Trajectory correction algorithms for a 3D underwater vehicle using affine transformations

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1. Introduction

Planning trajectories for 3D nonholonomic mobile robots such as underwater vehicles [5], quadrotor, satellites, or surgical needles [9, 1] is particularly difficult and time-consuming because of the nonholonomic nature and the high number of degrees of freedom associated with these systems [3, 4]. As a consequence, when such systems encounter on their ways an unexpected event (e.g. a random perturbation of the system state or of the target state, an unforeseen obstacle, etc.), it may be more efficient to *deform* in some manner the initially planned trajectory rather than to re-plan entirely a new one [2, 8, 7, 6].

Here we use a method based on affine geometry to derive trajectory correction algorithms for a class of 3D rear-propelled kinematic robots, which includes e.g. the underwater vehicle of [5]. In contrast with previous trajectory deformation methods [2, 8], affine-geometry-based algorithms are exact, can be executed in one step, and do not require any trajectory re-integration [6].

In section 2., we recall the basic definitions and results of the affine trajectory correction framework. In section 3., we derive trajectory correction algorithms for the general class of 3D rear-propelled kinematic robots whose angular velocities are required to be continuous. This generalizes the results obtained in [6], where no continuity constraints were imposed on the angular velocities. Finally, section 4. offers a brief conclusive discussion.

2. Background on affine trajectory corrections

We summarize here the basic definitions and results of the affine trajectory correction framework. For a more complete presentation, the reader is referred to [6].

2·1 Definitions

An affine space is a set $\mathbb{A}$ together with a group action of a vector space $\mathbb{W}$. An element $\mathbf{w} \in \mathbb{W}$ transforms a point $P \in \mathbb{A}$ into another point P' by $P' = P + \mathbf{w}$, which can also be noted $\overrightarrow{PP'} = \mathbf{w}$.

Given a point $O \in \mathbb{A}$ (the origin), an affine transformation $\mathcal{F}$ of the affine space can be defined by a couple $(\mathbf{w}, \mathcal{M})$ where $\mathbf{w} \in \mathbb{W}$ and $\mathcal{M}$ is a nonsingular endomorphism of $\mathbb{W}$ (i.e. a nonsingular linear application $\mathbb{W} \to \mathbb{W}$). The transformation $\mathcal{F}$ acts on $\mathbb{A}$ by

$$\forall P \in \mathbb{A} \quad \mathcal{F}(P) = O + \mathcal{M}(\overrightarrow{OP}) + \mathbf{w}.$$

If P_0 is a fixed-point of $\mathcal{F}$, then $\mathcal{F}$ can be written in the form

$$\forall P \in \mathbb{A} \quad \mathcal{F}(P) = P_0 + \mathcal{M}(\overrightarrow{P_0P}).$$

Note that the affine transformations of an n-dimensional space form a Lie group of dimension $n+n^2$ (n coordinates for the translation and n^2 coordinates for the endomorphism of the associated vector space). This Lie group is usually called the General Affine group and denoted GA_n.

Consider a commanded system of dimension N. A trajectory $\bar{\mathcal{C}}(t)_{t\in[0,T]}$ of the N system variables is *admissible* if one can find a set of admissible *commands* that generates $\bar{\mathcal{C}}$ (the definition of admissible commands depends on the system at hand, see section 2·2). Assume now that n out of the N system variables form an affine space – the *base space*. A base-space trajectory $\mathcal{C}(t)_{t\in[0,T]}$ is said to be admissible if there exists an admissible full-space trajectory $\bar{\mathcal{C}}$ whose projection on the base space coincides with $\mathcal{C}$.

Let $\mathcal{C}(t)_{t\in[0,T]}$ be a base-space trajectory and $\tau \in [0,T]$, a given time instant. We say that a transformation $\mathcal{F}$ occurring at τ *deforms* $\mathcal{C}(t)_{t\in[0,T]}$ into $\mathcal{C}'(t)_{t\in[0,T]}$ if

$$\begin{array}{ll} \forall t < \tau & \mathcal{C}'(t) = \mathcal{C}(t) \\ \forall t \geq \tau & \mathcal{C}'(t) = \mathcal{F}(\mathcal{C}(t)). \end{array} \tag{1}$$

Given an admissible base-space trajectory $\mathcal{C}$ and a time instant τ, an affine transformation $\mathcal{F}$ is said to be admissible if, at time τ, $\mathcal{F}$ deforms $\mathcal{C}$ into an admissible trajectory.

2·2 Affine trajectory corrections

In [6], we derived practical affine correction algorithms for several classes of nonholonomic robots by taking the following steps

1. Based on the definition of the admissible commands, identify the conditions for a base-space trajectory to be admissible. For instance, a base-space trajectory (x, y) of the unicycle is admissible if and only if x and y are C^2 and piecewise C^3; a base-space trajectory (x, y) of the bicycle is admissible if and only if x and y are C^2 and piecewise C^3 *and if* $\arctan2(\dot{y}, \dot{x})$ is C^1 and piecewise C^2; a base-space trajectory (x, y, z) of a 3D

underwater vehicle is admissible if and only if x, y and z are C^2 and piecewise C^3.

2. Based on the admissibility conditions for base-space trajectories, identify the set of admissible affine tranformations at a given time instant τ. We showed that this set is a Lie subgroup of GA_2 of dimension 2 for the unicycle, a Lie subgroup of GA_2 of dimension 1 for the bicycle and a Lie subgroup of GA_3 of dimension 6 for the 3D underwater vehicle.
3. Compute appropriate τs and appropriate admissible affine deformations at the given τs to achieve the desired correction. For instance, we derived practical algorithms for correcting the final position or orientation corrections of the unicycle, the bicycle and the 3D underwater vehicle as well as several other applications (obstacle avoidance, online feedback control, gap filling, etc.).

3. Kinematic 3D mobile robots with continuous angular velocities

3·1 Model description

A 3D rear-propelled robot can be modeled by the following kinematic equations [5]

$$\left\{ \begin{array}{ccc} \dot{v} & = & a \\ \begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} & = & \mathbf{R}(\phi,\theta) \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \\ \dot{x} & = & v\cos\psi\cos\theta \\ \dot{y} & = & v\sin\psi\cos\theta \\ \dot{z} & = & -v\sin\theta \end{array} \right. , \qquad (2)$$

where $(a, \omega_x, \omega_y, \omega_z)$ are the system control inputs (a is the robot's linear acceleration and $\omega_x, \omega_y, \omega_z$ are its angular velocities expressed in the local basis, see Fig. 1), $(x, y, z, \phi, \theta, \psi, v)$, the system variables (respectively: the robot's position, attitude and linear velocity), and

$$\mathbf{R}(\phi,\theta) = \begin{pmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta \end{pmatrix}.$$

Finally, as in [6], we choose (x, y, z) to be the base variables. Note that the general affine group GA_3 associated with this base space is of dimension 12.

In [6], we required $(x, y, z, \phi, \theta, \psi, v)$ to be continuous in any admissible trajectory. We then showed that the set of affine deformations that respect these constraints form at each time instant τ a Lie subgroup of GA_3 of dimension 6.

In practice however, the continuities of the *angular velocities* $(\omega_x, \omega_y, \omega_z)$ are also sometimes required. In a submarine or an aircraft for instance, the yaw velocity ω_z can only be changed by affecting the orientation of the rudders. Assuming that the orientation of the rudders cannot be changed instantaneously, the yaw velocity must thus be continuous.

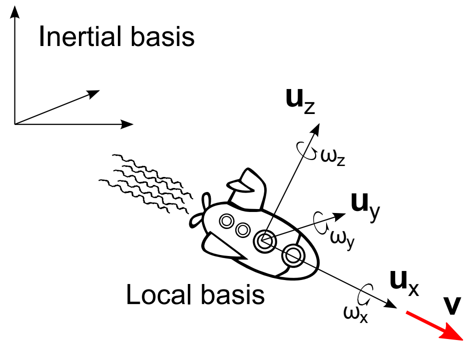

Fig.1 Inertial and local bases.

This bears resemblance to the case of a 2D bicycle: the bicycle angular velocity is a continuous function of the angle of the front wheel, which in turn is a continuous function of time, such that the angular velocity of the bicycle is required to be a continuous function of time in any admissible trajectory [6]. This constraint reduces the dimension of the set of admissible affine deformations from 2 in the case of the unicycle, where the angular velocity is *not* required to be continuous, to 1 in the case of the bicycle.

Our first objective here is thus to understand how the set of admissible affine deformations is reduced when continuity constraints are imposed on the angular velocities $(\omega_x, \omega_y, \omega_z)$.

3·2 Admissible affine deformations

Consider an affine transformation $\mathcal{F}$ occuring at time τ and let $\mathcal{M}$ denote its associated linear application. Recall from [6] that a necessary and sufficient condition for the continuity of $(x, y, z, \phi, \theta, \psi, v)$ is that $(x(\tau), y(\tau), z(\tau))$ is a fixed-point of $\mathcal{F}$ and that $\mathcal{M}(\mathbf{v}(\tau)) = \mathbf{v}(\tau)$. Assume from now that these conditions are fulfilled.

Following the notations of [5], let us consider the local basis $\mathcal{B}_L = \{\mathbf{u}_x, \mathbf{u}_y, \mathbf{u}_z\}$ (the local basis is attached to the robot as a rigid body and should not be confused with the Frenet-Serret basis) and let (a_x, a_y, a_z) be the coordinates of the acceleration vector $\mathbf{a}$ in this basis (see Fig. 2).

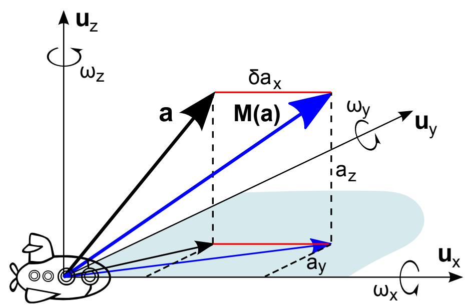

Fig.2 Acceleration vector and angular velocities in the local basis.

The system equations (2) imply, after some algebraic manipulations, that

$$\omega_y = -a_z/v, \quad \omega_z = a_y/v.$$

Since the linear velocity v is continuous, the continuities of ω_y and ω_z are thus equivalent to those of a_z and a_y respectively. Turning now to ω_x, remark that it can be expressed as

$$\omega_x = \dot{\phi} - \omega_y \sin\phi \tan\theta - \omega_z \cos\phi \tan\theta.$$

Thus ω_x is automatically guaranteed to be continous as long as ω_y and ω_z are continuous, since the continuities of ϕ, θ and $\dot{\phi}$ are already assumed.

Following the above development, a necessary and sufficient condition for the continuities of $(\omega_x, \omega_y, \omega_z)$ at τ is that a_y and a_z are continuous, but a_x is not required to be continuous. This is equivalent to

$$\exists\lambda \in \mathbb{R},\ \mathcal{M}(\mathbf{a}(\tau)) = \mathbf{a}'(\tau+) = \mathbf{a}'(\tau-) + \lambda\mathbf{u}_x = \mathbf{a}(\tau) + \lambda\mathbf{u}_x$$

In summary, a necessary and sufficient condition for $\mathcal{M}$ to be admissible is that it verifies

$$\begin{cases} \mathcal{M}(\mathbf{u}_x(\tau)) = \mathbf{u}_x(\tau) \\ \exists\lambda \in \mathbb{R},\quad \mathcal{M}(\mathbf{a}(\tau)) = \mathbf{a}(\tau) + \lambda\mathbf{u}_x(\tau) \end{cases} \qquad (3)$$

where the first equation equivalently replaces $\mathcal{M}(\mathbf{v}) = \mathbf{v}$ since $\mathbf{v}$ and $\mathbf{u}_x$ are collinear.

To define completely $\mathcal{M}$, let us attach to $\{\mathbf{u}_x, \mathbf{a}\}$ a third vector $\mathbf{p}$ in order to form a basis $\{\mathbf{u}_x, \mathbf{a}, \mathbf{p}\}$; this third vector can for instance be chosen as $\mathbf{p} = \mathbf{u}_x \times \mathbf{a}$. Let next $\mathbf{p}' = \mathcal{M}(\mathbf{p})$, and remark that $\mathbf{p}'$ can be arbitrary, as long as it is not in span$(\mathbf{u}_x, \mathbf{a})$, so as to ensure the nonsingularity of $\mathcal{M}$.

From the above development, it appears that the set of admissible affine transformations at time τ form a Lie subgroup of GA_3 of dimension 4, parameterized by λ and the three coordinates of $\mathbf{p}'$.

3·3 Trajectory correction

We now describe how to make a trajectory correction towards a desired final position $\mathcal{C}_d = (x_d, y_d, z_d)$, different from the initially planned final position $\mathcal{C}(T) = (x(T), y(T), z(T))$.

Let $\mathbf{M}$ be the matrix of the affine transformation $\mathcal{M}$ written in the basis $\mathcal{B}_L$. Let (p_x, p_y, p_z) and (p'_x, p'_y, p'_z) be the coordinates of $\mathbf{p}$ and $\mathbf{p}'$ in this basis. The conditions of equations (3) are then equivalent to

$$\mathbf{M}\begin{pmatrix} 1 & a_x & p_x \\ 0 & a_y & p_y \\ 0 & a_z & p_z \end{pmatrix} = \begin{pmatrix} 1 & a_x+\lambda & p'_x \\ 0 & a_y & p'_y \\ 0 & a_z & p'_z \end{pmatrix}$$

Thus $\mathbf{M}$ is given by

$$\mathbf{M} = \begin{pmatrix} 1 & a_x+\lambda & p'_x \\ 0 & a_y & p'_y \\ 0 & a_z & p'_z \end{pmatrix}\mathbf{Q}_1^{-1},$$

where

$$\mathbf{Q}_1 = \begin{pmatrix} 1 & a_x & p_x \\ 0 & a_y & p_y \\ 0 & a_z & p_z \end{pmatrix}$$

is the change-of-basis matrix from the basis $\{\mathbf{u}_x, \mathbf{a}, \mathbf{p}\}$ to the basis $\mathcal{B}_L$.

Let $\mathbf{Q}_2 = [\mathbf{u}_x, \mathbf{u}_y, \mathbf{u}_z]$ be the change-of-basis matrix from the basis $\mathcal{B}_L$ to the inertial basis. From the definition of trajectory deformations (cf. equations (1)), the desired correction for the final position is achieved if

$$\mathbf{Q}_2\mathbf{M}\mathbf{Q}_2^{-1}\begin{pmatrix} x(T)-x(\tau) \\ y(T)-y(\tau) \\ z(T)-z(\tau) \end{pmatrix} = \begin{pmatrix} x_d-x(\tau) \\ y_d-y(\tau) \\ z_d-z(\tau) \end{pmatrix}. \qquad (4)$$

Letting

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \mathbf{Q}_1^{-1}\mathbf{Q}_2^{-1}\begin{pmatrix} x(T)-x(\tau) \\ y(T)-y(\tau) \\ z(T)-z(\tau) \end{pmatrix} \qquad (5)$$

$$\text{and} \quad \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \mathbf{Q}_2^{-1}\begin{pmatrix} x_d-x(\tau) \\ y_d-y(\tau) \\ z_d-z(\tau) \end{pmatrix}$$

allows rewriting equation (4) as

$$\begin{pmatrix} 1 & a_x+\lambda & p'_x \\ 0 & a_y & p'_y \\ 0 & a_z & p'_z \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix},$$

which yields

$$\begin{cases} \lambda y_1 + p'_x z_1 = x_2 - x_1 - a_x y_1 \\ p'_y z_1 = y_2 - a_y y_1 \\ p'_z z_1 = z_2 - a_z y_1. \end{cases}$$

If $z_1 \neq 0$, one can finally write

$$\begin{cases} p'_z = (z_2 - a_z y_1)/z_1 \\ p'_y = (y_2 - a_y y_1)/z_1 \\ p'_x = (x_2 - x_1 - a_x y_1 - \lambda y_1)/z_1 \end{cases},$$

which gives the coefficients of the affine deformation that corrects $(x(T), y(T), z(T))$ towards (x_d, y_d, z_d) while respecting the nonholonomic constraints. Some examples of trajectory corrections are given in Fig. 3. Note from the last equation that one still has in hand an extra degree of freedom λ, which can be used to achieve various aims. Choosing for instance $\lambda = 0$ yields a deformation that also preserves the continuity of the linear acceleration. Also, by varying λ, it is possible to alter the final *attitude* of the robot ($\theta'(T)$ and $\psi'(T)$) according to need while keeping its final position unchanged (see Fig. 3).

"Degeneracy" case. Let us now examine more closely the condition $z_1 = 0$. From equation (5), it appears that z_1 is the coordinate of the vector $\overrightarrow{\mathcal{C}(\tau)\mathcal{C}(T)}$

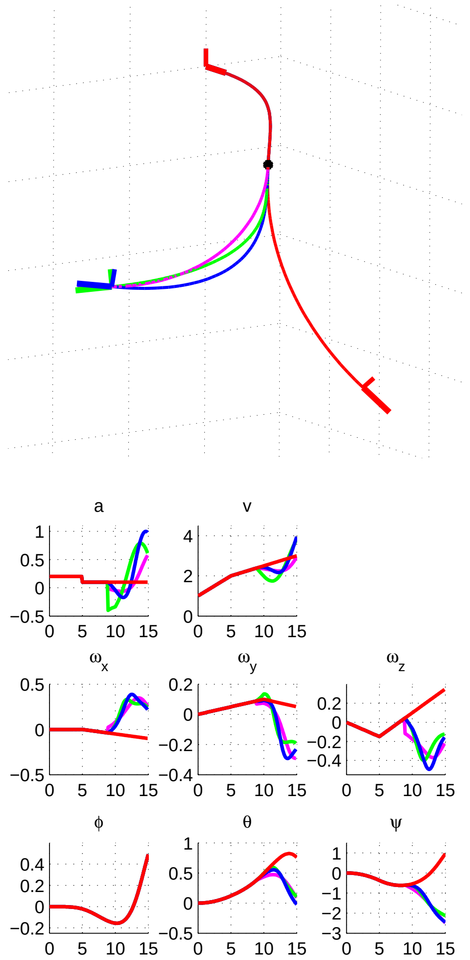

Fig.3 Examples of trajectory corrections for a 3D mobile robot whose angular velocities are required to be continuous. The original trajectory is in red. The blue and green trajectories result from corrections that respect the continuity of the angular velocities. The blue and green trajectories correspond to $\lambda = 0$ and $\lambda = -0.5$ respectively. Note that for $\lambda = 0$ (blue trajectory), the linear acceleration is continuous. Note also that, by varying λ, it is possible to alter the final attitude of the robot ($\theta(T)$ and $\psi(T)$) while keeping its final position unchanged. The magenta trajectory results from the correction that minimizes the distance of the linear application $\mathcal{M}$ from identity (see [6]), but which is not required to preserve the continuity of the angular velocities. Observe indeed the discontinuities of the mangenta lines in the plots of ω_y and ω_z.

along $\mathbf{p}$ in the basis $\{\mathbf{u}_x, \mathbf{a}, \mathbf{p}\}$. Thus $z_1 = 0$ means that $\overrightarrow{\mathcal{C}(\tau)\mathcal{C}(T)}$ is in span$(\mathbf{u}_x, \mathbf{a})$, or in other words, that $\mathcal{C}(T)$ is *on the osculating plane at* τ. This bears a strong resemblance to the case of the bicycle [6], for which a "degeneracy" arises whenever the tangent at τ goes through the initially planned final position $\mathcal{C}(T)$. Thus, as in the case of the bicycle where this "degeneracy" is leveraged to make corrections to the final orientation without changing the final positions, here this "degeneracy" can be leveraged to make corrections to the final *attitude* of the robot without modifying its final orientation. The development of this point is left to the reader.

4. Discussion

We have presented a method to deform the trajectories of 3D rear-propelled mobile robots in order to correct the final position of the robot while respecting the nonholonomic constraints. Using this method as a building block, it is possible to achieve more complex tasks, such as obstacle avoidance, online feedback control, or trajectory gap filling [6].

Regarding possible extensions of the present work, we are developping trajectory correction algorithms for the bevel needle, a minimally-invasive surgical tool which has recently attracted considerable attention from the robotics community [9, 1, 8]. The bevel needle can indeed be considered as a 3D rear-propelled robot where $\omega_z = 0$ and $\omega_y = \kappa v$, where κ is a constant curvature.

Acknowledgments

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