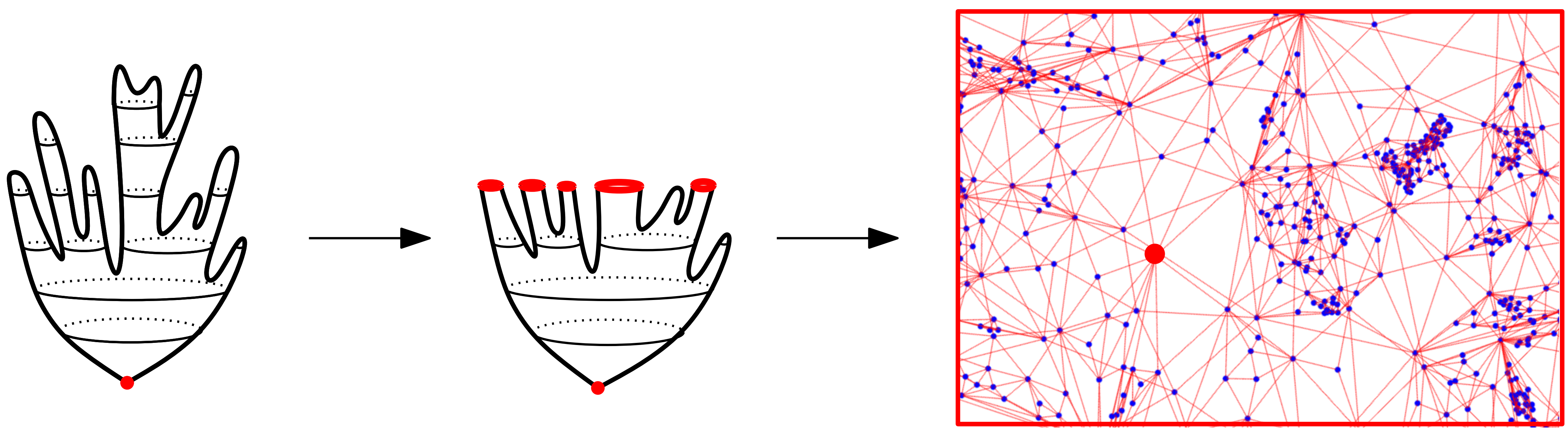
Random triangulations coupled with an Ising model

Laurent Ménard (Paris Nanterre)

joint work with **Marie Albenque** and **Gilles Schaeffer** (CNRS and LIX)



Bordeaux, November 2018

Outline

1. Introduction: 2DQG and planar maps
2. Local weak topology
3. Adding matter: Ising model
4. Combinatorics of triangulations with spins
5. Local limit of triangulations with spins

2D Quantum Gravity?

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Pick $a, b \in \mathbb{R}^2$, what does a random path $\gamma : [0,1] \to \mathbb{R}^2$ chosen "uniformly at random" between all paths from a to b look like?

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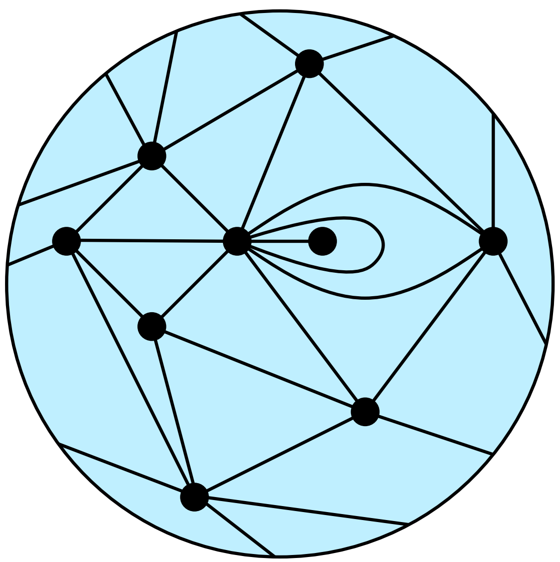
First idea: try discrete metric spaces (Donsker)

Planar Maps as discrete planar metric spaces

Definition:
A **planar map** is a proper embedding of a finite connected graph into the two-dimensional sphere (considered up to orientation-preserving homeomorphisms of the sphere).



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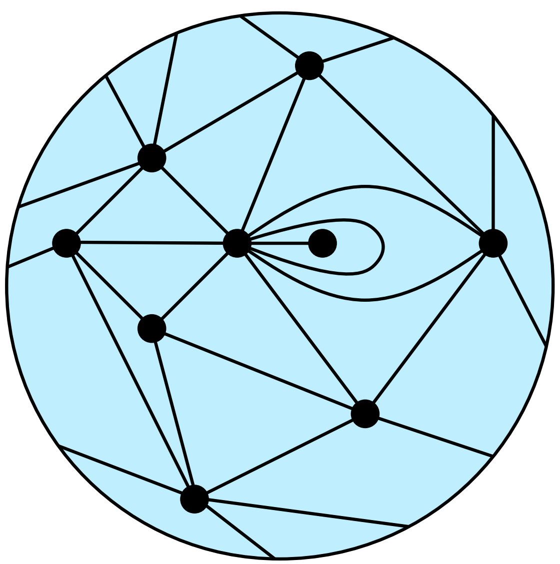

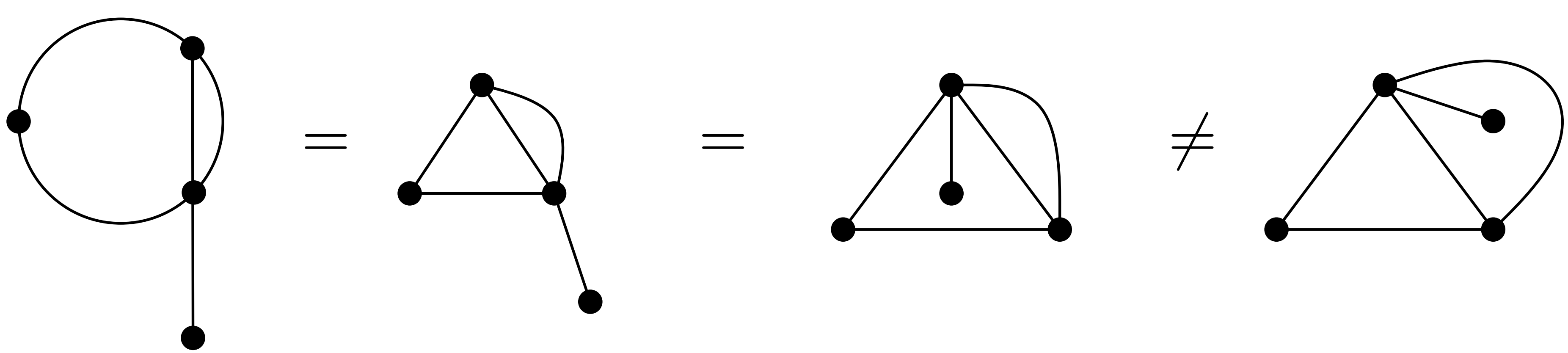

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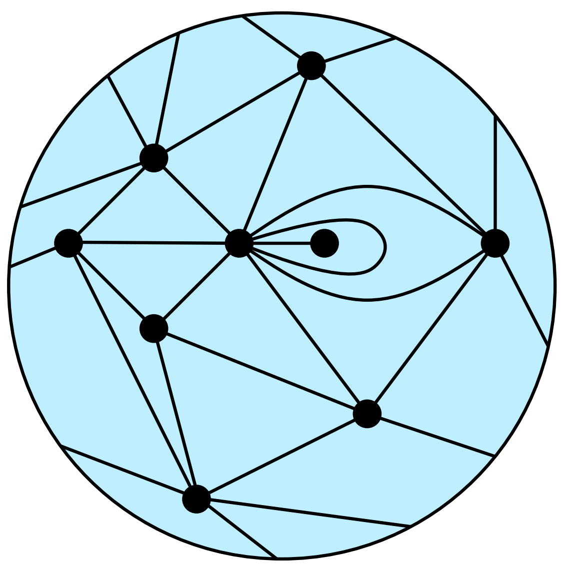
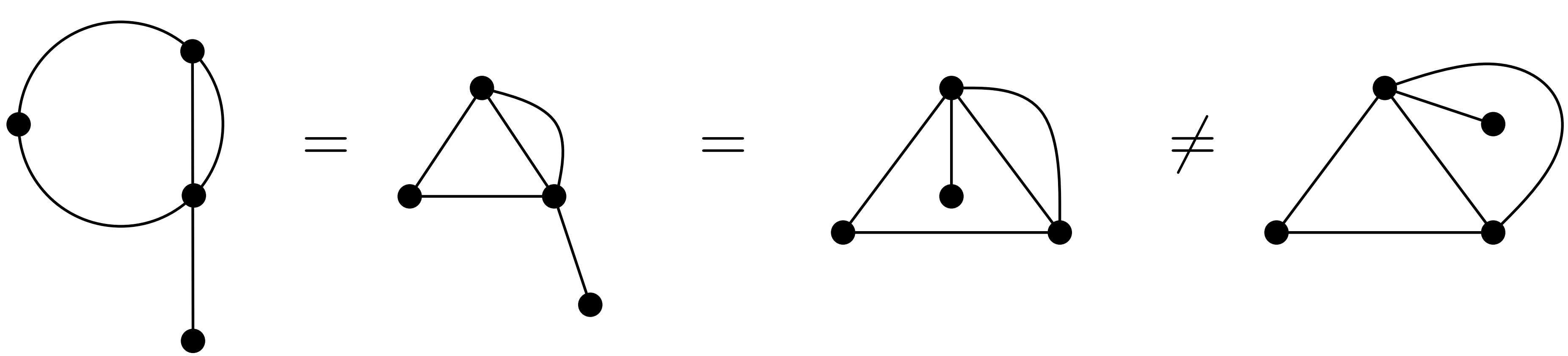


faces: connected components of the complement of edges

p-angulation: each face is bounded by p edges



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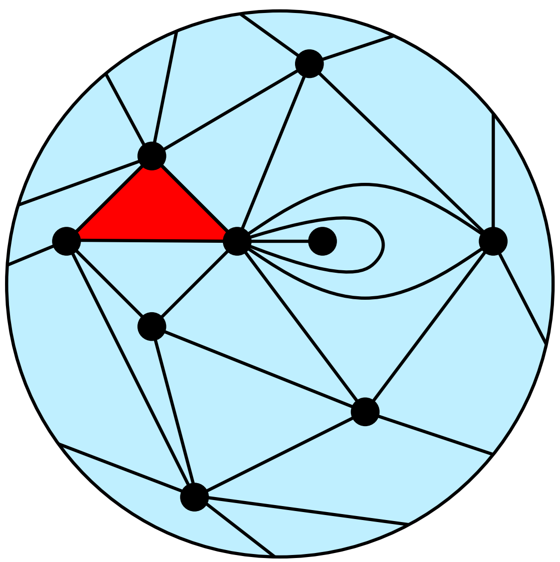

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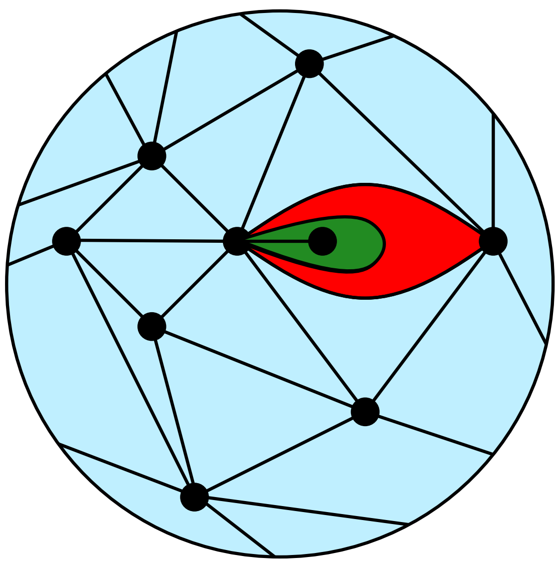

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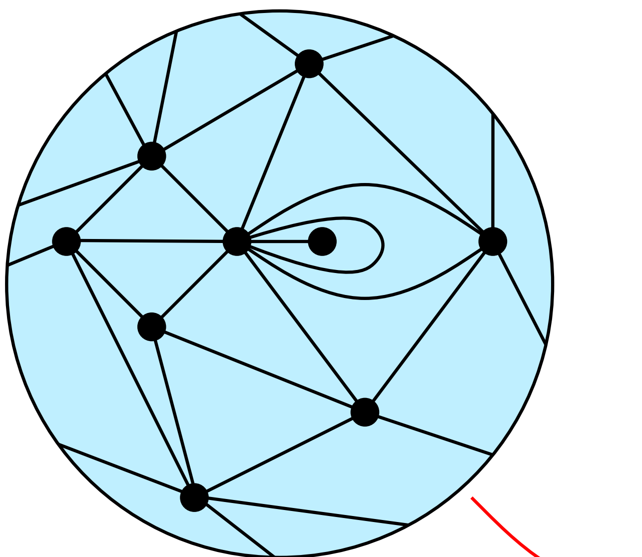

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This is a triangulation

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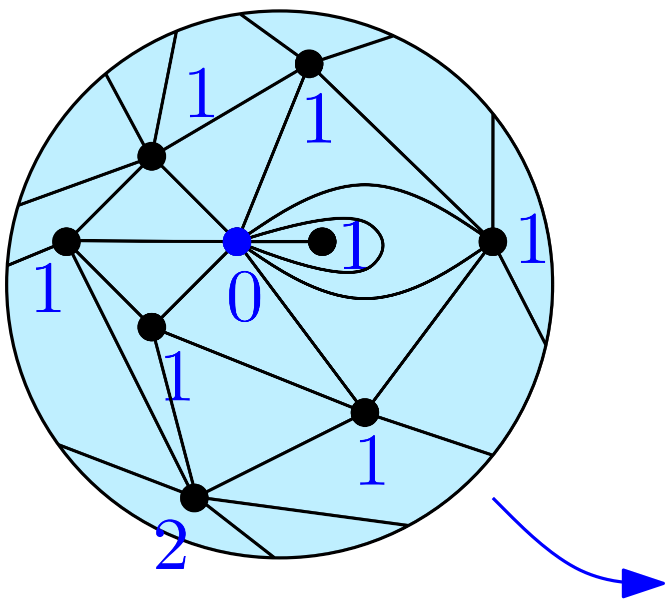

In blue, distances from •

M Planar Map:

- $V(M) :=$ set of vertices of M
- $d_{gr} :=$ graph distance on $V(M)$
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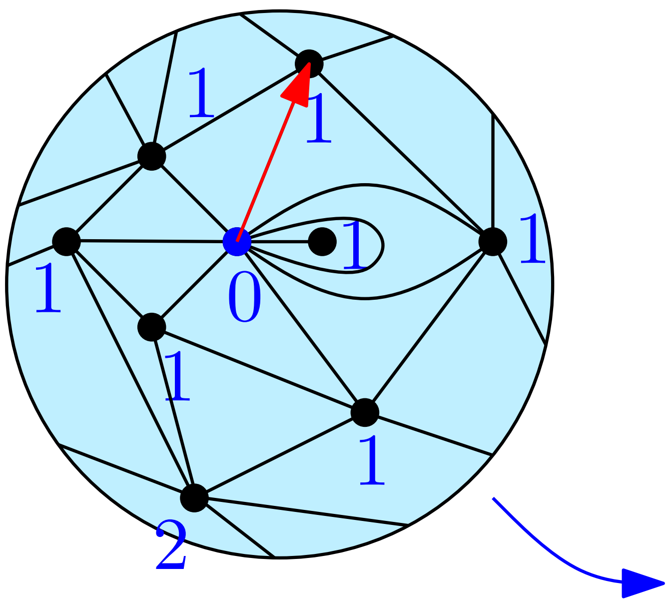

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Rooted map: mark an oriented edge of the map →

"Classical" large random triangulations

Euler relation in a triangulation: number of edges / vertices / faces linked

Take a triangulation of size n uniformly at random. What does it look like if n is large ?

Two points of view: global/local, continuous/discrete

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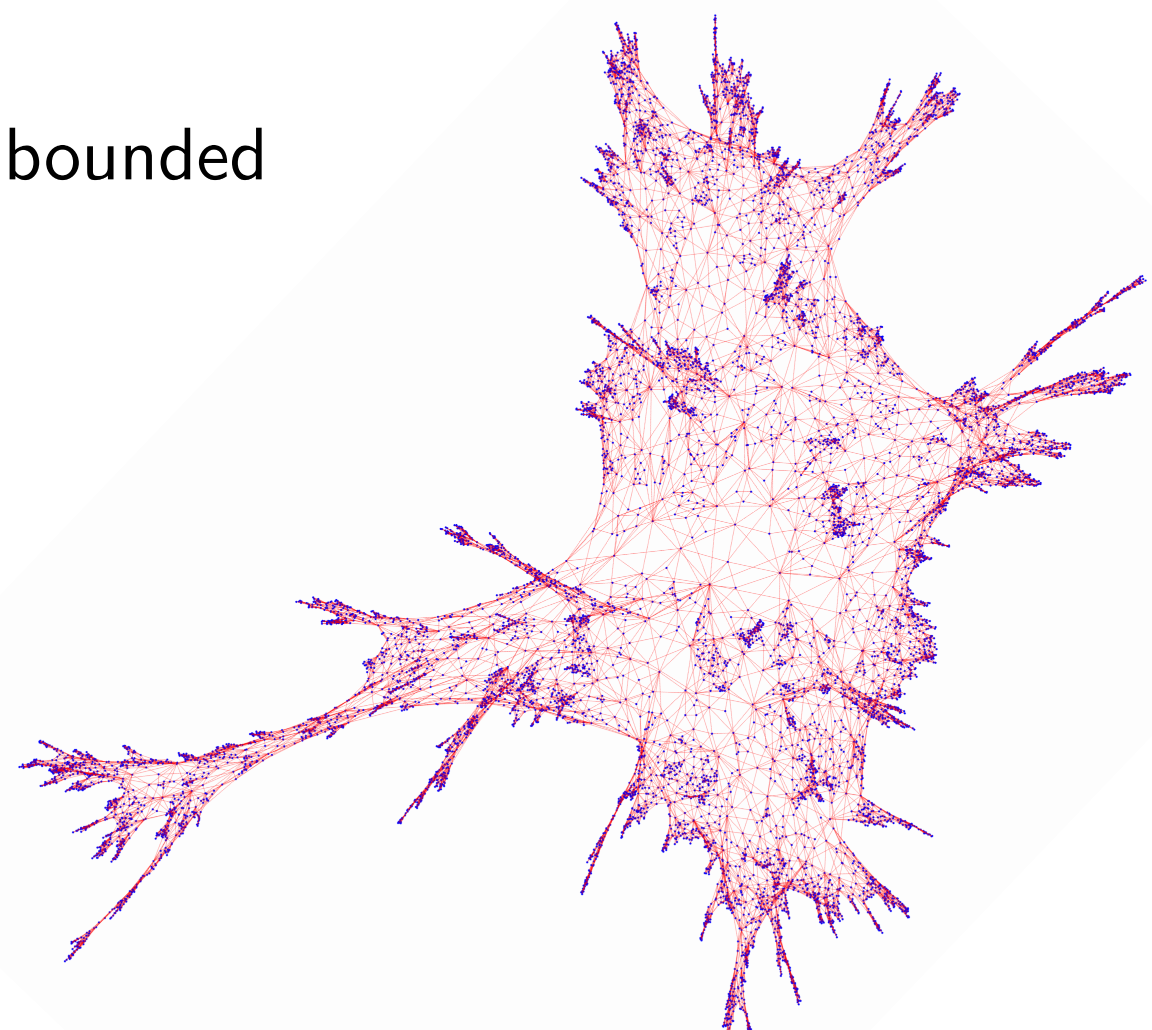
Global :

Rescale distances to keep diameter bounded

[Le Gall 13, Miermont 13]:

converges to the **Brownian map**.

- Gromov-Hausdorff topology
- Continuous metric space
- Homeomorphic to the sphere
- Hausdorff dimension 4
- **Universality**



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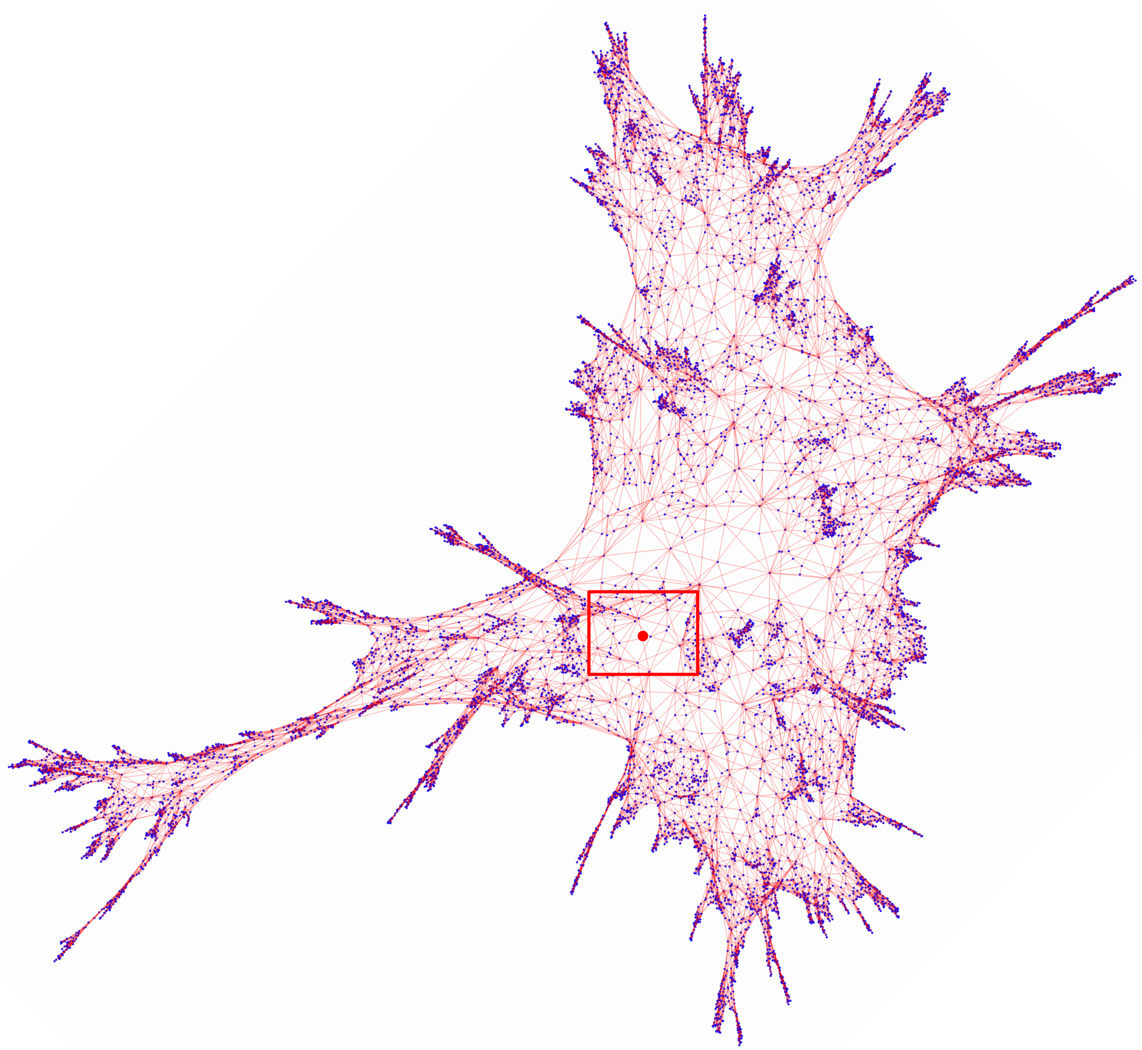
Don't rescale distances and look at neighborhoods of the root



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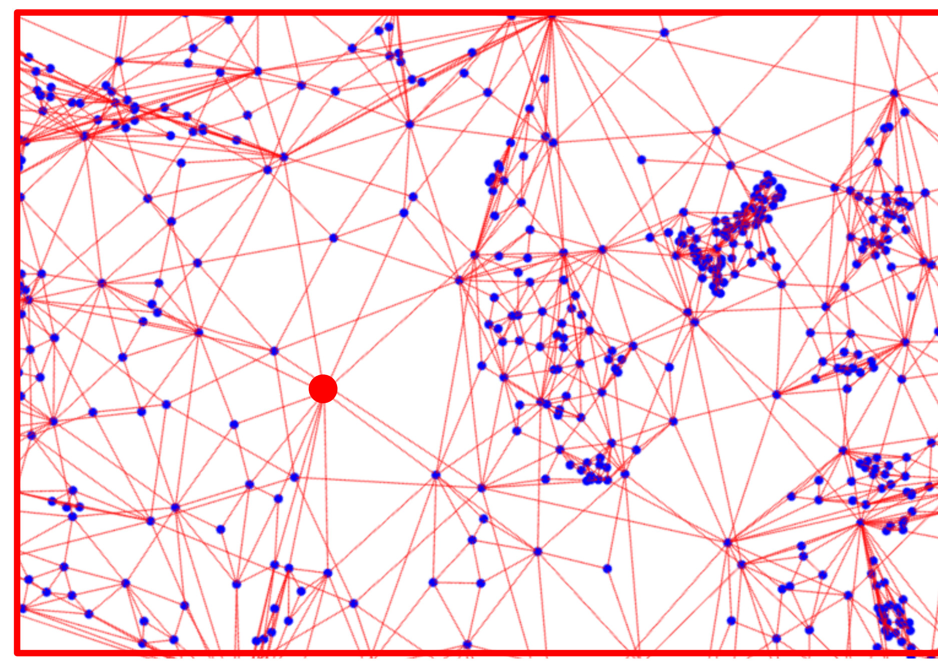
[Angel – Schramm 03, Krikun 05]:
Converges to the **Uniform Infinite Planar Triangulation**



- Local topology
- Metric balls of radius R grow like R^4
- **"Universality"** of the exponent 4.

Local Topology for planar maps

$$\mathcal{M}_f := \{\text{finite rooted planar maps}\}.$$

Definition:

The **local topology** on $\mathcal{M}_f$ is induced by the distance:

$$d_{loc}(m, m') := \left(1 + \max\{r \geq 0 : B_r(m) = B_r(m')\}\right)^{-1}$$

where $B_r(m)$ is the graph made of all the vertices and edges of m which are within distance r from the root.

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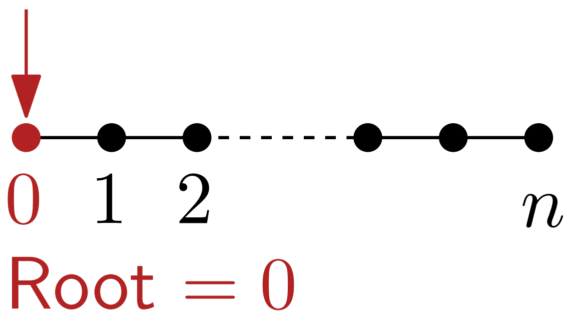
- $(\mathcal{M}, d_{loc})$: closure of $(\mathcal{M}_f, d_{loc})$. It is a **Polish** space (complete and separable).
- $\mathcal{M}_\infty := \mathcal{M} \setminus \mathcal{M}_f$ set of infinite planar maps.

Local convergence: simple examples



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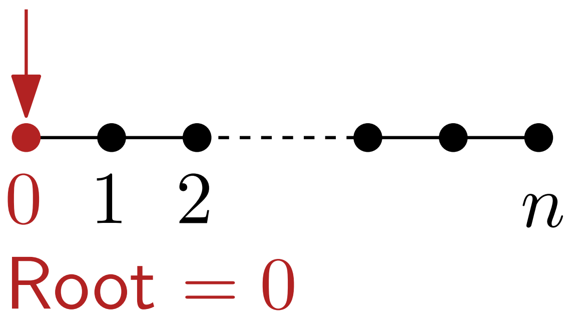

$\longrightarrow (\mathbb{Z}_+, 0)$

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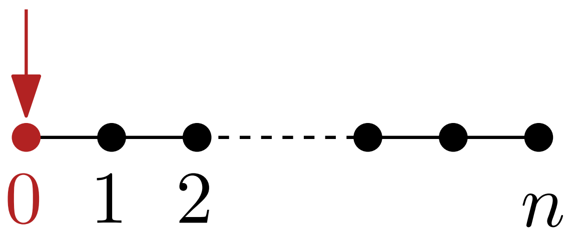

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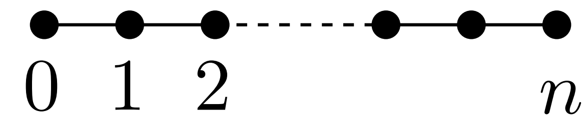
Root $= 0$



Uniformly chosen root

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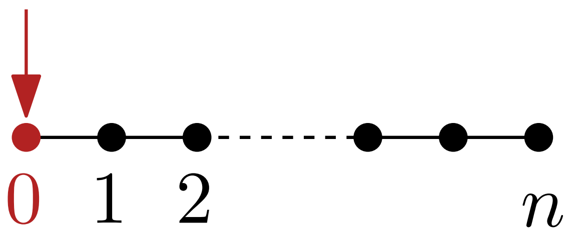

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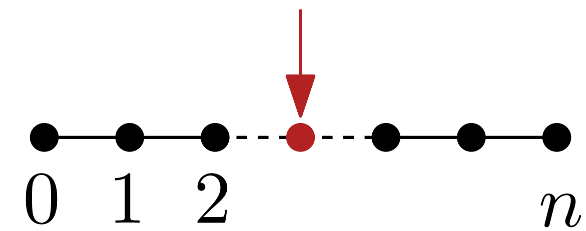

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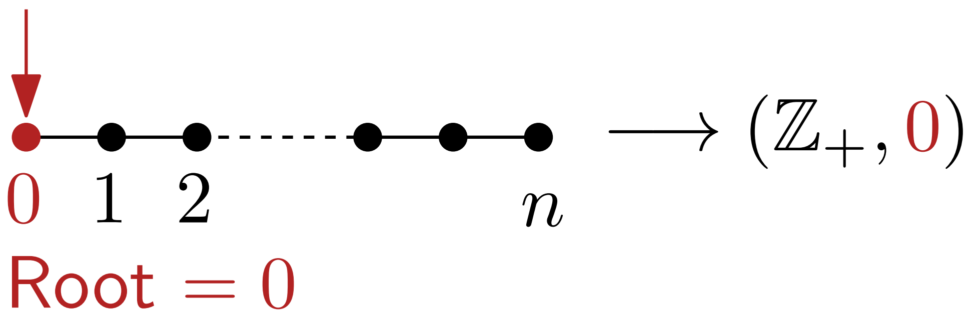

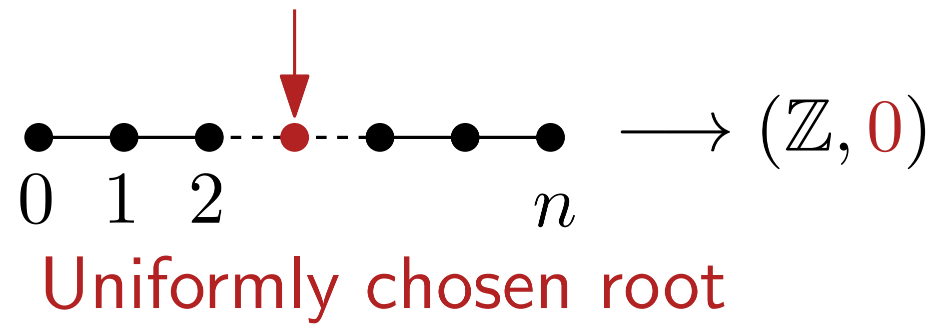

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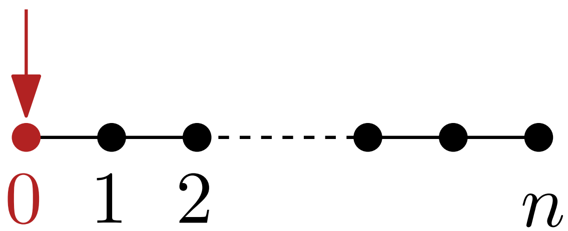

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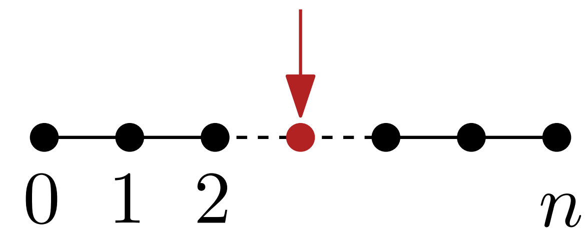

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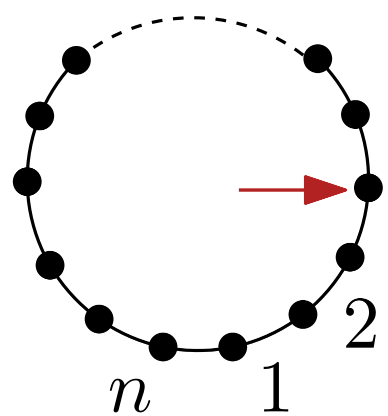

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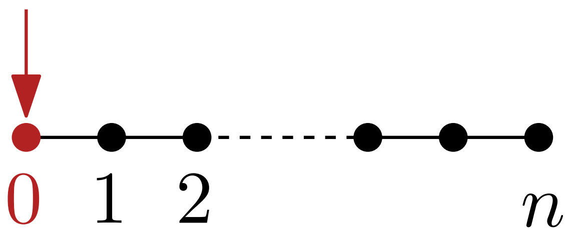

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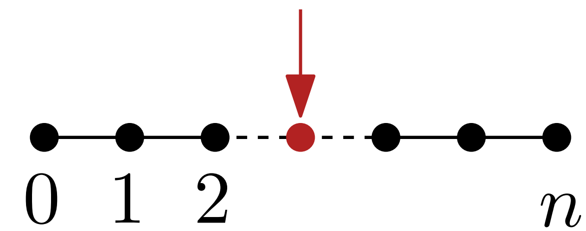

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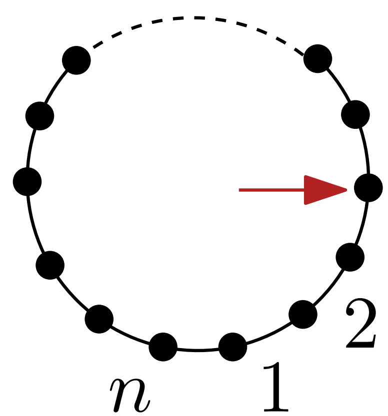

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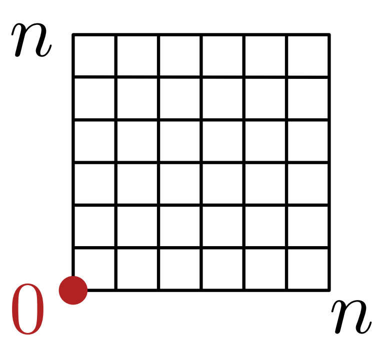

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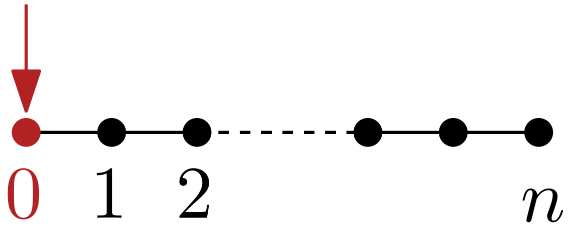

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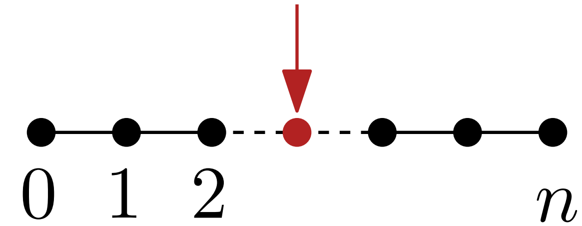

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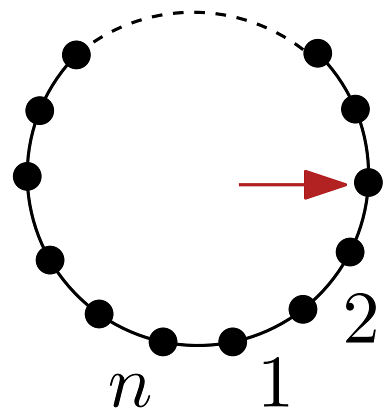

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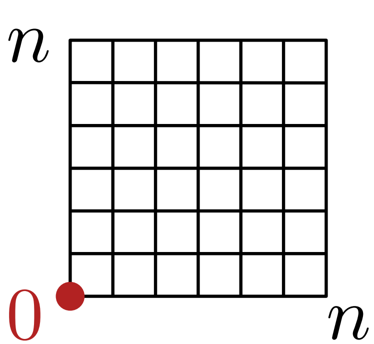

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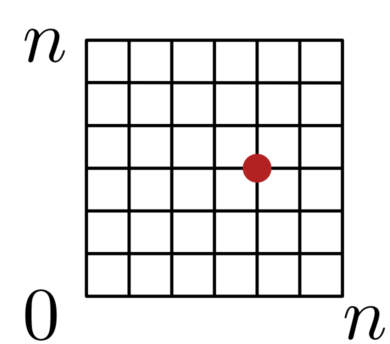

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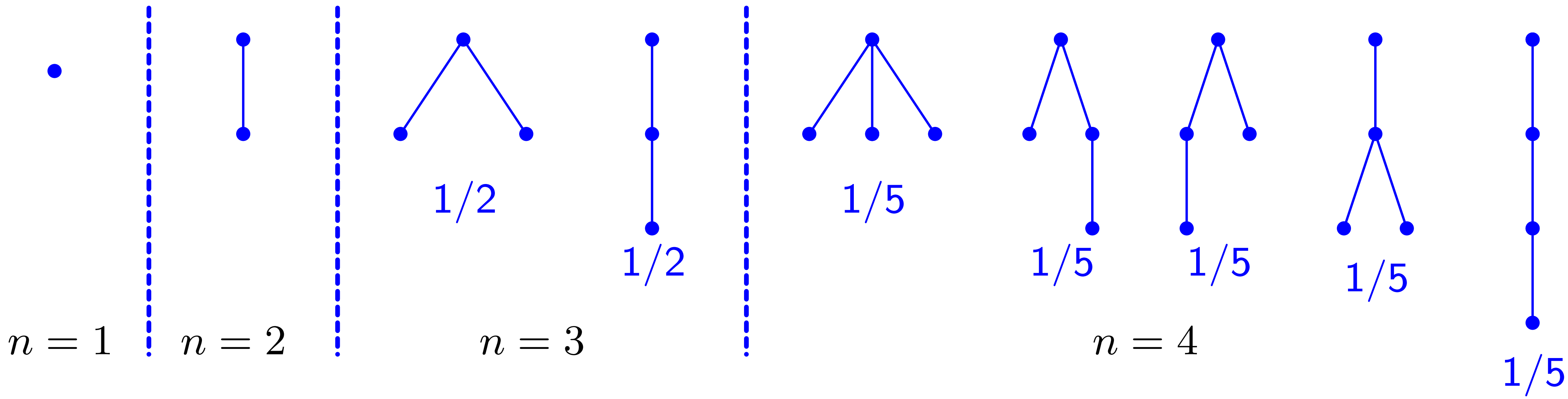
Local convergence: more complicated examples

Uniform plane rooted trees with n vertices:



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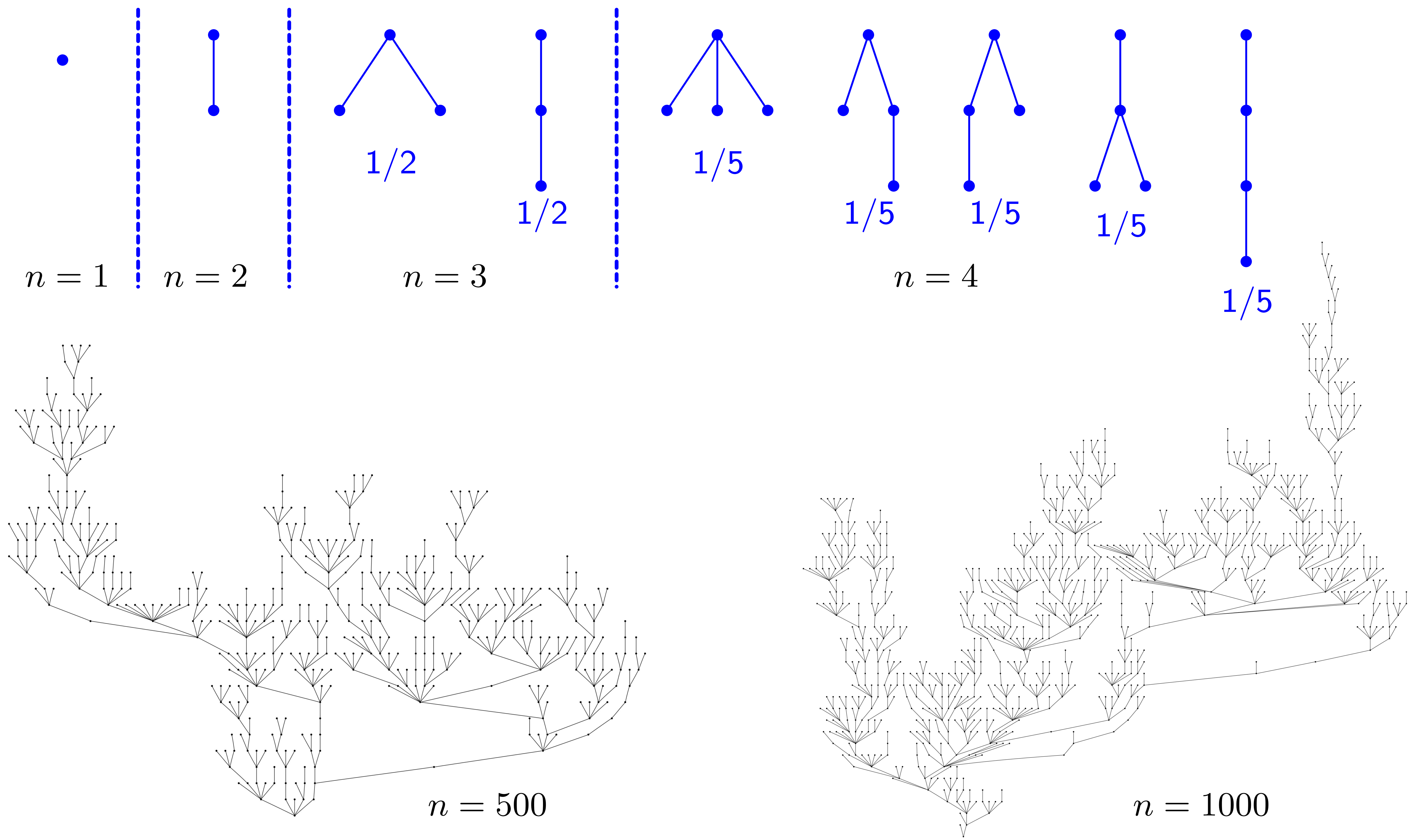

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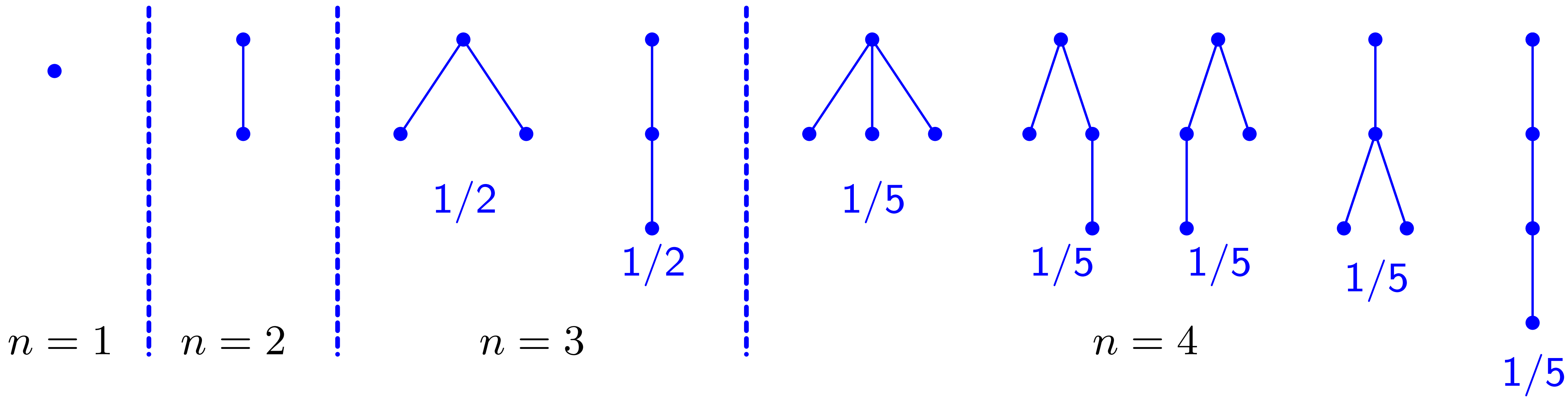
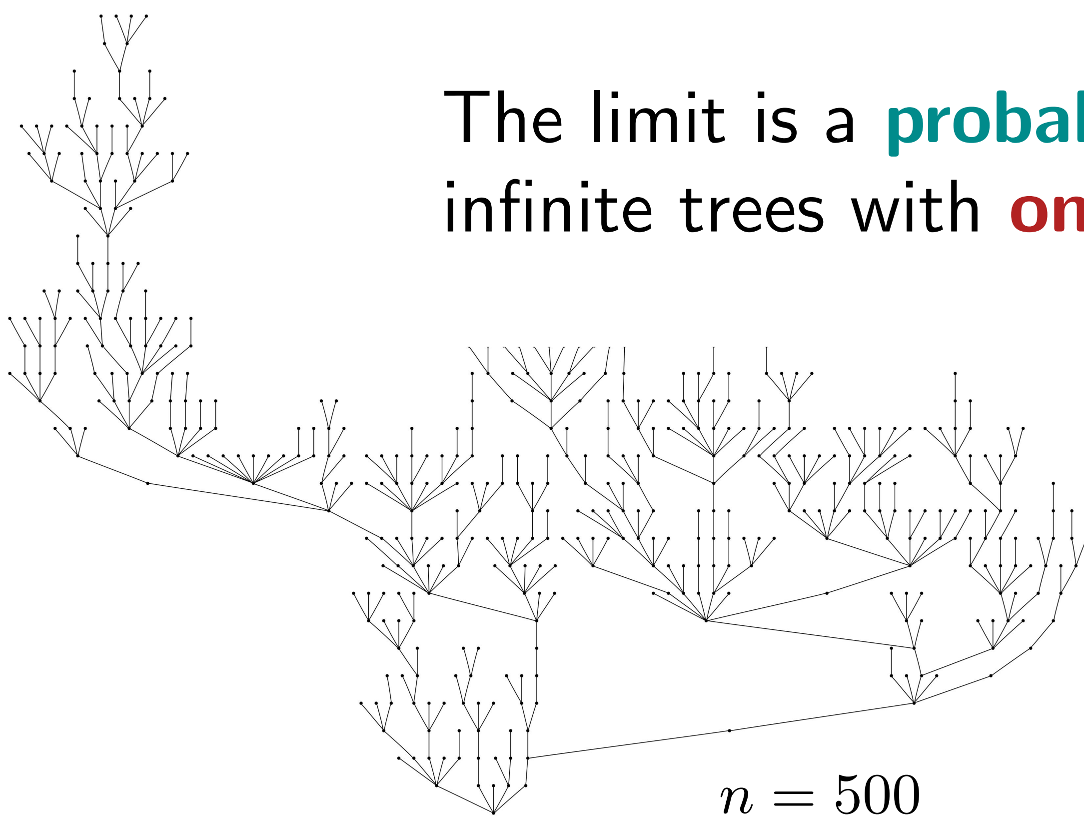
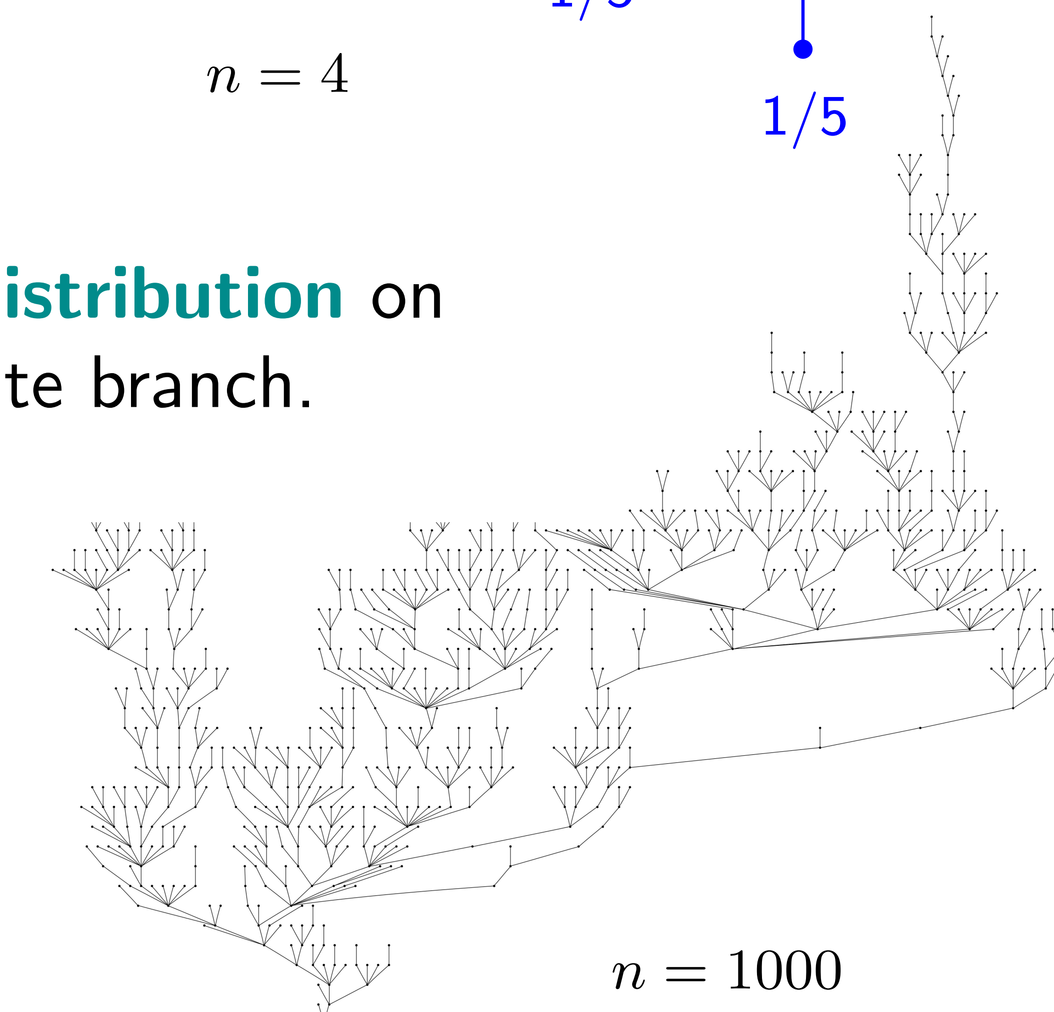


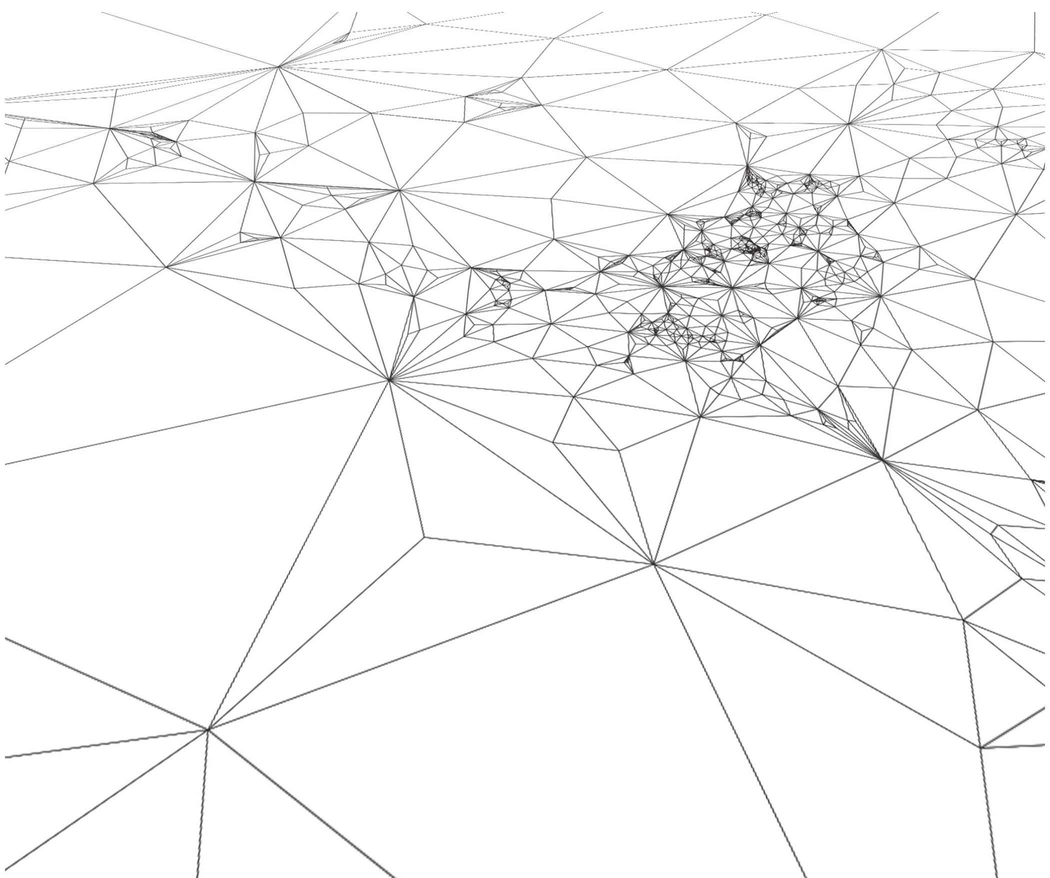
The limit is a **probability distribution** on infinite trees with **one** infinite branch.



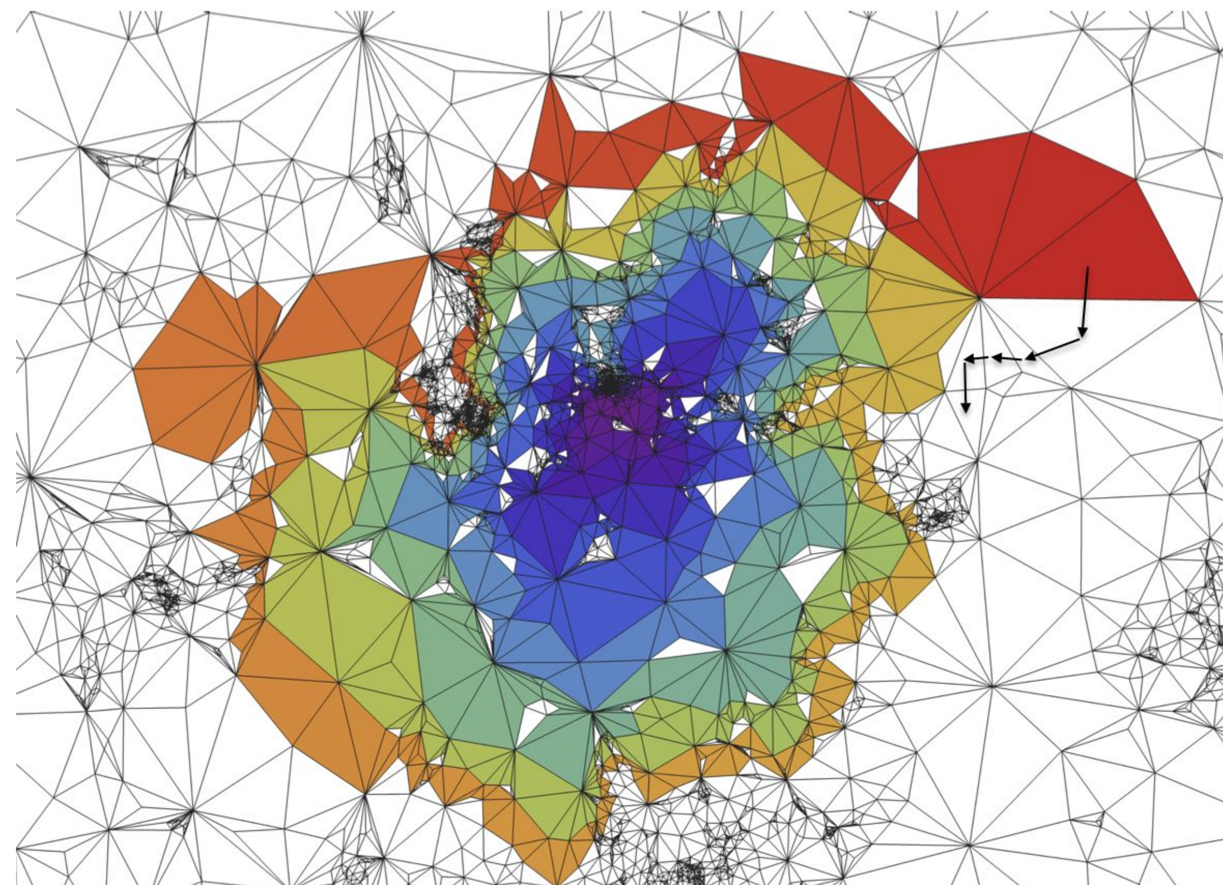


Local convergence of uniform triangulations

Theorem [Angel – Schramm, '03]
As $n \to \infty$, the uniform distribution on triangulations of size n converges weakly to a probability measure called the Uniform Infinite Planar Triangulation (or **UIPT**) for the **local topology**.



Courtesy of Igor Kortchemski



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Some properties of the UIPT:

- The UIPT has almost surely one end [Angel – Schramm, '03]
- Volume (nb. of vertices) and perimeters of balls known to some extent.

  For example $\mathbb{E}\left[|B_r(\mathbf{T}_\infty)|\right] \sim \frac{2}{7} r^4$ [Angel '04, Curien – Le Gall '12]
- Volume of hulls explicit [M. 16]
- "Uniqueness" of geodesic rays and horofunctions [Curien – M. 18]
- Bond and site percolation well understood [Angel, Angel–Curien, M.–Nolin]
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Universality: we expect the **same behavior** for slightly different models (e.g. quadrangulations, triangulations without loops, ...)

Adding matter: Ising model on triangulations

How does Ising model influence the underlying map?

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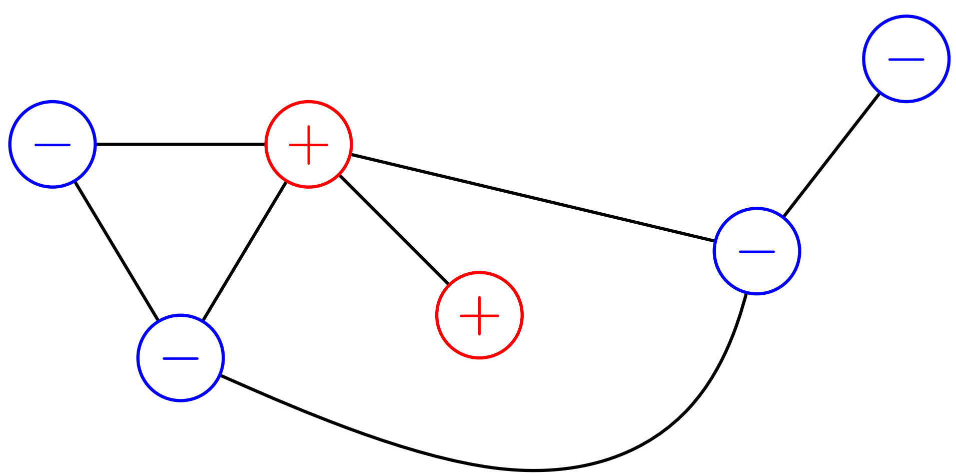
First, Ising model on a finite deterministic graph:

$G = (V, E)$ finite graph



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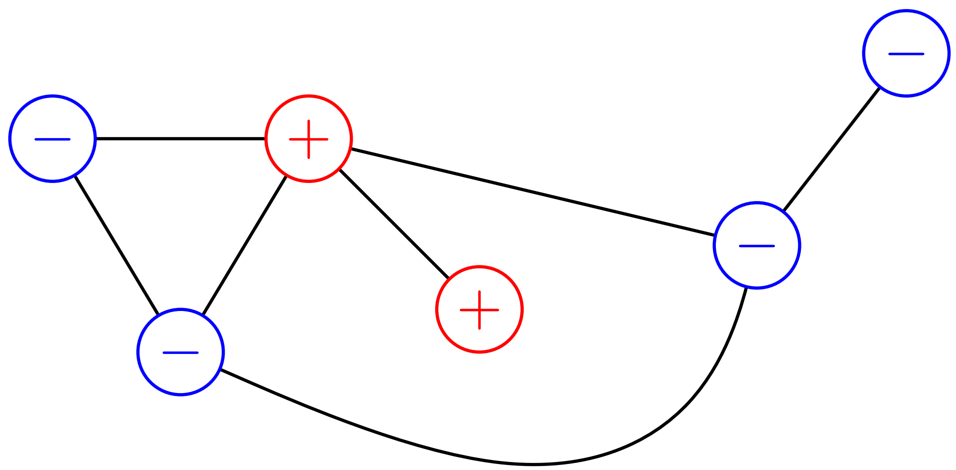

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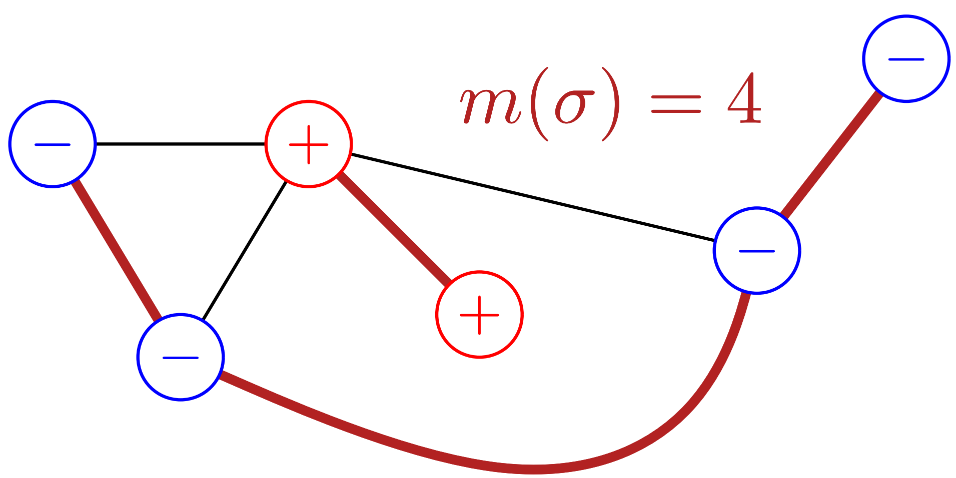

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Combinatorial formulation: $P(\sigma) \propto \nu^{m(\sigma)}$
with $m(\sigma) =$ number of monochromatic edges and $\nu = e^{\beta}$.

Adding matter: Ising model on triangulations

$\mathcal{T}_n = \{$rooted planar triangulations with $3n$ **edges**$\}$.

Random triangulation in $\mathcal{T}_n$ with probability $\propto \nu^{m(T,\sigma)}$?

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Generating series of **Ising-weighted triangulations**:

$$Q(\nu, t) = \sum_{T \in \mathcal{T}_f} \sum_{\sigma: V(T) \to \{-1,+1\}} \nu^{m(T,\sigma)} t^{e(T)}.$$

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Theorem [Bernardi – Bousquet-Mélou 11]
For every ν the series $Q(\nu,t)$ is algebraic, has $\rho_\nu > 0$ as unique dominant singularity and satisfies

$$[t^{3n}]Q(\nu,t) \underset{n\to\infty}{\sim} \begin{cases} \kappa\, \rho_{\nu_c}^{-n}\, n^{-7/3} & \text{if } \nu = \nu_c := 1 + \frac{1}{\sqrt{7}}, \\ \kappa\, \rho_{\nu}^{-n}\, n^{-5/2} & \text{if } \nu \neq \nu_c. \end{cases}$$

This suggests an unusual behavior of the underlying maps for $\nu = \nu_c$.
See also [Boulatov – Kazakov 1987], [Bousquet-Mélou – Schaeffer 03] and [Bouttier – Di Francesco – Guitter 04].

Adding matter: the model and Watabiki's predictions

Probability measure on triangulations of $\mathcal{T}_n$ with a spin configuration:

$$\mathbb{P}_n^{\nu}\Big(\{(T,\sigma)\}\Big) = \frac{\nu^{m(T,\sigma)}}{[t^{3n}]Q(\nu,t)}.$$

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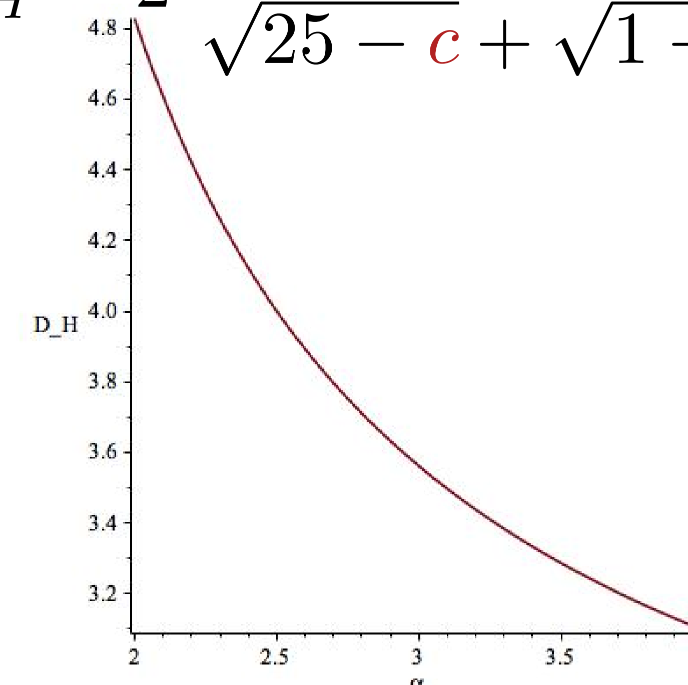
Counting exponent:
coeff $[t^n]$ of generating series of (decorated) maps $\sim \kappa\rho^{-n}n^{-\alpha}$

Central charge c:

$$\alpha = \frac{25 - c + \sqrt{(1-c)(25-c)}}{12}$$

Hausdorff dimension: [Watabiki 93]

$$D_H = 2\frac{\sqrt{25-c}+\sqrt{49-c}}{\sqrt{25-c}+\sqrt{1-c}}$$



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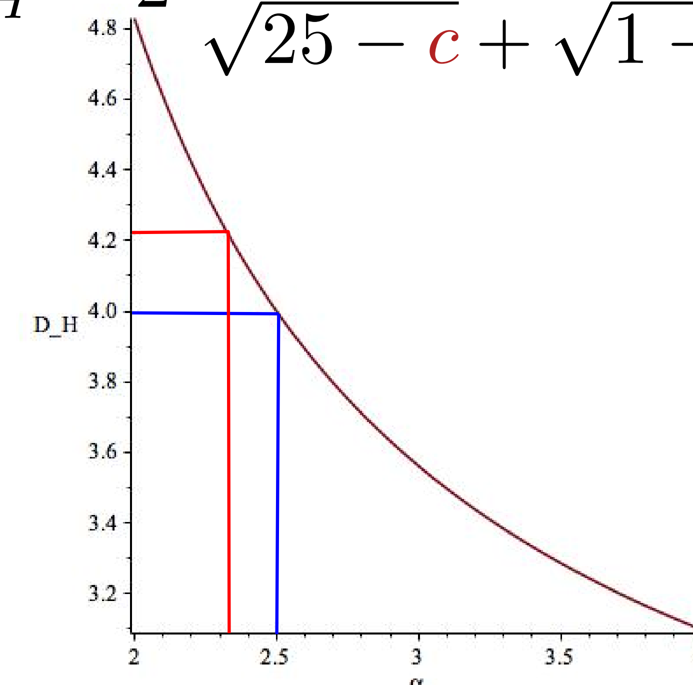
- $\alpha = 5/2$ gives $D_H = 4$
- $\alpha = 7/3$ gives $D_H = \frac{7+\sqrt{97}}{4} \approx 4.21$

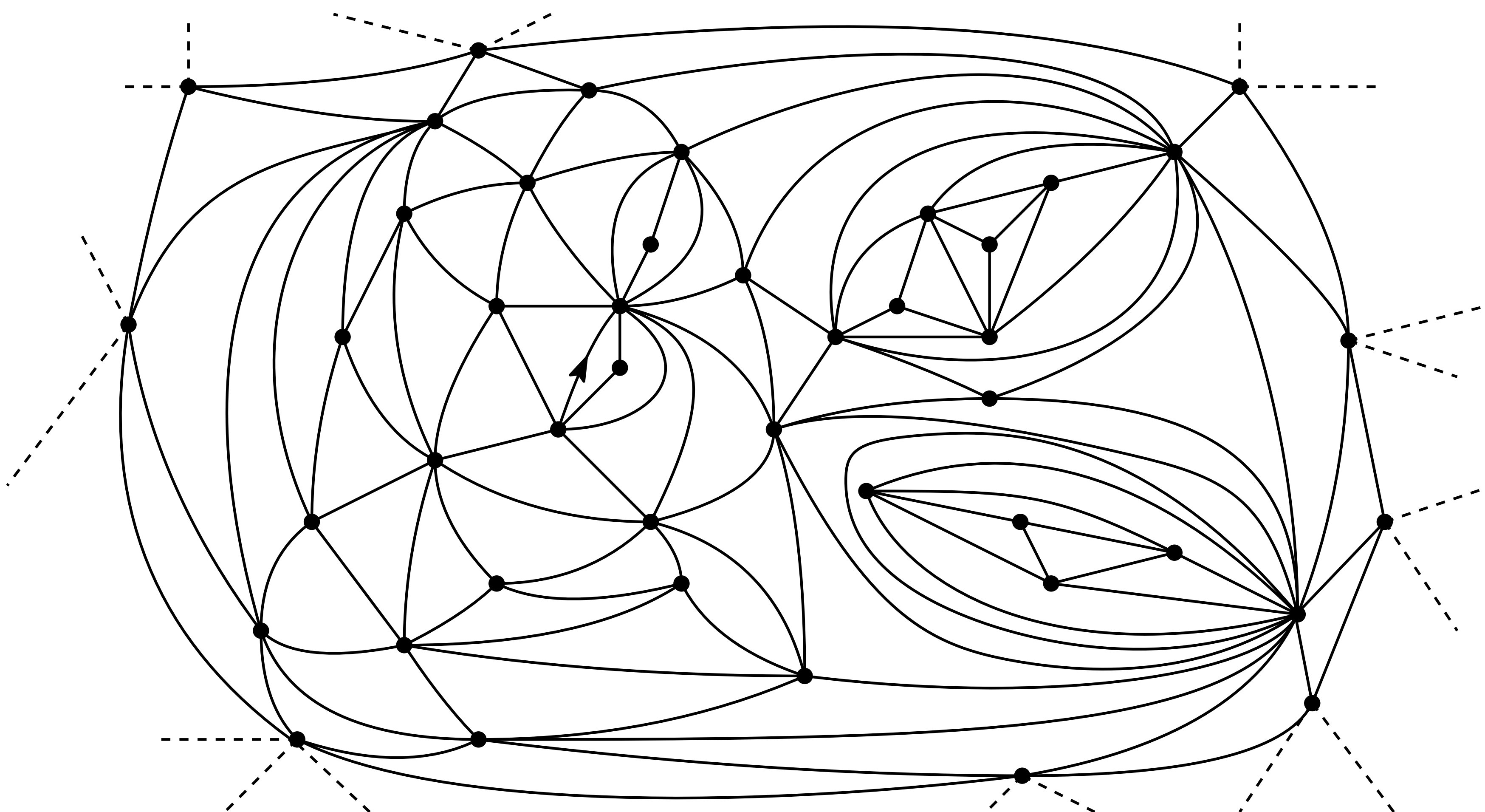


Local Topology for planar maps : balls

Definition:
The **local topology** on $\mathcal{T}_f$ is induced by the distance:

$$d_{loc}(T,T') := (1+\max\{r\geq 0 : B_r(T)=B_r(T')\})^{-1}$$

where $B_r(T)$ is the submap (with spins) of T composed by the **faces** of T with a vertex at distance $< r$ from the root.



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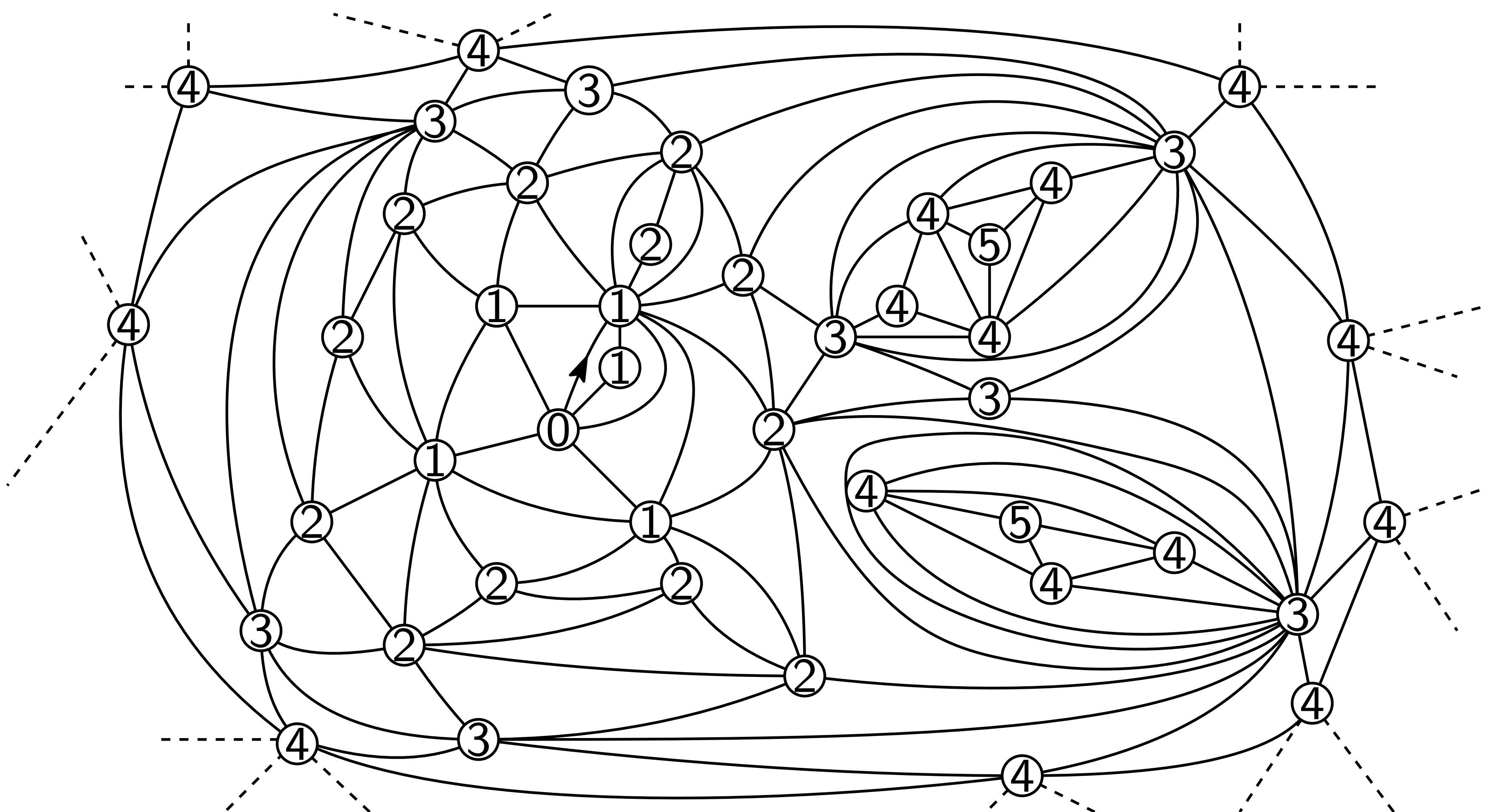

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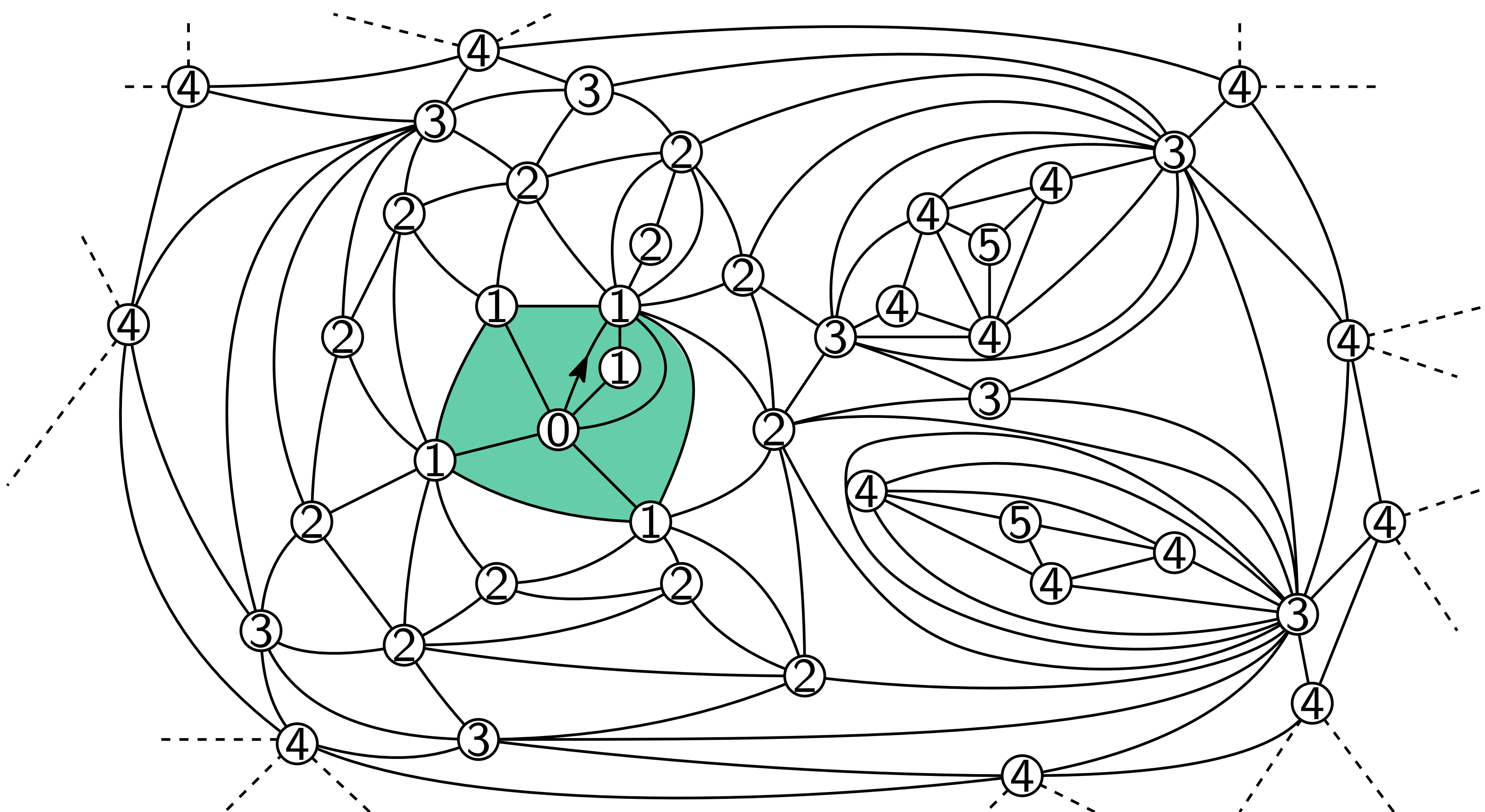

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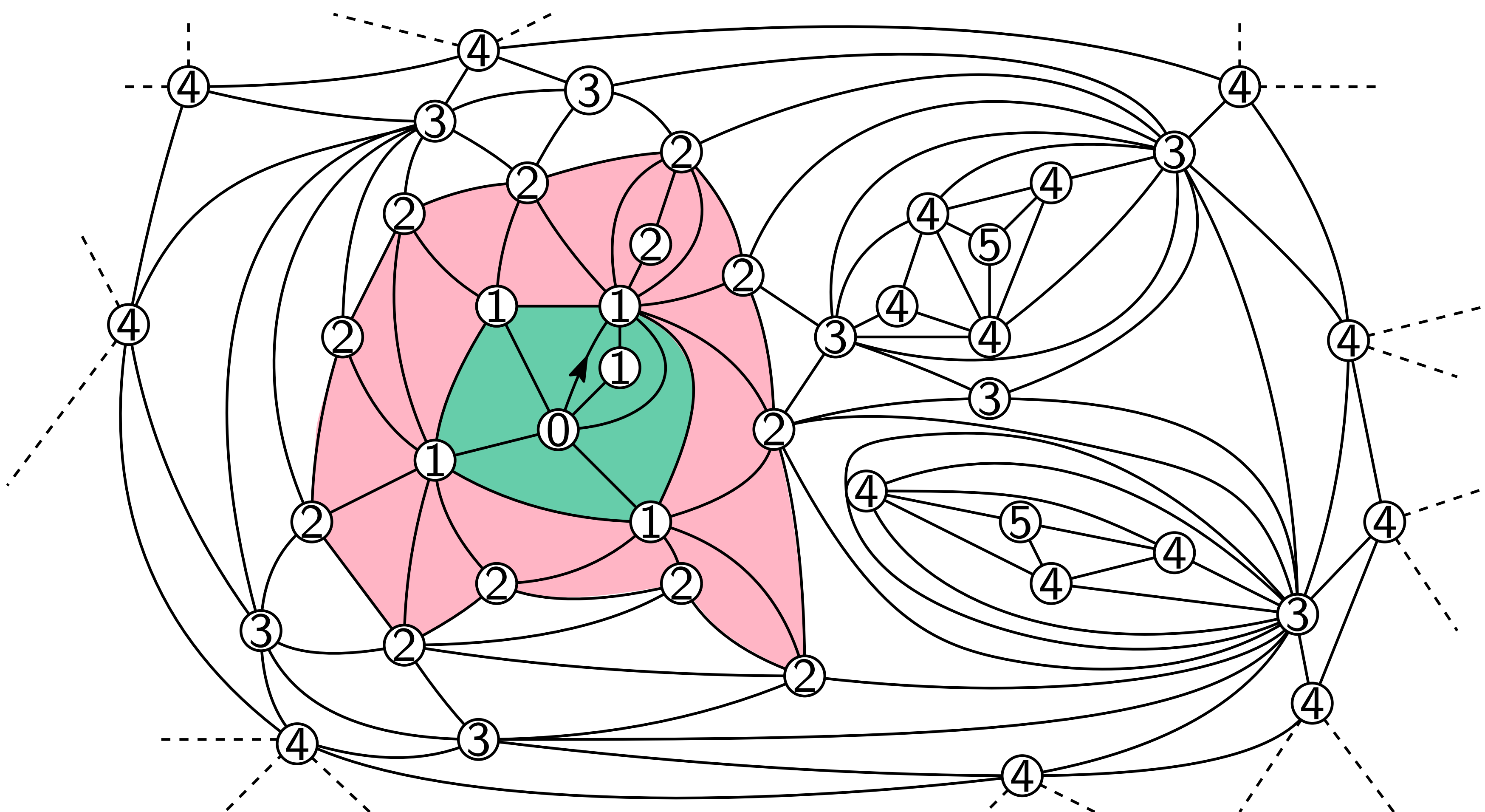

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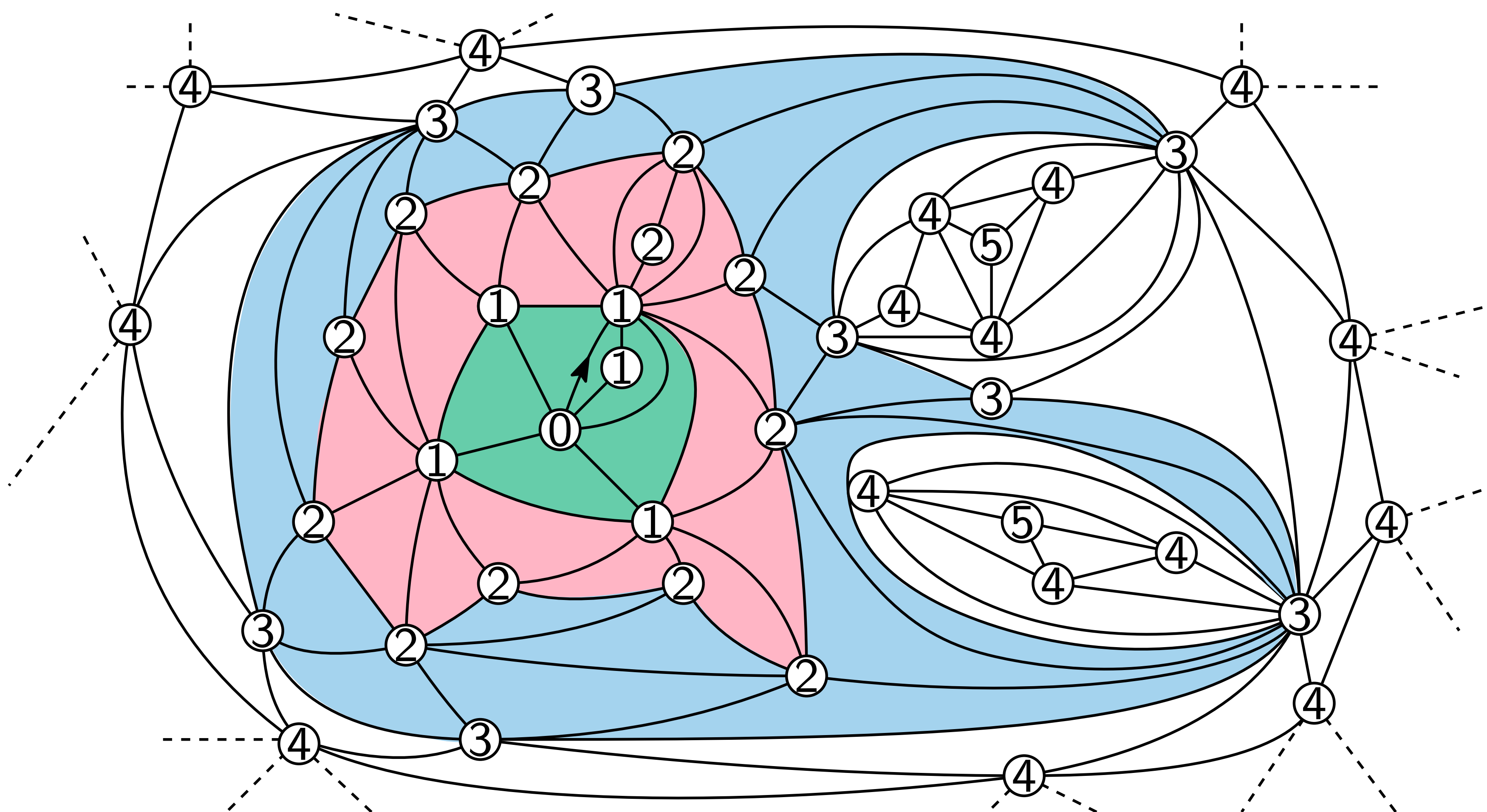

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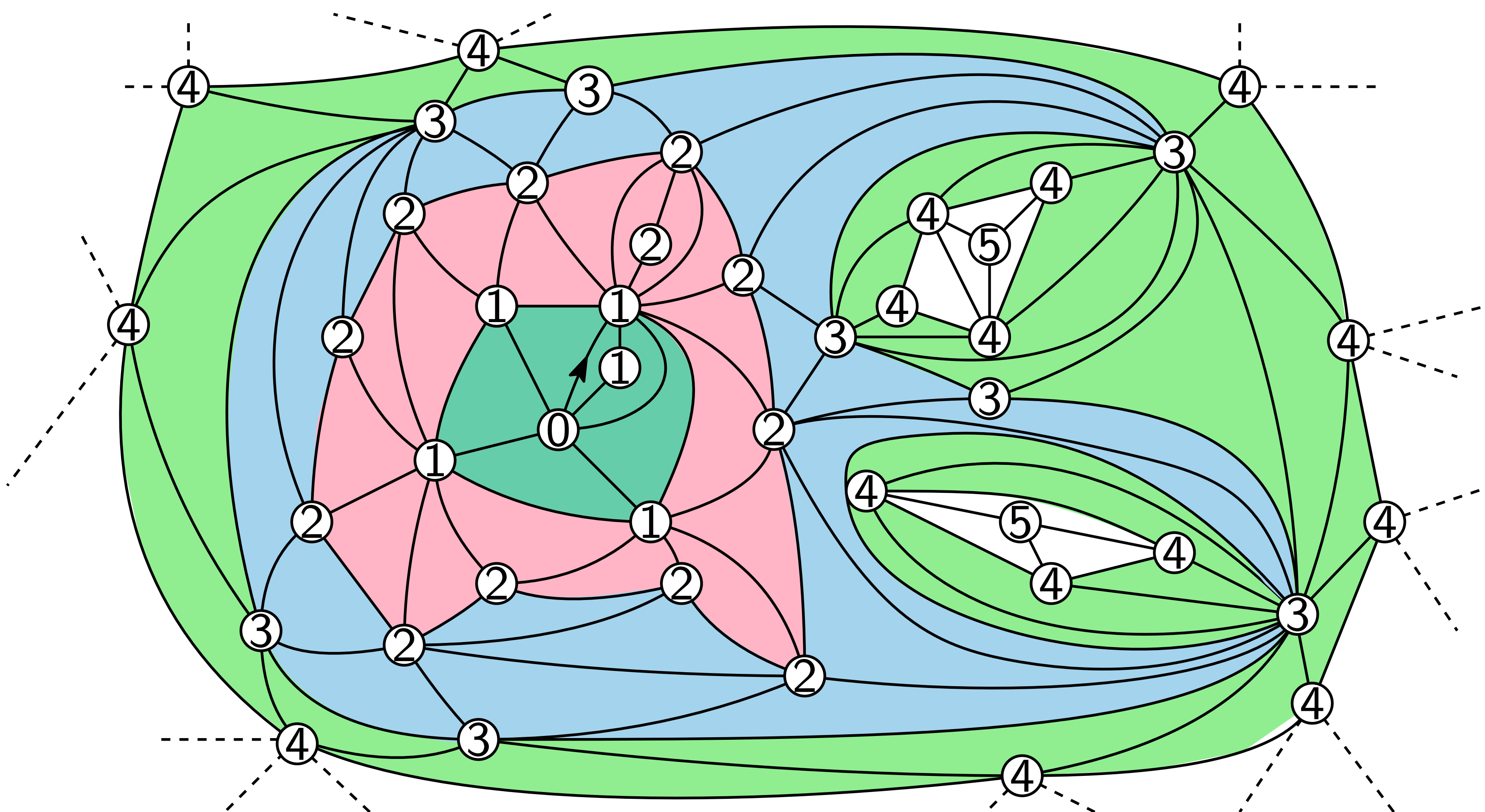

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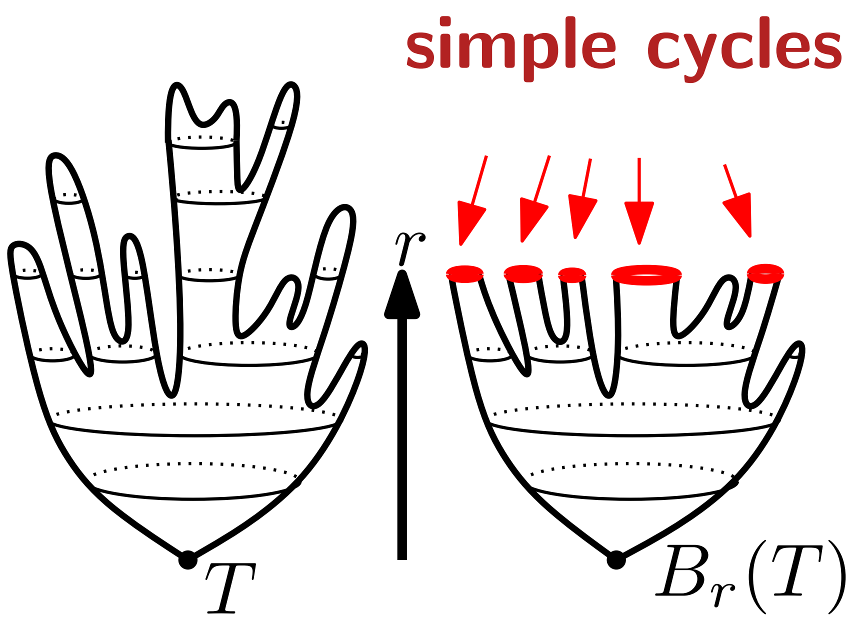

- $(\mathcal{T}, d_{loc})$: closure of $(\mathcal{T}_f, d_{loc})$.
  It is a **Polish** space.
- $\mathcal{T}_\infty := \mathcal{T} \setminus \mathcal{T}_f$ set of **infinite** planar triangulations.

Weak convergence for the local topology

Portemanteau theorem + Levy – Prokhorov metric:

A sequence of measures measures (P_n) on $\mathcal{T}_f$ converge weakly to a measure P on $\mathcal{T}_\infty$ if:

1. For every $r > 0$ and every possible r-ball Δ

$$P_n\Bigg(\{(T,v) \in \mathcal{T}_f \,:\, B_r(T,v) = \Delta\} \Bigg) \underset{n\to\infty}{\longrightarrow} P\Bigg(\{T \in \mathcal{T}_\infty \,:\, B_r(T) = \Delta\} \Bigg).$$

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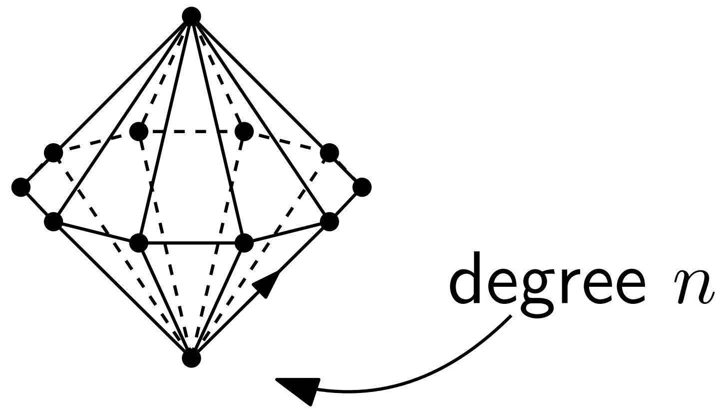
Problem: not sufficient since the space $(\mathcal{T}, d_{loc})$ is **not compact**!

Ex:



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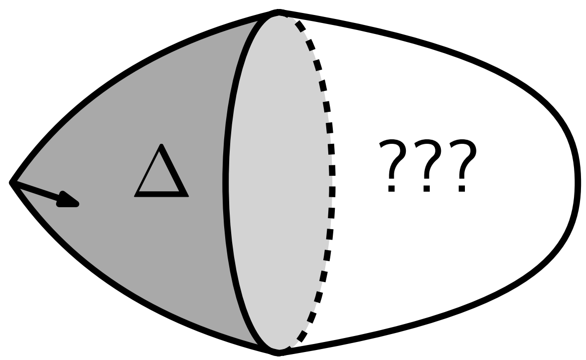
2. No loss of mass at the limit: Tightness of (P_n), **or**
the measure P defined by the limits in 1. is a probability measure.

- Vertex degrees are tight (at finite distance from the root)

- $$\forall r > 0, \qquad \sum_{r-\text{balls}\,\Delta} P\Big(\{T \in \mathcal{T}_\infty \, : \, B_r(T) = \Delta\} \Big) = 1.$$

Local convergence and generating series

Need to evaluate, for every possible ball Δ
(here, one boundary to keep it simple)

$\mathbb{P}_n\Bigg($  $\Bigg)$

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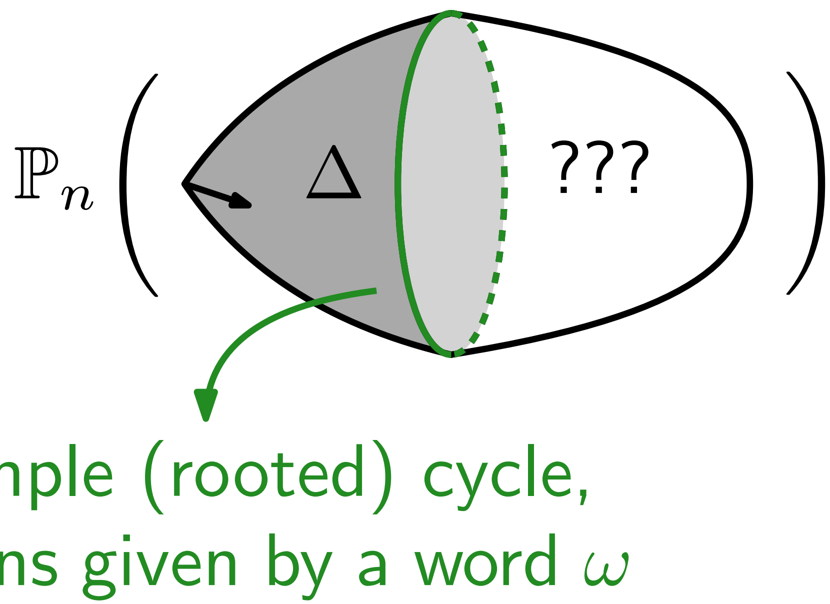

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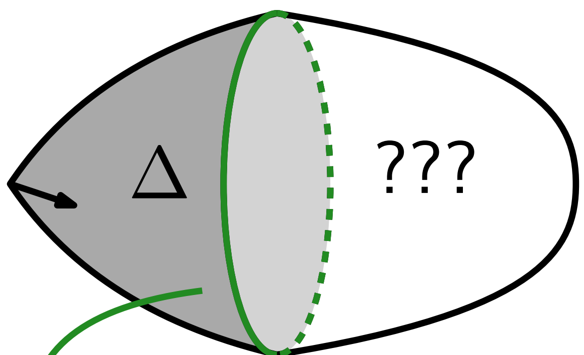
$\mathbb{P}_n\Big($  $\Big) = \dfrac{\nu^{m(\Delta)-m(\omega)}\,[t^{3n-e(\Delta)+|\omega|}]\mathbf{Z}_\omega(\nu,t)}{[t^{3n}]Q(\nu,t)}$

Simple (rooted) cycle,
spins given by a word ω

$\mathbf{Z}_\omega(\nu,t)$:= generating series of triangulations with simple boundary ω

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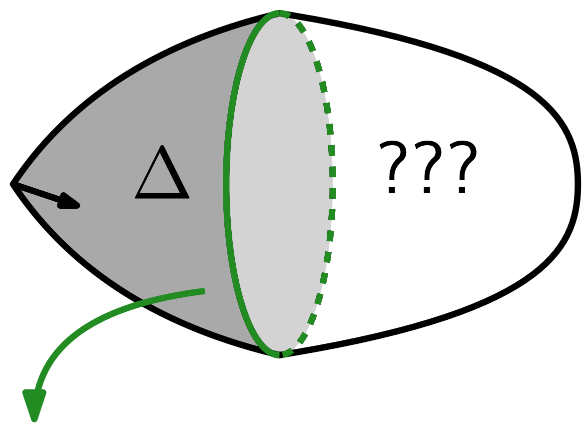

$$\mathbb{P}_n\left(\text{(diagram)} \right) = \frac{\nu^{m(\Delta)-m(\omega)}\,[t^{3n-e(\Delta)+|\omega|}]\mathbf{Z}_\omega(\nu,t)}{[t^{3n}]Q(\nu,t)}$$

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Theorem [Albenque – M. – Schaeffer 18+]
For every ω and ν, the series $t^{|\omega|}Z_\omega(\nu,t)$ is algebraic, has $\rho_\nu = t_\nu^3$ as unique dominant singularity and satisfies

$$[t^{3n}]t^{|\omega|}Z_\omega(\nu,t) \underset{n\to\infty}{\sim} \begin{cases} \kappa_\omega(\nu_c)\,\rho_{\nu_c}^{-n}\,n^{-7/3} & \text{if } \nu = \nu_c := 1+\frac{1}{\sqrt{7}}, \\ \kappa_\omega(\nu)\,\rho_\nu^{-n}\,n^{-5/2} & \text{if } \nu \neq \nu_c. \end{cases}$$

Triangulations with simple boundary

Fix a word ω, with injections from and into triangulations of the sphere:

$$[t^{3n}]t^{|\omega|}Z_\omega = \Theta\left(\rho_\nu^{-n} n^{-\alpha}\right), \text{ with } \alpha = 5/2 \text{ of } 7/3 \text{ depending on } \nu.$$

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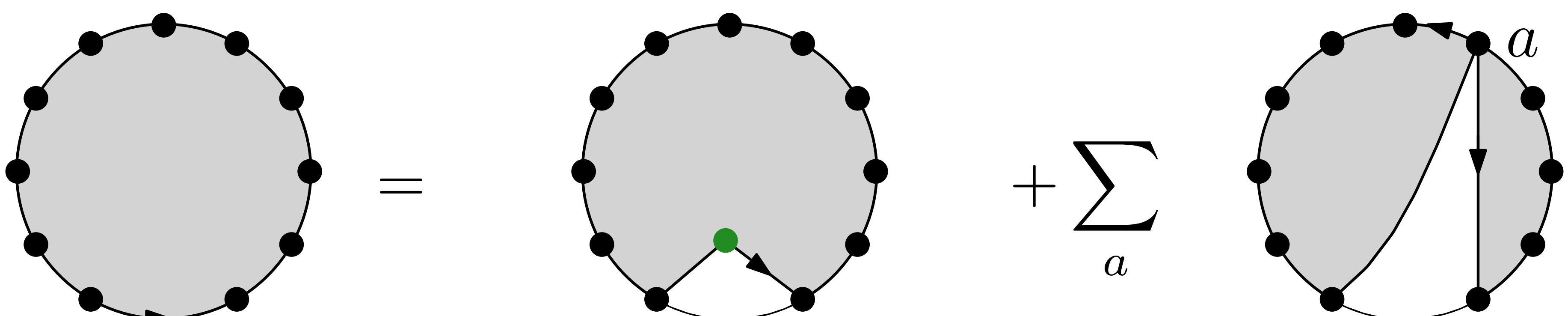
Tutte's equation (or peeling equation, or loop equation...):



$$Z_\omega = \left(Z_{\oplus\omega} + Z_{\ominus\omega} + \sum_{\omega=\omega_1 a \omega_2} Z_{a\omega_1} \cdot Z_{a\omega_2}\right) \times \nu^{\mathbf{1}_{\overleftarrow{\omega}=\overrightarrow{\omega}}} t$$

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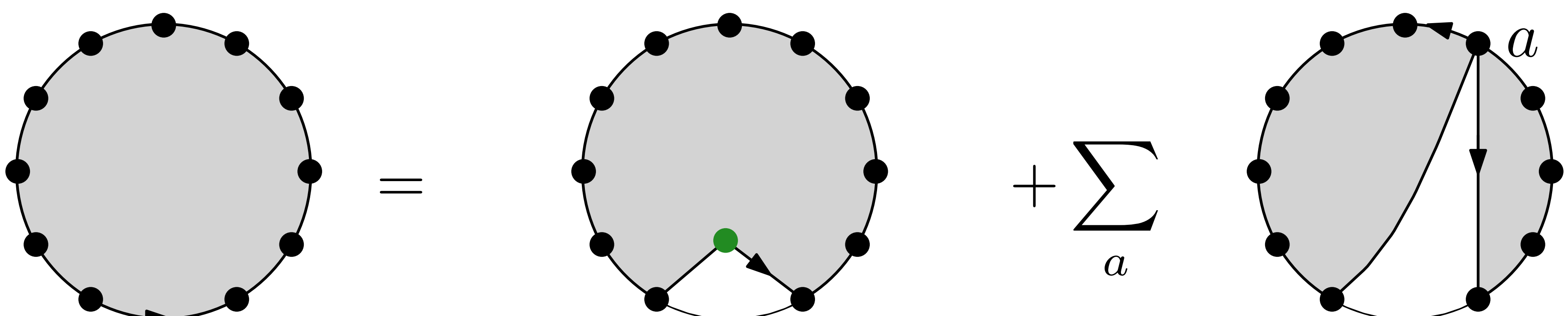

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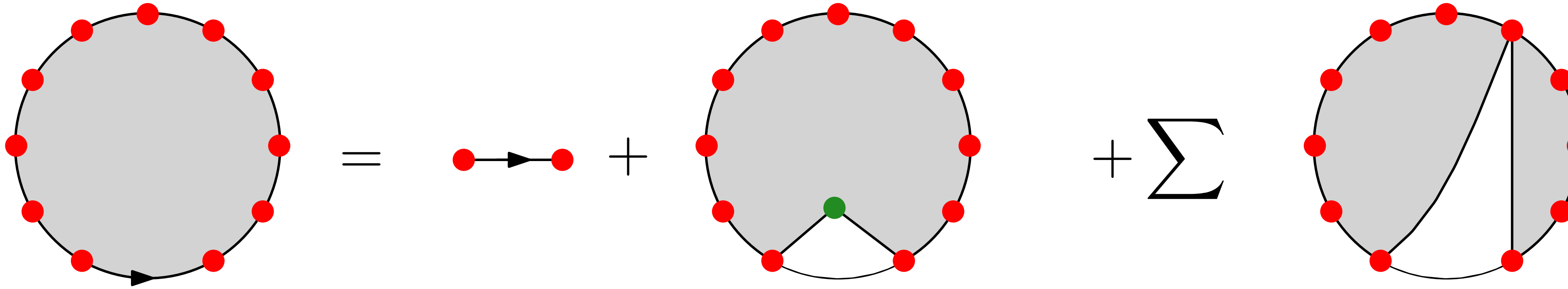
Double induction on $|\omega|$ and number of $\ominus$'s: enough to prove 1. and 2. for the $t^p Z_{\oplus^p}$'s

Positive boundary conditions: two catalytic variables



$$A(x) := \sum_{p\geq 1} Z_{\oplus^p} x^p = \nu t x^2 +$$

$$+\frac{\nu t}{x}(A(x))^2$$

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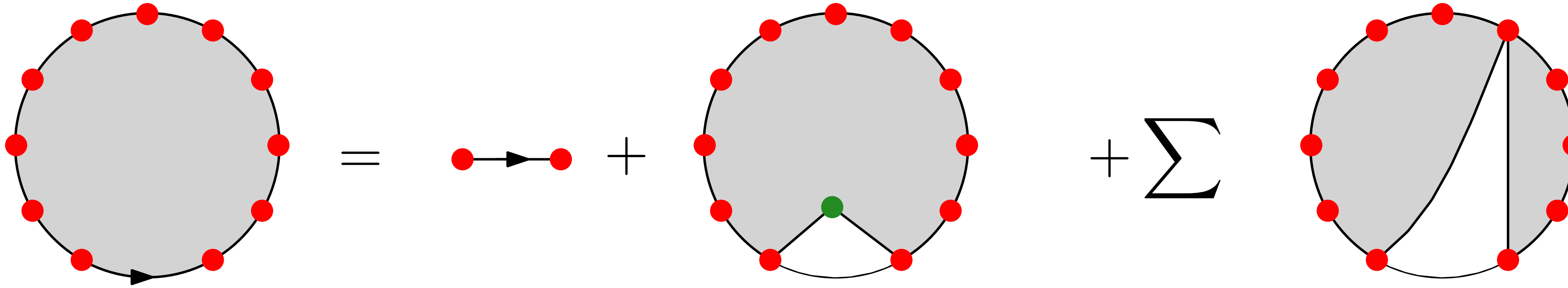

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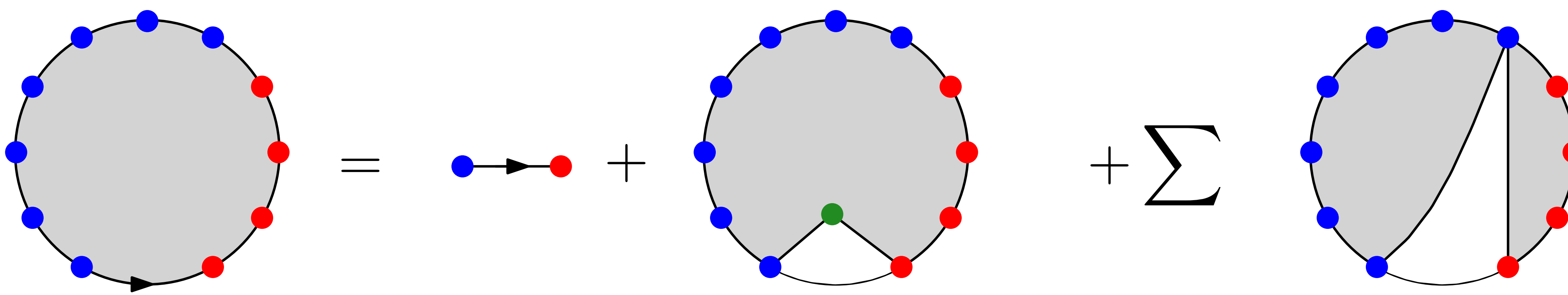
Peeling equation **at interface** $\ominus$–$\oplus$:



$$S(x,y) := \sum_{p,q\geq 1} Z_{\oplus^p \ominus^q} x^p y^q$$

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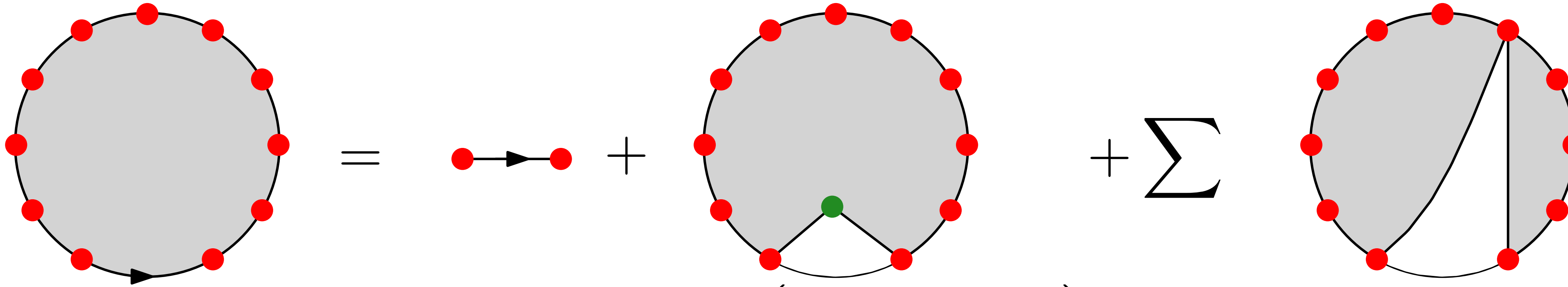

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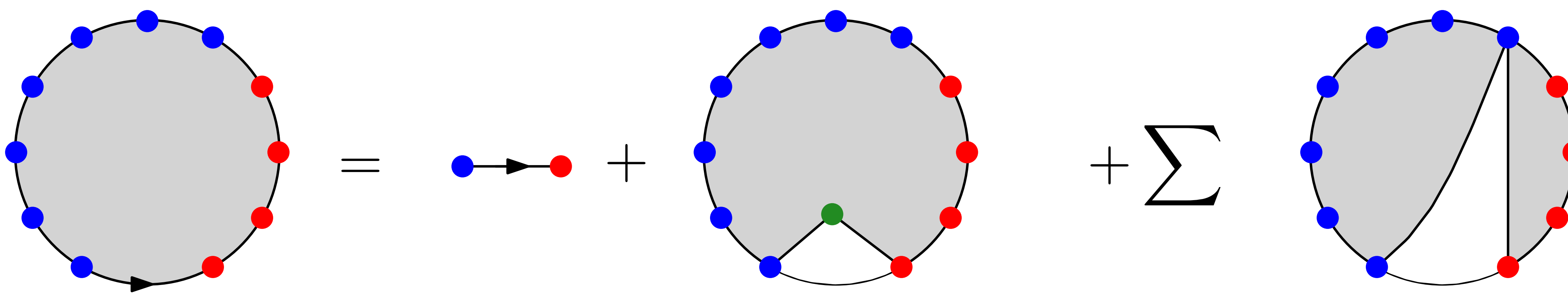

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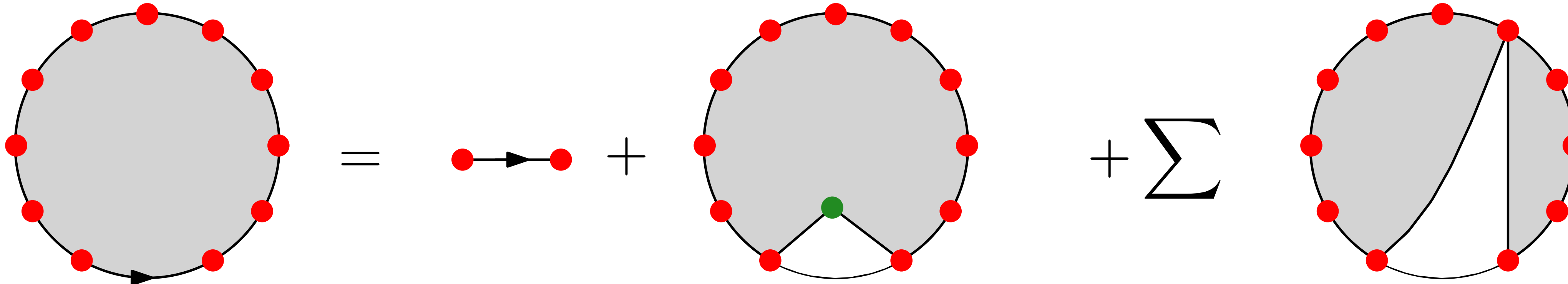

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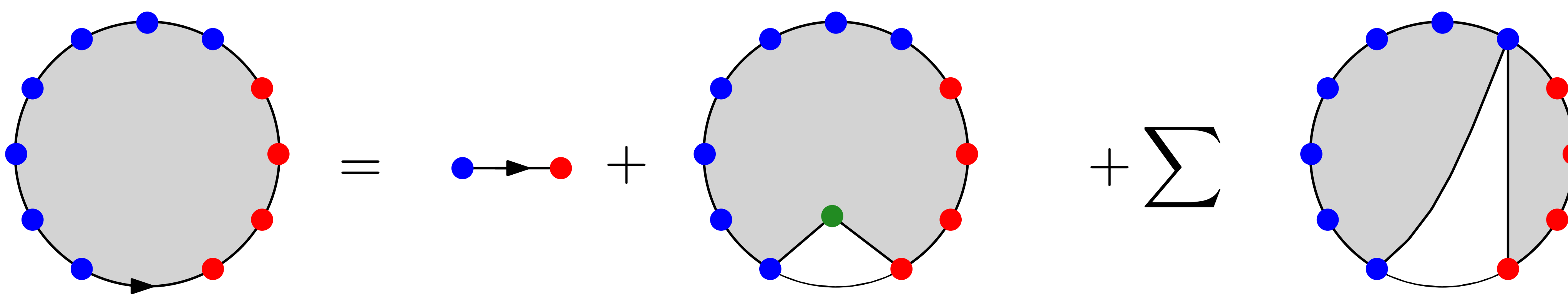

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$$= txy + \frac{t}{x}\Big(S(x,y) - x[x]S(x,y)\Big) + \frac{t}{y}\Big(S(x,y) - y[y]S(x,y)\Big)$$

$$+\frac{t}{x}S(x,y)A(x) + \frac{t}{y}S(x,y)A(y)$$

From two catalytic variables to one: Tutte's invariants

Kernel method: equation for S reads

$$K(x,y)\cdot S(x,y)=R(x,y)$$

where $$K(x,y)=1-\frac{t}{x}-\frac{t}{y}-\frac{t}{x}A(x)-\frac{t}{y}A(y).$$

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3. Prove that $J(y)=C_0(t)+C_1(t)I(y)+C_2(t)I^2(y)$ with C_i's explicit polynomials in $t,Z_{\oplus}(t)$ and $Z_{\oplus^2}(t)$.

Equation with one catalytic variable for $A(y)$ with $Z_{\oplus}$ and $Z_{\oplus^2}$!

Explicit solution for positive boundary conditions

Equation with one catalytic variable reads:

$$2t^2\nu(1-\nu)\left(\frac{A(y)}{y} - Z_{\oplus}\right) = y \cdot \mathrm{Pol}\left(\nu, \frac{A(y)}{y}, Z_{\oplus}, Z_{\oplus^2}, t, y\right)$$

[Bousquet-Mélou – Jehanne 06] gives **algebraicity** and strategy to solve this kind of equation.

Explicit solution for positive boundary conditions

Equation with one catalytic variable reads:

$$2t^2\nu(1-\nu)\left(\frac{A(y)}{y} - Z_{\oplus}\right) = y \cdot \mathrm{Pol}\left(\nu, \frac{A(y)}{y}, Z_{\oplus}, Z_{\oplus^2}, t, y\right)$$

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Maple: **rational (and Lagrangian) parametrization !**

$$t^3 = U \frac{P_1(\mu, U)}{4(1-2U)^2(1+\mu)^3}$$

$$y = V \frac{P_2(\mu, U, V)}{(1-2U)(1+\mu)^2(1-V)^2}$$

$$t^3 A(t, ty) = \frac{V P_3(\mu, U, V)}{4(1-2U)^2(1+\mu)^3(1-V)^3}$$

with $\nu = \frac{1+\mu}{1-\mu}$ and P_i's explicit polynomials.

Going back to local convergence

1. Fix $r \geq 0$ and take Δ a r-ball with boundary spins $\partial\Delta = (\omega_1, \ldots, \omega_k)$:

$$\mathbb{P}_n\left(B_r(T,v) = \Delta\right) = \frac{\nu^{m(\Delta)-m(\partial\Delta)}\left[t^{3n-e(\Delta)+|\partial\Delta|}\right]\left(\prod_{i=1}^{k} Z_{\omega_i}(\nu,t)\right)}{[t^{3n}]Q(\nu,t)}$$

$$\underset{n\to\infty}{\to} \left(\prod_{i=1}^{k} Z_{\omega_i}(\nu,t_\nu)\right) \cdot \sum_{j=1}^{k} \frac{\nu^{m(\Delta)-m(\partial\Delta)}\, t_\nu^{|\Delta|-|\omega|}\ \kappa_{\omega_j}}{\kappa\, t_\nu^{|\omega_j|} Z_{\omega_j}(\nu,t_\nu)}.$$

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   - We show that expected degree at the root under $\mathbb{P}_n$ is bounded with n

A simple tightness argument

We want to study the degree of the root vertex δ:

Mark a uniform edge conditionally on the triangulation

$$\overline{\mathbb{P}_n}\left(\delta \in e\right) = \sum_{k=1}^{3n} \overline{\mathbb{P}}\left(\delta \in e \middle| \deg(\delta) = k\right) \cdot \overline{\mathbb{P}_n}\left(\deg(\delta) = k\right)$$

$$\geq \sum_{k=1}^{3n} \frac{k}{2 \cdot 3n} \overline{\mathbb{P}_n}\left(\deg(\delta) = k\right) = \frac{1}{6n} \mathbb{E}_n\left[\deg(\delta)\right]$$

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$$\overline{\mathbb{P}_n}\left(\delta \in e\right) \leq \max\left\{\frac{1}{\nu}, 1\right\}^2 \frac{[t^{3n+2}](Z_4 + Z_2^2 + Z_1^2 + Z_1^2 Z_2 + Z_1 Z_3)}{3n\,[t^{3n}]\mathcal{Z}}$$

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$$\boxed{\mathbb{E}_n\left[\deg(\delta)\right] = \mathcal{O}(1).}$$

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Summer school **Random trees and graphs**

July 1 to 5, 2019 in Marseille France

Org. M. Albenque, J. Bettinelli, J. Rué and L.Menard

Summer school **Random walks and models of complex networks**

July 8 to 19, 2019 in Nice

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Thank you for your attention!