Modularity, synchronization, and noise: a view from nonlinear contraction theory

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Plan

1. Nonlinear contraction theory
2. Stable synchronization, concurrent synchronization
3. Stochastic contraction
4. Synchronization and protection against noise

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Stability and modularity

- Biological systems (e.g. neuronal networks) are complex, contain multiple feedback loops
- The probability for a network to be stable decreases with the network's size (Grey Walter, 1951)
- Evolution = accumulation of stable components?
- Question: how accumulation can preserve stability?

Contraction theory: a tool to analyze stability

Consider the dynamical system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$$

If there exist a metric $\boldsymbol{\Theta}(\mathbf{x}, t)^\top \boldsymbol{\Theta}(\mathbf{x}, t)$ such that

$$\forall \mathbf{x}, t \quad \lambda_{\mathrm{max}}(\mathbf{J}_s) < -\lambda$$

where

$$\mathbf{J} = \left(\dot{\boldsymbol{\Theta}} + \boldsymbol{\Theta} \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right) \boldsymbol{\Theta}^{-1} \qquad \boldsymbol{\Theta}(\mathbf{x}, t)^\top \boldsymbol{\Theta}(\mathbf{x}, t) > 0$$

then all system trajectories converge exponentially towards a unique trajectory, independently of initial conditions (Lohmiller & Slotine, *Automatica*, 1998)

Proof: Consider a smooth path between each pair of trajectories and differentiate its length

Interesting properties

- Exact and global analysis (in contrast with linearization techniques)
- Converse theorem: global exponential stability $\Rightarrow$ contraction in some metric
- Combination properties
  - Parallel combination
  - Hierarchical
  - Negative feedback
  - Small gains

Example: negative feedback

- Consider the combination

$$\begin{cases} d\mathbf{x}_1 = \mathbf{f}_1(\mathbf{x}_1, \mathbf{x}_2, t)dt \\ d\mathbf{x}_2 = \mathbf{f}_2(\mathbf{x}_1, \mathbf{x}_2, t)dt \end{cases}$$

  where system $\mathbf{x}_i$ est contracting with rate λ_i in the metric $\mathbf{M}_i = \mathbf{\Theta}_i^T \mathbf{\Theta}_i$

- Assume that the combination is negtive feedback, i.e.

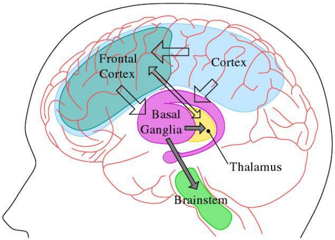
$$\mathbf{\Theta}_1 \mathbf{J}_{12} \mathbf{\Theta}_2^{-1} = -k \mathbf{\Theta}_2 \mathbf{J}_{21}^T \mathbf{\Theta}_1^{-1}$$

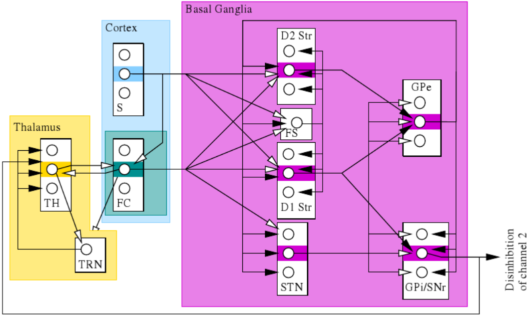
- Then the combined system is contracting with rate $\min(\lambda_1, \lambda_2)$ in the metric $\mathbf{M} = \mathbf{\Theta}^T \mathbf{\Theta}$ where

$$\mathbf{\Theta} = \begin{pmatrix} \mathbf{\Theta}_1 & \mathbf{0} \\ \mathbf{0} & \sqrt{k}\mathbf{\Theta}_2 \end{pmatrix}$$

Application: modeling the basal ganglia

- Basal ganglia: role in motor action selection
- Multiple hierarchical and feedback loops physiologically identified
- Robotics application: action selection in a survival task





Girard, Tabareau, Pham, Berthoz & Slotine, *Neural Networks*, 2008

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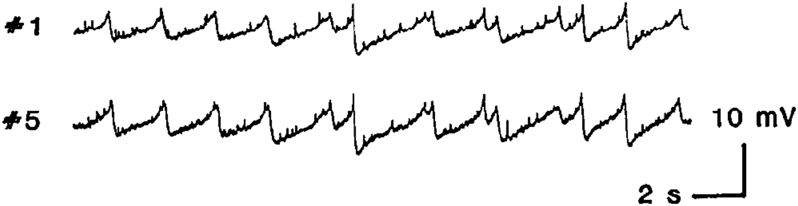
Synchronization phenomena

- In neuronal networks
  - Observation: similar behavior of different neurons in time



Christie et al, *J Neurosci*, 1989

  - Proposed mechanisms: connections of neurons through chemical and electrical connections, network effects
- Elsewhere
  - Flocking (birds), schooling (fishes),...
  - Quorum sensing in cells
  - Multi-robots deployment
  - ...

Roles of synchronization in neuronal networks

- Allow different distant sites to communicate, example:
  - in the "binding problem": for instance, relate different attributes (computed in different brain areas) – color, form, movement – of the same object (Engel et Singer, *Trends Cog Sci*, 2001)
  - between hippocampus and prefrontal cortex in memory consolidation (Peyrache et al, *Nat Neurosci*, 2009)
- Signal amplication or protection against noise (see later)
- ...

Synchronization and contraction

Synchronization = convergence towards a linear subspace of the global state space

Example:

- consider a system of 4 oscillators $\widehat{\mathbf{x}} = (\mathbf{x}_1, \ldots, \mathbf{x}_4)$
- then full synchronization corresponds to the subspace $\mathcal{M} = \{\widehat{\mathbf{x}} : \mathbf{x}_1 = \mathbf{x}_2 = \mathbf{x}_3 = \mathbf{x}_4\}$ (of dimension 1)

Convergence to a linear flow-invariant space

Consider a system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$ (not contracting in general)

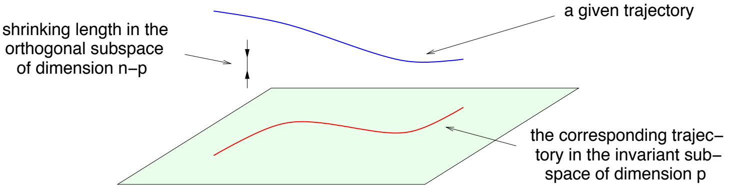
- Assume that there exists a flow-invariant linear subspace $\mathcal{M}$, i.e. :

$$\forall t : \mathbf{f}(\mathcal{M}, t) \subset \mathcal{M}$$

- Consider an orthonormal "projection" on $\mathcal{M}^\perp$, described by a matrix $\mathbf{V}$ and construct the auxiliary system

$$\dot{\mathbf{y}} = \mathbf{V}\mathbf{f}(\mathbf{V}^\top \mathbf{y} + \mathbf{U}^\top \mathbf{U}\mathbf{x}, t)$$

- If the $\mathbf{y}$-system is contracting then all solutions of the $\mathbf{x}$-system converge to $\mathcal{M}$.



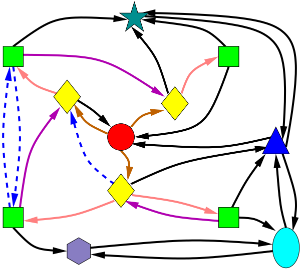
Interesting properties

Naturally inherits the properties of standard contraction theory

- Exact and global analysis
- Combination properties
    - Hierarchy
    - Negative feedback
    - Small gains

Concurrent synchronization

Multiple groups of oscillators synchronized within a group but not across groups



Pham & Slotine, *Neural Networks*, 2007

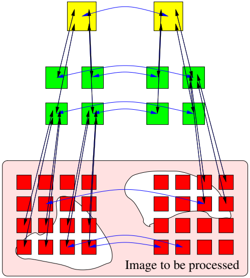
⇒ Accumulation and cohabitation of multiple ensembles of synchronized neurons

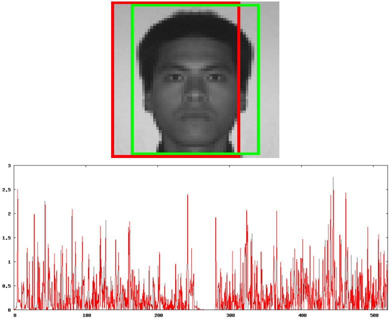
Concurrent synchronization

Concurrent synchronization can be treated by the same framework as previously. Example:

- consider a system of 4 oscillators $\mathbf{x}_1, \ldots, \mathbf{x}_4$
- a state where $\mathbf{x}_1 = \mathbf{x}_2$ and $\mathbf{x}_3 = \mathbf{x}_4$ but where $\mathbf{x}_1 \neq \mathbf{x}_3$ is a concurrent synchronization state
- this concurrent synchronization corresponds to the linear subspace $\mathcal{M} = \{\widehat{\mathbf{x}} : \mathbf{x}_1 = \mathbf{x}_2\} \cap \{\widehat{\mathbf{x}} : \mathbf{x}_3 = \mathbf{x}_4\}$ (of dimension 2)

Example: symmetry detection





Pham & Slotine, *Neural Networks*, 2007

Other examples:

- CPG-based control of a turtle-like underwater vehicle (Seo, Chung & Slotine, *Autonomous Robots*, 2010)
- Quorum sensing (Russo & Slotine, *Physical Review E*, 2010)

Plan

Motivations

- Biological or artificial systems are often subject to random perturbations
- Benefits from the interesting properties of contraction theory
    - Exact and global analysis
    - Combination properties
    - Concurrent synchronization

How to model perturbations?

In physics, engineering, finance, neuroscience,... random perturbations are traditionnally modelled with Itô stochastic differential equations (Itô SDE)

$$d\mathbf{x} = \mathbf{f}(\mathbf{x}, t)dt + \sigma(\mathbf{x}, t)dW$$

- $\mathbf{f}$ is the dynamics of the noise-free version of the system
- σ is the noise variance matrix (noise intensity)
- W is a Wiener process ($dW/dt =$ "white noise")

The stochastic contraction theorem

- If the noise-free system is contracting

$$\lambda_{\max}(\mathbf{J}_s) \leq -\lambda$$

Pham, Tabareau & Slotine, *IEEE Trans Aut Contr*, 2009

The stochastic contraction theorem

- If the noise-free system is contracting

$$\lambda_{\max}(\mathbf{J}_s) \leq -\lambda$$

- and the noise variance is upper-bounded

$$\mathrm{tr}\left(\sigma(\mathbf{x},t)^T\sigma(\mathbf{x},t)\right) \leq C$$

Pham, Tabareau & Slotine, *IEEE Trans Aut Contr*, 2009

The stochastic contraction theorem

- If the noise-free system is contracting

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- and the noise variance is upper-bounded

$$\mathrm{tr}\left(\sigma(\mathbf{x},t)^T\sigma(\mathbf{x},t)\right) \leq C$$

- Then

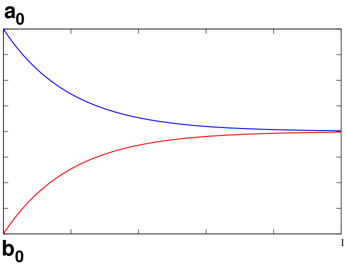
$$\forall t \geq 0 \quad \mathbb{E}\left(\|\mathbf{a}(t)-\mathbf{b}(t)\|^2\right) \leq \frac{C}{\lambda} + \|\mathbf{a}_0 - \mathbf{b}_0\|^2 e^{-2\lambda t}$$

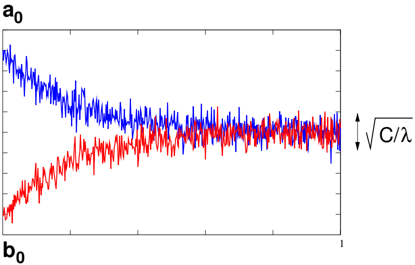
Pham, Tabareau & Slotine, *IEEE Trans Aut Contr*, 2009

Practical meaning

After exponential transients of rate λ, we have

$$\mathbb{E}\left(\|\mathbf{a}(t) - \mathbf{b}(t)\|\right) \leq \sqrt{\frac{C}{\lambda}}$$





Combinations of stochastically contracting systems

Combinations results in deterministic contraction can be adapted very naturally for stochastic contraction

- Parallel combinations
- Hierarchical combinations
- Negative feedback combinations
- Small gains

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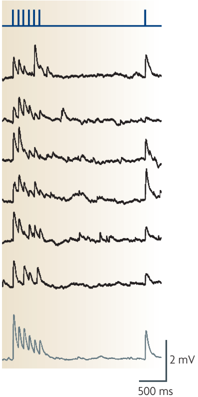
Noise in the nervous system

- Observation: variable response to identical stimulations



- Possible causes: canal noise, synaptic noise, etc. (cf. Faisal et al, *Nat Rev Neurosci*, 2008)

Effect of noise on synchronization

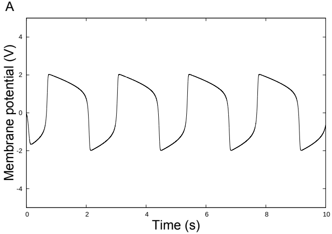
- Noise can destroy la synchronisation
- or, on contrary, enable synchronization: Mainen et Sejnowski (*Science* 1995), Teramae et Tanaka (*PRL* 2004)

Our study

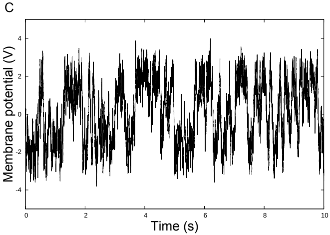
- We study here the converse relation: the effect of synchronization on noise
- By doing so, we indentify another functional role for synchronization

Noise perturbs the trajectories of the oscillators

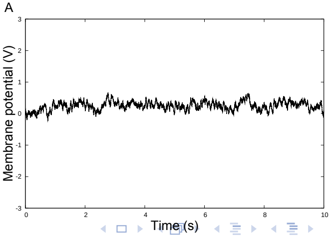
Noiseless oscillator



Noisy oscillator



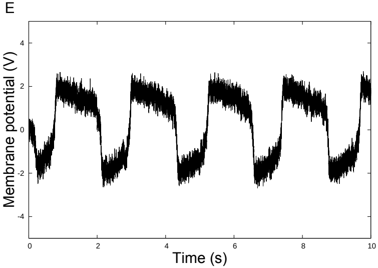
Average value of 100 noisy oscillators



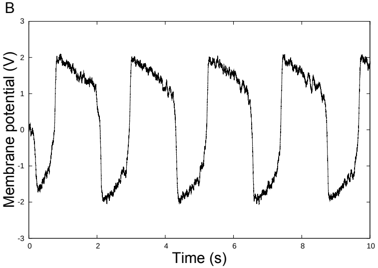
Averaging gets rid of the noise, but also of the signal!

Synchronization protects from noise

Noisy synchronized oscillator



Average value of 100 noisy synchronized oscillators



Tabareau, Slotine & Pham, *PLoS Comput Biol*, 2010

Hypothesis (I)

Network of n noisy oscillators and diffusively coupled (Itô SDE):

$$d\mathbf{x}_i = \left(\mathbf{f}(\mathbf{x}_i, t) + \sum_{j \neq i} \mathbf{K}_{ji}(\mathbf{x}_j - \mathbf{x}_i) \right) dt + \sigma dW_i, \;\; i = 1 \ldots n$$

or in the "physicist's way"

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i, t) + \sum_{j \neq i} \mathbf{K}_{ji}(\mathbf{x}_j - \mathbf{x}_i) + \sigma \xi_i, \;\; i = 1 \ldots n$$

with ξ representing a "white noise"

Hypothesis (II)

For assumptions:

- A1 The network is balanced: for each oscillator $\sum_j \mathbf{K}_{ji} = \sum_j \mathbf{K}_{ij}$)
- A2 The nonlinearity of $\mathbf{f}$ is bounded: $|\lambda_{\max}(\mathbf{H}_j)| \leq \frac{1}{\sqrt{d}}\mathbf{H}_{bd}$
- A3 The dynamics of $\mathbf{f}$ is robust to small perturbations
- A4 The oscillators are synchronized:

$$\mathbb{E}\left(\sum_{i<j} \|\mathbf{x}_i - \mathbf{x}_j\|^2\right) \leq \rho$$

Results

Under the previous hypotheses, when $\rho/n^2 \to 0$ and $n \to \infty$, the effect of noise on each oscillator evolves as

$$\frac{\rho \mathbf{H}_{bd}}{2n^2} + \frac{\sigma}{\sqrt{n}}$$

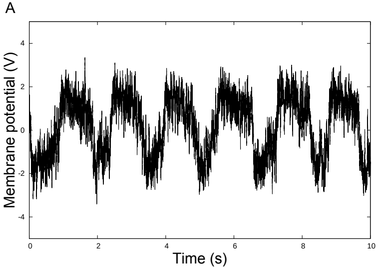
Remark: when the systems are linear, we have $\mathbf{H}_{bd} = 0$ and thus the known result for linear systems are recovered

Elements of proof

- Compare the mean trajectory to a noiseless trajectory
- Since the trajectories are close to each other (A4), they are close to the mean trajectory

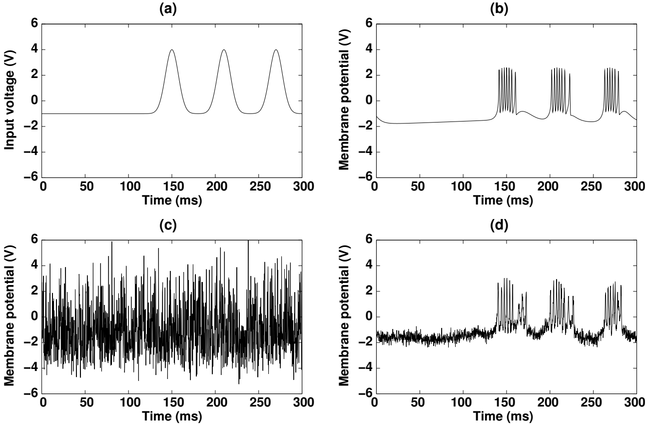
Extension: probabilistic network

- 200 oscillators
- each pair of oscillators has probability 0.1 to be connected



(no formal proof at the present time)

Hindmarsh-Rose + time-varying inputs



- a Input
- b Output – unperturbed oscillators
- c Output – noisy oscillators
- d Output – noisy synchronized oscillators

End

Thank you for your attention!
I'll be happy to answer questions.