belongs to $\mathcal{C}(X)$. In particular, if the integral converges for every x, then $x \mapsto \int_{\mathbb{R}^m} g(x,u)\,du \in \mathcal{C}(X)$.*

This stability under integration is the key closure property: constructible functions are the smallest class of functions that contains all globally subanalytic functions and is closed under parametric integration.

Recall 2.7 (Cell preparation [CM11, Thm. 3.11]). *After a definable coordinate change (centring $\tilde{x}_j = x_j - \theta_j$ at a subanalytic centre θ), on each cell of a finite subanalytic cell decomposition, every constructible function takes the form $\sum_i u_i(\tilde{x}) \prod_j |\tilde{x}_j|^{\alpha_{ij}} (\log|\tilde{x}_j|)^{\ell_{ij}}$ with $\alpha_{ij} \in \mathbb{Q}$, $\ell_{ij} \in \mathbb{N}_0$, u_i subanalytic units, and $\tilde{x}$ the prepared coordinates.*

The restriction to *rational* exponents is a structural feature: constructible functions exhibit power-law behaviour $|x|^{p/q}$ but never irrational exponents such as $|x|^{\pi}$. This fact is used in §3.6 to exhibit measures outside $\mathrm{DefStoch}(\mathbb{R}_{\mathrm{an}})$.

3 Definable stochastic kernels

We now construct $\mathrm{DefStoch}(\mathcal{M})$, a Markov category whose morphisms are stochastic kernels with constructible densities on definable latent spaces. The definition is axiomatic: we state four hypotheses on a function class $\mathcal{C}_{\mathcal{M}}$ and derive the Markov category structure, then instantiate for the Cluckers–Miller constructible functions.

3.1 Admissible function classes

Definition 3.1 (Admissible function class). An *admissible function class* for $\mathcal{M}$ is an assignment $X \mapsto \mathcal{C}_{\mathcal{M}}(X)$ of an $\mathbb{R}$-algebra of Borel measurable functions on X, for each definable set X, satisfying:

- **(H1) Algebra structure.** $\mathcal{C}_{\mathcal{M}}(X)$ is an $\mathbb{R}$-subalgebra of the algebra of all Borel measurable functions $X \to \mathbb{R}$.
- **(H2) Definable functions.** Every function $f : X \to \mathbb{R}$ definable in $\mathcal{M}$ lies in $\mathcal{C}_{\mathcal{M}}(X)$.
- **(H3) Pullback stability.** If $h : X \to Y$ is definable and $g \in \mathcal{C}_{\mathcal{M}}(Y)$, then $g \circ h \in \mathcal{C}_{\mathcal{M}}(X)$.
- **(H4) Integration closure.** If $U \subseteq X \times \mathbb{R}^m$ is definable and $g \in \mathcal{C}_{\mathcal{M}}(U)$ is fibrewise integrable, then $x \mapsto \int_{\mathbb{R}^m} \tilde{g}(x,u)\,du \in \mathcal{C}_{\mathcal{M}}(X)$, where $\tilde{g}$ extends g by zero outside U (note $\tilde{g} = g \cdot \mathbf{1}_U \in \mathcal{C}_{\mathcal{M}}(X \times \mathbb{R}^m)$ when $\mathbf{1}_U$ is in $\mathcal{C}_{\mathcal{M}}$, which holds whenever U is definable, by (H2)).

Remark 3.2. We list the four hypotheses separately because different results invoke different subsets: the composition theorem (Theorem 3.7) uses only (H1) and (H3), whereas stabilisation (Proposition 3.6) additionally requires (H4). Among the four, only (H2) prescribes which functions must belong to $\mathcal{C}_{\mathcal{M}}(X)$; (H1), (H3) and (H4) are closure properties.

3.2 Presented kernels

Definition 3.3 (Presented kernel). Let X, Y be definable sets. A *presentation* of a kernel $X \to Y$ consists of a finite index set I, integers $m_i \geq 0$, definable sets $U_i \subseteq X \times \mathbb{R}^{m_i}$, nonnegative functions $g_i \in \mathcal{C}_{\mathcal{M}}(U_i)$, and definable maps $h_i : U_i \to Y$, such that for every $x \in X$:

$$\sum_{i \in I} \int_{(U_i)_x} g_i(x,u)\,du = 1, \qquad (U_i)_x := \{u \in \mathbb{R}^{m_i} : (x,u) \in U_i\}. \tag{N}$$

The *induced kernel* is $\kappa(x,B) := \sum_{i \in I} \int_{(U_i)_x} \mathbf{1}_B(h_i(x,u))\, g_i(x,u)\,du$ for Borel $B \subseteq Y$.

A *morphism* $X \to Y$ in $\mathrm{DefStoch}_{\mathcal{C}_{\mathcal{M}}}(\mathcal{M})$ is a Markov kernel admitting such a presentation.

Remark 3.4. Presentations are not unique: a single kernel may admit many presentations with different index sets and latent dimensions. A morphism is the underlying kernel, not the presentation. The latent variables $u \in \mathbb{R}^{m_i}$ are auxiliary random inputs — hidden coordinates that are integrated out — whose conditional distribution given x is governed by the density g_i, after which h_i deterministically produces an output in Y. Singular measures (concentrated on lower-dimensional subsets) require no special treatment: they arise when h_i drops dimension. The map $x \mapsto \kappa(x,B)$ is Borel measurable for each Borel B: it is a finite sum of parametric integrals of measurable functions, hence measurable by Tonelli's theorem.

3.3 Composition

Lemma 3.5 (Integration against a presented kernel)**.** *Let $\kappa : X \to Y$ be presented as $(U_i, g_i, h_i)_{i\in I}$. For any nonnegative Borel measurable $\varphi : Y \to [0,\infty]$, resp. any κ_x-integrable $\varphi : Y \to \mathbb{R}$:*

$$\int_Y \varphi(y)\,\kappa(x,dy) = \sum_{i\in I} \int_{(U_i)_x} \varphi(h_i(x,u))\, g_i(x,u)\,du. \tag{IL}$$

Proof. For $\varphi = \mathbf{1}_B$ this is the definition. Extend to simple functions by linearity and to general nonnegative measurable functions by monotone convergence. By splitting $\varphi = \varphi^+ - \varphi^-$, the identity extends to any κ_x-integrable φ. □

Proposition 3.6 (Stabilisation)**.** *If $\kappa : X \to Y$ is a presented kernel and $\varphi \in \mathcal{C}_{\mathcal{M}}(Y)$ satisfies $\int_Y |\varphi|\,d\kappa_x < \infty$ for all x, then $x \mapsto \int_Y \varphi\,d\kappa_x \in \mathcal{C}_{\mathcal{M}}(X)$.*

Proof. By (IL), $\int_Y \varphi\,d\kappa_x = \sum_{i\in I}\int_{(U_i)_x} \varphi(h_i(x,u))\,g_i(x,u)\,du$, where I is the index set of the presentation $(U_i, g_i, h_i)_{i\in I}$ of κ. Each summand lies in $\mathcal{C}_{\mathcal{M}}(X)$: the pullback $\varphi \circ h_i$ is in $\mathcal{C}_{\mathcal{M}}(U_i)$ by (H3), the product $(\varphi\circ h_i)\cdot g_i \in \mathcal{C}_{\mathcal{M}}(U_i)$ by (H1), and $|\varphi\circ h_i|\cdot g_i$ integrates to at most $\int_Y |\varphi|\,d\kappa_x < \infty$, so the parametric integral lies in $\mathcal{C}_{\mathcal{M}}(X)$ by (H4). The finite sum is in $\mathcal{C}_{\mathcal{M}}(X)$ by (H1). □

Theorem 3.7 (Composition closure)**.** *Let $\kappa : X \to Y$ have presentation $(U_i, g_i, h_i)_{i\in I}$ with latent dimensions m_i, and $\lambda : Y \to Z$ have presentation $(V_j, k_j, p_j)_{j\in J}$ with latent dimensions n_j. The Kleisli composite $\lambda \circ \kappa$ has presentation $(W_{ij}, G_{ij}, H_{ij})_{(i,j)\in I\times J}$ where:*

$$W_{ij} := \{(x,u,v) : (x,u) \in U_i,\ (h_i(x,u), v) \in V_j\} \subseteq X \times \mathbb{R}^{m_i+n_j}, \tag{C1}$$
$$G_{ij}(x,u,v) := g_i(x,u)\cdot k_j(h_i(x,u), v), \tag{C2}$$
$$H_{ij}(x,u,v) := p_j(h_i(x,u), v). \tag{C3}$$

The hypotheses used are (H1) *and* (H3) *for constructibility of G_{ij}; closure under composition follows from the explicit fibre-product construction without invoking* (H4).

Proof. We verify that (W_{ij}, G_{ij}, H_{ij}) is a valid presentation inducing $(\lambda\circ\kappa)(x,C) = \int_Y \lambda(y,C)\,\kappa(x,dy)$.

Definability and constructibility. W_{ij} is definable (intersection of preimages under definable maps). $G_{ij} = (g_i \circ \pi)\cdot(k_j \circ \psi) \in \mathcal{C}_{\mathcal{M}}(W_{ij})$ where $\pi(x,u,v) = (x,u)$ and $\psi(x,u,v) = (h_i(x,u), v)$, using (H3) for pullbacks and (H1) for the product. $H_{ij} = p_j \circ \psi$ is definable.

Induced kernel. By (IL) applied to κ with $\varphi(y) = \lambda(y,C)$:

$$(\lambda\circ\kappa)(x,C) = \sum_{i\in I}\int_{(U_i)_x} \lambda(h_i(x,u), C)\, g_i(x,u)\,du.$$

Substituting the presentation of λ and applying Tonelli (the integrand is nonnegative, index sets finite):

$$= \sum_{(i,j)\in I\times J} \int_{\mathbb{R}^{m_i+n_j}} \mathbf{1}_{(U_i)_x}(u)\,\mathbf{1}_{(V_j)_{h_i(x,u)}}(v)\,\mathbf{1}_C(p_j(h_i(x,u),v))\,k_j(h_i(x,u),v)\,g_i(x,u)\,d(u,v)$$
$$= \sum_{(i,j)\in I\times J} \int_{(W_{ij})_x} \mathbf{1}_C(H_{ij}(x,u,v))\,G_{ij}(x,u,v)\,d(u,v).$$

Normalisation. Setting $C = Z$: $\sum_{(i,j)\in I\times J} \int_{(W_{ij})_x} G_{ij} = (\lambda \circ \kappa)(x,Z) = 1$. Composition is well-defined on kernels (the Kleisli integral depends only on κ and λ, not on their presentations), and associativity follows from Fubini. □

3.4 Markov category structure

Every definable map $f : X \to Y$ defines a morphism via the Dirac kernel $\delta_f(x,B) = \mathbf{1}_B(f(x))$, presented with $|I| = 1$, $m_1 = 0$, $g_1 \equiv 1$, $h_1 = f$. In particular, $\mathrm{Def}(\mathcal{M})$ embeds as the deterministic sub-Markov-category.

Proposition 3.8 (Tensor product)**.** *Let $\kappa : X_1 \to Y_1$ and $\lambda : X_2 \to Y_2$ be presented kernels. The product measure $(\kappa\otimes\lambda)((x_1,x_2),-) := \kappa(x_1,-)\otimes\lambda(x_2,-)$ is a presented kernel $X_1\times X_2 \to Y_1\times Y_2$, with presentation obtained by taking products of latent spaces and multiplying densities.*

Proof. With presentations $(U_i, g_i, h_i)_{i\in I}$ and $(V_j, k_j, p_j)_{j\in J}$, set

$$W_{ij} := \{((x_1,x_2),u,v) : (x_1,u) \in U_i, (x_2,v) \in V_j\}, \quad G_{ij} := g_i \cdot k_j, \quad H_{ij} := (h_i, p_j).$$

W_{ij} is definable, $G_{ij} \in \mathcal{C}_{\mathcal{M}}(W_{ij})$ by (H3) (pullbacks to the product) and (H1) (product), and H_{ij} is definable. Normalisation holds since the total mass factors: $\sum_{(i,j)\in I\times J} \int G_{ij} = \big(\sum_{i\in I}\int g_i\big)\big(\sum_{j\in J}\int k_j\big) = 1\cdot 1 = 1$. On rectangles $B_1 \times B_2$:

$$(\kappa\otimes\lambda)((x_1,x_2), B_1\times B_2) = \kappa(x_1,B_1)\cdot\lambda(x_2,B_2),$$

which matches the presentation by Fubini. Agreement on rectangles extends to all Borel sets by the π–λ theorem. □

Lemma 3.9 (Interchange law)**.** *For morphisms $f_1 : X_1 \to Y_1$, $f_2 : Y_1 \to Z_1$, $g_1 : X_2 \to Y_2$, $g_2 : Y_2 \to Z_2$:*

$$(f_2 \otimes g_2) \circ (f_1 \otimes g_1) = (f_2 \circ f_1) \otimes (g_2 \circ g_1).$$

Proof. Both sides are kernels $X_1 \times X_2 \to Z_1 \times Z_2$. On rectangles $C_1 \times C_2$, the LHS equals

$$\int_{Y_1\times Y_2} f_2(y_1,C_1)\,g_2(y_2,C_2)\,d(f_1(x_1,-)\otimes g_1(x_2,-)),$$

which factors by Fubini into $(f_2\circ f_1)(x_1,C_1)\cdot(g_2\circ g_1)(x_2,C_2)$, matching the RHS. Extend by π–λ. □

Proposition 3.10 (Markov category)**.** $\mathrm{DefStoch}_{\mathcal{C}_{\mathcal{M}}}(\mathcal{M})$ *is a Markov category in the sense of [Fri20], with monoidal product $X\otimes Y := X\times Y$, unit $I := \{*\}$, $\mathrm{copy}_X(x,B) = \mathbf{1}_B(x,x)$, $\mathrm{del}_X(x,\{*\}) = 1$, and symmetry $\sigma_{X,Y}((x,y),B) = \mathbf{1}_B(y,x)$.*

Proof. All structure morphisms are Dirac kernels of definable maps. The associator, unitors, and symmetry are Dirac kernels of the underlying definable bijections; their coherence equations are inherited from $\mathrm{Def}(\mathcal{M})$, while their naturality against arbitrary kernels reduces to the Fubini factorisation of the independent product measure, as in Lemma 3.9. We verify the axioms of [Fri20, Def. 2.1]:

(2.2)–(2.3): Commutative comonoid. Coassociativity, counitality, and commutativity hold since copy_X and del_X are Dirac kernels of the diagonal $x \mapsto (x,x)$ and the unique map $x \mapsto *$.

(2.4): Monoidal compatibility. $\mathrm{del}_{X\times Y}$ sends $(x,y) \mapsto \delta_*$, agreeing with $\mathrm{del}_X \otimes \mathrm{del}_Y$. $\mathrm{copy}_{X\times Y}$ sends $(x,y) \mapsto \delta_{(x,y,x,y)}$, while the composite $(\mathrm{id}_X \otimes \sigma_{X,Y} \otimes \mathrm{id}_Y) \circ (\mathrm{copy}_X \otimes \mathrm{copy}_Y)$ sends $(x,y) \mapsto (x,x,y,y) \mapsto (x,y,x,y)$.

(2.5): Naturality of del. For any morphism f, $\mathrm{del}_Y \circ f = \mathrm{del}_X$ holds by the normalisation condition (N).

$\mathrm{id}_X \otimes \mathrm{id}_Y = \mathrm{id}_{X\times Y}$ (both are the Dirac kernel of the identity). Bifunctoriality is Lemma 3.9. Associativity and unitality of composition hold at the level of Markov kernels by Fubini; Theorem 3.7 ensures that the composites admit presentations, so these operations are well-defined in $\mathrm{DefStoch}_{\mathcal{C}_\mathcal{M}}(\mathcal{M})$. □

3.5 Instantiation and examples

We now verify that the Cluckers–Miller constructible functions form an admissible function class for $\mathbb{R}_{\mathrm{an}}$. The only hypothesis requiring an argument beyond the definitions is pullback stability (H3), recorded in the following lemma; (H1) is the definition of $\mathcal{C}$ as an $\mathbb{R}$-algebra and (H4) is the Cluckers–Miller stability theorem (Recall 2.6).

Lemma 3.11 (Pullback stability)**.** *Let $h : X \to Y$ be globally subanalytic and $g \in \mathcal{C}(Y)$ a constructible function. Then $g \circ h \in \mathcal{C}(X)$.*

Proof. By definition, $g(y) = \sum_i f_i(y) \prod_j \log f_{ij}(y)$ with $f_i : Y \to \mathbb{R}$ and $f_{ij} : Y \to (0,\infty)$ globally subanalytic. Pre-composing with h: each $f_i \circ h$ is subanalytic (composition of definable maps), each $f_{ij} \circ h$ is positive subanalytic, and $\log(f_{ij} \circ h) = (\log f_{ij}) \circ h$. So $g \circ h = \sum_i (f_i \circ h) \prod_j \log(f_{ij} \circ h) \in \mathcal{C}(X)$. □

Proposition 3.12. *The Cluckers–Miller constructible functions [CM11] form an admissible function class for $\mathbb{R}_{\mathrm{an}}$:* (H1) *since $\mathcal{C}(X)$ is an $\mathbb{R}$-algebra by construction,* (H2) *since it contains all globally subanalytic (= $\mathbb{R}_{\mathrm{an}}$-definable) functions by definition,* (H3) *by Lemma 3.11, and* (H4) *by [CM11, Thm. 1.3].*

Definition 3.13. $\mathrm{DefStoch}(\mathbb{R}_{\mathrm{an}})$ denotes $\mathrm{DefStoch}_{\mathcal{C}}(\mathbb{R}_{\mathrm{an}})$ with $\mathcal{C}$ the Cluckers–Miller constructible functions. More generally, $\mathrm{DefStoch}(\mathcal{M})$ denotes $\mathrm{DefStoch}_{\mathcal{C}_\mathcal{M}}(\mathcal{M})$ for any o-minimal $\mathcal{M}$ equipped with an admissible function class.

For strict expansions of $\mathbb{R}_{\mathrm{an}}$ (e.g., $\mathbb{R}_{\mathrm{an,exp}}$), the Cluckers–Miller class $\mathcal{C}$ need not be admissible: (H2) may fail since the exponential function is $\mathbb{R}_{\mathrm{an,exp}}$-definable but not constructible, and (H3) may fail for pullbacks along non-subanalytic definable maps. Extending Definition 3.1 to richer structures requires a correspondingly richer function class.

Remark 3.14 (Initiality)**.** For $\mathbb{R}_{\mathrm{an}}$, the Cluckers–Miller class is the *smallest* admissible function class: it is the smallest $\mathbb{R}$-algebra containing all globally subanalytic functions and closed under parametric integration [CM11, p. 3]. Any admissible class $\mathcal{C}'$ contains the subanalytic functions by (H2) and is closed under parametric integration by (H4), so $\mathcal{C} \subseteq \mathcal{C}'$, hence $\mathrm{DefStoch}_{\mathcal{C}}(\mathbb{R}_{\mathrm{an}}) \subseteq \mathrm{DefStoch}_{\mathcal{C}'}(\mathbb{R}_{\mathrm{an}})$. Thus $\mathrm{DefStoch}(\mathbb{R}_{\mathrm{an}})$ is the smallest sub-Markov-category of BorelStoch obtainable from $\mathrm{Def}(\mathbb{R}_{\mathrm{an}})$ by the axiomatic framework.

Proposition 3.15 (Kernel types)**.** *The following kernel types are morphisms in* $\mathrm{DefStoch}_{\mathcal{C}_\mathcal{M}}(\mathcal{M})$*:*

1. ***Deterministic:*** *$\delta_f(x,B) = \mathbf{1}_B(f(x))$ for definable $f : X \to Y$. Presented with $|I| = 1$, $m_1 = 0$, $g_1 \equiv 1$, $h_1 = f$.*
2. ***Absolutely continuous:*** *$\kappa(x,B) = \int_B g(x,y)\,dy$ for $g \in \mathcal{C}_{\mathcal{M}}(X \times Y)$ with $g \geq 0$ and $\int_Y g(x,y)\,dy = 1$, where $Y \subseteq \mathbb{R}^k$ has dimension k. Presented with $U = X \times Y$, $h = \pi_Y$.*
3. ***Singular:*** *$\kappa(x,-)$ is the pushforward of a density in $\mathcal{C}_{\mathcal{M}}$ on a definable set $U_x \subseteq \mathbb{R}^m$ through a definable map $h : U \to Y$ with $\dim h(U_x) < \dim Y$. This is a single-component presentation.*
4. ***Finite mixture:*** *concatenation of presentations with weight functions in $\mathcal{C}_{\mathcal{M}}(X)$ summing to 1. (Given κ_k with weights $w_k \in \mathcal{C}_{\mathcal{M}}(X)$, $w_k \geq 0$, $\sum_k w_k = 1$, the mixture has presentation $(w_k \cdot g_{k,i})$ on the disjoint union of latent spaces.)*

Crucially, type labels are *not* preserved under composition:

Example 3.16 (Dimension jumping)**.** Let $X = (0,1)$, $Y = Z = \mathbb{R}^2$. Define singular kernels $\kappa_1 : X \to Y$ by $\kappa_1(x,-) = \mathrm{Uniform}(\{(t,0) : t \in [0,x]\})$ and $\kappa_2 : Y \to Z$ by $\kappa_2(y,-) = \mathrm{Uniform}(\{(y_1,s) : s \in [0,1]\})$ (ignoring y_2). Both have 1-dimensional support in $\mathbb{R}^2$.

By Theorem 3.7, the composite has latent space $W = \{(x,t,s) : 0 \leq t \leq x, 0 \leq s \leq 1\}$ with density $G(x,t,s) = 1/x$ and output map $H(x,t,s) = (t,s)$. The induced kernel is $(\kappa_2 \circ \kappa_1)(x,-) = \mathrm{Uniform}([0,x] \times [0,1])$, with density

$$f(x; z_1, z_2) = \frac{1}{x} \cdot \mathbf{1}_{[0,x]}(z_1) \cdot \mathbf{1}_{[0,1]}(z_2),$$

which is absolutely continuous (2-dimensional support) and constructible (subanalytic). In particular, type labels (singular, absolutely continuous) are properties of presentations, not morphism invariants.

Example 3.17 (Iterated uniforms)**.** The kernel $k : (0,1] \to (0,1]$ given by $k(x,-) = \mathrm{Uniform}([0,x])$ has presentation $g(x,t) = 1/x$ on $U = \{0 < t \leq x\}$ with $h(x,t) = t$. By induction, its n-fold composition has density

$$k^{\circ n}(x,t) = \frac{(\log(x/t))^{n-1}}{(n-1)!\,x} \cdot \mathbf{1}_{(0,x]}(t).$$

The base kernel is subanalytic; the first composition introduces log, illustrating the Lion–Rolin phenomenon [LR98]. All iterates remain constructible, with log-degree growing by 1 per composition.

3.6 Properties and comparison

Every presented kernel is a Markov kernel on standard Borel spaces, giving a faithful identity-on-objects functor $\mathrm{DefStoch}(\mathbb{R}_{\mathrm{an}}) \to \mathrm{BorelStoch}$ that preserves the Markov category structure. Thus $\mathrm{DefStoch}(\mathbb{R}_{\mathrm{an}})$ is a sub-Markov-category of BorelStoch, selecting those kernels admitting a constructible latent-variable presentation.

Remark 3.18 (Deterministic morphisms)**.** A morphism $\kappa : X \to Y$ of $\mathrm{DefStoch}(\mathbb{R}_{\mathrm{an}})$ is deterministic (in the sense of [Fri20, Def. 10.1]) if and only if it is a Dirac kernel δ_f for some definable $f : X \to Y$. Indeed, determinism forces each κ_x to be a point mass $\delta_{f(x)}$; it remains to see that f is definable. Decompose each U_i of a presentation into cells on which h_i is C^1, g_i is continuous, and the fibre dimension is constant. Fix x. On a cell fibre that is open in $\mathbb{R}^{m_i}$, a positive measure dominated by a point mass concentrates on it, so $h_i(x,\cdot) = f(x)$ almost everywhere — hence, by continuity, everywhere — on the open set $\{g_i(x,\cdot) > 0\}$, and $\nabla_u h_i$ vanishes there; cells with lower-dimensional fibres carry no du-mass, and components with $m_i = 0$ contribute point masses at the already definable $h_i(x)$, which we include among the c_j below. Decomposing the definable set $\{\nabla_u h_i = 0\}$ into cells with definably connected fibres, h_i is constant along each fibre (definable arcs are piecewise C^1, so the chain rule applies along them), with value a definable partial function $c_j(x)$ (definable choice). Hence $f(x) \in \{c_1(x), \ldots, c_N(x)\}$ for finitely many definable c_j. Finally, $w_j(x) := \kappa(x, \{c_j(x)\})$ is constructible ($\mathbf{1}_{h_i = c_j} \cdot g_i \in \mathcal{C}$ by (H1)–(H2), then integrate by (H4)) and $\{0,1\}$-valued; being continuous on the cells of a suitable decomposition, w_j is the indicator of a definable set A_j, on which $f = c_j$. The A_j cover X, so f is definable.

Remark 3.19. DefStoch$(\mathbb{R}_{\text{an}})$ is *positive* [Fri20, Def. 11.22] and *causal* [Fri20, Def. 11.31]: positivity is inherited via the faithful Markov embedding DefStoch$(\mathbb{R}_{\text{an}}) \hookrightarrow$ BorelStoch [Fri20, Rem. 11.26], and causality likewise transfers along faithful Markov functors (it is a universal Horn condition); BorelStoch is causal by [Fri20, Prop. 11.34] since it has conditionals. Thus DefStoch$(\mathbb{R}_{\text{an}})$ is positive and causal but does not have conditionals: the conditionals exist in BorelStoch (regular conditional distributions), but there are presented joints whose regular conditionals are not morphisms in DefStoch$(\mathbb{R}_{\text{an}})$ (Proposition 3.20).

We now turn to the main negative result of this section.

Proposition 3.20 (No conditionals)**.** DefStoch$(\mathbb{R}_{\text{an}})$ *does not have conditionals in the sense of [Fri20, Def. 11.5] (see also [CJ19] for the connection between conditionals and Bayesian inversion).*

Proof sketch. Consider the joint $f : \{*\} \to (0,1) \times (0,1)$ with density $\frac{4}{3}(-\log y) \cdot \mathbf{1}_{0<y<x<1}$; this is a presented kernel (the density is constructible and the domain is definable). The X-marginal $f_X(dx) = \frac{4}{3}x(1 - \log x)\,dx$ has full support on $(0,1)$. If any presented kernel κ satisfied the conditional equation [Fri20, Def. 11.5], it would agree with the measure-theoretic conditional for a.e. x [Fri20, Prop. 13.7]. Stabilisation (Proposition 3.6) with $\varphi(y) = y$ would then give a constructible function $F(x) = \int y\,\kappa(x, dy)$ equal a.e. to

$$\frac{x(1 - 2\log x)}{4(1 - \log x)}.$$

Since constructible functions are continuous on the cells of a decomposition adapted to their subanalytic data (Definition 2.5), F equals this expression on some interval $(0, \delta)$. But $F(x) - x/2 = -x/\big(4(1 - \log x)\big) \sim -x \cdot (-\log x)^{-1}/4$ as $x \to 0^+$. Cell preparation (Recall 2.7) requires all log-exponents $\ell_{ij} \geq 0$, so a term $(-\log x)^{-1}$ cannot appear in any cell preparation of a constructible function. Thus F is not constructible — contradiction. The full computation is given in Appendix A. □

Lemma 3.21 (Density lemma)**.** *If μ is a presented probability measure on $\mathbb{R}^n$ that is absolutely continuous with respect to Lebesgue measure, then its Radon–Nikodym density $d\mu/d$Leb lies in $\mathcal{C}(\mathbb{R}^n)$.*

Proof sketch. Decompose the presentation by rank of h_i on cells. Rank $< n$ components are singular (zero contribution). Equal-dimensional components use the change-of-variables formula with definable inverse. Submersive components use the implicit function theorem and the integration stability theorem (Recall 2.6). See Appendix B for the full proof. □

The density lemma is also used in the conclusion to rule out conditional products: if the conditional-independence joint were presented, its density would be constructible, but cell preparation forbids the required terms.

Remark 3.22 (Which laws are presentable)**.** By the density lemma, every absolutely continuous presented law has a constructible density; conversely, any nonnegative constructible function on a full-dimensional definable set that integrates to 1 is the density of a presentable kernel (Proposition 3.15). So, among absolutely continuous laws, the presentable ones are exactly those whose densities are built from globally subanalytic functions and logarithms. This includes the uniform, triangular, and Beta(p,q) laws ($p,q \in \mathbb{Q}$) on bounded supports, but also algebraic-tailed laws on unbounded support such as the Cauchy law $\frac{1}{\pi}(1+x^2)^{-1}$, together with definable pushforwards and finite mixtures of all these. It excludes laws whose densities require the unrestricted exponential, such as the Gaussian $e^{-x^2/2}$ on $\mathbb{R}$ or the Gamma laws on $(0,\infty)$. Such densities are not constructible: by cell preparation (Recall 2.7) a nonzero constructible function is asymptotic to $cx^{\alpha}|\log x|^{\ell}$ ($c \neq 0$, $\alpha \in \mathbb{Q}$, $\ell \in \mathbb{N}_0$) at each end of its domain, whereas $e^{-x^2/2}$ and $x^{k-1}e^{-x}$ decay faster than every power of x at infinity. Lemma 3.21 then excludes these laws (though any of them *truncated* to a bounded definable set is presentable, since exp is restricted-analytic on a bounded box). The dividing line is algebraic-and-logarithmic tameness, not boundedness of support; the failure of conditionals (Proposition 3.20) is a statement about this tame fragment.

Remark 3.23 (Tame measures)**.** Kaiser [Kai12] introduces $\mathcal{M}$-*tame* measures: a Borel measure μ on a definable set X is $\mathcal{M}$-tame if there is an o-minimal expansion $\mathcal{M}^*$ of $\mathcal{M}$ such that, for every $\mathcal{M}$-definable family $\{f_t\}_{t\in T}$ of integrable functions, the parametric integral $t \mapsto \int_X f_t\, d\mu$ is definable in $\mathcal{M}^*$. Every presented kernel κ_x in DefStoch$(\mathbb{R}_{\text{an}})$ is $\mathbb{R}_{\text{an}}$-tame: integrating an $\mathbb{R}_{\text{an}}$-definable family against κ_x yields a constructible function by Proposition 3.6, and constructible functions are $\mathbb{R}_{\text{an,exp}}$-definable.

The converse fails: the measure μ with density $(\pi+1)x^{\pi}$ on $(0,1)$ witnesses that the inclusion is strict. It is not presentable: if it were, the density lemma (Lemma 3.21) would force its density $(\pi+1)x^{\pi}$ to be constructible, but cell preparation (Recall 2.7) allows only rational power-exponents, and $\pi \notin \mathbb{Q}$. However, μ *is* $\mathbb{R}_{\text{an}}$-tame with $\mathcal{M}^* = \mathbb{R}_{\text{an,exp}}$. After Lion–Rolin preparation [LR97], integrating an $\mathbb{R}_{\text{an}}$-definable family against μ reduces to finitely many cell integrals $\int_{a(t)}^{b(t)} U(\varphi(t,x))\,x^{q+\pi}\,dx$ ($q \in \mathbb{Q}$, φ bounded definable, $U \circ \varphi$ a subanalytic unit). Rescaling $x = b(t)\sigma$ factors each as the real power $b(t)^{q+\pi+1}$ times $\int_{a(t)/b(t)}^{1} U(\varphi(t,b(t)\sigma))\,\sigma^{q+\pi}\,d\sigma$, whose integrand is restricted analytic in σ and the bounded parameters; since $\pi \notin \mathbb{Q}$ the exponent $q+\pi+1 \neq 0$, so $\sigma^{q+\pi}$ integrates to a power, not a logarithm. Summing the cells, the parametric integral is $\mathbb{R}_{\text{an,exp}}$-definable. The irrationality of π thus does double duty: it excludes μ from DefStoch$(\mathbb{R}_{\text{an}})$, yet secures its $\mathbb{R}_{\text{an}}$-tameness.

4 The possibilistic case

We now construct DefRel$_+(\mathcal{M})$, a possibilistic analogue of DefStoch$(\mathcal{M})$ in which probability measures are replaced by nonempty definable subsets.

4.1 Definition

Definition 4.1. DefRel$_+(\mathcal{M})$ is the category whose objects are nonempty definable sets in $\mathcal{M}$ and whose morphisms $X \to Y$ are definable relations $R \subseteq X \times Y$ with nonempty fibres: $R_x = \{y \in Y : (x,y) \in R\} \neq \varnothing$ for all $x \in X$. Composition is relational:

$$(S \circ R)_x = \bigcup_{y \in R_x} S_y = \pi_Z\{(y,z) : (x,y) \in R \wedge (y,z) \in S\}.$$

The identity on X is the diagonal Δ_X.

Composition is well-defined: the set $R \times_Y S = \{(x,y,z) : (x,y) \in R, (y,z) \in S\}$ is definable, and its projection $\pi_{X\times Z}(R \times_Y S)$ is definable by o-minimality. This is the possibilistic analogue of the integration closure (H4) that makes DefStoch$(\mathcal{M})$ work: projection closure plays the role of integration closure.

Remark 4.2. The nonempty powerset monad P_+ on Set sends $X \mapsto 2^X \setminus \{\varnothing\}$. Its Kleisli category $\mathrm{Kl}(P_+)$ has morphisms $X \to Y$ given by functions $X \to P_+(Y)$, i.e. total relations with nonempty fibres. $\mathrm{DefRel}_+(\mathcal{M})$ is the subcategory of $\mathrm{Kl}(P_+)$ on definable objects, with morphisms restricted to definable relations. It is, however, *not* the Kleisli category of any monad on $\mathrm{Def}(\mathcal{M})$. Such a monad P would require $P(Y)$ to be a single definable set $\subseteq \mathbb{R}^N$ whose points name the admissible fibres R_y; but the fibres occurring in $\mathrm{DefRel}_+(\mathcal{M})$ are *arbitrary* nonempty definable subsets of Y, and these cannot be enumerated by a definable set of bounded dimension — already for $Y = \mathbb{R}$ they comprise all finite unions of points and intervals, a family requiring unboundedly many real parameters. This is exactly the obstruction that makes $P_+(Y)$ non-definable.

One might ask whether a smaller, genuinely definable powerset would suffice — for instance the finite powerset monad P_{fin}. Two obstacles remain. First, $P_{\mathrm{fin}}(Y)$ is itself non-definable for infinite Y: only the subsets of cardinality at most k form a definable set (coded by their increasing enumerations under a definable linear order on Y, such as the lexicographic order), and there is no uniform bound on k. Second, and more fundamentally, P_{fin} would capture only relations with *finite* fibres, whereas the morphisms of $\mathrm{DefRel}_+(\mathcal{M})$ have fibres of positive dimension, such as intervals — the possibilistic analogue of a continuous distribution. A family of subsets of bounded combinatorial complexity *does* form a definable set, but no such family is closed under the relational composition of Definition 4.1, which can increase the number of components without bound (composing a k-point relation with one whose fibres are disjoint intervals already yields a fibre with k components). We therefore construct $\mathrm{DefRel}_+(\mathcal{M})$ directly, with projection closure (o-minimality) playing the role that the monad multiplication would.

4.2 Markov category structure

Proposition 4.3. $\mathrm{DefRel}_+(\mathcal{M})$ *is a Markov category with monoidal product* $X \otimes Y := X \times Y$*, unit* $I := \{*\}$*,* $\mathrm{copy}_X(x) = \{(x,x)\}$*,* $\mathrm{del}_X(x) = \{*\}$*, and symmetry* $\sigma_{X,Y}(x,y) = \{(y,x)\}$*.*

Proof. All structure morphisms are deterministic (singleton-fibre) and definable. The monoidal product of morphisms is $(R \otimes S)_{(x_1,x_2)} = R_{x_1} \times S_{x_2}$, which is definable with nonempty fibres. The axioms of [Fri20, Def. 2.1] are verified as in Proposition 3.10: comonoid laws hold since copy and del are given by the diagonal and terminal maps, monoidal compatibility is direct, and naturality of del holds since every fibre is nonempty ($\mathrm{del}_Y(R_x) = \{*\} = \mathrm{del}_X(x)$). □

A morphism $R : X \to Y$ is *deterministic* iff $|R_x| = 1$ for all x, i.e., R is the graph of a definable function. The deterministic sub-Markov-category is $\mathrm{Def}(\mathcal{M})$.

$\mathrm{DefRel}_+(\mathcal{M})$ is *not* cartesian: a morphism $I \to X \times Y$ is a nonempty definable subset $S \subseteq X \times Y$, which need not be a product. For instance, the simplex $\{(x,y) \in [0,1]^2 : x + y \le 1\}$ expresses correlated nondeterminism.

Proposition 4.4. $\mathrm{DefRel}_+(\mathcal{M})$ *has conditionals, and hence is positive and causal [Fri20, Lem. 11.24, Prop. 11.34].*

Proof. Given $f : A \to X \times Y$ in $\mathrm{DefRel}_+(\mathcal{M})$, define $f|_X : X \times A \to Y$ by

$$f|_X(x,a) = \begin{cases} \{y : (x,y) \in f(a)\} & \text{if } x \in \pi_X(f(a)), \\ Y & \text{otherwise.} \end{cases}$$

When $x \in \pi_X(f(a))$, the fibre is nonempty by definition of projection. When $x \notin \pi_X(f(a))$, we use Y (nonempty by assumption on objects). Both cases are definable, so $f|_X$ is a morphism in $\mathrm{DefRel}_+(\mathcal{M})$.

For the chain rule, note that composing with the X-marginal $f_X(a) = \pi_X(f(a))$ restricts to $x \in f_X(a)$, so the default case is never reached: $\{(x,y) : x \in f_X(a),\ y \in f|_X(x,a)\} = f(a)$. □

4.3 Definable-specific features

Beyond the generic Markov category structure (which $\mathrm{Kl}(P_+)$ on Set already has), the definable setting provides:

Cell decomposition bounds complexity. Every morphism $R : X \to Y$ decomposes into finitely many cells by [Dri98, Ch. 3]. The fibres R_x have finitely many connected components and bounded dimension. The function $x \mapsto \dim(R_x)$ takes finitely many values, and its level sets form a definable partition of X — a possibilistic analogue of the kernel-type classification (Proposition 3.15).

Definable choice. By [Dri98, Ch. 6], every morphism $R : X \to Y$ admits a deterministic section: a definable function $g : X \to Y$ with $g(x) \in R_x$ for all x. This property is specific to o-minimal structures and fails in general topological or measurable settings.

Remark 4.5. Let us compare the two Markov categories constructed in this paper:

	$\mathrm{DefStoch}(\mathbb{R}_{\mathrm{an}})$	$\mathrm{DefRel}_+(\mathcal{M})$
Morphisms	presented kernels	definable total relations
Composition	fibre product of latent spaces	relational (projection)
Closure mechanism	integration closure (H4)	projection (o-minimality)
Conditionals	no (Proposition 3.20)	fibre construction
Positive, causal	yes (Remark 3.19)	yes (via conditionals)
Randomness pushback	open (§5)	yes (definable choice)
Deterministic	definable functions	definable functions

The contrast between the two rows "Conditionals" and "Closure mechanism" is not coincidental: possibilistic conditioning is a set-theoretic operation (taking fibres of a relation, which preserves definability), while probabilistic conditioning requires division by a marginal density, which can leave the constructible function class.

The *support map* $\mathrm{supp} : \mathrm{DefStoch}(\mathbb{R}_{\mathrm{an}}) \to \mathrm{DefRel}_+(\mathcal{M})$, sending each kernel to the definable relation $\mathrm{supp}(\kappa) = \{(x,y) : y \in \mathrm{supp}(\kappa_x)\}$, preserves identities, copy, delete, tensor, and symmetry (all structure morphisms are Dirac, whose support is a function graph). For composition, the inclusion $\mathrm{supp}(\lambda \circ \kappa) \subseteq \overline{\mathrm{supp}(\lambda) \circ \mathrm{supp}(\kappa)}$ holds (fibrewise closure in the codomain), but the closure cannot be removed: a kernel $\kappa = \mathrm{Uniform}(0,1)$ composed with $\lambda_y = \delta_{1/y}$ into $[1,\infty)$ has $\mathrm{supp}(\lambda \circ \kappa) = [1,\infty)$ while $\mathrm{supp}(\lambda) \circ \mathrm{supp}(\kappa) = (1,\infty)$. Conversely, equality can fail even with closure: a measure-zero set in $\mathrm{supp}(\kappa_x)$ can affect the relational composite without affecting the probabilistic one.

5 Conclusion

We have constructed two Markov categories of definable morphisms: $\mathrm{DefStoch}(\mathbb{R}_{\mathrm{an}})$, whose morphisms are stochastic kernels with constructible latent-variable presentations, and $\mathrm{DefRel}_+(\mathcal{M})$, whose morphisms are definable total relations. Both exploit the same model-theoretic ingredients — projection closure and cell decomposition — and the axiomatic framework (H1)–(H4) isolates what is needed from the function class.

The two categories exhibit an asymmetry in their Markov-theoretic properties. The possibilistic category $\mathrm{DefRel}_+(\mathcal{M})$, a sub-Markov-category of $\mathrm{Kl}(P_+)$ on Set restricted to definable objects, has conditionals, positivity, and causality (Proposition 4.4). The probabilistic category $\mathrm{DefStoch}(\mathbb{R}_{an})$, by contrast, does *not* have conditionals (Proposition 3.20): the obstruction is that conditioning introduces reciprocals of constructible functions, and the cell preparation theorem forces all log-exponents to be nonnegative integers, ruling out terms of the form $(-\log x)^{-1}$. This is a *definability* obstruction, not a measure-theoretic one — the conditionals exist in BorelStoch but do not restrict to $\mathrm{DefStoch}(\mathbb{R}_{an})$. The same obstruction rules out *conditional products* in the sense of [Fri20, Prop. 12.9]: a pair of presented kernels with common marginal $r(w) = \frac{4}{3}w(1-\log w)$ has a conditional-independence joint that is absolutely continuous on $(0,1)^3$ (since both conditionals have densities on open subsets of $(0,1)$) with density involving $1/(w(1-\log w))$; the density lemma (Lemma 3.21) forces this density to be constructible if the joint is presented. But the constructible representative agrees with this density off a lower-dimensional definable set (continuity on cells), so a generic definable slice $x = \lambda w$, $y = \mu w$ avoids it; along the slice, the asymptotic expansion as $w \to 0^+$ contains a *subleading* term $\frac{4}{3}(1+\log\lambda)(1+\log\mu)\,w^{-1}(-\log w)^{-1}$ that cell preparation forbids — at $\lambda = \mu = \frac{1}{2}$ the restriction is $\frac{4}{3}(\log(2/w))^2/(w(1-\log w))$, with subleading coefficient $\frac{4}{3}(1-\log 2)^2$. Thus even the strictly weaker notion of conditional products for specific well-behaved pairs fails.

The dilations framework of Fritz–Gonda et al. [FGHL+23] gives a reformulation. Positivity is equivalent to the existence of initial dilations for all *deterministic* morphisms [FGHL+23, Prop. 4.12]. Conditionals guarantee initial dilations for *all* morphisms [FGHL+23, Prop. 4.13]. Since $\mathrm{DefStoch}(\mathbb{R}_{an})$ is positive but lacks conditionals, it has initial dilations for deterministic morphisms (definable maps). For non-deterministic kernels, the situation is open: the standard construction of initial dilations via the input-output copy [FGHL+23, Prop. 4.13] requires the conditional, which would introduce the forbidden $(-\log x)^{-1}$ terms, but an alternative construction not passing through conditionals has not been ruled out.

This failure mode is distinct from that of quasi-Borel spaces [HKSY17]: in QBStoch, the privacy equation forces non-positivity [SSSW21, Props. 5.2, 5.8], so the obstruction is global (positivity itself fails). In $\mathrm{DefStoch}(\mathbb{R}_{an})$, the failure is localised: conditionals fail already at pairs of intervals, but positivity and causality are fully preserved (Remark 3.19).

Moreover, the obstruction is specific to the *conditional independence* condition, not to marginal matching per se. For the same counterexample pair, the *comonotone coupling* — the presented kernel on latent space $U = \{(x,w) : 0 < x < w < 1\}$ with density $g(x,w) = \frac{4}{3}(-\log x)$ and output map $h(x,w) = (x,w,x)$ — is a valid morphism $I \to X \otimes W \otimes Y$ in $\mathrm{DefStoch}(\mathbb{R}_{an})$ whose $X \otimes W$ and $W \otimes Y$ marginals match those of the counterexample pair. This coupling sets $Y = X$ almost surely (maximal dependence, violating conditional independence) and is singular with respect to Lebesgue measure on $(0,1)^3$, so the density lemma does not apply. The existence of such definable couplings, in the absence of conditional products, suggests that the space of definable couplings for a given marginal pair may carry useful categorical structure, even when the unique conditional-independence coupling is not presentable.

We close with four open directions.

Support map. The map $\mathrm{supp} : \mathrm{DefStoch}(\mathbb{R}_{an}) \to \mathrm{DefRel}_+(\mathcal{M})$ preserves all Markov category structure except composition, where fibrewise closure is needed and even then the inclusion can be strict (Remark 4.5). Whether supp can be made a strict or lax Markov functor via a modified relational composition is open.

Tame measures and the presentation gap. Every presented kernel in $\mathrm{DefStoch}(\mathbb{R}_{an})$ is $\mathbb{R}_{an}$-tame in Kaiser's sense (Remark 3.23), but the converse fails: the measure with density $(\pi+1)x^{\pi}$ on $(0,1)$ is $\mathbb{R}_{an}$-tame (with uniform witness $\mathbb{R}_{an,exp}$) yet not presentable (Remark 3.23). This strict gap between semantic tameness and syntactic presentability remains to be understood; in particular, whether every tame measure is a limit of presentable ones is open.

Randomness pushback. Fritz [Fri20, Def. 11.19] asks whether every morphism factors through a deterministic map and a randomness source. For $\mathrm{DefRel}_+(\mathcal{M})$ the answer is yes, using definable choice for sections. For $\mathrm{DefStoch}(\mathbb{R}_{\mathrm{an}})$, the question reduces to a transport problem: same-dimensional subanalytic maps can only produce subanalytic densities, but higher-dimensional maps can produce constructible densities (e.g., $T(x,y) = xy$ pushes Leb^2 to $-\log t$). Monomial densities from cell preparation are achievable via products of uniforms; the remaining gap is whether cross-coordinate subanalytic unit factors also admit subanalytic transports.

Beyond $\mathbb{R}_{\mathrm{an}}$. The axiomatic framework applies to any function class satisfying (H1)–(H4), but we have only verified the hypotheses for Cluckers–Miller constructible functions over $\mathbb{R}_{\mathrm{an}}$, where the constructible class is the *smallest* admissible class (Remark 3.14). Other candidates include logarithmic-exponential functions in the sense of Lion–Rolin [LR97], which form a strictly larger algebra than the constructible class and might give a larger category with weaker closure properties, and function classes over polynomially bounded o-minimal structures. The conditionals obstruction (Proposition 3.20) is specific to the Cluckers–Miller class and its nonnegative log-exponent constraint; a function class admitting $(-\log x)^{-1}$ might support conditionals, at the cost of losing the clean cell preparation theorem.

Acknowledgements

I thank the anonymous referees for their reading and constructive suggestions, which sharpened the o-minimal preliminaries, the discussion of related work, and the placement of several results. This work was supported by a Coefficient Giving grant. Large language models were used extensively in preparing this paper, including in its mathematical development and exposition; I take full responsibility for all results and any errors.

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A Proof of Proposition 3.20

We show that a specific morphism in $\mathrm{DefStoch}(\mathbb{R}_{\mathrm{an}})$ admits no presented kernel satisfying Fritz's conditional equation.

The counterexample. Define $f:\{*\}\to(0,1)\times(0,1)$ by the probability measure with density

$$g(x,y) \;=\; \tfrac{4}{3}(-\log y)\cdot\mathbf{1}_{0<y<x<1}$$

with respect to Lebesgue measure on $(0,1)^2$. This is a presented kernel: $|I|=1$, latent dimension $m=2$, domain $U=\{(x,y)\in\mathbb{R}^2 : 0<y<x<1\}$ (definable), density $\frac{4}{3}\log(1/y)$ (constructible, since $1/y$ is positive subanalytic), output map $h(x,y)=(x,y)$ (definable), and $g\geq 0$ on U.

Normalisation. $\int_0^x(-\log y)\,dy = x(1-\log x)$ by integration by parts (using $\lim_{y\to 0^+} y\log y = 0$), and $\int_0^1 x(1-\log x)\,dx = \frac{1}{2}+\frac{1}{4}=\frac{3}{4}$, giving $\frac{4}{3}\cdot\frac{3}{4}=1$.

The conditional. The X-marginal is $f_X(dx)=\frac{4}{3}x(1-\log x)\,dx$, which is strictly positive on $(0,1)$. By the disintegration theorem, the conditional $f|_X(x,dy)=\frac{(-\log y)}{x(1-\log x)}\cdot\mathbf{1}_{(0,x)}(y)\,dy$ is uniquely determined for f_X-a.e. x.

Stabilisation test. Suppose for contradiction that some presented kernel κ satisfies Fritz's conditional equation for f. The conditional equation determines $\kappa(x,-)$ for f_X-a.e. x [Fri20, Prop. 13.7], hence for Lebesgue-a.e. x (since f_X is equivalent to Lebesgue). Taking $\varphi(y) = y \in \mathcal{C}\big((0,1)\big)$, Proposition 3.6 gives $\Phi(x) := \int y\,\kappa(x,dy) \in \mathcal{C}\big((0,1)\big)$. Since Φ is constructible, it is continuous on each cell of a finite cell decomposition; since Φ equals the continuous function $F(x) := \int y\, f|_X(x,dy)$ for a.e. x, we have $\Phi = F$ on $(0,\delta)$ for some $\delta > 0$ (two continuous functions agreeing a.e. on an interval agree everywhere). In particular, $F \in \mathcal{C}\big((0,\delta)\big)$. Define $G(x) := F(x) - x/2$, which lies in $\mathcal{C}$ by (H1).

Computing F. Integration by parts yields $\int_0^x y(-\log y)\,dy = x^2\big(\frac{1}{4} - \frac{\log x}{2}\big)$, so

$$F(x) \;=\; \frac{x^2(1/4 - \log x/2)}{x(1-\log x)} \;=\; \frac{x(1-2\log x)}{4(1-\log x)}.$$

Non-constructibility. Since $\mathcal{C}\big((0,1)\big)$ is an $\mathbb{R}$-algebra containing all subanalytic functions,

$$G(x) \;:=\; F(x) - \tfrac{x}{2} \;=\; \frac{-x}{4(1-\log x)}$$

would also be constructible. Setting $L = -\log x > 0$ for $x \in (0,1)$:

$$G(x) \;=\; \frac{-x}{4(1+L)} \;\sim\; -\frac{x}{4L} \;=\; -\frac{x}{4}\cdot(-\log x)^{-1} \quad \text{as } x \to 0^+.$$

By cell preparation (Recall 2.7), on some interval $(0,\delta)$, every constructible function takes the form $\sum_{i=1}^N u_i(x)\cdot x^{\alpha_i}\cdot(-\log x)^{\ell_i}$ with $\alpha_i \in \mathbb{Q}$, $\ell_i \in \mathbb{N}_0$, and u_i subanalytic units (bounded, nonvanishing). Since $G(x)/x = -1/(4(1-\log x)) \to 0$ as $x \to 0^+$:

- Terms with $\alpha_i < 1$ would make $G(x)/x$ diverge as $x \to 0^+$, contradicting $G(x)/x \to 0$.
- Terms with $\alpha_i > 1$ contribute $o(1)$ to $G(x)/x$.
- Terms with $\alpha_i = 1$ contribute $u_i(0)\cdot(-\log x)^{\ell_i}$ to $G(x)/x$. Since $\ell_i \geq 0$, each such term either diverges ($\ell_i > 0$) or approaches a nonzero constant ($\ell_i = 0$).

More precisely, grouping by leading $(-\log x)$-exponent, the asymptotic behaviour of any nonzero constructible function on $(0,\delta)$ is $Cx^\beta(-\log x)^\ell$ with $C \neq 0$, $\beta \in \mathbb{Q}$, and $\ell \in \mathbb{N}_0$; matching $G(x)/x \sim -\frac{1}{4}(-\log x)^{-1}$ would require $\ell = -1$, which cell preparation forbids. Thus G is not constructible, so F is not constructible, contradicting stabilisation. □

B Proof of Lemma 3.21

Let $\mu = \sum_i (h_i)_*(g_i \cdot \mathrm{Leb}|_{U_i})$ be a presentation with $U_i \subseteq \mathbb{R}^{m_i}$, $g_i \geq 0$, and $h_i : U_i \to \mathbb{R}^n$ definable. Each summand $\nu_i = (h_i)_*(g_i \cdot \mathrm{Leb}|_{U_i})$ is a positive measure. Since μ is absolutely continuous and $\mu = \sum_i \nu_i$ with $\nu_i \geq 0$, each ν_i must be absolutely continuous (positive singular parts cannot cancel). We may assume each U_i is full-dimensional ($\dim U_i = m_i$); otherwise $\mathrm{Leb}|_{U_i} = 0$, so $\nu_i = 0$ and the summand contributes nothing. We show each ν_i has a constructible density, by C^1 cell decomposition of U_i [Dri98, Ch. 7] into cells on which h_i has constant rank, so that the Jacobian and the local inverses used below are well defined.

Case 1: rank$h_i < n$ on a cell C. The pushforward $(h_i|_C)_*(g_i \cdot \mathrm{Leb}|_C)$ is supported on $h_i(C)$, which has dimension $< n$, hence is singular with respect to Lebesgue measure on $\mathbb{R}^n$. Since ν_i is absolutely continuous, this component must be zero. It contributes nothing to the density.

Case 2: rank$h_i = n$, dim$U_i = n$. By further cell decomposition, $h_i|_C$ is injective on each sub-cell C_k. The change-of-variables formula gives pushforward density

$$f_{i,k}(z) = g_i\big((h_i|_{C_k})^{-1}(z)\big) \cdot \big|\det J_{(h_i|_{C_k})^{-1}}(z)\big|$$

at $z \in h_i(C_k)$. The inverse $(h_i|_{C_k})^{-1}$ is definable (o-minimal inverse function theorem), so $g_i \circ (h_i|_{C_k})^{-1} \in \mathcal{C}$ by (H3). The Jacobian determinant of a definable map is definable, hence in $\mathcal{C}$ by (H2). The product is in $\mathcal{C}$ by (H1).

Case 3: rank$h_i = n$, dim$U_i = m_i > n$. After further cell decomposition by all $\binom{m_i}{n}$ minors of the Jacobian Jh_i, each resulting cell either has rank $< n$ (Case 1) or has a fixed nonvanishing $n \times n$ minor. On a cell of the second type, choose the coordinate subset I with $|I| = n$ corresponding to the nonvanishing minor, write $u = (u_I, u_{I^c})$, and define $\Phi(u) = (h_i(u), u_{I^c})$, a definable map $U_i \to \mathbb{R}^n \times \mathbb{R}^{m_i - n}$. Since $\det(\partial h_i / \partial u_I) \neq 0$ on this cell, Φ is a local C^1-diffeomorphism by the o-minimal inverse function theorem [Dri98, Ch. 7]; after further decomposition, assume $\Phi|_{C_k}$ is injective. By change of variables and Fubini:

$$f_{i,k}(z) = \int g_i(\Phi^{-1}(z,v)) \cdot |\det J\Phi^{-1}(z,v)|\, dv.$$

The integrand is in $\mathcal{C}$: $g_i \circ \Phi^{-1} \in \mathcal{C}$ by (H3), and $|\det J\Phi^{-1}|$ is definable, hence in $\mathcal{C}$ by (H2). The integral over definable fibres gives $f_{i,k} \in \mathcal{C}$ by Recall 2.6, whose zero-convention covers the Lebesgue-null set of z where the fibre integral diverges (a density may be modified on a null set).

The total density $d\mu/d\mathrm{Leb} = \sum_{i,k} f_{i,k}$ is a finite sum of elements of $\mathcal{C}$, hence in $\mathcal{C}$. □