An elementary proof of the contraction theorem

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Abstract

Nonlinear contraction theory was introduced by Lohmiller and Slotine in [1]. The development of this theory relies heavily on differential notions, especially on the so-called "infinitesimal displacement at fixed time" $\delta\mathbf{x}$. However, these notions were never explicitly defined, and as a consequence, some of the proofs remained rather obscure. In these notes, we aim at providing rigorous proofs of the basic theorems of nonlinear contraction theory, using only elementary differential calculus.

Consider a smooth nonlinear dynamical equation

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) \tag{1}$$

Consider two trajectories starting respectively at $\mathbf{a}$ and $\mathbf{b}$. At time $t = 0$, consider a smooth path $\gamma(0)$ that connects $\mathbf{a}$ and $\mathbf{b}$. The intermediate values along that path are parameterized by $u \in [0, 1]$, with $\mathbf{x}(0, 0) = \mathbf{a}$ and $\mathbf{x}(0, 1) = \mathbf{b}$. For all $t > 0$ define now the path $\gamma(t) = \{\mathbf{x}(t, u) | u \in [0, 1]\}$, where $\mathbf{x}(t, u)$ is the position at time t of a particle that starts at $\mathbf{x}(0, u)$ at $t = 0$. Standard results in differential calculus guarantee that for all t, $\gamma(t)$ is a smooth path whose $\mathbf{M}$-length is given by

$$L_{\mathbf{M}}(\gamma(t)) = \int_0^1 \sqrt{\left(\frac{\partial \mathbf{x}}{\partial u}(t, u)\right)^T \mathbf{M}(\mathbf{x}(t, u), t) \left(\frac{\partial \mathbf{x}}{\partial u}(t, u)\right)} du$$

Define now

$$\phi(t) = \int_0^1 \left(\frac{\partial \mathbf{x}}{\partial u}(t,u)\right)^T \mathbf{M}(\mathbf{x}(t,u),t) \left(\frac{\partial \mathbf{x}}{\partial u}(t,u)\right) du$$

By Schwarz's inequality, one has $L^2_{\mathbf{M}}(\gamma(t)) \leq \phi(t)$. The time derivative of ϕ is next given by

$$\begin{array}{rcl} \frac{d\phi}{dt} &=& \frac{d}{dt}\int_0^1 \left(\frac{\partial \mathbf{x}}{\partial u}(t,u)\right)^T \mathbf{M}(\mathbf{x}(t,u),t) \left(\frac{\partial \mathbf{x}}{\partial u}(t,u)\right) du \\ &=& \int_0^1 \frac{\partial}{\partial t}\left(\left(\frac{\partial \mathbf{x}}{\partial u}(t,u)\right)^T \mathbf{M}(\mathbf{x}(t,u),t) \left(\frac{\partial \mathbf{x}}{\partial u}(t,u)\right)\right) du \\ &=& \int_0^1 \left(\frac{\partial}{\partial t}\left(\frac{\partial \mathbf{x}}{\partial u}(t,u)\right)\right)^T \mathbf{M}(\mathbf{x}(t,u),t) \left(\frac{\partial \mathbf{x}}{\partial u}(t,u)\right) \\ &+& \left(\frac{\partial \mathbf{x}}{\partial u}(t,u)\right)^T \left(\frac{\partial}{\partial t}\mathbf{M}(\mathbf{x}(t,u),t)\right) \left(\frac{\partial \mathbf{x}}{\partial u}(t,u)\right) \\ &+& \left(\frac{\partial \mathbf{x}}{\partial u}(t,u)\right)^T \mathbf{M}(\mathbf{x}(t,u),t) \left(\frac{\partial}{\partial t}\left(\frac{\partial \mathbf{x}}{\partial u}(t,u)\right)\right) du \end{array}$$

Using successively Schwarz's theorem, equation (1) and the chain rule yields

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathbf{x}}{\partial u}(t,u)\right) = \frac{\partial}{\partial u}\left(\frac{\partial \mathbf{x}}{\partial t}(t,u)\right) = \frac{\partial}{\partial u}\left(\mathbf{f}(\mathbf{x}(t,u),t)\right) = \mathbf{J}(\mathbf{x}(t,u),t)\cdot\frac{\partial \mathbf{x}}{\partial u}(t,u)$$

where $\mathbf{J}(\mathbf{x}(t,u),t)$ denotes the Jacobian matrix of $\mathbf{f}$ computed at $(\mathbf{x}(t,u),t)$. Thus

$$\begin{array}{rcl} \frac{d\phi}{dt} &=& \int_0^1 \left(\frac{\partial \mathbf{x}}{\partial u}(t,u)\right)^T \left(\mathbf{J}^T\mathbf{M} + \dot{\mathbf{M}} + \mathbf{M}\mathbf{J}\right) \left(\frac{\partial \mathbf{x}}{\partial u}(t,u)\right) du \\ &\leq& -2\lambda \int_0^1 \left(\frac{\partial \mathbf{x}}{\partial u}(t,u)\right)^T \mathbf{M} \left(\frac{\partial \mathbf{x}}{\partial u}(t,u)\right) du \\ &=& -2\lambda\phi(t) \end{array}$$

where λ verifies

$$\forall \mathbf{x}, t \quad \mathbf{J}^T\mathbf{M} + \dot{\mathbf{M}} + \mathbf{M}\mathbf{J} \leq -2\lambda\mathbf{M}$$

Applying Gronwall's lemma then yields

$$\forall t \geq 0 \quad \phi(t) \leq \phi(0)e^{-2\lambda t}$$

Thus $\phi(t)$ tends exponentially to 0 with rate 2λ, which implies that $L_{\mathbf{M}}(\gamma(t))$ tends exponentially to 0 with rate λ.

References

[1] W. Lohmiller, J.-J. Slotine. On Contraction Analysis for Nonlinear Systems. *Automatica*, **34** (6):671–682, 1998.