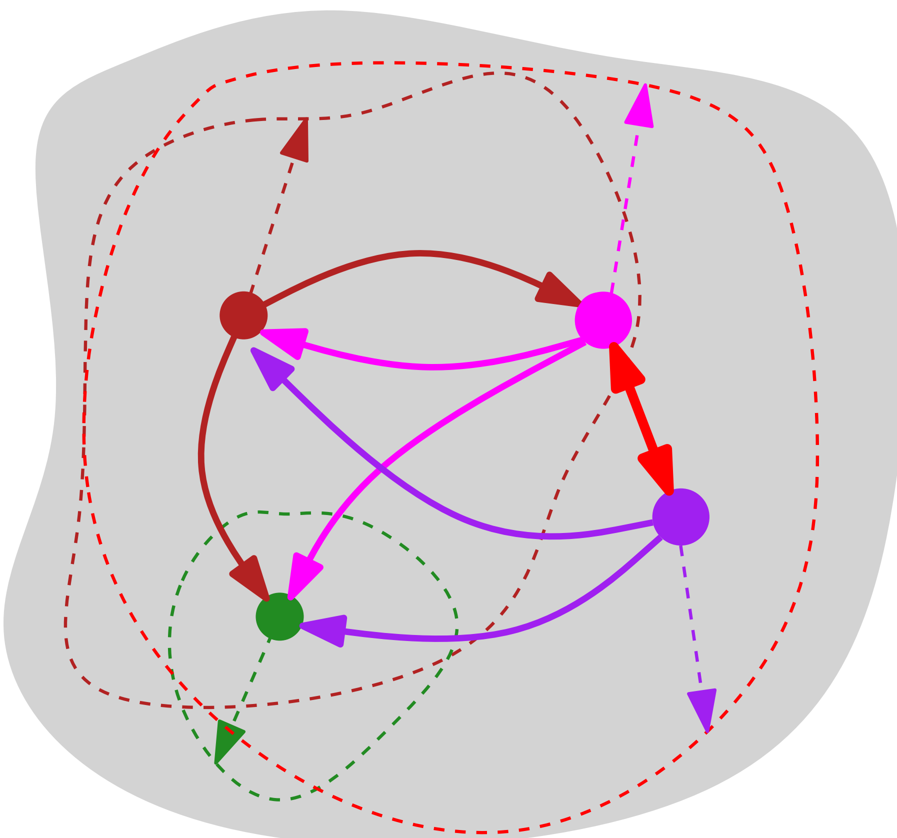
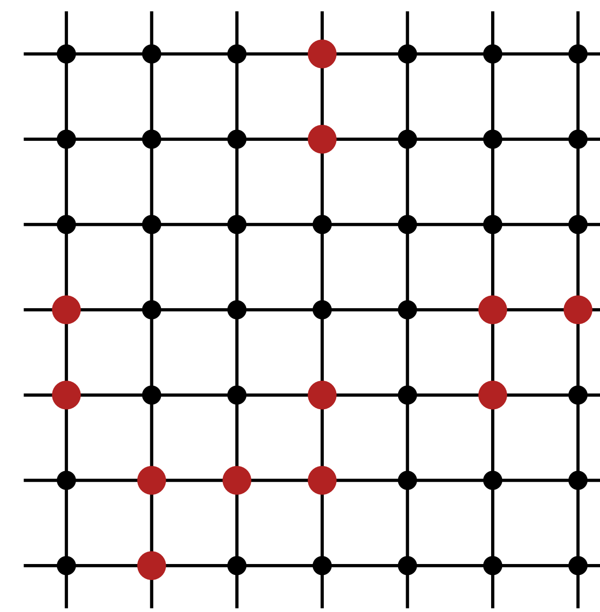
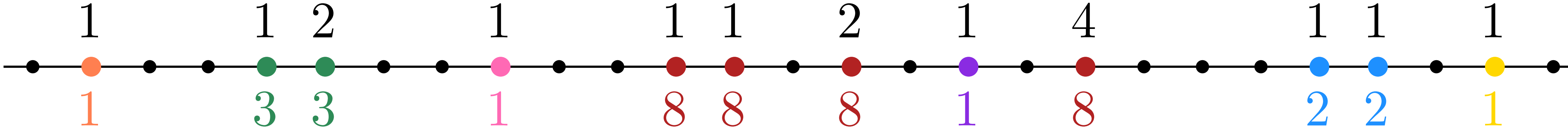
Percolation by cumulative merging and phase transition of the contact process

Laurent Ménard (Modal'X)

joint work with **Arvind Singh** (Orsay)





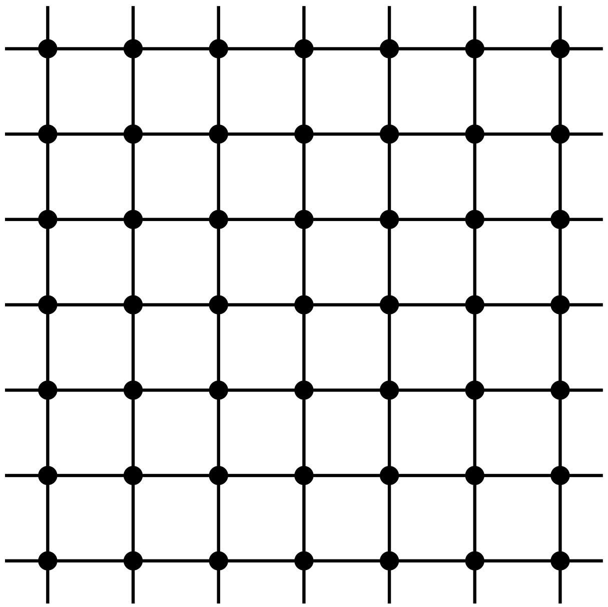


Outline

1. The contact process and main result
2. Heuristics for the contact process
3. Cumulative merging
4. Phase transition for cumulative merging percolation
5. Link with the contact process

The contact process (Susceptible-Infected-Susceptible)

Epidemic model on graphs introduced by [Harris 74]



$G = (V, E)$ locally finite graph, $\lambda > 0$.

- Vertices are either **healthy** or **infected**.
- An infected site **recovers** at rate 1.
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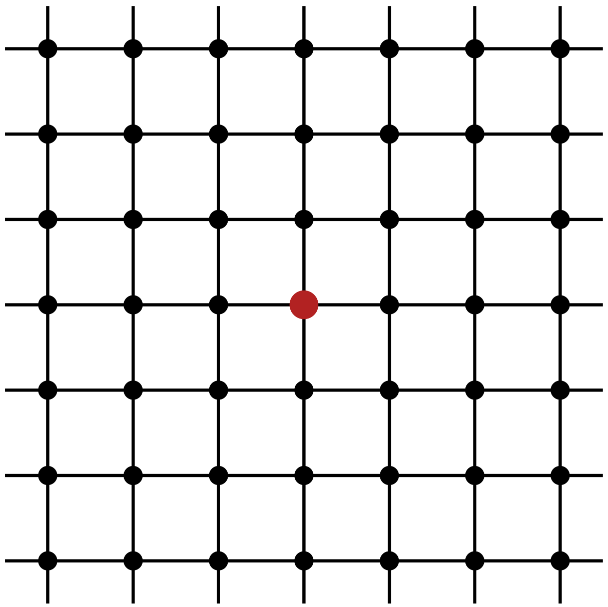

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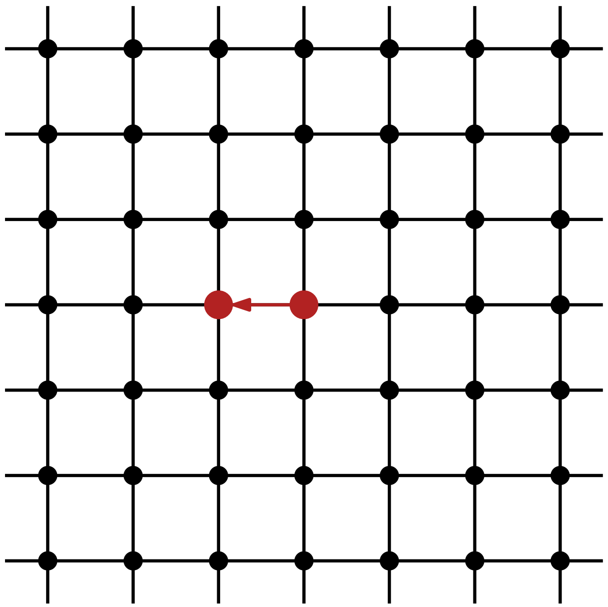

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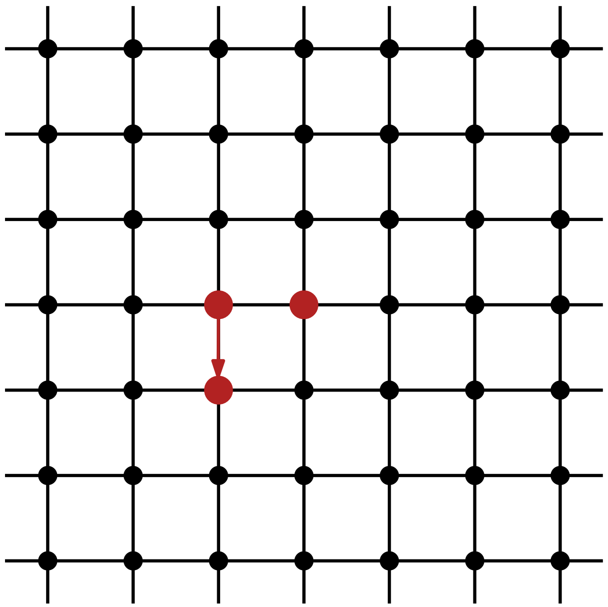

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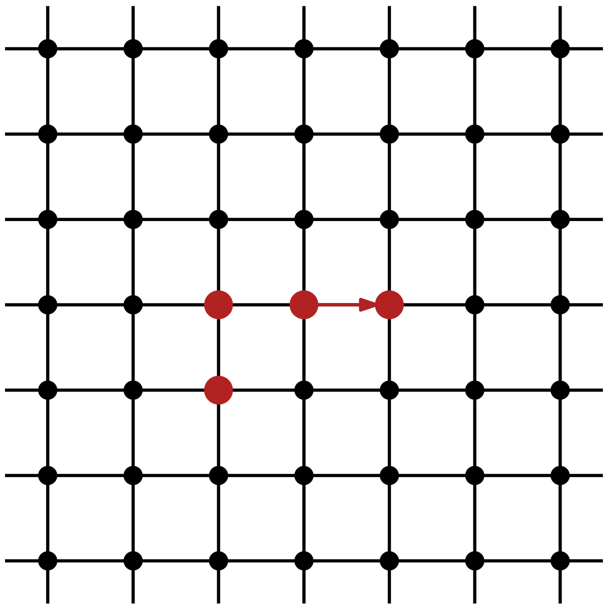

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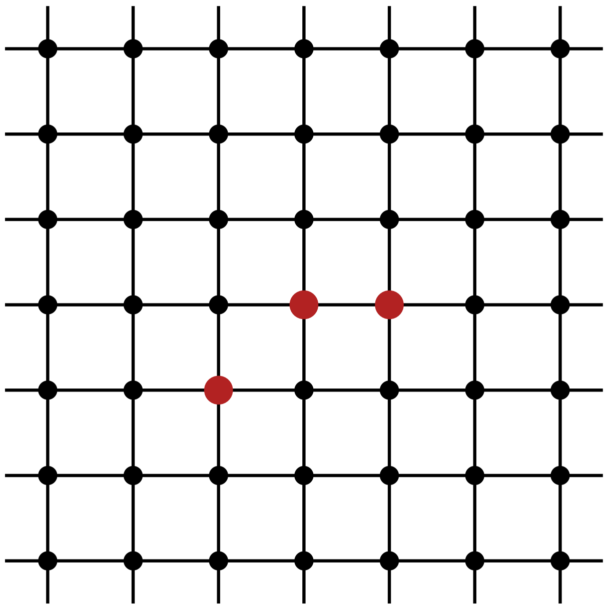

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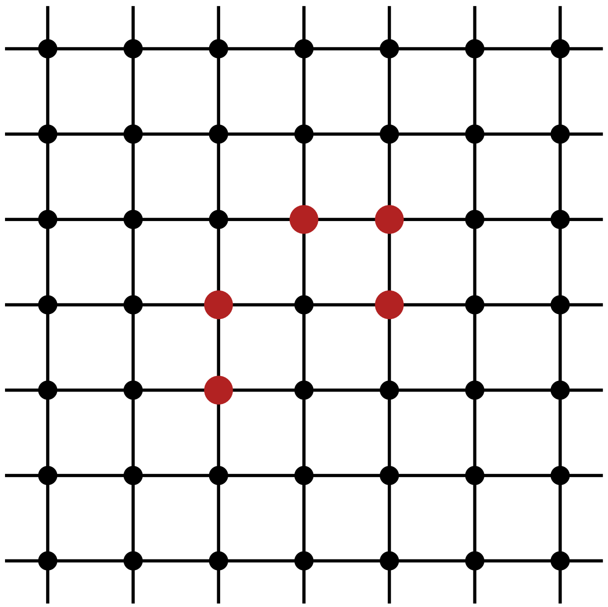

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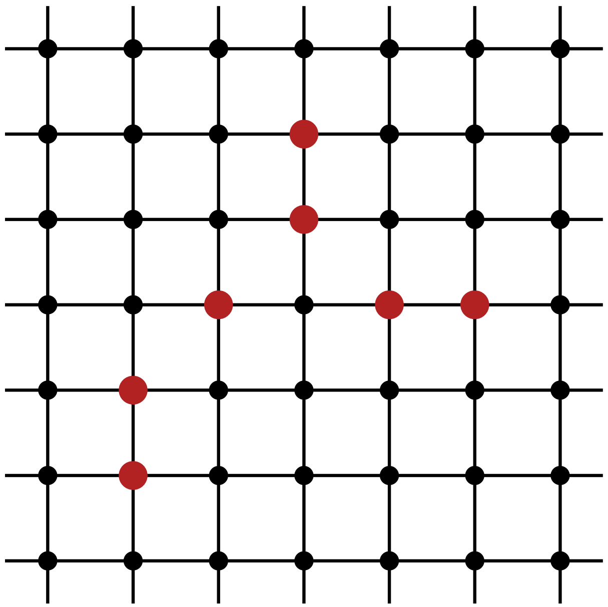

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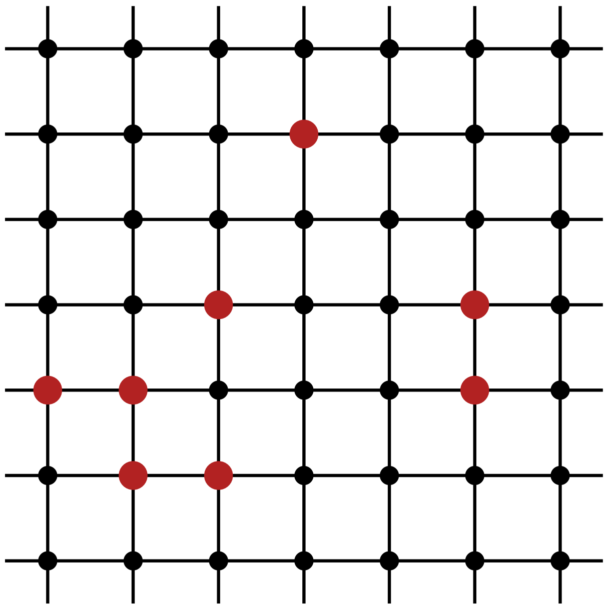

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- if $\lambda < \lambda_c$, the infection **dies out** *a.s.*;
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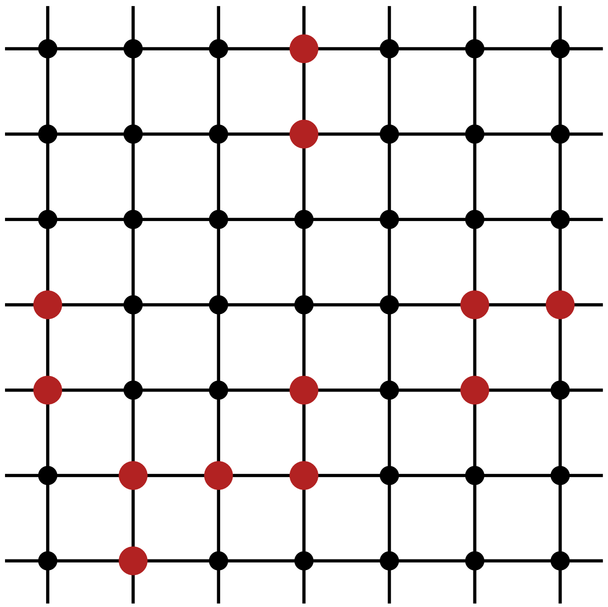

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Question: condition on G to ensure $\lambda_c > 0$?

The contact process on a graph with bounded degrees

If G has **bounded** degrees, then $\lambda_c > 0$.

Compare with **branching random walk**:

- **No interaction** between particles;
- particles die at rate 1;
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- No other method to prove that contact process dies out.
- No example of graph with unbounded degrees for which we know $\lambda_c > 0$.

Main result

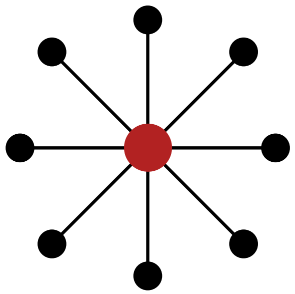
Theorem
Let G be either a
- (supercritical) random geometric graph
- Delaunay triangulation

constructed from a Poisson point process on $\mathbb{R}^d$ with Lebesgue intensity. Then one has $\lambda_c > 0$.

Proof:
Criterion on G for $\lambda_c > 0$ in terms of a percolation model, **Cumulative Merging**.

Heuristics for the contact process

Contact process on a star graph of large degree d:



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- If $\lambda > \lambda_c(d)$, survival time of the process is $\approx \exp(d)$.

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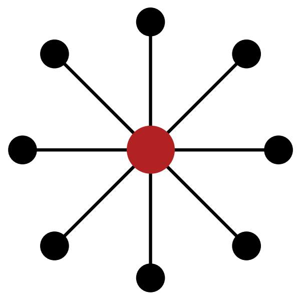

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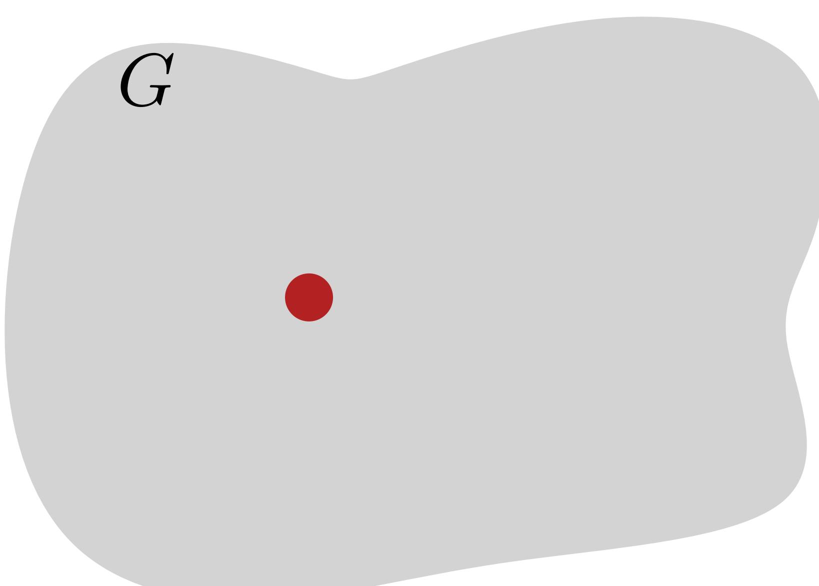
Now fix $\lambda > 0$ and consider contact process on a gaph G *s.t.*

- One vertex (●) has large degree d_0 with $\lambda > \lambda_c(d_0)$;
- all other vertices have small degrees d with $\lambda \ll \lambda_c(d)$.



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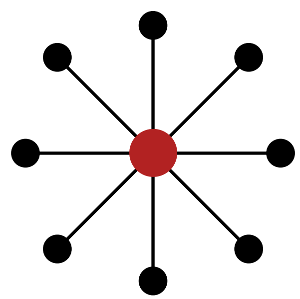

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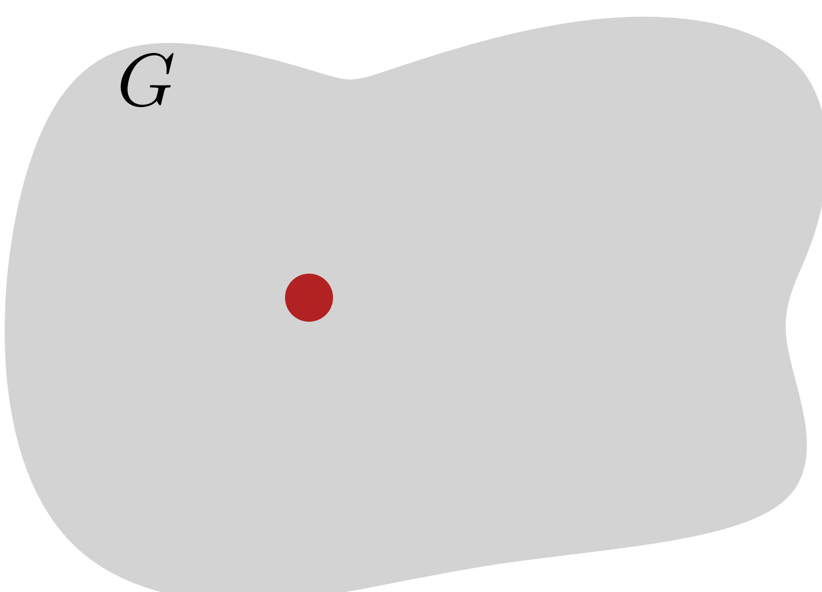

- Start with only ● infected.
- Force ● to stay infected a time $\exp(d_0)$.
- After that time, force the whole star around ● to recover.

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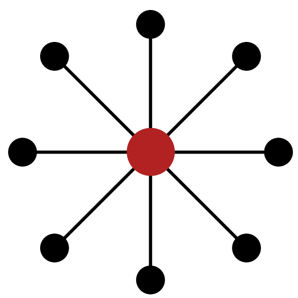

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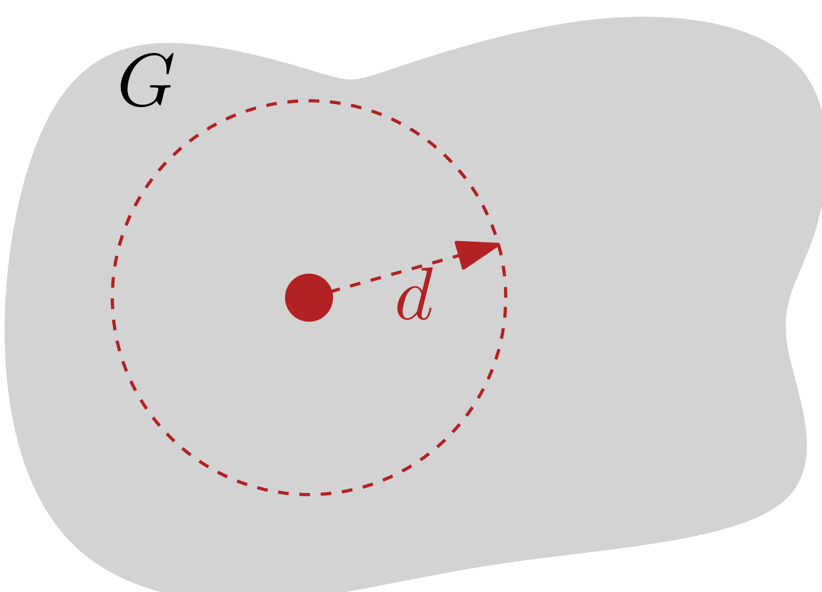

- Start with only ● infected.
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Maximal distance reached by infection is $\approx d_0$.

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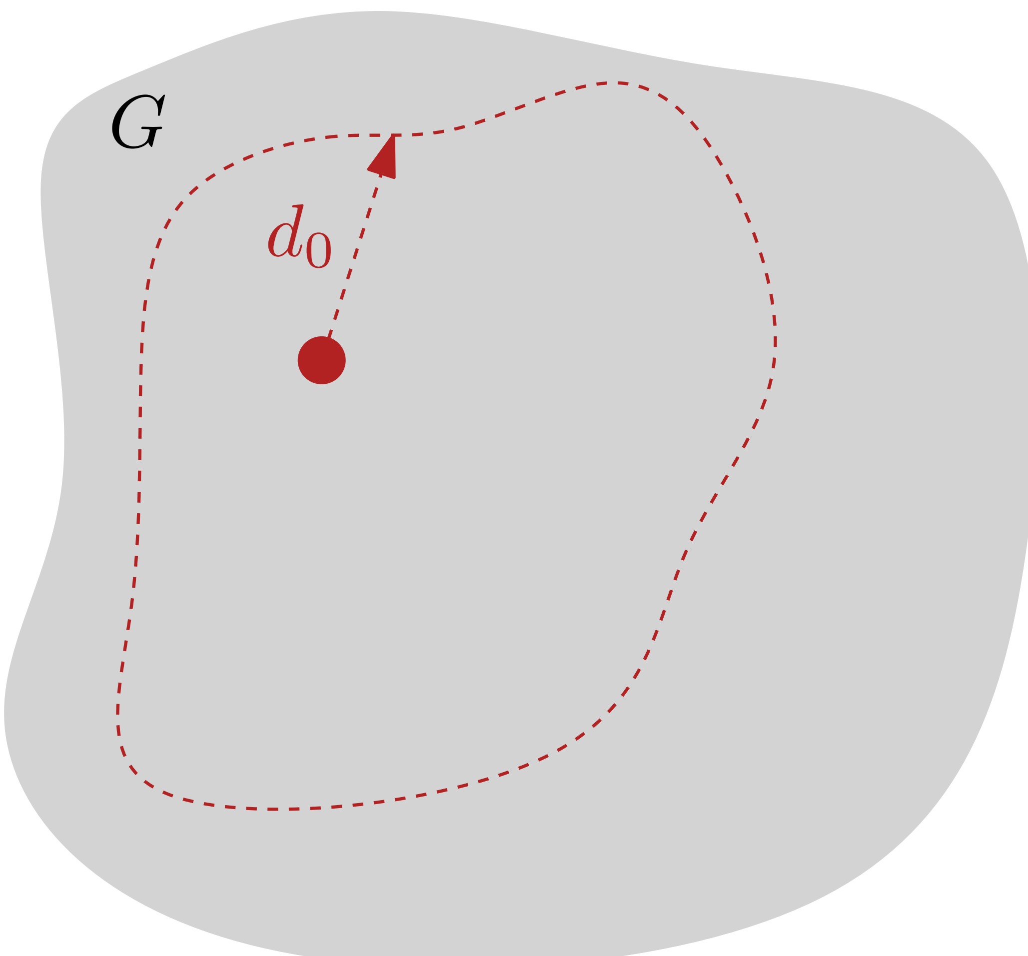
Same as before for ● and start of the infection.



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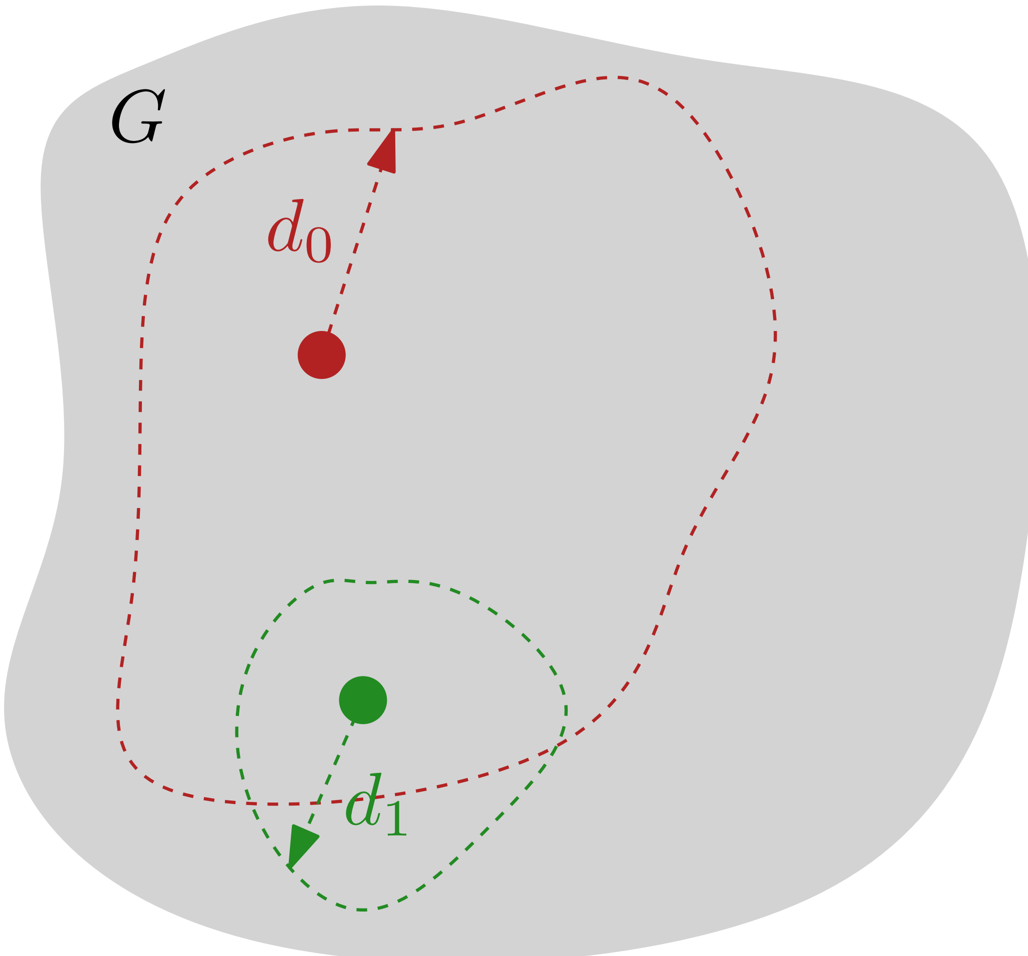

In addition suppose:

- At distance $< d$ from ● , there is another vertex (●) with large degree d_1 *s.t.* $\lambda > \lambda_c(d_1)$.
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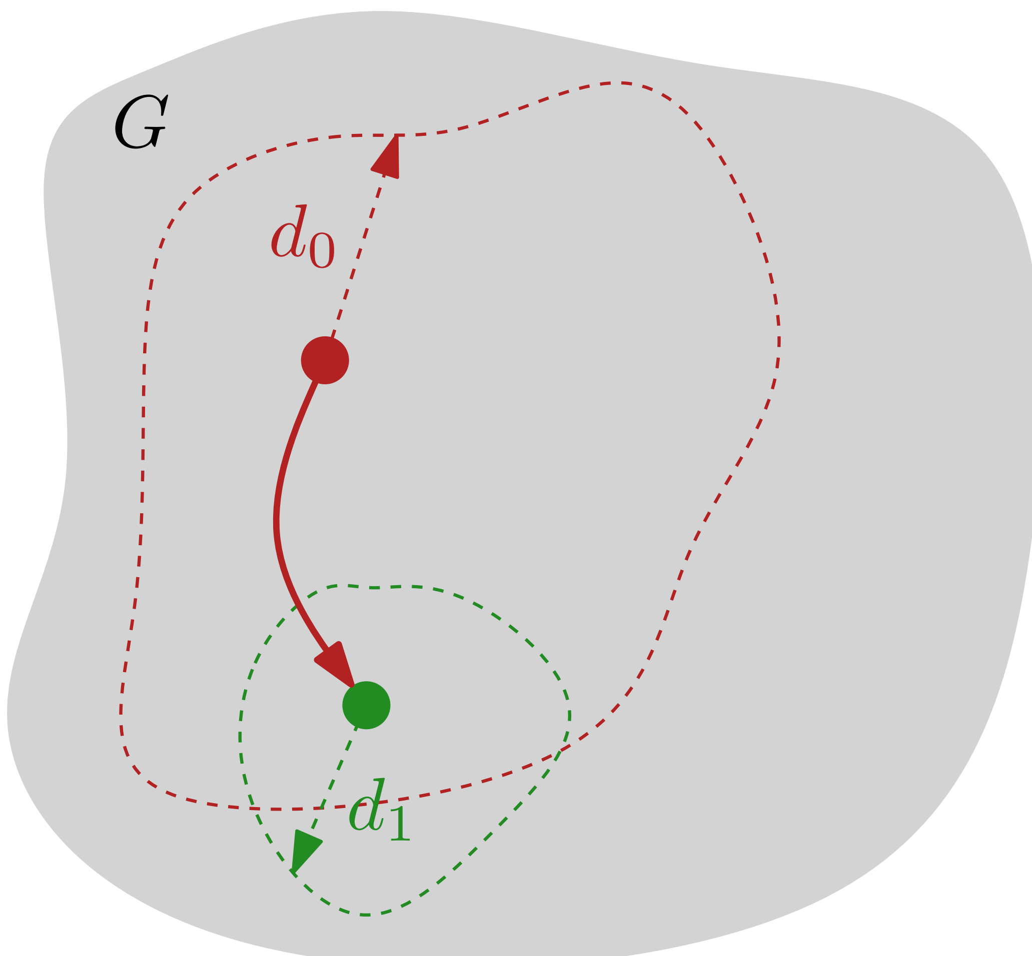

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● cannot send infections to ● and the survival time of the process is $\approx \exp(d_0) + \exp(d_1) \approx \exp(d_0)$.

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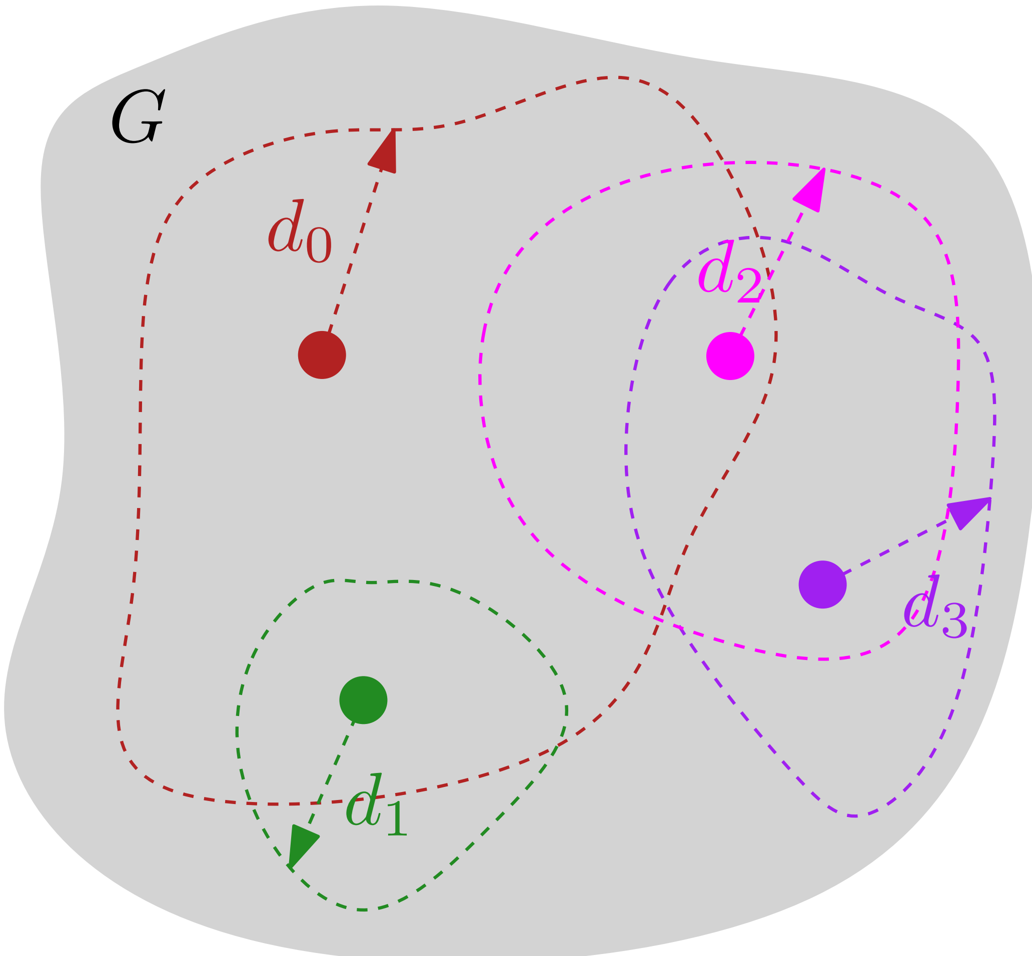

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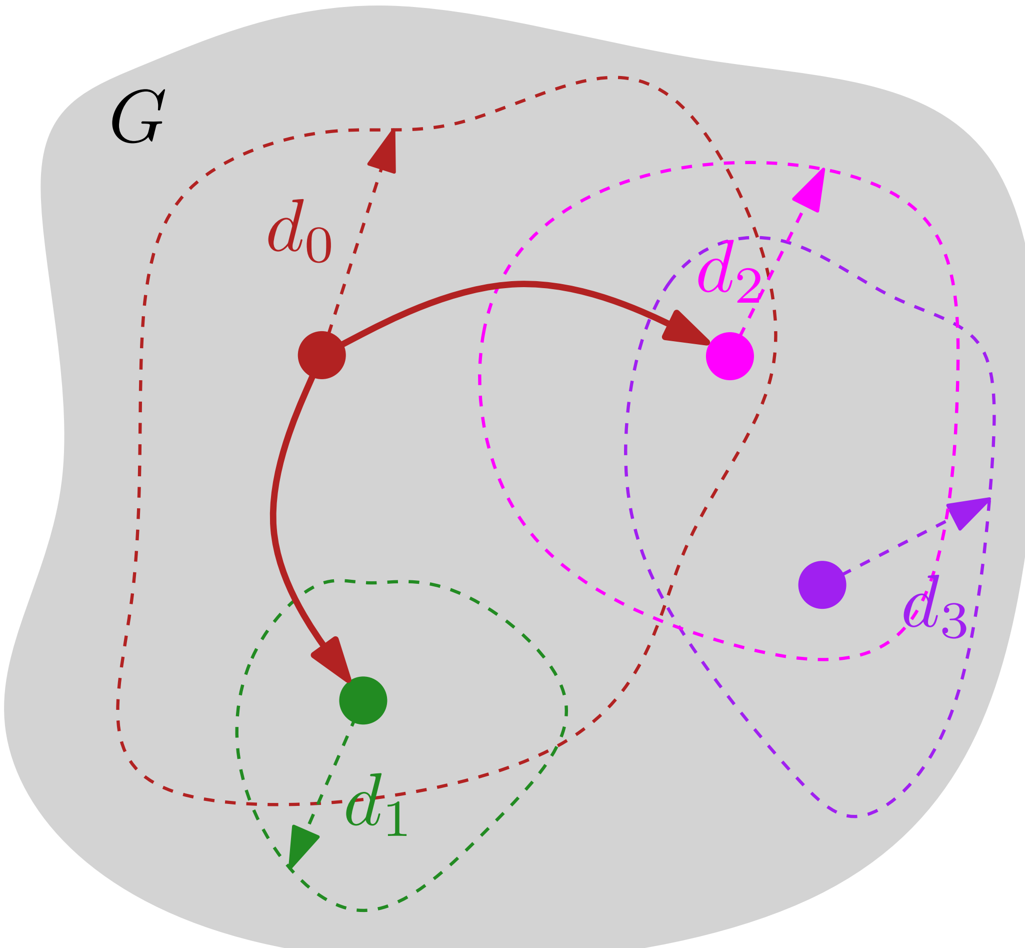

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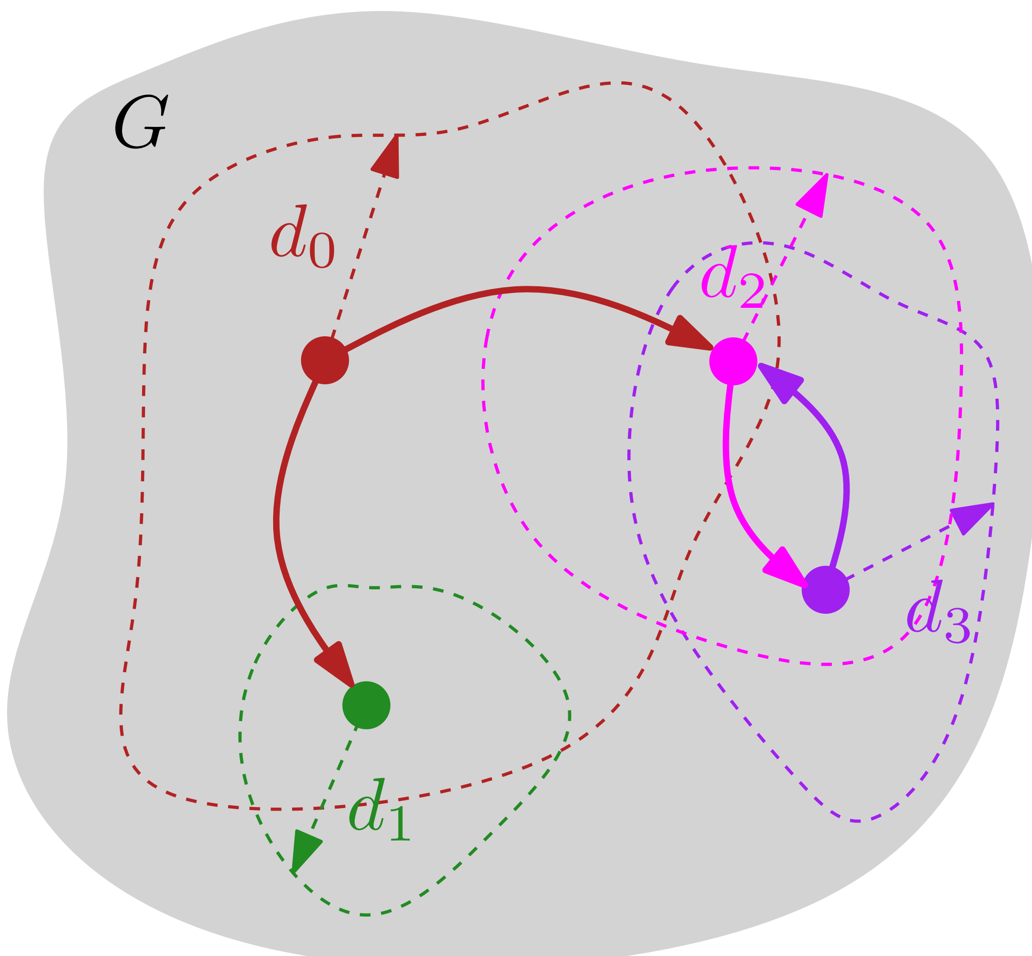

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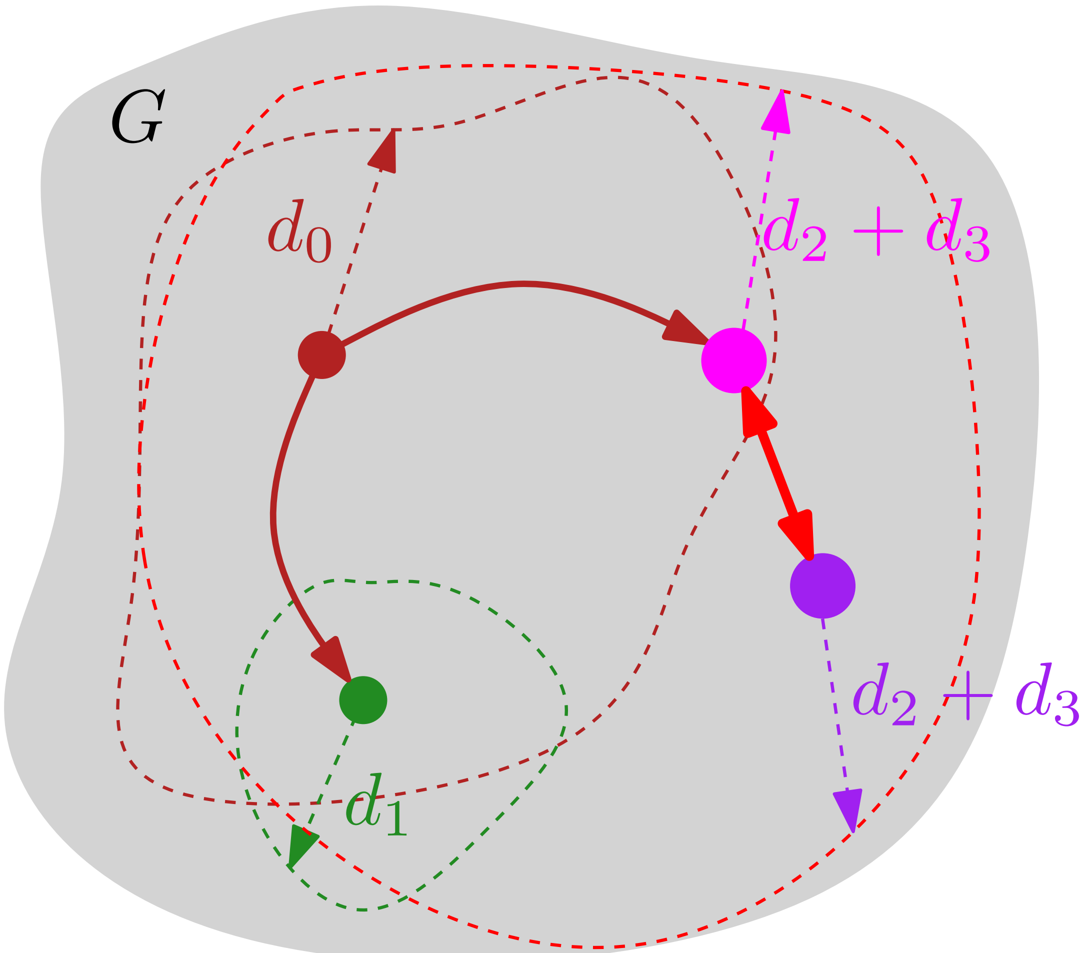

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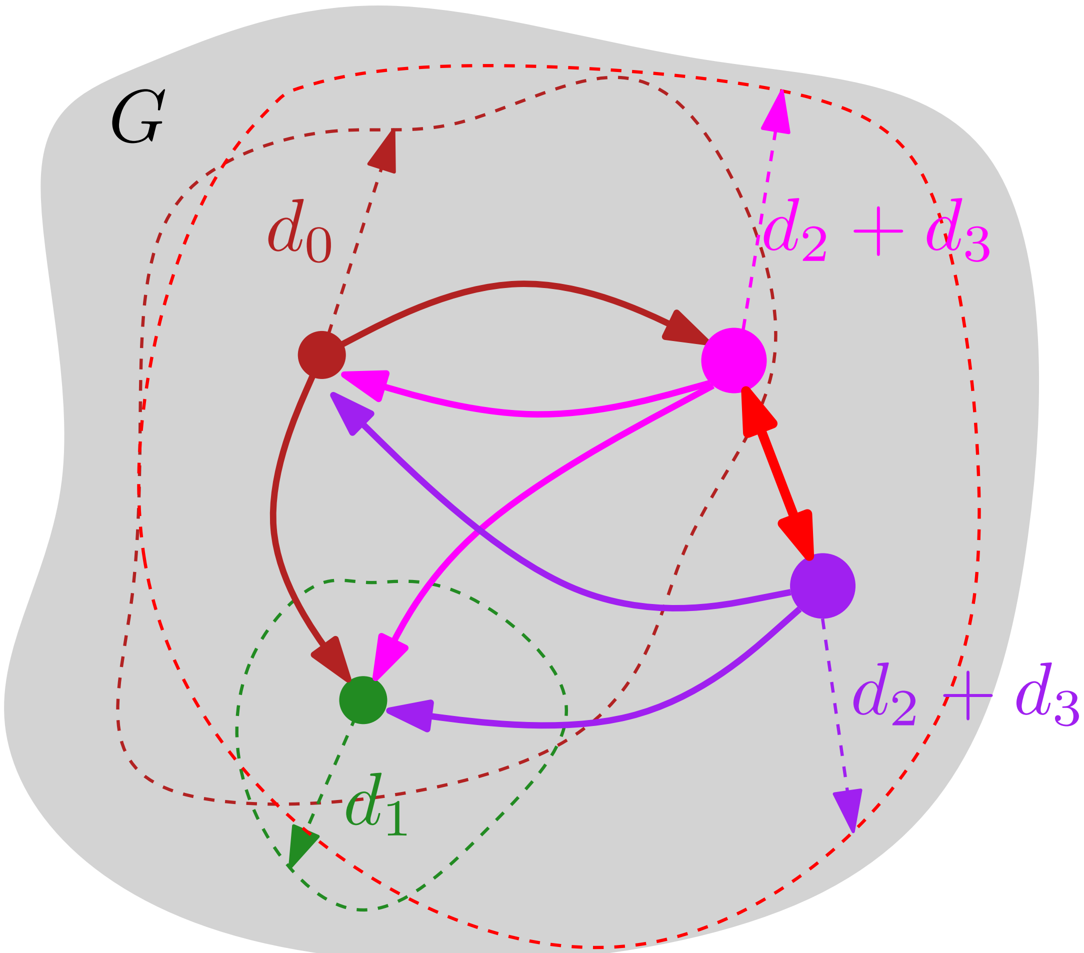

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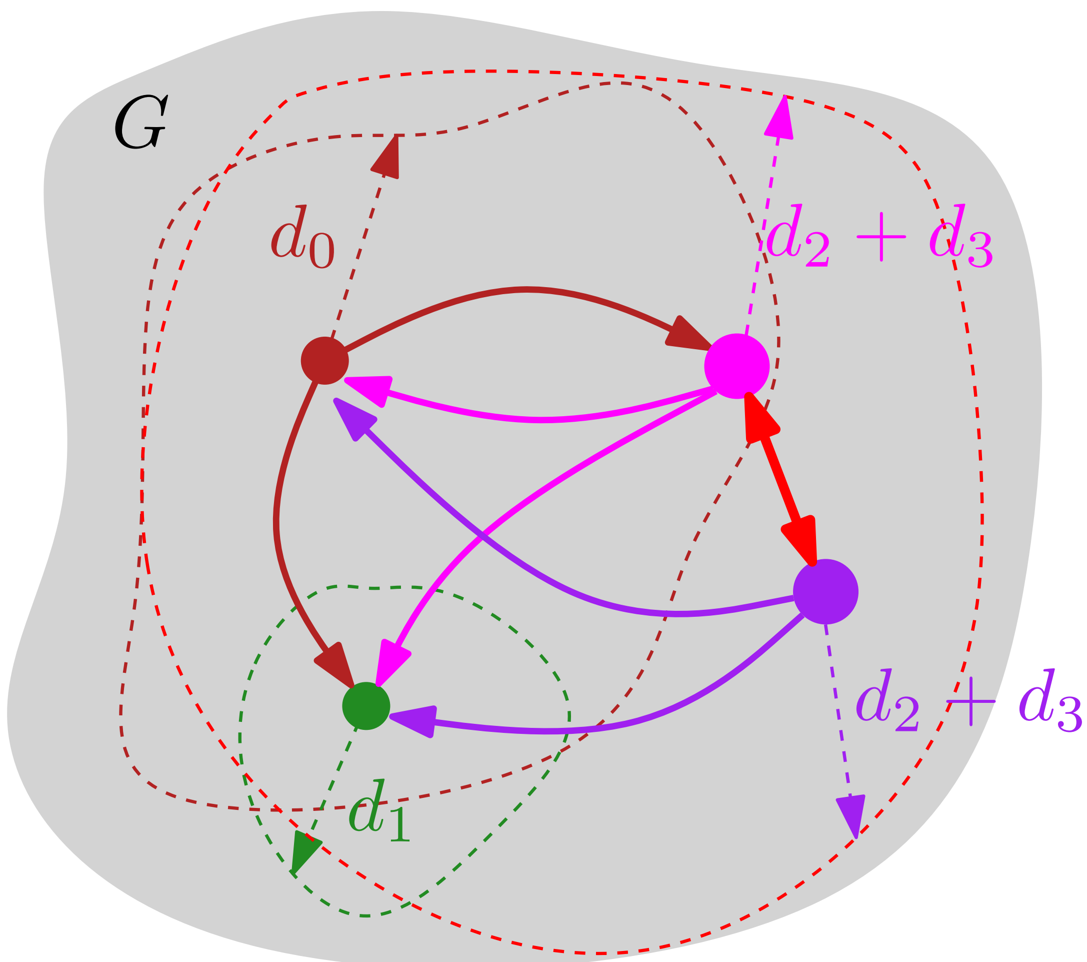

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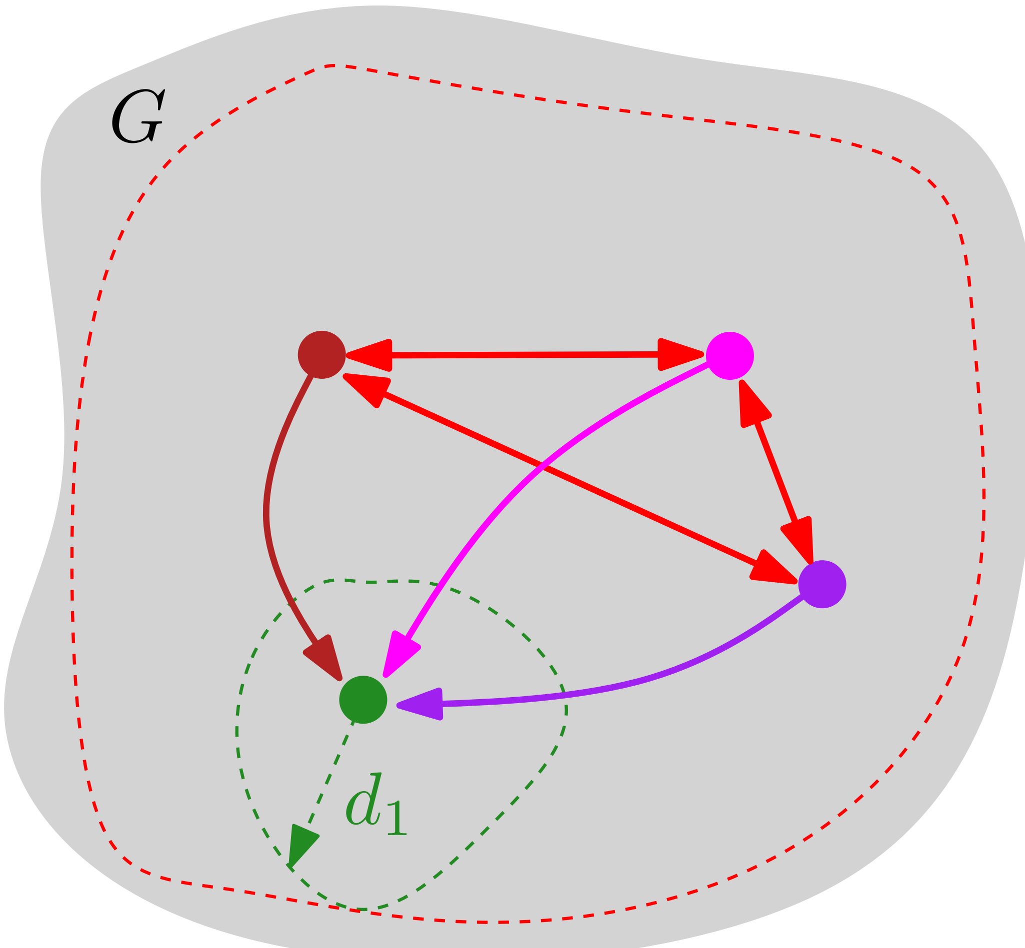

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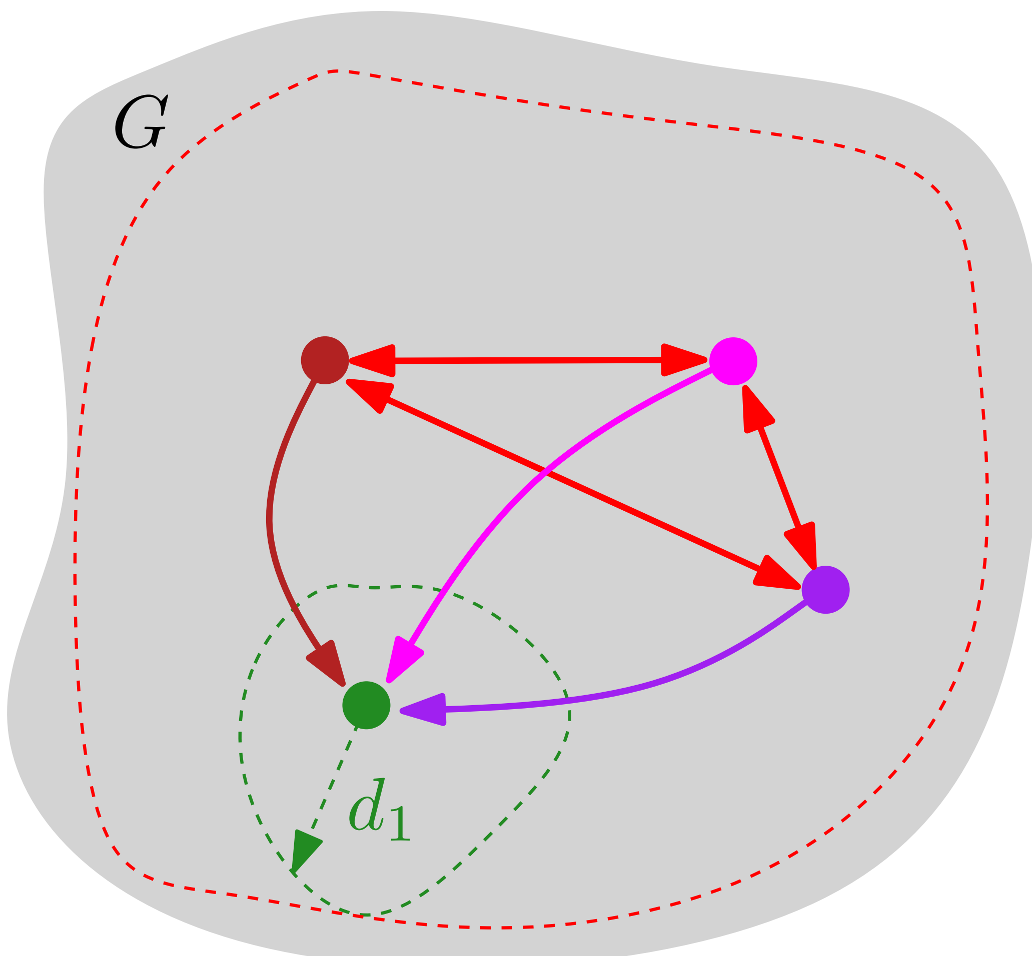

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⟶ ● still cannot reach the other 3 vertices to interact.

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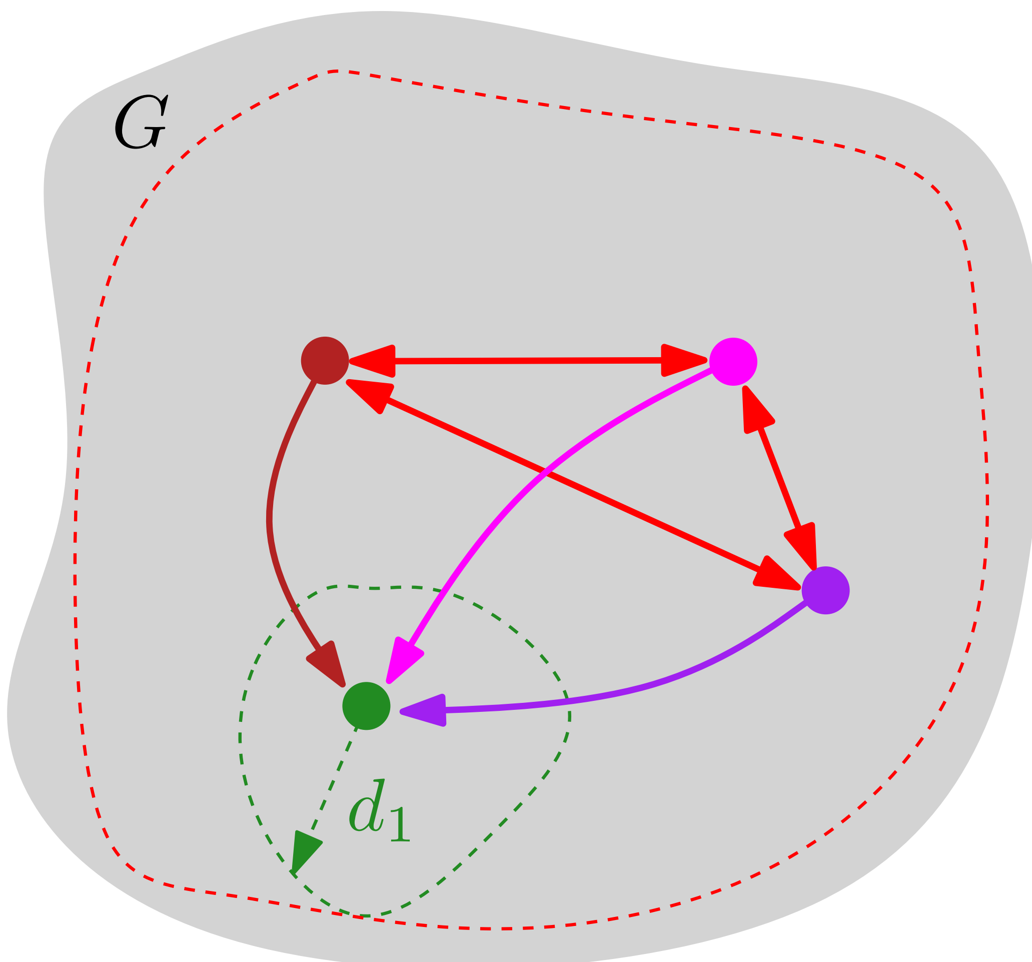


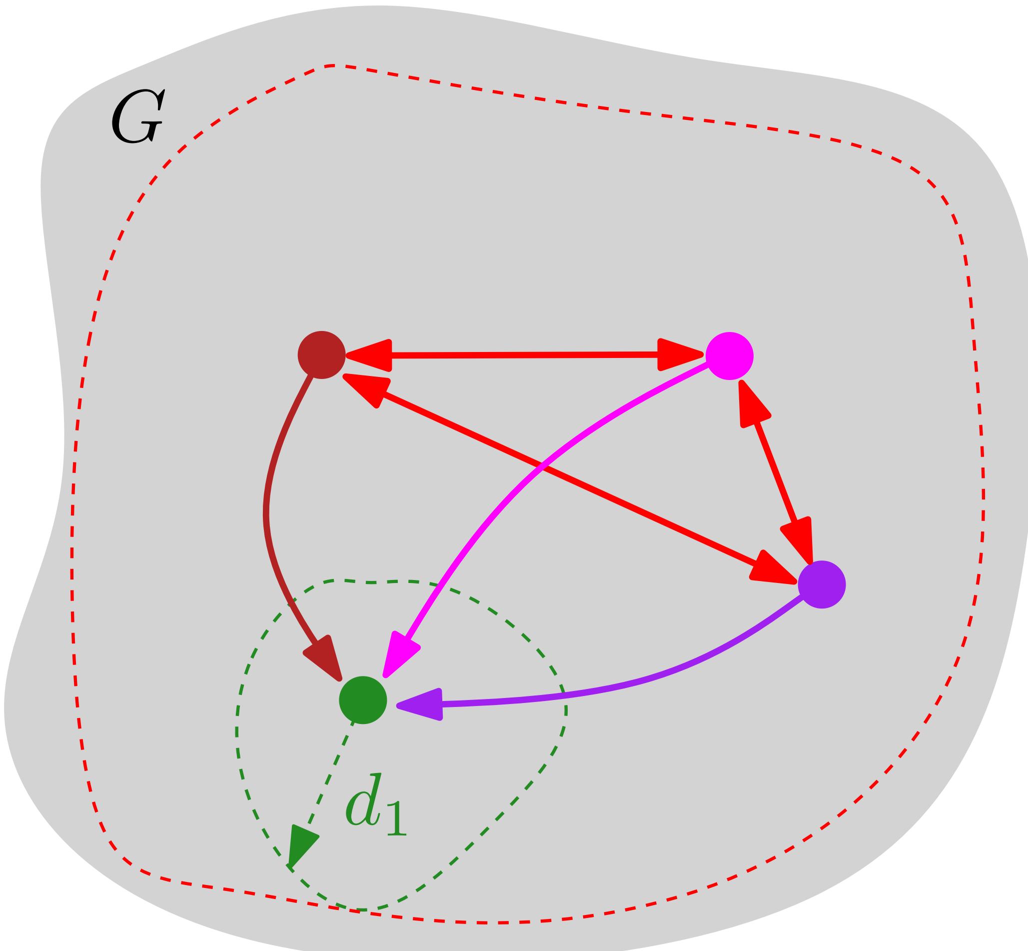
Questions:

- Can we recursively group vertices in classes such that for any two different classes A and B we have:

  $d(A,B) > \min\{deg(A); deg(B)\}$?

- Is all this hand waving valid?

Heuristics for the contact process



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Cumulative Merging

Cumulative Merging: admissible partitions

Consider a weighted graph $G = (V, E, r)$ with $r : V \to [0, \infty]$.

Definition

A partition $\mathcal{P}$ of V is **admissible** *iff* $\forall A \neq B \in \mathcal{P}$:

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$$\mathcal{P}_1 \cap \mathcal{P}_2 := \{C_1 \cap C_2 : C_1 \in \mathcal{P}_1, C_2 \in \mathcal{P}_2\}$$

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Definition

$$\mathscr{C}(G, r) := \bigcap_{\text{admissible } \mathcal{P}} \mathcal{P} \qquad \text{(finest admissible partition)}$$

Merging operators

Definition: Merging operators

For $x \neq y \in V$, $M_{x,y} : \{\text{partitions of } V\} \to \{\text{partitions of } V\}$ defined by

$$M_{x,y}(\mathcal{P}) := \begin{cases} (\mathcal{P} \setminus \{\mathcal{P}_x, \mathcal{P}_y\}) \cup \{\mathcal{P}_x \cup \mathcal{P}_y\} & \text{if } \mathcal{P}_x \neq \mathcal{P}_y \text{ and} \\ & d(x,y) \leq r(\mathcal{P}_x) \wedge r(\mathcal{P}_y), \\ \mathcal{P} & \text{otherwise.} \end{cases}$$

for every partition $\mathcal{P}$, where $\mathcal{P}_x$ is the **cluster** of x in $\mathcal{P}$.

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Proposition:

The merging operators are monotone: for every $x \neq y \in V$ and every partitions $\mathcal{P}$ and $\mathcal{P}'$

- $\mathcal{P}$ is finer than $M_{x,y}(\mathcal{P})$;
- If $\mathcal{P}$ is finer than $\mathcal{P}'$, then $M_{x,y}(\mathcal{P})$ is finer than $M_{x,y}(\mathcal{P}')$.

Cumulative Merging

Proposition:
Take $(x_n, y_n) \in V^{\mathbb{N}} \times V^{\mathbb{N}}$ such that for every $x \neq y \in V$:

$$\{x_n, y_n\} = \{x, y\} \text{ for infinitely many } n.$$

Then

$$\mathscr{C}(V, E, r) = \lim_{n \to \infty} \uparrow \; M_{x_n, y_n} \circ \cdots \circ M_{x_1, y_1}(\bar{V})$$

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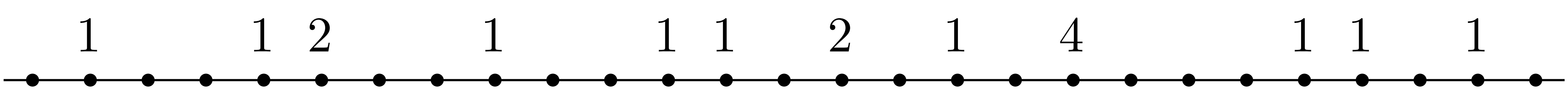

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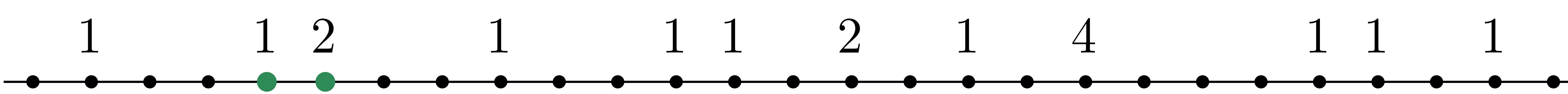

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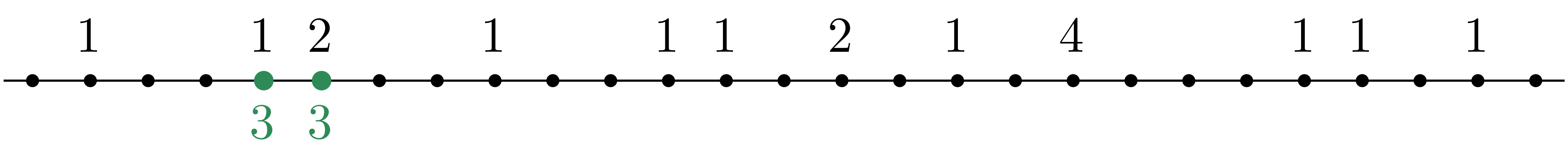

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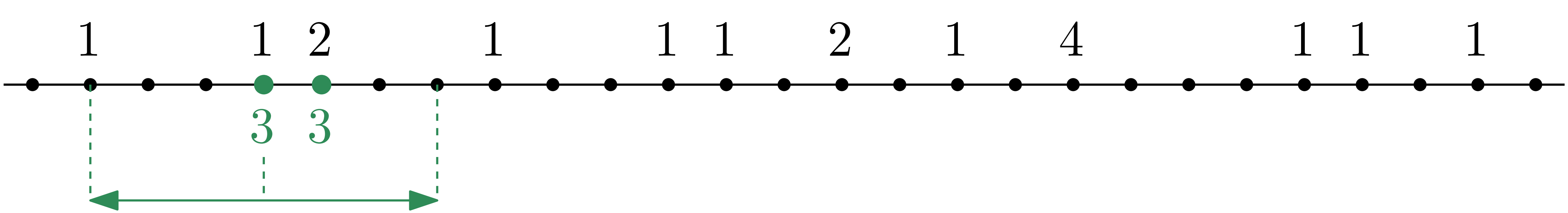

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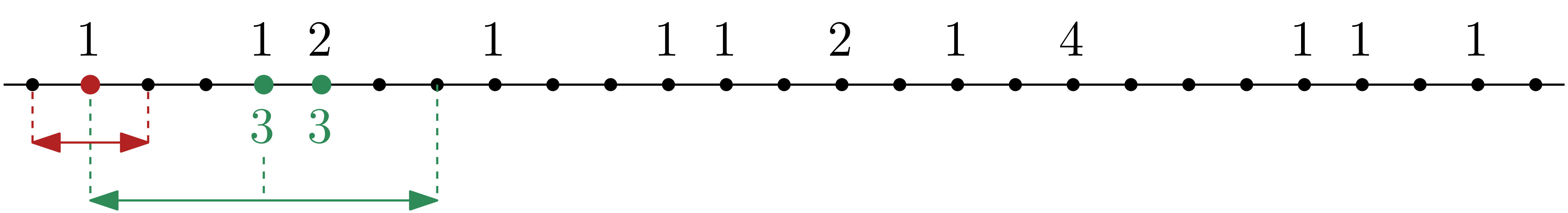

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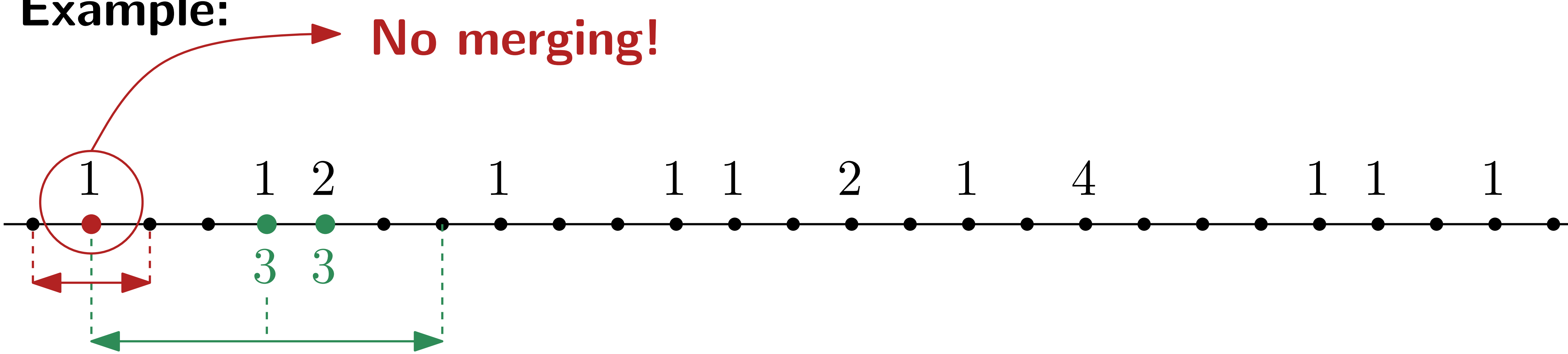

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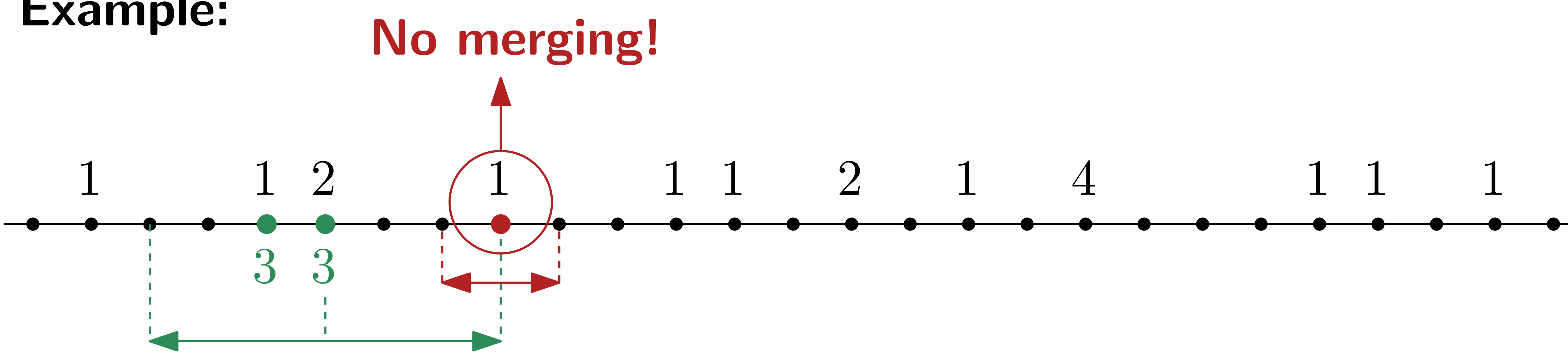

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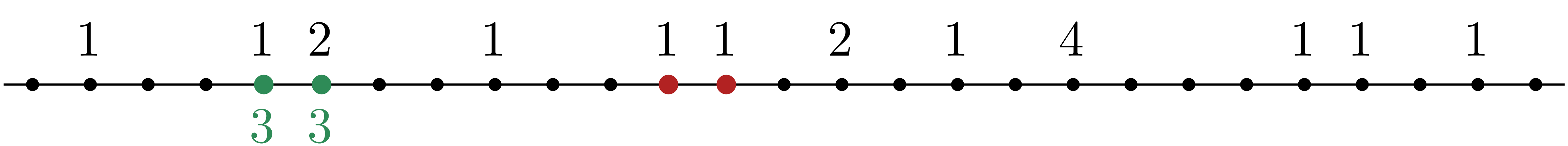

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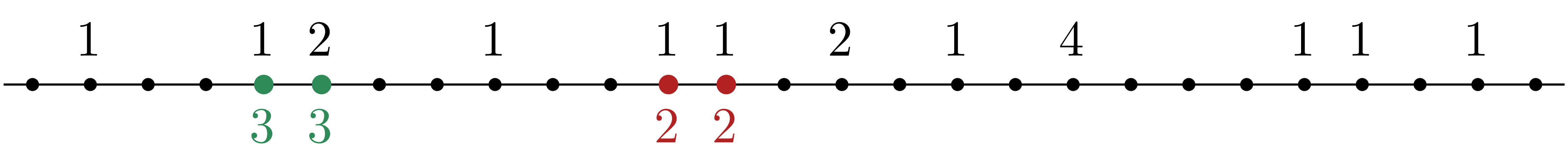

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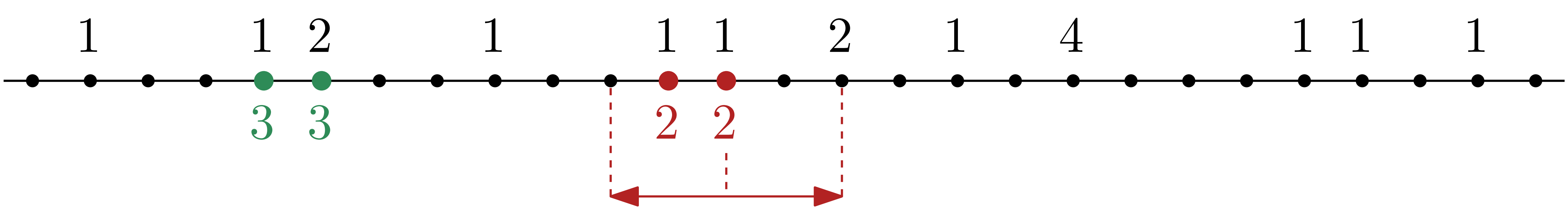

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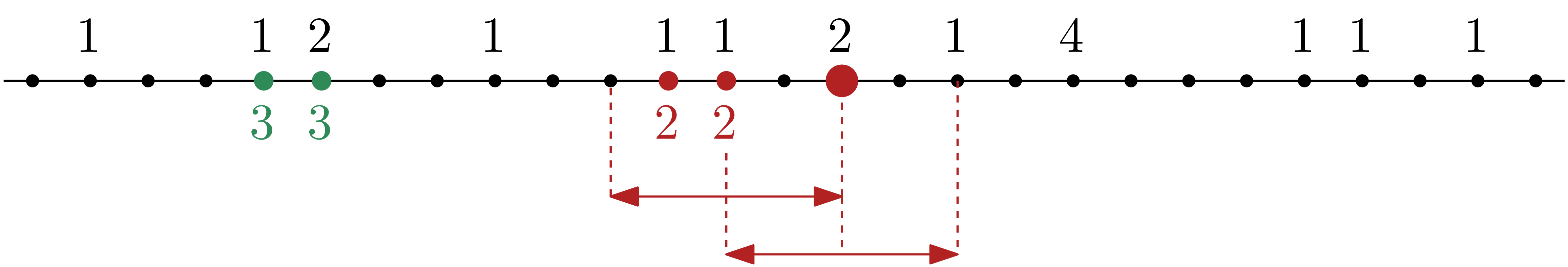

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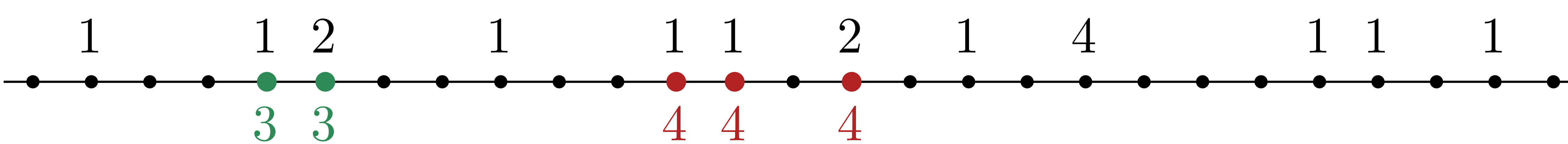

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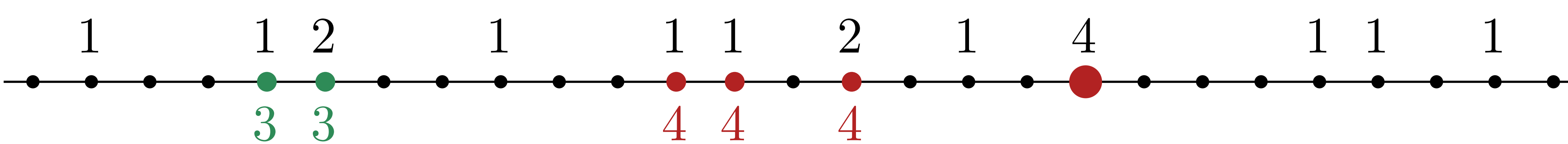

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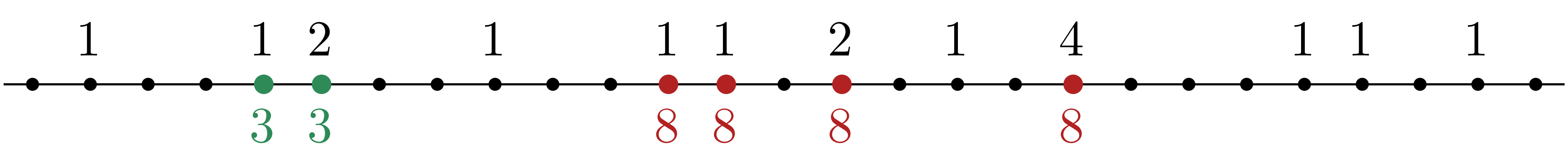

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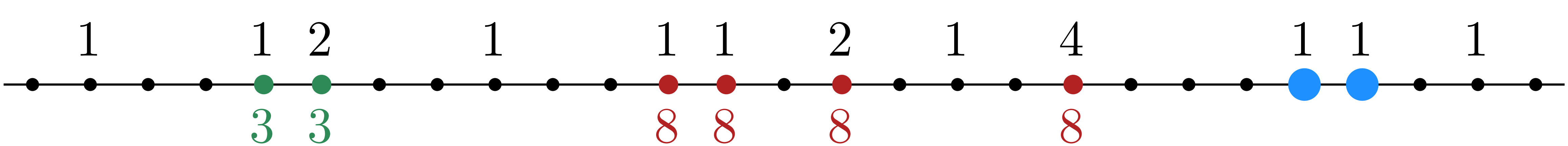

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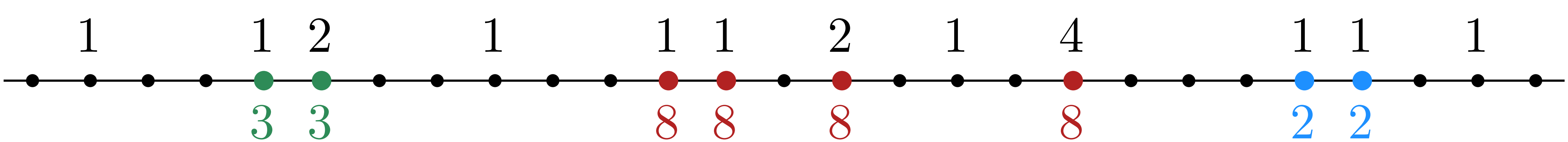

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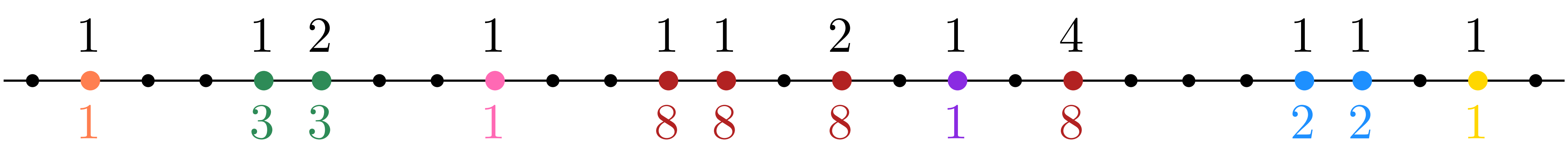

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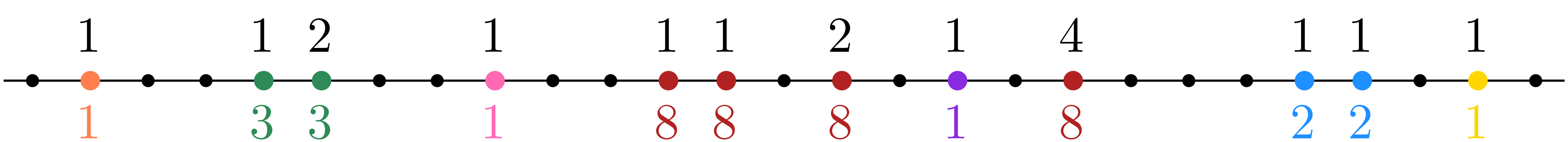
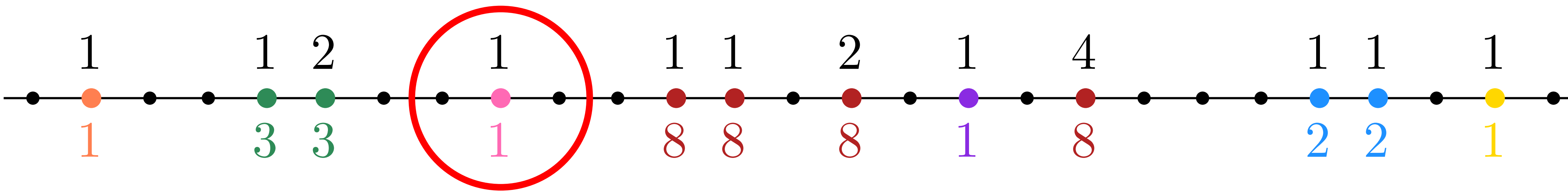


Remark: clusters in $\mathscr{C}$ are not necessarily connected sets!

CMP: stable sets



These 3 vertices will never merge with anything!

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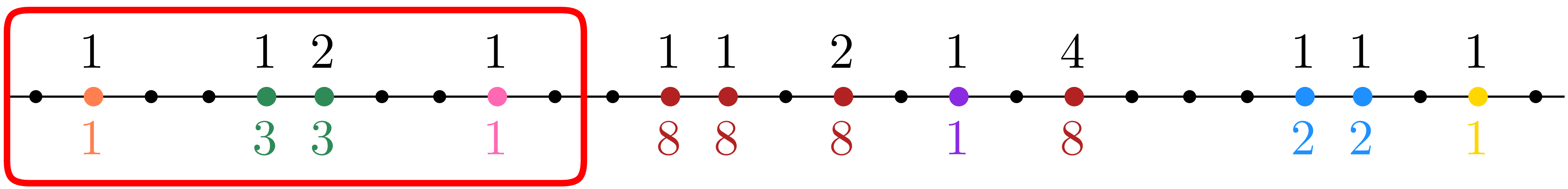

Vertices inside that box will never merge again.

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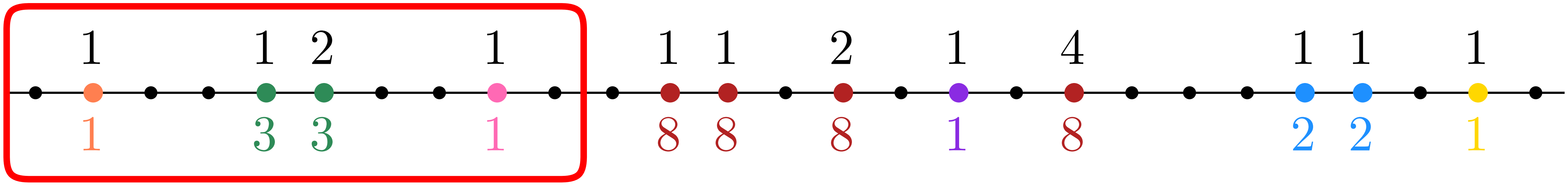

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Definition:

Fix $H \subset V$. We say that H is a **stable set** *iff*:

$$\forall C \in \mathscr{C}(H, E_H, r) \text{ one has } B\left(C, r(C)\right) \subset H.$$

Remark:

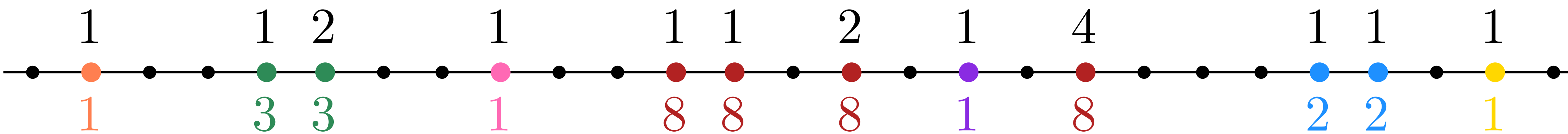
- Unions and intersections of stable sets are stable.
- Being stable is a **local** porperty.
- If H is stable, then

$$\mathscr{C}(G) = \mathscr{C}(H) \sqcup \mathcal{C}(G \setminus H).$$

CMP: stabilisers

Definition

Let $H \subset V$ the **stabiliser** of H, $\mathcal{S}_H$, is the smallest stable set of vertices containing H.



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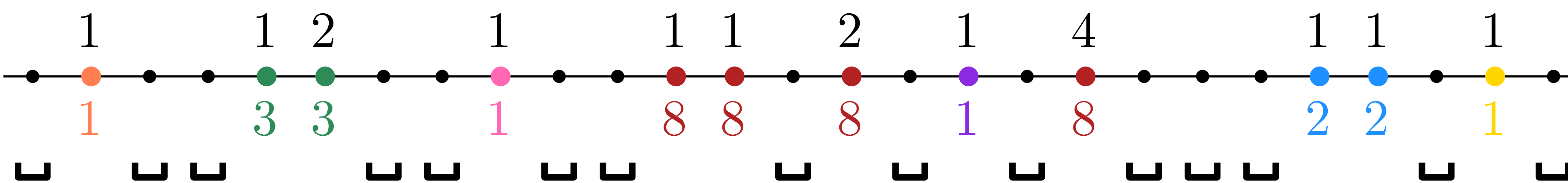

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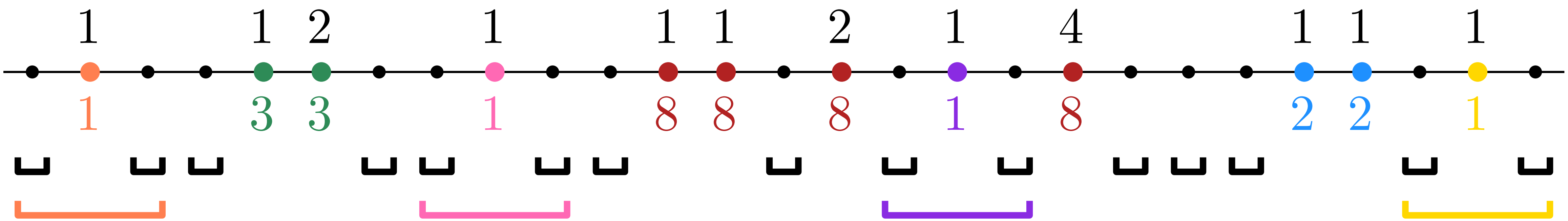

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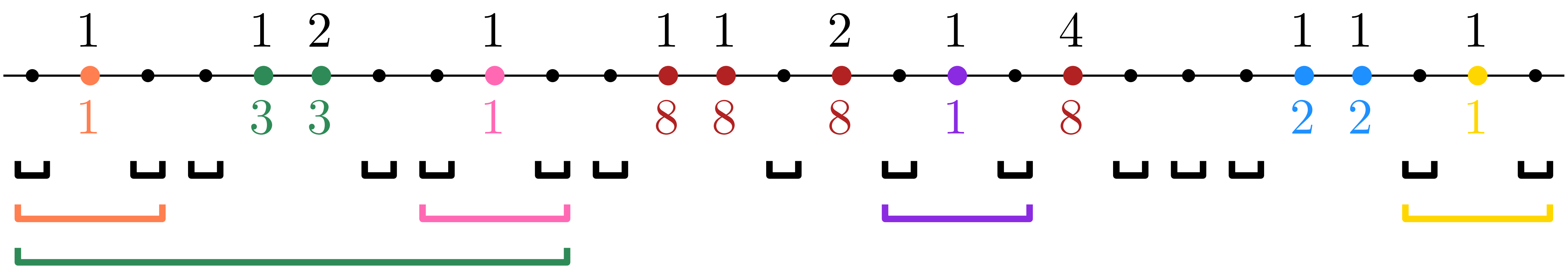

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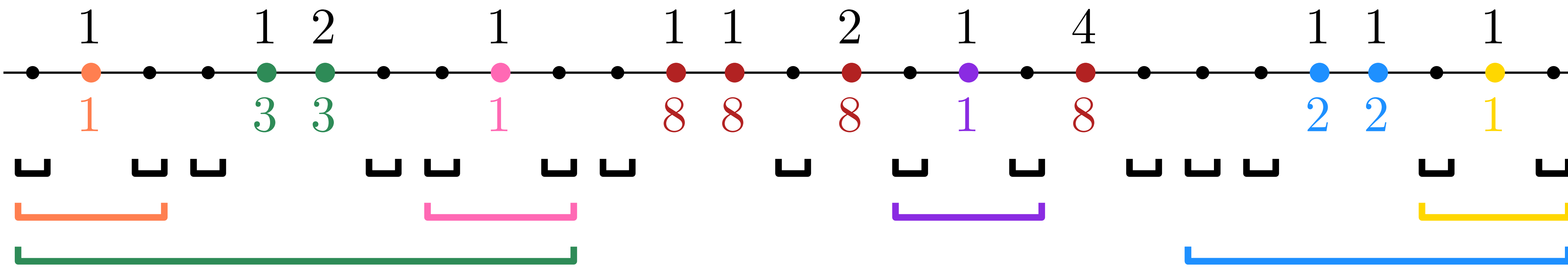

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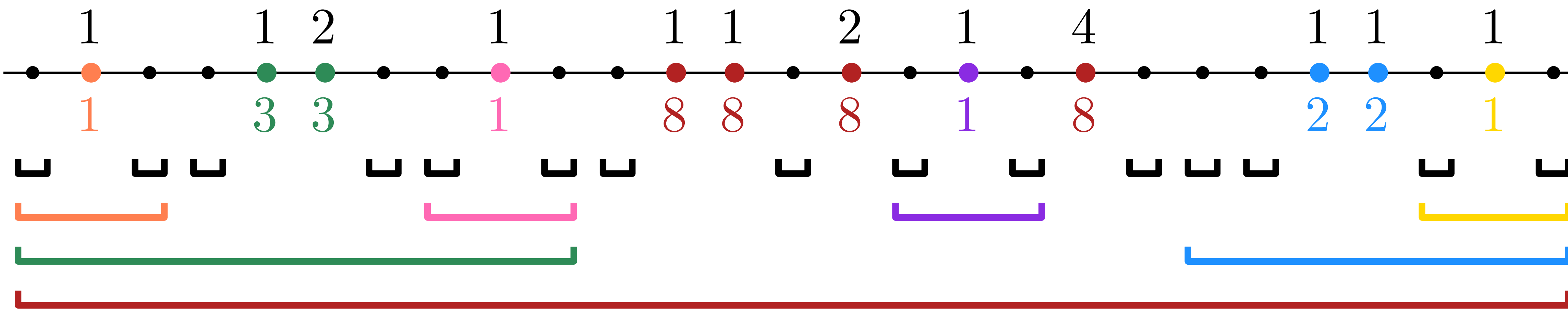

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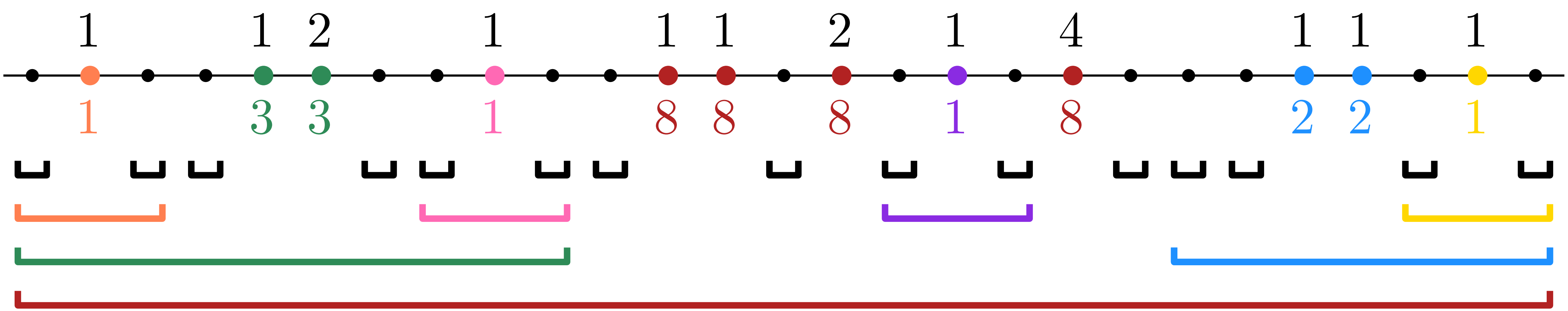

- Stabilisers are connected sets in G.
- Sabilisers are **nested**.
- Nice oriented graph structure on clusters and on stabilisers.

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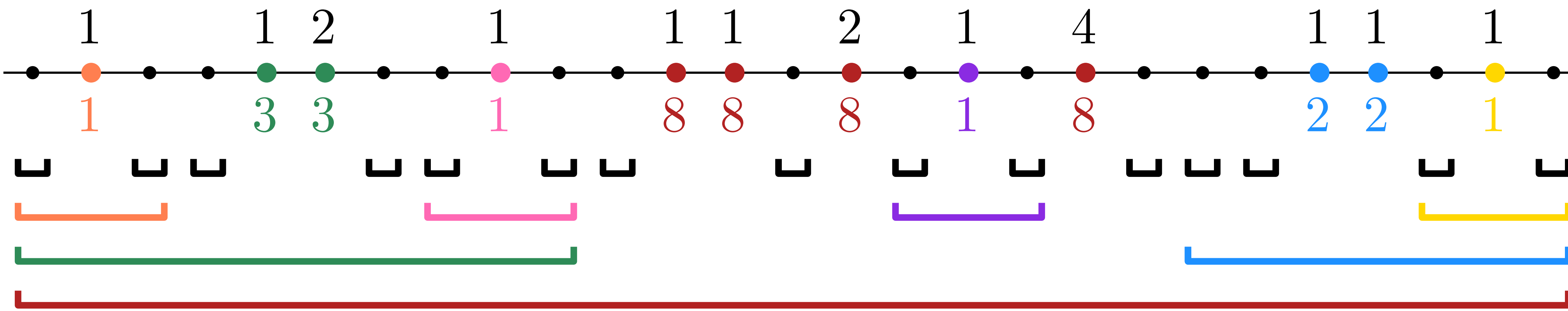

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Theorem

Suppose G infinite:

1. $\forall x \in V: |\mathscr{C}_x| = \infty \Leftrightarrow |\mathcal{S}_x| = \infty \Leftrightarrow \mathcal{S}_x = V$.
2. $\mathscr{C}$ has no infinite cluster *iff* there exists an increasing sequence of stable sets S_n *s.t.* $\lim \uparrow S_n = V$.

Cumulative Merging: phase transitions

Several natural choices of weights for a locally finite graph (V,E):

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- $(r(x))_{x \in V}$ *i.i.d.* Bernoulli *r.v.* with parameter p.

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- $(r(x))_{x \in V}$ *i.i.d.* Bernoulli *r.v.* with parameter p.
- $(r(x))_{x \in V}$ *i.i.d.* *r.v.* with law $\lambda \times Z$, $\lambda \geq 0$ deterministic.

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Several natural choices of weights for a locally finite graph (V,E):

- $(r(x))_{x\in V}$ *i.i.d.* Bernoulli *r.v.* with parameter p.
- $(r(x))_{x\in V}$ *i.i.d.* *r.v.* with law $\lambda \times Z$, $\lambda \geq 0$ deterministic.
- $\forall x \in V$, $r(x) = \begin{cases} \deg(x) & \text{if } \deg(x) > \Delta; \\ 0 & \text{otherwise.} \end{cases}$

Cumulative Merging: phase transitions

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Critical parameters for the existence of an infinite cluster in $\mathscr{C}(V,E,r)$.

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Theorem:

1. CMP on $\mathbb{Z}^d$: $p_c \in (0,1)$.
2. CMP on $\mathbb{Z}^d$: if $E\left[Z^\beta\right]$ for $\beta > (4d)^2$, then $\lambda_c \in (0,\infty)$.
3. CMP on d-dimensional Delaunay triangulation or geometric graph: $\Delta_c < \infty$.

Cumulative Merging: link with the contact process

Need to change the definition of admissible partition:

$\mathcal{P}$ is admissible *iff* $\forall A, B \in \mathcal{P}$

$$d_G(A, B) > \ r(A) \wedge r(B) \quad .$$

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Thank you!