Robustness of noise-induced synchronization

Ngày 16 tháng 4 năm 2008

Outline

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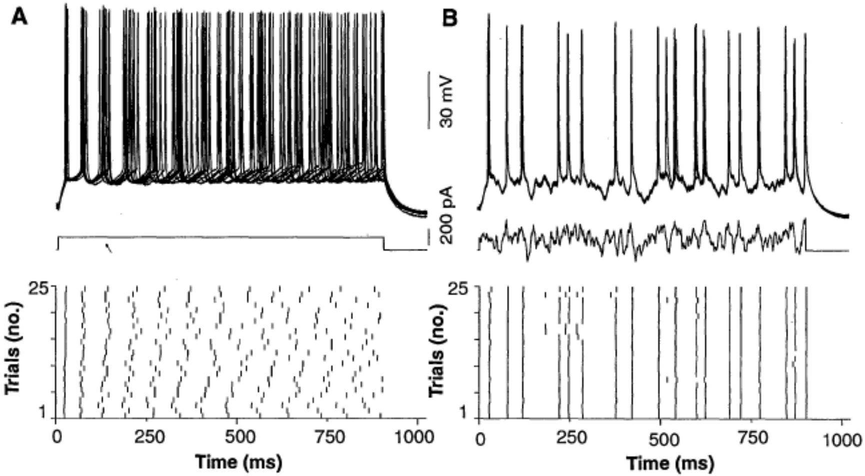
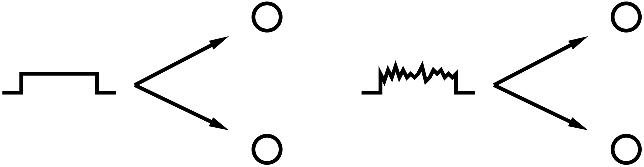
Mainen & Sejnowski experiment



[Mainen & Sejnowski, 1995]

Synchronization interpretation

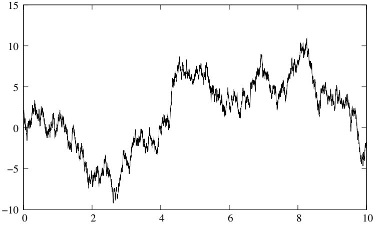


Outline

1. Problem statement
2. Modelling "noise": Stochastic Differential Equations
3. The proof
4. Limitations of the analysis

Random walk and Wiener process

- Random walk (discrete-time): $\mathbf{x}_{t+\Delta t} = \mathbf{x}_t + \xi_t \Delta t$
  where $(\xi_t)_{t\in\mathbb{N}}$ are Gaussian and mutually independent
- If one is interested in very rapidly varying perturbations, Δt has to be very small
- Wiener process (or Brownian motion) (continuous-time): limit of the random walk when $\Delta t \to 0$



Wiener process and "white noise"

- Problem: a Wiener process is not differentiable (why?), thus it is not the solution of any ordinary differential equation
- Define formally ξ_t ("white noise") = "derivative" of the Wiener process
- Formally: $W(t) - W(0) = \int_0^t \xi_t dt$ or $dW/dt = \xi_t$ or $dW = \xi_t dt$

Stochastic Differential Equations

- We consider processes driven by “white noise”
- We would like to write (but it’s not correct, because ξ does not exist)

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\xi$$

- In integral form, it may be more correct

$$\mathbf{x}(t) - \mathbf{x}(0) = \int_0^t \mathbf{f}(\mathbf{x})dt + \int_0^t \mathbf{g}(\mathbf{x})dW$$

  where the last term is a Stieltjes integral against W (which does exist)
- The integral form can also be written in differential form

$$d\mathbf{x} = \mathbf{f}(\mathbf{x})dt + \mathbf{g}(\mathbf{x})dW$$

Definition of Itô and Stratonovich integrals

- For deterministic function α, the Stieltjes integral (which generalizes Riemann integrals) against α is defined as

$$\int_0^T \beta(t)d\alpha = \lim_{N\to\infty} \sum_1^{N-1} \beta(t_i)\left[\alpha(t_{i+1}) - \alpha(t_i)\right]$$

- Thus one can define, by analogy

$$\int_0^T \beta(t)dW = \lim_{N\to\infty} \sum_1^{N-1} \beta(t_i)\left[W(t_{i+1}) - W(t_i)\right]$$

  which is the Itô integral
- But one can also define

$$\int_0^T \beta(t)dW = \lim_{N\to\infty} \sum_1^{N-1} \beta\left(\frac{t_i + t_{i+1}}{2}\right)\left[W(t_{i+1}) - W(t_i)\right]$$

  which is the Stratonovich integral

Independance properties

- The two above definitions lead to the same result in the deterministic case (probably, C^1 is required)
- But there are differences in the stochastic case:
- Since $\beta(t_i)$ (present) is independent of $W(t_{i+1}) - W(t_i)$ (future), one has, for Itô integrals

$$\mathbb{E}\left(\beta(t_i)[W(t_{i+1}) - W(t_i)]\right) = \mathbb{E}(\beta(t_i))\mathbb{E}(W(t_{i+1}) - W(t_i)) = 0$$

leading to

$$\mathbb{E}\left(\int_0^T \beta(t) dW\right) = 0$$

This explains Teramae claim "In the Itô formulation, [...], the correlation between ϕ and ξ vanishes"

Variable transformation

- The variable transformation ("changement de variable" in French) formula is also different for Itô and Stratonovich integrals
- Consider the function $y(x)$. In the deterministic case, one has, for instance

$$\frac{dy}{dt} = \frac{\partial y}{\partial x} \cdot \frac{dx}{dt} \qquad \text{or} \qquad dy = \frac{\partial y}{\partial x} dx$$

- The same rule is valid for Stratonovich integrals (Teramae's "conventional variable transformation"): if

$$dx = f(x)dt + g(x)dW$$

then

$$dy = \frac{\partial y}{\partial x} \left(f(x)dt + g(x)dW \right)$$

Itô's formula for variable transformation

- Consider the Iô SDE

$$dx = f(x)dt + g(x)dW$$

- Then for a function $y(x)$, one has (Itô's formula)

$$dy = \left(\frac{\partial y}{\partial x} f(x) + \frac{1}{2} \frac{\partial^2 y}{\partial x^2} g(x)^2 \right) dt + \frac{\partial y}{\partial x} g(x) dW$$

- This will explain Teramae's "the disappeared correlation is compensated by the new extra drift term $Z'DZ$"

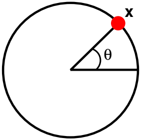
Outline

Phase reduction

- Consider the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ which has a limit-cycle.
- We would like to find a phase variable $\phi(\mathbf{x})$ such that:

$$\frac{d\phi}{dt} = \omega \qquad \omega = \text{constant}$$

Example: a mobile travelling on a circle with constant velocity



- General case (using the chain rule):

$$\frac{d\phi}{dt} = \frac{\partial\phi}{\partial\mathbf{x}}(\mathbf{x}) \cdot \frac{d\mathbf{x}}{dt} = \frac{\partial\phi}{\partial\mathbf{x}}(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}) = \omega$$

- One then has to solve the above PDE to find ϕ

Phase reduction(continued)

- Consider now a small perturbation ξ

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \xi$$

- Then the equation on the phase becomes

$$\frac{d\phi}{dt} = \frac{\partial\phi}{\partial\mathbf{x}}(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}) + \frac{\partial\phi}{\partial\mathbf{x}}(\mathbf{x}) \cdot \xi = \omega + \frac{\partial\phi}{\partial\mathbf{x}}(\mathbf{x}) \cdot \xi$$

- This can be converted into a ϕ-only equation using some approximations

$$\frac{d\phi}{dt} = \omega + Z(\phi)\xi$$

- This was equation (2) in [Teramae & Tanaka, 2004]

Stratonovich to Itô switch

- Actually, the authors could have done everything in Itô!
- Let us compute the phase equation obtained previously but using now Itô's formula (with $D = \frac{1}{2}g^2$)

$$d\phi = \left(\frac{\partial \phi}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x}) + D \frac{\partial^2 \phi}{\partial \mathbf{x}^2} \right) dt + \frac{\partial \phi}{\partial x} dW$$

- As above, let $Z(\phi) = \frac{\partial \phi}{\partial x}$. Then

$$\frac{\partial^2 \phi}{\partial \mathbf{x}^2} = \frac{\partial}{\partial \mathbf{x}} Z(\phi) = \frac{\partial Z}{\partial \phi} \frac{\partial \phi}{\partial \mathbf{x}} = Z'(\phi) Z(\phi)$$

- Thus

$$d\phi = (\omega + Z'(\phi) D Z(\phi)) dt + Z(\phi) dW$$

which is equation (3) (after formal division by dt)

Linearization

- Consider $\dot{\phi}_1 = f(\phi_1)$ and $\dot{\phi}_2 = f(\phi_2)$
- Then (using the Taylor expansion assuming $\phi_1 - \phi_2$ very small)

$$\begin{aligned}
\dot{\phi}_1 - \dot{\phi}_2 &= f(\phi_1) - f(\phi_2) \\
&= f(\phi_1) - (f(\phi_1) + (\phi_2 - \phi_1)f'(\phi_1)) \\
&= f'(\phi_1)(\phi_1 - \phi_2)
\end{aligned}$$

- This explains equation (4) if we set $\psi = \phi_1 - \phi_2$

Lyapunov exponent

- Consider two infinitesimally close trajectories. The Lyapunov exponent λ verifies (intuitively)

$$\|\delta\phi(t)\| \simeq e^{\lambda t}\|\delta\phi_0\|$$

  - If $\lambda > 0$, then nearby trajectories diverge = instability
  - If $\lambda < 0$, then nearby the trajectories converge = stability

- (Remark: if a system is contracting, then $\lambda < 0$)

Lyapunov exponent (continued)

- Let us manipulate the above expression:

$$e^{\lambda t} \simeq \frac{\|\delta\phi(t)\|}{\|\delta\phi_0\|} \qquad \lambda t = \ln\left(\frac{\|\delta\phi(t)\|}{\|\delta\phi_0\|}\right) \qquad \lambda = \frac{1}{t}\ln\left(\frac{\|\delta\phi(t)\|}{\|\delta\phi_0\|}\right)$$

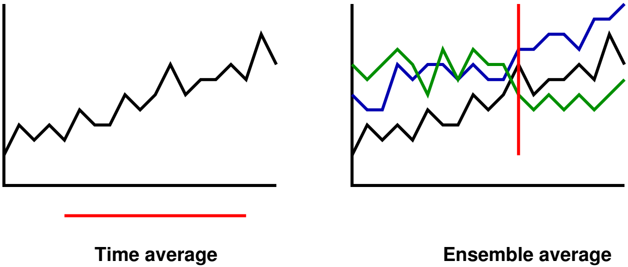
- Actually, the Lyapunov exponent is defined as (because we are interested in long-time behaviour)

$$\lambda = \lim_{t\to\infty}\frac{1}{t}\ln\left(\frac{\|\delta\phi(t)\|}{\|\delta\phi_0\|}\right)$$

Ergodic hypothesis

- Consider a stochastic process $\mathbf{x}(\omega, t)$
- Any physicist knows that (ergodic hypothesis): $\forall \omega, t$

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \mathbf{x}(\omega, t')dt' = \int_\Omega \mathbf{x}(\omega', t)d\omega' = \mathbb{E}\left(\mathbf{x}(t)\right)$$



Ergodic hypothesis (continued)

- Remark now that y in equation (5) is defined as

$$y = \ln(\psi) = \ln(\phi_1 - \phi_2) = \ln(\delta\phi)$$

- Remark that

$$\int_0^T \dot{y} = y(T) - y(0) = \ln(\delta\phi(T)) - \ln(\delta\phi(0)) = \ln \frac{\delta\phi(T)}{\delta\phi(0)}$$

thus

$$\frac{1}{T} \lim_{T\to\infty} \int_0^T \dot{y} = \frac{1}{T} \lim_{T\to\infty} \ln \frac{\delta\phi(T)}{\delta\phi(0)} = \lambda$$

By the ergodic hypothesis, we then have

$$\lambda = \mathbb{E}(\dot{y})$$

which explains the first line in equation (6).

Probability density

- Let $P(\phi, t)$ denotes the time-dependent probability density of the random variable $\phi \in [0, 2\pi]$
- P is constant intuitively means that ϕ has equal probability of being anywhere in $[0, 2\pi]$

Outline

Some limitations

- A lot of unproven statements (ergodicity, uniform distribution of ϕ in steady state,...). Perhaps those statements are evident for physicists!
- There is a mistake in the computation of the phase equation, as pointed out by [Yoshimura & Arai, 2007] (Thank you, Francis!). However, this mistake does not alter the result.
- The analysis is only valid when Z is continuously differentiable up to the second-order, which is not verified for e.g. resetting neuron models (Integrate and Fire, Izhikevich,...)