Duality and canonical modules

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1 Duality functors

1.1 The case of vector spaces

Let k be a field, and Vect_k be the category of finite-dimensional vector spaces over k. Write D for the functor $V \mapsto V^*$, where V^* denotes the linear dual $\mathrm{Hom}_k(V, k)$ of V.

The we have the following properties:

- $D(D(V))$ is isomorphic to V in a functorial way (we say that $D \circ D$ and id are isomorphic as *functors*)
- if $0 \to A \to B \to C \to 0$ is an exact sequence, then so is $0 \to D(C) \to D(B) \to D(A) \to 0$
- $\mathrm{Hom}(D(V), D(W))$ and $\mathrm{Hom}(W, V)$ are canonically isomorphic (the isomorphism is called *transposition*).

Definition. A functor D having these properties is called a *dualising functor*.

1.2 Naïve duality for modules over algebras

Denote by $M \to M^*$ the naïve duality functor for A-modules of finite type, $M^* = \mathrm{Hom}(M, A)$. Then we only have the following properties:

- if $A \to B \to C \to 0$ is exact, then $0 \to C^* \to B^* \to A^*$ is exact;
- the natural transformation $M \to M^{**}$ can be neither injective nor surjective;
- transposition $\mathrm{Hom}(M, N) \to \mathrm{Hom}(N^*, M^*)$ is no longer injective nor surjective.

However, it is sometimes true that the naïve duality functor shares properties of D. For example, if $A = \mathbb{Z}/p^n\mathbb{Z}$, remember that any A-module of finite type can be written as a direct sum $M = \bigoplus \mathbb{Z}/p^i\mathbb{Z}$ where $i \leqslant n$. Now observe that $\mathrm{Hom}(\mathbb{Z}/p^k\mathbb{Z}, \mathbb{Z}/p^n\mathbb{Z}) \simeq \mathbb{Z}/p^k\mathbb{Z}$ (if f is such a morphism, $f(1)$ is p^{n-k} times some element a of $\mathbb{Z}/p^k\mathbb{Z}$, and we say $f = f_a$).

Moreover, given a morphism $g : \mathbb{Z}/p^l\mathbb{Z} \to \mathbb{Z}/p^k\mathbb{Z}$ ($l \geqslant k$), and $f_a : \mathbb{Z}/p^k\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z}$, the induced morphism $g \circ f_a$ takes 1 to $ag(1)p^{n-k} = ag(1)p^{l-k}p^{n-l}$. So the transpose morphism $g^* : \mathbb{Z}/p^k\mathbb{Z} \to \mathbb{Z}/p^l\mathbb{Z}$ is multiplication by $g(1)p^{l-k}$. Thus transposition acts as an involution.

When $M \to M^*$ is a dualising functor, we say the ring A has *the Gorenstein property*.

Example. The rings $\mathbb{Z}/p^n\mathbb{Z}$, $k[\varepsilon]/(\varepsilon^n)$ have the Gorenstein property.

Now let A be the ring $k[x,y]/(x^2, xy, y^2)$, and M be the A-module $k \simeq A/(x,y)$. Then $\mathrm{Hom}_A(M,A) \simeq k^2$, and $\mathrm{Hom}_A(\mathrm{Hom}_A(M,A),A) \simeq k^4$, which is not isomorphic to k.

The following property says that dualising functors nevertheless have a very simple form.

Proposition 1. Let D be a dualizing functor on the category of finitely generated modules over A. For any module M of finite type, $D(M)$ is given by the formula

$$D(M) = \mathrm{Hom}_A(M, D(A))$$

Proof. This is because $D(M) = \mathrm{Hom}_A(A, D(M)) = \mathrm{Hom}_A(D(D(M)), D(A))$. □

Definition. If $D = \mathrm{Hom}_A(\bullet, \Delta)$ is a dualising functor, we say $\Delta = D(A)$ is a *dualising module*.

1.3 Lifting duality over fields to algebras

There are simple cases where morphisms of vector spaces correspond to morphisms of modules. Let $\iota : k \to A$ be the inclusion of the field k in a finite-dimensional k-algebra A. There are canonical functors

$$\Gamma_* : \mathfrak{Mod}_A \to \mathrm{Vect}_k \qquad \Gamma^* : \mathrm{Vect}_k \to \mathfrak{Mod}_A$$

defined by $\Gamma_*(M) = M$ and $\Gamma^*(V) = V \otimes_k A$.

Proposition 2. These functors have the *adjunction* property :

$$\mathrm{Hom}_k(V, \Gamma_* M) \simeq \mathrm{Hom}_A(\Gamma^* V, M)$$

But we are not interested in linear maps from vector spaces to modules, but in $\mathrm{Hom}_k(M,k)$, which has all properties required for a dualising functor.

Conjecture 3. *Is there a functor $\Gamma^!$ having the adjunction property:*

$$\mathrm{Hom}_k(\Gamma_* M, V) \simeq \mathrm{Hom}_A(M, \Gamma^! V) \text{ ?}$$

Since all vector spaces are direct sums of copies of k, we are looking for a A-module $\omega_A = \Gamma^!(k)$ having the property

$$\mathrm{Hom}_k(M,k) \simeq \mathrm{Hom}_A(M, \omega_A).$$

Proposition 4. The only A-module giving the adjunction property is $\omega_A = \mathrm{Hom}_k(A,k)$.

Grothendieck's duality theory looks for functors $\Gamma^!$ in more general settings. In general, the definition of $\Gamma^!$ is **complicated**.

2 Local rings of dimension zero

2.1 Injective modules over a k-algebra

Here, A is an arbitrary k-algebra, for some field k.

Definition (Injective module)**.** An injective A-module is a module I such that for any injective map of modules $M \to N$, any map $M \to I$ can be extended to a map $N \to I$.

Lemma 5. A dualising module is necessarily injective.

Proof. Remember that the dualising functor $\mathrm{Hom}_A(M, \omega_A)$ has to be exact: for any injection $M \to N$, $\mathrm{Hom}_A(N, \omega_A) \to \mathrm{Hom}_A(M, \omega_A)$ is a surjective map.

This exactly means that any map $M \to \omega_A$ can be extended to a map $N \to \omega_A$ along an injective $M \to N$. □

Arbitrary modules can be embedded into injective modules:

Proposition 6. Any A-module is contained in an injective module.

Proof. Notice that $\mathrm{Hom}_k(A, M)$ is a A-module containing M, and that a morphism of A-modules $f : V \to \mathrm{Hom}_k(A, M)$ is equivalent to a morphism of vector spaces $f' : V \to M$ (set $f'(v) = f(v)(1)$, and given f', set $f(v)(a) = f'(av)$.

Of course, morphism of vector spaces do extend along injections, so $\mathrm{Hom}_k(A, M)$ is injective. □

We now turn to the important notion of *injective hull*, which allows to define the dualising module. This notion arise naturally from the following fact.

Lemma 7. A dualising module for A contains k as a submodule. More generally, if M is a simple A-module, then any dualising module contains M.

Proof. This is because $\mathrm{Hom}_A(k, \omega_A)$ (resp. $\mathrm{Hom}_A(M, \omega_A)$) is nonzero, and because such maps are always injective. □

Definition. An injective hull for a A-module M is an injective module I_M containing M such that any submodule $J \subset I_M$ intersects M (i.e. I_M is an *essential extension*).

Theorem 8. *Injective hulls exist and are unique up to isomorphism.*

Proof. Suppose I_M and J_M exist and are injective hulls for M. Then the injection $M \to J_M$ extends to $I_M \to J_M$. This morphism is injective (otherwise some element of M would be in the kernel), so the identity $I_M \to I_M$ can be extended to a morphism $J_M \to I_M$ with is a retraction: $J_M = I_M \oplus K$ for some K, and K does not intersect M, so $I_M = J_M$.

Now let $M \subset J_M$ be an inclusion of M in an injective A-module. By Zorn's lemma there exists some maximal essential extension $M \subset I_M \subset J_M$ (the limit of a chain of essential extensions is again essential). Let N be a maximal submodule of J_M not intersecting M (equivalently, not intersecting I_M). Then J_M/N is an essential extension of M, since if it were not, there would be N'/N not intersecting M nor I_M, contradicting the maximality of N.

Not if X has a morphism to I_M, it can be extended to $Y \to J_M \to I_M$, so I_M is injective itself, and an essential extension. □

2.2 The dualising module

Let $(A, \mathfrak{m})$ be a local ring, with residue field k.

Definition. The *top* (or *fibre*) of a A-module M is the quotient $M/\mathfrak{m}M$, which is a k-vector space. The *socle* of M is the maximal submodule annihilated by $\mathfrak{m}$ in M. It is again a k-vector space.

Proposition 9. Suppose A is a finite-dimensional graded local k-algebra. Let D denote the dualising functor $M \mapsto \mathrm{Hom}_k(M, k)$. If M is a finite-dimensional graded vector space, the socle of M naturally corresponds to the top of $D(M)$.

It follows from the previous section that dualising modules need be injective modules containing simple A-modules (which are only k if A is a local ring).

We will prove the following theorem:

Theorem 10. *Suppose A is a zero-dimensional local ring, with residue field k. A module ω_A defines a duality functor* $\mathrm{Hom}(\bullet, \omega_A)$ *if and only if ω_A is isomorphic to the injective hull of k in the category of A-modules of (finite length).*

In particular, if A is a k-algebra which is finite-dimensional, the dualising module ω_A is $\mathrm{Hom}_k(A, k)$.

Proof. Suppose D is a dualising functor. We already know that $D(M) = \mathrm{Hom}(M, \omega_A)$ and that ω_A is injective. Since A has dimension zero, for any nonzero module M, there is a last nonzero $\mathfrak{m}^k M$, which is contained in the socle of M.

Now let M be any nonzero submodule of ω_A: then M has a nonzero socle, which is a k-vector space, and the inclusion of M in ω_A defines morphisms from k^r to $k \subset \omega_A$. Since $D(k) = \mathrm{Hom}_A(k, \omega_A)$ is a k-vector space and $\mathrm{Hom}(D(k), k) \subset \mathrm{Hom}(D(k), \omega_A) \simeq k$, $D(k) = k$, and $M \subset \omega_A$ maps the socle of M onto k.

It follows that ω_A is an essential extension of k, hence its injective hull.

Conversely, if ω_A is the injective hull of k, we show that $D(M) = \mathrm{Hom}(M, \omega_A)$ is a dualizing functor. First remark that D is exact, since ω_A is injective, and that $\mathrm{Hom}(k, \omega_A) = k$, since ω_A is an essential extension of k. Then D has the duality property on k-vector spaces.

For general M, there exists a finite filtration

$$M \supset \mathfrak{m}M \supset \mathfrak{m}^2 M \supset \cdots$$

with k-vector spaces as graded components. By exactness,

$$D^2(M) \supset D^2(\mathfrak{m}M) \supset D^2(\mathfrak{m}^2 M) \supset \cdots$$

is also a decreasing filtration, and the morphism $X \to D^2(X)$ is an isomorphism on the graded components of these filtrations, and this implies by *dévissage* that $M \to D^2(M)$ is an isomorphism. □

2.3 Functoriality property

Theorem 11. *Let A be a zero-dimensional local ring, and let $f : A \leftarrow B$ be an finite type local morphism, where B is a local ring. If I_B is the injective hull of k_B,*

$$\omega_A = \mathrm{Hom}_B(A, I_B)$$

Proof. Recall that there is a bijective correspondance between $f \in \mathrm{Hom}_A(M, \mathrm{Hom}_B(A, I_B))$ and $f_1 \in \mathrm{Hom}_B(M, I_B)$. For given some f, the formula $f_1(m) = f(m)(1)$ is B-linear snice $f_1(bm) = f(bm)(1) = f(m)(b) = bf(m)(1)$. The identity $f(am)(x) = f(m)(ax)$ proves that $f(m)(a) = f_1(am)$, so f is determined by f_1.

It follows that $\mathrm{Hom}_B(A, I_B)$ is injective as a A-module, and contains $k_A \simeq \mathrm{Hom}_B(k_A, k_B) \subset \mathrm{Hom}_B(k_A, I_B)$ (use the fact that k_A is finite-dimensional).

Let M be a A-submodule of $\mathrm{Hom}_B(A, I_B)$. Then M has a nonzero socle S, and S consists of morphisms $f : A \to I_B$ such that $f(mx) = 0$ for $m \in \mathfrak{m}_A$, thus $mf(x) = 0$ for $m \in \mathfrak{m}_B$. So f is actually a morphism from k_A to the B-socle of I_B, which k_B, so it intersects non-trivially $k_A \simeq \mathrm{Hom}_B(k_A, k_B)$.

Hence we are looking at the injective hull of k_A, which is ω_A. □

2.4 The residue map

Definition. Let A be a local zero-dimensional k-algebra. Then $\mathrm{Hom}_k(A, k)$ has a canonical map

$$f \in \omega_A \mapsto f(1) \in k$$

which is A-linear, since $(af)(1) = f(\bar{a}) = \bar{a}f(1)$ where $\bar{a}$ is the residue of a in k.

This map is called the *residue map*.

In the case of zero-dimensional quotients of polynomial rings, this is easily understood as the traditional residue map. Let $R = k[[x_1, \ldots, x_d]]$ be a power series ring, with maximal ideal $\mathfrak{m}$ and define K_R to be the R-module $k((x_1, \ldots, x_r))/\mathfrak{m}$ whose all elements have torsion.

Theorem 12 (Thm 21.6 in Eisenbud)**.** *Given an ideal I defining a zero-dimensional quotient R/I, the submodule $K_{R/I}$ of K_R annihilated by I ($K_{R/I} = \mathrm{Hom}_R(R/I, K_R)$) is isomorphic to $\omega_{R/I}$. This defined a bijection between quotients R/I and finite type submodules of K_R.*

Proof. It is obvious that $K_{R/I}$ contains $R/\mathfrak{m}$ as a submodule, so $K_{R/I}$ is a (R/I)-module containing k. It is easy to see that K_R, hence $K_{R/I}$, is an essential extension of k.

To see that $K_{R/I}$ is injective, let p be a large integer such that $\mathfrak{m}^p$ contains I. Then $K_{R/\mathfrak{m}^p}$ is isomorphic to $\mathrm{Hom}_k(R/\mathfrak{m}^p, k)$ (by $\kappa \mapsto (f \mapsto (f\kappa)_0)$ and is injective. Then the diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathrm{Hom}(N/M, K_{R/I}) & \longrightarrow & \mathrm{Hom}(N, K_{R/I}) & \longrightarrow & \mathrm{Hom}(M, K_{R/I}) & \\
& \| & & \| & & \| & \\
0 \longrightarrow & \mathrm{Hom}(N/M, K_{R/\mathfrak{m}^p}) & \longrightarrow & \mathrm{Hom}(N, K_{R/\mathfrak{m}^p}) & \longrightarrow & \mathrm{Hom}(M, K_{R/\mathfrak{m}^p}) & \longrightarrow 0
\end{array}$$

is commutative, and since the second line is exact, the first one is also exact. □

This gives a classical interpretation of the dualising module as residues : if $A = k[z]/(z^{n+1})$ is the ring of Taylor expansions

$$f = a_0 + a_1 z + \cdots + a_n z^n + o(z^{n+1})$$

then $\omega_A \simeq \mathrm{Hom}(A, k((z))/zk[[z]])$ is the A-module of differential forms

$$\omega = \big(\frac{b_n}{z^n} + \cdots + \frac{b_1}{z^2} + b_0 + o(1)\big) \times \frac{dz}{z}$$

and the pairing $A \otimes \omega_A \to k$ is given by

$$(f, \omega) \to \frac{1}{2i\pi} \int_C f(z)\omega = \sum_{i=0}^{n} a_i b_i$$

which is a non-degenerate bilinear form.

A Appendix: injective modules over arbitrary rings

We study first the case of $\mathbb{Z}$-modules.

Proposition 13. Abelian divisible groups are injective as $\mathbb{Z}$-modules.

Proof. Let I be an abelian divisible group, and $M \to I$ a morphism of abelian groups, and N an abelian group containing M. Let $f_0; M \to I$, and construct morphisms $f_i : M_i \to I$ in the following way: let n be an element of $N - M_i$, and $M_{i+1} = M_i + \mathbb{Z}n$. Then if k is the smallest integer such that $kn \in M_i$, choose $f_{i+1}(n)$ such that $kf_{i+1}(n) = f_i(kn)$, and $f_{i+1}(m) = f_i(m)$ for $m \in M_i$.

By transfinite induction (Zorn's lemma), this defines a morphism $N \to I$, for some transfinite ordinal i. □

Now this can be used as a basis for the more general case.

Proposition 14. If A does not contain a field, it is still true that any A-module is contained in an injective module.

Proof. Let I be the quotient *as abelian groups* of $\mathbb{Q}^{(M)}$ by the relations $e_m + e_{m'} = e_{m+m'}$. Then I is an injective $\mathbb{Z}$-module, because it is divisible.

Now $I_M = \mathrm{Hom}_{\mathbb{Z}}(A, I)$ contains M as a A-module, where m is identified with $a \to e_{am}$. Since for any A-module N, $\mathrm{Hom}_A(N, I_M) \simeq \mathrm{Hom}_{\mathbb{Z}}(N, I)$, morphisms to I_M extend along any injective morphism. □

References

[Eis95] David Eisenbud, *Commutative algebra*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995. With a view toward algebraic geometry.