# Attractors

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## 1 Linear attractors

Consider general deterministic systems of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) \tag{1}$$

Assume that there exists a $\mathbf{f}$-invariant linear subspace $\mathcal{M}$, i.e. $\forall t : \mathbf{f}(\mathcal{M}, t) \subset \mathcal{M}$ (in particular, it implies that any trajectory starting in $\mathcal{M}$ remains in $\mathcal{M}$). Assume furthermore that $\dim(\mathcal{M}) = p$ and consider an orthonormal basis $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ where the first $p$ vectors form a basis of $\mathcal{M}$ and the last $n-p$ a basis of $\mathcal{M}^\perp$. Inspired by the ideas in [4], we consider an $n \times (n-p)$ matrix $\mathbf{V}$ whose columns are $\mathbf{e}_{p+1}, \ldots, \mathbf{e}_n$. $\mathbf{V}^\top$ may be regarded as a projection on $\mathcal{M}^\perp$, and it verifies the following properties :

$$\forall \mathbf{x} \in \mathcal{M} : \mathbf{V}^\top \mathbf{x} = \mathbf{0}, \quad \mathbf{V}^\top \mathbf{V} = \mathbf{I}_{n-p}, \quad \mathbf{V}\mathbf{V}^\top + \mathbf{U}\mathbf{U}^\top = \mathbf{I}_n$$

where $\mathbf{U}$ is the matrix formed by the first $p$ vectors.

We consider now the *grounded* state $\mathbf{z} = \mathbf{V}^\top \mathbf{x}$ . By construction, $\mathbf{x}$ converges to the subspace $\mathcal{M}$ if and only if $\mathbf{z}$ converges to $\mathbf{0}$. Multiplying (1) by $\mathbf{V}^\top$ on the left, we get

$$\dot{\mathbf{z}} = \mathbf{V}^\top \mathbf{f}(\mathbf{V}\mathbf{z} + \mathbf{U}\mathbf{U}^\top \mathbf{x}, t)$$

Construct the auxiliary system

$$\dot{\mathbf{y}} = \mathbf{V}^\top \mathbf{f}(\mathbf{V}\mathbf{y} + \mathbf{U}\mathbf{U}^\top \mathbf{x}, t) \tag{2}$$

By construction, a particular solution of system (2) is $\mathbf{y}(t) = \mathbf{z}(t)$. In addition, since $\mathbf{U}\mathbf{U}^\top \mathbf{x} \in \mathcal{M}$ and $\mathcal{M}$ is $\mathbf{f}$-invariant, $\mathbf{f}(\mathbf{U}\mathbf{U}^\top \mathbf{x}) \in \mathcal{M} = \mathrm{Ker}(\mathbf{V}^\top)$. Thus $\mathbf{y}(t) = \mathbf{0}$ is another particular solution of system (2). If furthermore system (2) is contracting with respect to $\mathbf{y}$ ($\mathbf{x}$ being regarded as an external input), $\mathbf{z}(t)$ converges exponentially to $\mathbf{0}$ ([12], theorem 1).

We can state the following theorem :

**Theorem 1** *If a vector subspace $\mathcal{M}$ is $\mathbf{f}$-invariant and if $\mathbf{V}^\top \mathbf{J}_{\mathbf{f}s} \mathbf{V}$ is uniformly negative definite (where $\mathbf{V}^\top$ is an orthogonal projection on $\mathcal{M}^\perp$ as defined above), then all solutions of system (1) converge exponentially to $\mathcal{M}$.*

## 2 Synchronization

### 2.1 Global synchronization

Using the results stated in the previous section, we study some aspects of synchronization in networks with diffusive couplings.

Consider a network containing $n$ elements with diffusive couplings

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i, t) + \sum_{j \in \mathcal{N}_i} \mathbf{K}_{i,j}(\mathbf{x}_j - \mathbf{x}_i) \qquad i = 1, \ldots, n$$

Let

$$\mathbf{x} = \begin{pmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_n \end{pmatrix}, \quad \mathbf{F}(\mathbf{x}, t) = \begin{pmatrix} \mathbf{f}(\mathbf{x}_1, t) \\ \vdots \\ \mathbf{f}(\mathbf{x}_n, t) \end{pmatrix}, \quad \mathbf{L} = \begin{pmatrix} \sum_{j \neq 1} \mathbf{K}_{1,j} & \cdots & -\mathbf{K}_{1,n} \\ & \vdots & \\ -\mathbf{K}_{n,1} & \cdots & \sum_{j \neq n} \mathbf{K}_{n,j} \end{pmatrix}$$

$\mathbf{L}$ is the Laplacian of the network. Note that $\mathbf{K}_{i,j} = 0$ if $j \notin \mathcal{N}_i$.

The above equation can be rewritten

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, t) - \mathbf{L}\mathbf{x} \tag{3}$$

Consider the vector $\mathbf{1} = (1, \ldots, 1)^\top$ and let $\text{span}(\mathbf{1})$ be its span subspace. Note that $\mathbf{x} \in \text{span}(\mathbf{1})$ if and only if $\mathbf{x}_1 = \ldots = \mathbf{x}_n$, which means that the individual elements are in synchrony. One can also easily check that $\text{span}(\mathbf{1})$ is invariant by $\mathbf{F} - \mathbf{L}$. Let $\mathbf{V}_g^\top$ denote an orthogonal projection on $\text{span}(\mathbf{1})^\perp$ as in section 1. Using theorem 1 we can deduce the following proposition

**Proposition 1** *Regardless of initial conditions, all the elements within a generally coupled network will reach synchrony if*

- $\lambda_{max}(\mathbf{J}_{\mathbf{f}s})$ *is upper-bounded for the isolated dynamics* $\mathbf{f}$*.*
- $\mathbf{V}_g^\top \mathbf{L}_s \mathbf{V}_g$ *is sufficiently positive definite. More precisely*

$$\lambda_{min}(\mathbf{V}_g^\top \mathbf{L}_s \mathbf{V}_g) > \sup_{\mathbf{x},t} \lambda_{max}(\mathbf{J}_{\mathbf{f}s}(\mathbf{x}, t))$$

A few remarks :

**Multidimensional case :** the proof is straightforward, it suffices to take the Kronecker product with $\mathbf{I}_d$ where $d$ is the dimension of each element.

**Semi-definite couplings :** see [12].

**Nonlinear couplings :** see [12].

**Slightly more general linear couplings :** as in [13] we consider the following dynamics

$$\begin{cases} \dot{\mathbf{x}}_1 = \mathbf{f}_1(\mathbf{x}_1,t) + \mathbf{A}^\top(\mathbf{B}\mathbf{x}_2 - \mathbf{A}\mathbf{x}_1) \\ \dot{\mathbf{x}}_2 = \mathbf{f}_2(\mathbf{x}_2,t) + \mathbf{B}^\top(\mathbf{A}\mathbf{x}_1 - \mathbf{B}\mathbf{x}_2) \end{cases} \tag{4}$$

Here $\mathbf{x}_1$ and $\mathbf{x}_2$ could be of different dimensions, say $n$ and $m$. The Jacobian of the overall system is

$$\begin{pmatrix} \frac{\partial \mathbf{f}_1}{\partial \mathbf{x}_1} & \\ & \frac{\partial \mathbf{f}_2}{\partial \mathbf{x}_2} \end{pmatrix} - \begin{pmatrix} \mathbf{A}^\top\mathbf{A} & -\mathbf{A}^\top\mathbf{B} \\ -\mathbf{B}^\top\mathbf{A} & \mathbf{B}^\top\mathbf{B} \end{pmatrix}$$

Denote by $\mathbf{L}$ the second matrix in the above formula. $\mathbf{L}$ is symmetric and positive semi-definite, indeed

$$\begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{pmatrix} \mathbf{L} \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} = \mathbf{x}_1^\top\mathbf{A}^\top\mathbf{A}\mathbf{x}_1 - \mathbf{x}_2^\top\mathbf{B}^\top\mathbf{A}\mathbf{x}_1 - \mathbf{x}_1^\top\mathbf{A}^\top\mathbf{B}\mathbf{x}_2 + \mathbf{x}_2^\top\mathbf{B}^\top\mathbf{B}\mathbf{x}_2$$

Let $\mathbf{z} = \mathbf{A}\mathbf{x}_1 - \mathbf{B}\mathbf{x}_2$, then the above quantity equals $\mathbf{z}^\top\mathbf{z} \geq 0$. Consider now the linear subspace of $\mathbb{R}^n \times \mathbb{R}^m$ defined by

$$\mathcal{M} = \left\{ \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} \in \mathbb{R}^n \times \mathbb{R}^m : \mathbf{A}\mathbf{x}_1 - \mathbf{B}\mathbf{x}_2 = 0 \right\}$$

Consider as above a projection $\mathbf{V}^\top$ on $\mathcal{M}^\perp$, then $\mathbf{V}^\top\mathbf{L}\mathbf{V}$ is positive definite. If we assume furthermore that $\mathcal{M}$ is an invariant set of $\mathbf{f}$ and that the individual dynamics are upper-bounded, then strong enough couplings could ensure exponential convergence to the manifold $\mathcal{M}$.

Two remarks :

- If each subsystem is contracting, then the whole system is also contracting since $\mathbf{L}$ is positive.
- If the subsystems have the same dimension and if $\mathbf{A} = \mathbf{B}$ are non singular, then we are in presence of classical diffusion couplings.

## 2.2 Influence of network topology

In this section, we discuss the influence of the network topology on synchronization behaviour. Since the results stated in the previous section are quite general, they can be used to address various cases. Indeed, we show in the sequel that many network configuration encountered in the literature can be easily understood within our general framework.

- Symmetric connected networks : in this case, $\mathbf{L}$ has a unique zero eigenvalue corresponding to the eigenspace span($\mathbf{1}$). Thus $\mathbf{V}^\top\mathbf{L}\mathbf{V}$ is positive definite and strong enough coupling gain could ensure exponential convergence to span($\mathbf{1}$), i.e. synchronization occurs.

- Balanced networks : in [9], the authors consider balanced networks. Note that the symmetric part of the Laplacian of a balanced network is simply the Laplacian of a certain symmetric network.

- Ring networks or poursuite strategy : in [7] the authors consider the poursuite strategy to solve the rendez-vous problem. The underlying network is a ring, which is a particular case of balanced networks.

- Networks with globally reachable node :

- Leader-followers networks :

- Switching topology : the problem of switching topologies has already been adressed in previous works [12], [5], [9], [7]. Here, we believe to shed a new light by giving a simpler and more intuitive proof.

### 2.3 Partial synchronization

We study in this section *partial* synchronization. Partial synchronization occurs when $\mathbf{x}_i = \mathbf{x}_j$ for some states $i, j$ but all states are not necessarily equal. Note that partial synchronization is pervasive if the network is disconnected, but here, we investigate partial synchronization in the more interesting case in which the network is *connected*.

In the sequel, we develop the original ideas found in [10]. The assumptions, the results and sometimes even the methodology are roughly the same as in [10]. But by using the concise yet powerful contraction theory rather than direct Lyapunov method, we hope to shed a new light on this interesting phenomenon.

Recall that if the network possesses certain symmetry, this symmetry must be present in the Laplacian matrix $\mathbf{L}$. Indeed, consider a permutation matrix $\mathbf{\Pi}_\sigma$ associated with some permutation $\sigma \in \mathfrak{S}_n$. We know that $\mathbf{\Pi}_\sigma$ commutes with $\mathbf{L}$ if and only if the network is globally invariant by $\sigma$. In such a case, $\mathrm{Ker}(\mathbf{I}_n - \mathbf{\Pi}_\sigma)$ is $(\mathbf{F} - \mathbf{L})$-invariant. Indeed, let $\mathbf{x} \in \mathrm{Ker}(\mathbf{I}_n - \mathbf{\Pi}_\sigma)$, i.e. $(\mathbf{I}_n - \mathbf{\Pi}_\sigma)\mathbf{x} = 0$, we have

$$(\mathbf{I}_n - \mathbf{\Pi}_\sigma)(\mathbf{F}(\mathbf{x}, t) - \mathbf{L}\mathbf{x}) = \mathbf{F}(\mathbf{x}, t) - \mathbf{\Pi}_\sigma\mathbf{F}(\mathbf{x}, t) + \mathbf{L}\mathbf{x} - \mathbf{\Pi}_\sigma\mathbf{L}\mathbf{x}$$

Since $\mathbf{\Pi}_\sigma$ is a permutation matrix, we have $\mathbf{\Pi}_\sigma\mathbf{F}(\mathbf{x}, t) = \mathbf{F}(\mathbf{\Pi}_\sigma\mathbf{x}, t) = \mathbf{F}(\mathbf{x}, t)$. Thus the first two terms annihilate each other. So do the last two terms since we have assumed that $\mathbf{\Pi}_\sigma$ commuted with $\mathbf{L}$. Hence $\mathbf{F}(\mathbf{x}, t) - \mathbf{L}\mathbf{x} \in \mathrm{Ker}(\mathbf{I}_n - \mathbf{\Pi}_\sigma)$, q.e.d.

Let $\mathbf{V}_\sigma^\top$ denote an orthogonal projection on $\mathrm{Ker}(\mathbf{I}_n - \mathbf{\Pi}_\sigma)^\perp$ as in section 1. Using once again theorem 1, we can state the following proposition

**Proposition 2** *Assume that the network is invariant by $\sigma$. Let $C_1 \circ \ldots \circ C_k$ be the decomposition of $\sigma$ into disjoint cycles. Then, regardless of initial conditions, all the elements within each $C_i$ will synchronize together if*

- $\lambda_{max}(\mathbf{J_f}_s)$ *is upper-bounded for the isolated dynamics* $\mathbf{f}$.
- $\mathbf{V}_\sigma^\top \mathbf{L}_s \mathbf{V}_\sigma$ *is sufficiently positive definite. More precisely*

$$\lambda_{min}(\mathbf{V}_\sigma^\top \mathbf{L}_s \mathbf{V}_\sigma) > \sup_{\mathbf{x},t} \lambda_{max}(\mathbf{J_f}_s(\mathbf{x}, t))$$

**Proof :** Clearly, $\mathrm{Ker}(\mathbf{I}_n - \mathbf{\Pi}_\sigma) = \bigcap_{i=1}^k \{\mathbf{x}_{j_1} = \ldots = \mathbf{x}_{j_p} : C_i = (j_1, \ldots, j_p)\}$. Using theorem 1, we can show that all trajectory converge exponentially to $\mathrm{Ker}(\mathbf{I}_n - \mathbf{\Pi}_\sigma)$. Thus, asymptotically, $\forall i : \mathbf{x}_{j_1} = \ldots = \mathbf{x}_{j_p}$ where $C_i = (j_1, \ldots, j_p)$. $\Box$

**Remark :** This proposition may be regarded as an extension of proposition 1. Indeed, for all $\sigma$, $\mathrm{Ker}(\mathbf{I}_n - \mathbf{\Pi}_\sigma)$ contains $\mathrm{span}(\mathbf{1})$.

Let us go further with this analysis. Recall that in order to construct $\mathbf{V}_\sigma$, we had to choose $n - p$ orthonormal vectors in $\mathrm{Ker}(\mathbf{I}_n - \mathbf{\Pi}_\sigma)^\perp$ to form its columns. Since $\mathrm{span}(\mathbf{1})^\perp \supset \mathrm{Ker}(\mathbf{I}_n - \mathbf{\Pi}_\sigma)^\perp$, we can complete those $n-p$ vectors with $p-1$ other vectors to form the columns of $\mathbf{V}_g$. Hence $\mathbf{V}_\sigma^\top \mathbf{L}_s \mathbf{V}_\sigma$ is a principal submatrix of $\mathbf{V}_g^\top \mathbf{L}_s \mathbf{V}_g$ and therefore $\lambda_{min}(\mathbf{V}_\sigma^\top \mathbf{L}_s \mathbf{V}_\sigma) \geq \lambda_{min}(\mathbf{V}_g^\top \mathbf{L}_s \mathbf{V}_g)$.

In some cases, one might have $\lambda_{min}(\mathbf{V}_g^\top \mathbf{L}_s \mathbf{V}_g) < \sup_{\mathbf{x}} \lambda_{max}(\mathbf{J_f}_s(\mathbf{x})) < \lambda_{min}(\mathbf{V}_\sigma^\top \mathbf{L}_s \mathbf{V}_\sigma)$. Then, according to propositions 1 and 2, the system does not necessarily synchronize globally, but it does partially.

**Example 1 (Partial synchronization of coupled Lorentz oscillators)**

# 3 Further extensions of linear attractors

## 3.1 Affine attractors

A first, straightforward extension of linear attractors is affine attractors. Suppose we would like to show that $\mathbf{x}$ converges asymptotically to an affine subspace $\mathbf{c} + \mathcal{M}$, where $\mathbf{c}$ is a *constant* vector and $\mathcal{M}$ is a linear subspace. Let $\tilde{\mathbf{x}} = \mathbf{x} - \mathbf{c}$, so that $\tilde{\mathbf{x}}$ converges to $\mathcal{M}$ if and only if $\mathbf{x}$ converges to $\mathbf{c} + \mathcal{M}$. Next, $\tilde{\mathbf{x}}$ verifies

$$\dot{\tilde{\mathbf{x}}} = \mathbf{f}(\tilde{\mathbf{x}} + \mathbf{c}, t)$$

Using theorem 1, it suffices to show that $\mathcal{M}$ is $\mathbf{f}(\cdot + \mathbf{c}, t)$-invariant (i.e. : $\forall \mathbf{m} \in \mathcal{M}, \forall t > 0 : \mathbf{f}(\mathbf{m} + \mathbf{c}, t) \in \mathcal{M}$, and that $\mathbf{V}^\top \mathbf{J_f}_s \mathbf{V}$ is uniformly negative definite.

**Example 2 (Formation stabilization and tracking)**

As in [8] we consider the problem of formation stabilization. Assume that the interesting pattern is given by a vector $\mathbf{c} = (\mathbf{c}_1^\top, \ldots, \mathbf{c}_n^\top)^\top$. For example, a square with side length $a$ and aligned with the axis may be described by $(0, 0, 0, a, a, 0, a, a)^\top$ (see figure 1). Let $\mathbf{c}' = \mathbf{Lc}$. Keeping the same notations as in section 2.1 we consider now the system

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x} - \mathbf{c}) + \mathbf{c}' - \mathbf{Lx} \tag{5}$$

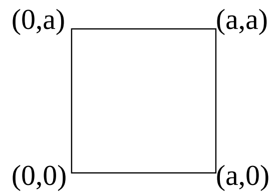


Figure 1: Description of a square formation

For $\mathbf{m} \in \mathrm{span}(\mathbf{1})$, we have $\mathbf{F}(\mathbf{m}+\mathbf{c}-\mathbf{c})+\mathbf{c}'-\mathbf{L}(\mathbf{m}+\mathbf{c}) = \mathbf{F}(\mathbf{m})+\mathbf{Lm}+\mathbf{Lc}-\mathbf{c}' = \mathbf{F}(\mathbf{m}) \in \mathrm{span}(\mathbf{1})$.

Thus $\mathbf{c} + \mathrm{span}(\mathbf{1})$ is an invariant set of system 5. We can then conclude that, under the same assumptions of contractivity as in section 2.1, all trajectories converge exponentially to a trajectory of the form

$$\begin{pmatrix} \mathbf{a}(t) + \mathbf{c}_1 \\ \vdots \\ \mathbf{a}(t) + \mathbf{c}_n \end{pmatrix}$$

which means that the formation described by $\mathbf{c}$ is achieved exponentially fast.

One interesting feature is that $\mathbf{a}(t)$ might be time-varying. In fact, $\mathbf{a}(t)$ is the nominal trajectory driven by the isolated dynamics. Thus, after a transient period during which they strive to achieve the formation, the individual dynamics move collectively while maintaining the formation, even in presence of pertubations.

Yet another interesting feature is that $\mathbf{c}$ might be time-varying too. Indeed, if the time constant of $\mathbf{c}(t)$ is small enough (with respect to the smallest eigenvalue of the contracting quantity), the overall system can evolve and track the time-varying formation $\mathbf{c}(t)$.

**Example 3 (Kuramoto model)**

Consider the Kuramoto model of nonidentical and nonlinearly-coupled oscillators of the form

$$\dot{\theta}_i = \omega_i + \sum_{j=1}^{n} k_{ij} \sin(\theta_j - \theta_i)$$

Once again, let $\mathcal{M} = \text{span}(\mathbf{1})$. We have to look for a $\mathbf{c} = (c_1, \ldots, c_n)$, such that $\forall \mathbf{m} \in \mathcal{M} : \mathbf{f}(\mathbf{m} + \mathbf{c}) \in \mathcal{M}$, i.e. :

$$\omega_1 + \sum_{j=1}^{n} k_{1j} \sin(c_j - c_1) = \ldots = \omega_n + \sum_{j=1}^{n} k_{nj} \sin(c_j - c_n)$$

This is a system of $n-1$ equations and $n$ variables, which has solutions as soon as the $(k_{ij})_{ij}$ are large enough.

Next, we compute the Jacobian of the system :

$$\mathbf{J} = \begin{pmatrix} -\sum_{j=2}^{n} k_{1j} \cos(\theta_j - \theta_i) & \cdots & k_{1n} \cos(\theta_n - \theta_1) \\ & \vdots & \\ k_{n1} \cos(\theta_1 - \theta_n) & \cdots & -\sum_{j=2}^{n} k_{nj} \cos(\theta_j - \theta_n) \end{pmatrix}$$

If, similarly to [4], we make the stupid assumption that $\forall i : |\theta_i| < \frac{\pi}{4}$ uniformly, then all the $\cos(\theta_i - \theta_j)$ will be strictly positive. In such case, using the same proof as in section 2.1 we can show that for strong enough couplings, $\mathbf{V}^\top \mathbf{J}_s \mathbf{V}$ is negative definite.

**Example 4 (Threshold-linear networks)**

In [1] the authors consider threshold-linear networks which are described by the following dynamics

$$\dot{\mathbf{x}} = [\mathbf{W}\mathbf{x} + \mathbf{b}]^+ - \mathbf{x} \tag{6}$$

Using theorem 1 we can show the following proposition

**Proposition 3** *Assume that*

- $\mathbf{W}$ *is nonnegative (i.e. all couplings are excitatory)*
- $\mathbf{I} - \mathbf{W}$ *is positive semi-definite (denote by* $\mathcal{M}$ *its null space)*
- $\mathbf{b}$ *is in range space of* $\mathbf{I} - \mathbf{W}$ *(let* $\mathbf{c}$ *be a vector such that* $(\mathbf{I} - \mathbf{W})\mathbf{c} = \mathbf{b}$*)*

*Then all trajectories converge exponentially to the affine vector space* $\mathbf{c} + \mathcal{M}$*.*

First we need a useful lemma

**Lemma 1** *Assume that*

- $\mathbf{W}$ *is nonnegative and*
- $\mathbf{I} - \mathbf{W}$ *is positive semi-definite*

*Then* $\mathbf{I} + \mathbf{W}$*, and all their principle submatrices are positive semi-definite.*

**Proof :** Since $\lambda_i(\mathbf{I}-\mathbf{W}) = 1 - \lambda_i(\mathbf{W})$, we have

$$\forall\ i = 1, \ldots, n : \lambda_i(\mathbf{W})\ \leq 1 \quad \text{with } \lambda_{max}(\mathbf{W})\ =\ 1$$

According to extended Perron's theorem [2](page 503), if $\mathbf{W}$ is nonnegative, then its spectral radius $\rho(\mathbf{W})$ is an eigenvalue of $\mathbf{W}$, which implies that $\rho(\mathbf{W}) = 1$ in this case. Thus $\lambda_i(\mathbf{I}+\mathbf{W}) = 1 + \lambda_i(\mathbf{W})\mathbf{e}0$, i.e., $\mathbf{I}+\mathbf{W}$ is positive semi-definite. Furthermore, all submatrices of $\mathbf{I}-\mathbf{W}$ and $\mathbf{I}+\mathbf{W}$ are positive semi-definite, which can be concluded from Interlacing Theorem. □

**Proof of proposition 3 :** Actually, we shall only consider trajectories with initial position laying in the nonnegative orthant as in [1].

First, let $\mathbf{m}$ be any element of $\mathcal{M}$ such that $\mathbf{m}+\mathbf{c}$ is nonnegative. We have

$$[\mathbf{W}(\mathbf{m}+\mathbf{c})+\mathbf{b}]^+ - (\mathbf{m}+\mathbf{c}) = [\mathbf{W}\mathbf{m}+(\mathbf{W}\mathbf{c}+\mathbf{b})]^+ - (\mathbf{m}+\mathbf{c}) = [\mathbf{m}+\mathbf{c}]^+ - \mathbf{m}+\mathbf{c} = 0$$

□

## 3.2 Nonlinear attractors

In this section, we deal with the more tricky case of nonlinear attractive manifolds.

### 3.2.1 Hypersurface attractors

$\mathcal{M}$ is now a differentiable manifold of codimension 1, or a hypersurface if you prefer. Assume that there exists a constant vector $\mathbf{u}$ which is never tangent to $\mathcal{M}$. We can thus define a projection on $\mathcal{M}$ with respect to $\mathbf{u}$ (see figure 2). Note that $\mathbf{J}_r(\mathbf{x})$ is constant along the dashed line and that it is orthogonal to the manifold. Let $z = r(\mathbf{x})$. Taking the derivative of this equality with respect to time yields

$$\dot{z} = \mathbf{J}_r(\mathbf{x})\dot{\mathbf{x}} = \mathbf{J}_r(\mathbf{x})\mathbf{f}(\mathbf{u}z + p(\mathbf{x}), t)$$

As in section 1, we construct the auxiliary system

$$\dot{y} = \mathbf{J}_r(\mathbf{x})\mathbf{f}(\mathbf{u}y + p(\mathbf{x}), t) \tag{7}$$

By construction $y(t) = z(t)$ is a particular solution of system 7. Let us show that $y(t) = 0$ is another particular solution. Indeed, since $\mathcal{M}$ is $\mathbf{f}$-invariant, $\mathbf{f}(p(\mathbf{x}), t)$ must be in the tangent hyperplane to $\mathcal{M}$ at $p(\mathbf{x})$. But $\mathbf{J}_r(\mathbf{x})$ is orthogonal to the manifold as we have noticed, thus $\mathbf{J}_r(\mathbf{x})\mathbf{f}(p(\mathbf{x}), t) = 0$.

As a consequence, if system 7 is contracting with respect to $y$, $z(t)$ converges exponentially to 0, i.e. $\mathbf{x}$ converges exponentially to the manifold $\mathcal{M}$. To check the contraction condition, one has to compute the Jacobian of system 7 with respect to $y$, which is $\mathbf{J}_r(\mathbf{x})\mathbf{J}_\mathbf{f}(\mathbf{x}, t)\mathbf{u}$.

A further step is to relax the condition of $\mathbf{u}$ being constant. Indeed, the only conditions we need are

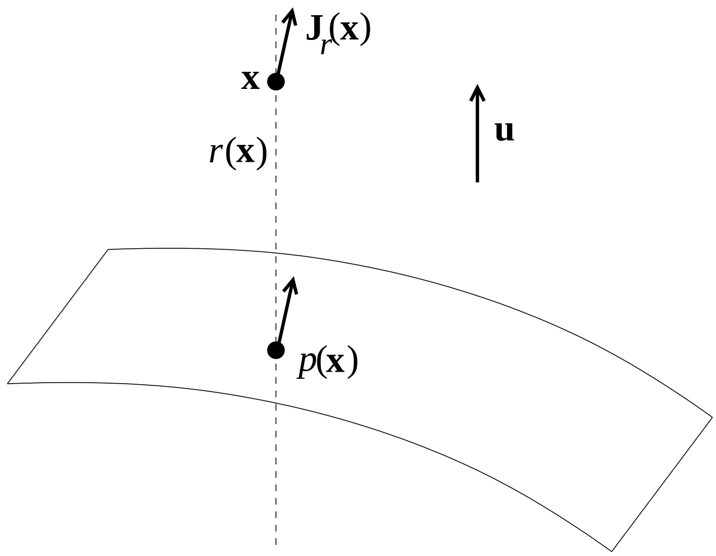


Figure 2: Projection on a hypersurface

1. $p(\mathbf{x})$ can be defined without ambiguity
2. $\mathbf{J}_r(\mathbf{x})$ is orthogonal to the manifold at $p(\mathbf{x})$ for all $\mathbf{x}$ in the equivalence class of $p(\mathbf{x})$.

These conditions are verified in diverse constructions, for instance the "orthogonal" projection on a hypersphere (see figure 3).

**Remark :** The analogy with the linear case is straightforward. In the linear case, $\mathbf{J}_r(\mathbf{x})$ is $\mathbf{V}^\top$ itself, $\mathbf{u}z$ stands for $\mathbf{Vz}$ and $p(\mathbf{x})$ replaces $\mathbf{UU}^\top\mathbf{x}$.

**Example 5 (Andronov-Hopf oscillator)**

In [3], the author considers the oscillator of Andronov-Hopf, whose dynamics is described by

$$\begin{aligned} \dot{a} &= a - b - a^3 - ab^2 \\ \dot{b} &= a + b - b^3 - ba^2 \end{aligned} \tag{8}$$

First, observe that the unit circle is an invariant set for the system. Next, let $\rho = \sqrt{a^2 + b^2}$, then $\mathbf{u} = \begin{pmatrix} a/\rho \\ b/\rho \end{pmatrix}$, $r = \rho - 1$, $\mathbf{J}_r = \begin{pmatrix} a/\rho \\ b/\rho \end{pmatrix}$

The Jacobian of the system is

$$\begin{pmatrix} 1 - 3a^2 - b^2 & -2ab - 1 \\ -2ab + 1 & 1 - 3b^2 - a^2 \end{pmatrix}$$

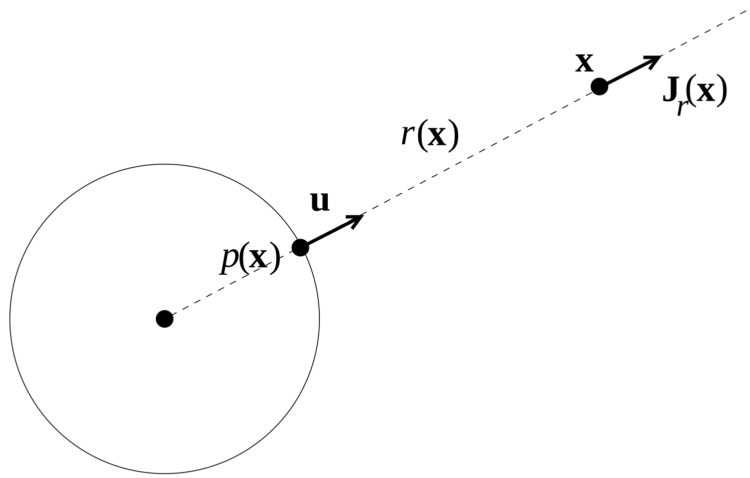


Figure 3: Projection on a hypersphere

Such that $\mathbf{J}_r\mathbf{J}_\mathbf{f}\mathbf{u} = 1 - 3\rho^2$. Thus, for any $\eta > 1/3$, $C_\eta = \{(a,b) \in \mathbb{R}^2 : a^2 + b^2 \geq \eta\}$ is a contracting region. It is also easy to check that any trajectory starting in $C_\eta$ remains in $C_\eta$. One can then conclude that, any trajectory starting outside of the disk $\{(a,b) \in \mathbb{R}^2 : a^2 + b^2 \leq 1/3\}$ converges exponentially to the unit circle.

#### 3.2.2 Nonlinear attractive manifolds of arbitrary dimension

Now, it is easy to extend the hypersurface attractor to the more general case of a nonlinear attractive manifold of arbitrary dimension. Assume that $\mathcal{M}$ is a manifold of codimension $p$. We define an "orthogonal" linear space of dimension $p$ by finding a set of $p$ independant vectors which are never tangent to $\mathcal{M}$ (or we can proceed more generally as in the case of the hypersphere). See figure 4.

### 3.3 Linear attractors in complex vector spaces and phase-locking

In [12] the authors derive conditions that ensure *synchronization* or *anti-synchronization* of two dynamical systems coupled together. One can view these phenomena as special cases of theorem 1 where the invariant linear subspace is the diagonal subspace $\mathcal{M} = \{\mathbf{x}_1 = \mathbf{x}_2\}$ or the the anti-diagonal subspace $\mathcal{M} = \{\mathbf{x}_1 = -\mathbf{x}_2\}$, respectively.

Contraction theory can be extended straightforwardly to complex vector spaces [6]. Instead of computing the symmetric part of the Jacobian matrix, one has now to deal with its *Hermitian* part $\frac{1}{2}(\mathbf{A} + \mathbf{A}^*)$.

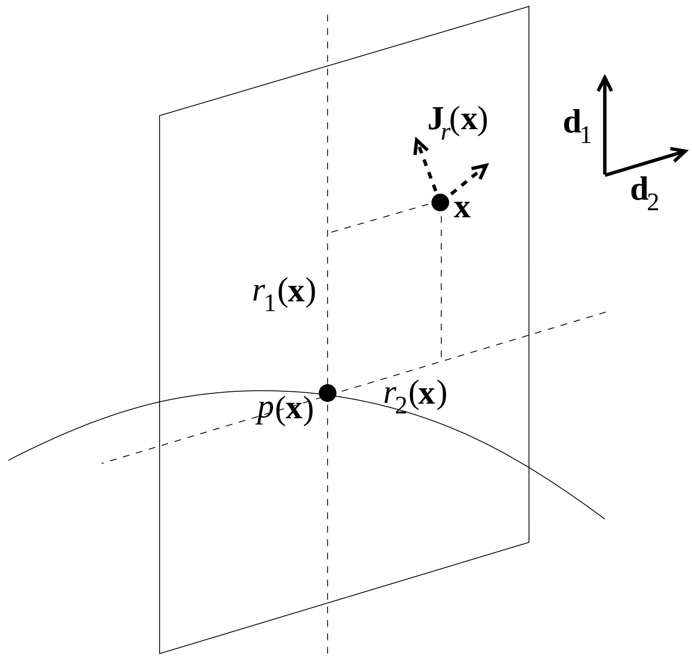


Figure 4: Projection on a manifold of arbitrary dimension

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