Algorithms for the ~~topology~~ cohomology of compact arithmetic manifolds and Hecke operators

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§1 ~~Intro~~ Arithmetic groups

An <u>arithmetic group</u> is a subgroup $\Gamma \subseteq \mathbb{G}(\mathbb{Z})$ of finite index where $\mathbb{G} \subseteq SL_n$ is an algebraic group $/\mathbb{Q}$ (semisimple)

ex: $\Gamma = SL_n(\mathbb{Z})$, $SO(Q, \mathbb{Z})$
Q: quadratic form $/\mathbb{Q}$

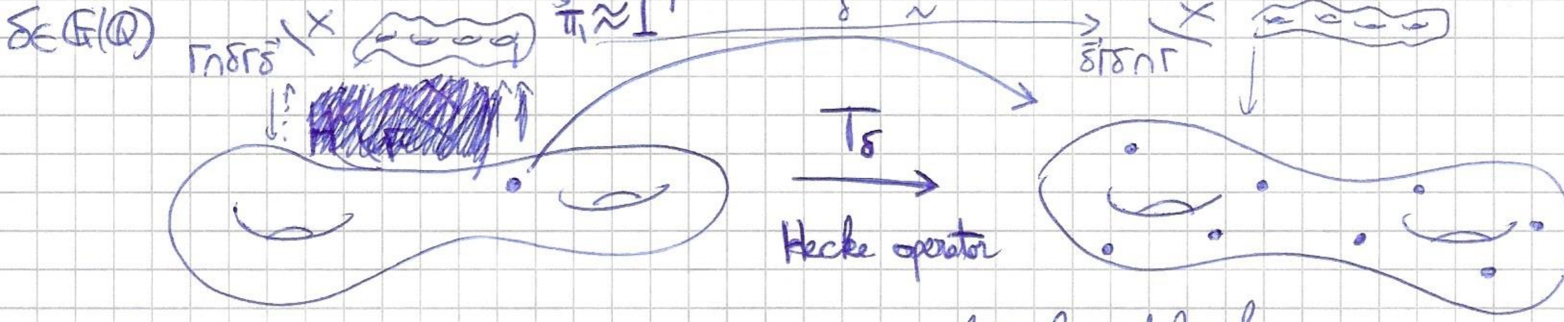
Γ usually infinite, but has a finite presentation (Borel, Harish-Chandra)

$\Gamma \circlearrowright X = \mathbb{G}(\mathbb{R})/K$ $\quad K \subseteq \mathbb{G}(\mathbb{R})$ maximal compact
X: contractible : symmetric space

arithmetic manifold:

$\Gamma \backslash X \to$ $\approx$ compact manifold $\quad$ (For simplicity, assume this is really the case.)

$\Gamma \backslash X \to \pi_1 \approx \Gamma$



$T \circlearrowright H^i(\Gamma\backslash X)$
$\to$ recovers classical modular forms $\Gamma \subseteq SL_2(\mathbb{Z})$ $\quad X = \mathcal{H}$
$\to$ automorphic forms
$\to$ torsion
$\Big\}$ $\leftrightsquigarrow$ Galois representations ℓ-adic / mod ℓ

<u>Q</u> Given Γ, can we compute those objects? How fast?

- **input**: equations for $\mathbb{G}$ + membership test for Γ
- measure of complexity : ~~size of input~~
  $V = \mathrm{vol}(\Gamma\backslash X)$

II

What speed can we hope for?

Thm (Gelander)

Fix X. Then Γ is generated by $\overset{\leq c_1 \cdot V}{O(V)}$ elements.

[in many cases, presentation length]

Thm (Gromov)

Fix X.

$$\sum_i b_i(_\Gamma X) \leq c_2 \cdot V$$

[Gel: in many cases, homotopy type]

Thm (Grunewald–Segal '80)

There exists an algorithm which, given Γ, computes a presentation for it.

Complexity? ~~Complexity~~ Completely impractical.

§2. A simpler case: dim 0

Let
- Q positive-definite quadratic form on $V = \mathbb{Q}^n$
- $L_1 \subseteq V$ lattice
- $\mathbb{G} = SO(Q)$
- $\widetilde{X} = \text{genus}(L_1) = \left\{ L \subseteq V \text{ s.t. } (L, Q) \underset{\mathbb{Z}_p}{\cong} (L_1, Q) \ \forall \underset{\text{prime}}{p} \right\}$

<u>Thm</u> $\mathbb{G}(\mathbb{Q}) \backslash \widetilde{X}$ is finite. (arithmetic manifold) [dim 0]

$\hookleftarrow$ class set.

<u>Q:</u> Compute $\mathbb{G}(\mathbb{Q}) \backslash \widetilde{X}$?

measure of complexity: $V = \# \left(\mathbb{G}(\mathbb{Q}) \backslash \widetilde{X} \right)$.

1) test equivalence
2) produce enough elements

1) Given L, L' lattices. $\exists \gamma \in \mathbb{G}(\mathbb{Q})$ s.t. $\gamma L = L'$?

<u>Algorithm</u> (Plesken–Souvignier)

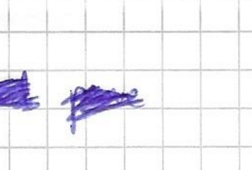
$b_1, \dots, b_n$ basis of L

Q positive-definite $\Longrightarrow$ $\gamma b_i \in$ finite computable set

$\longrightarrow$ enumeration

<u>Neighbours</u> Let $p \nmid 2 \operatorname{disc}(L_1, Q)$

<u>Def:</u> L, L' are <u>p-neighbours</u> if



$\rightsquigarrow$ $\widetilde{X}$ becomes a graph $\widetilde{X}_p$.

<u>Thm</u> (Kneser) $\mathbb{G}(\mathbb{Q}) \backslash \widetilde{X}_p$ is connected.

<u>Algorithm</u> explore $\widetilde{X}_p$ from L_1

$\longrightarrow$ test equivalence with previous points.

Complexity $O_{\dim}(V^2)$

IV

§3 – A new algorithm in dim > 0

(orally?)

~~Previous algorithms~~ Known algorithms rely on some reduction theory
- Siegel sets (Grunewald–Segal) ⟶ impractical
- Voronoï / Minkowski reduction theory ⟶ only symmetric cones
- Dirichlet domains ⟶ mostly hyperbolic space
- some ad hoc constructions.

~~so~~ Let's try not to rely on properties of special symmetric spaces.

Most basic properties: points, distance.
(Riemannian)

Def $F \subseteq Y$ metric space, $R > 0$
- F is <u>R-dense</u> if $Y = \bigcup_{x \in F} B_R(x)$
- F is <u>R-separated</u> if $d(x,y) \geq R \quad \forall x \neq y \in F$.

~~Remark: F maximal R-separated ⟹ F R-dense~~

<u>Čech complex</u> $\mathcal{C}_R(F)$: simplicial complex
- vertices: ~~points~~ elements of F.
- $\{x_0, \dots, x_k\}$ k-simplex $\iff \bigcap_i B_R(x_i) \neq \emptyset$

Thm
Y compact Riemannian manifold
$F \subseteq Y$ R-dense with $R \ll 1$
Then $\mathcal{C}_R(F) \simeq Y$ (homotopy equivalence)

⟶ (Find a finite F), approximate $\Gamma \backslash X$ by a finite set F of points, imitate dim 0 methods.

V

Q. How do we produce a set of points R-dense in a manifold you don't know?

(can skip) First idea: use Hecke operators

As $\deg T \longrightarrow \infty$, Tx equidistributes in $\Gamma\backslash X$. (effective)

Problem too many points.

Classical argument:

Rk $F \subseteq Y$ maximal R-separated $\Rightarrow F$ is R-dense.

Problem non-effective! keeps F small.

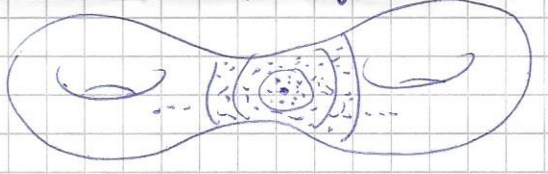
~~More effective~~ More effective version: if F' $R/2$-dense in Y

Let $F \subseteq F'$ $R/2$-separated maximal (computable)

$\Rightarrow F$ $R/2$-dense in F'

$\Rightarrow F$ R-dense in Y.

Apply this to an expanding set in Y:



Pb I did as if I could really compute in the quotient.

Q Given $x, y \in X$, test $\exists \gamma \in \Gamma$ s.t. $d(x, \gamma y) \leq R$?

Use $X \hookrightarrow X_{SL_n} = \{\text{positive quadratic forms on } \mathbb{R}^n \ \det = 1\}$

$Q, Q' \in X_{SL_n}$

Observation

$\forall x \in \mathbb{R}^n \setminus 0, \quad |\log Q(x) - \log Q'(x)| \leq d(Q, Q')$

If $\gamma \in G(\mathbb{Z})$ and $d(Q', \gamma Q) \leq R$ then

$\forall x \in \mathbb{Q}^n, \quad Q'(x) e^{-R} \leq Q(\gamma x) \leq Q'(x) e^{R}$

$\gamma : (\mathbb{Z}^n, Q) \longrightarrow (\mathbb{Z}^n, Q') \quad e^R$-quasi-isometry.

$\longrightarrow$ Enumeration algorithm.

Summary

- generate points locally
- prune using quasi-isometry testing
- stop when no more point can be added.
- compute $\mathcal{C}_R(F) \simeq Y$

Thm (Lipnowski–P.)

There exists an algorithm ~~which~~ that, given Γ s.t. $\Gamma\backslash X$ is a compact manifold, computes

- a simplicial complex $S \simeq \Gamma\backslash X$ with $O_{\dim}(V)$ simplices
- an explicit isomorphism $\pi_1(S) \longrightarrow \Gamma$

It terminates in time $O_{\dim}(V^2)$

can do orally.

There exists an algorithm ~~which~~ that, given a ~~chain~~ chain $\sigma \in C^\bullet(S)$ and a Hecke operator T, computes a ~~cha~~ chain in $C^\bullet(S)$ that is homologous to $T\sigma$, in time $O_{\dim}(V \cdot {}^{(\deg T)^2 +} \underbrace{\deg T}_{\text{degree}})$

$\cancel{\in}$ $C^\bullet(S)$

degree: measure of complexity of a Hecke operator.

Hecke

1) Geometry: F finite R-dense + $2R$-adjacency graph
2) Finite complex: $\mathcal{C}_R(F)$ $\cancel{\circlearrowleft}$ Hecke
   4) $\uparrow\downarrow$
3) Infinite complex: $\mathcal{C}_R(\Gamma\backslash X)$ $\circlearrowleft$ Hecke
4) Equivalence: $\mathcal{C}_R(\Gamma\backslash X) \longrightarrow \mathcal{C}_{R'}(F) \supseteq \mathcal{C}_R(F)$

Implementation (orally: cohom. looks like manifold of correct dim, simplification)

dim X	#local cover	time (s)	$\#S^0$	$\#S^1$	$\#S^2$	$\#S^3$	$\#S^4$
2	~2000	<1	3	23	48	50	26
3	~40 000	850	13	~200	1400	4000	6500
4	~~[illegible]~~						
4	~400 000	15 000	61	~3300	40 000	3.10^5	2.10^6
5	$\sim 2.10^6$	$>10^5$					