```
> restart;
> with(algcurves) : with(gfun) : with(plots) : with(CurveFitting) :
```

Site Percolation on the UIPT

Generating series of triangulations with a boundary

Algebraic equations for T(p,t,ty) and T_1(p,t) (Section 2)

```
> eqfunT := y + x^2 * z * T^2 + (y-1) * z * (T-y)^2 / (y * x * T) + z / (y * x) * (T-y-x * T1)
     -T;
```

$$eqfunT := y + x^2 z T^2 + \frac{(y-1) z (T-y)^2}{y x T} + \frac{z (-x T1 + T - y)}{y x} - T \quad \mathbf{(1.1.1.1)}$$

Functional equation for the series T=T(t,y,p), with T1=[y^1]T(t,y,p). t=edges, y = perimeter, p=outer edges

```
> eqfunT:=p+y^2*t*T^2+(p-1)*t*(T-p)^2/(p*y*T)+t/(p*y)*(T-p-y*T1)-T;
```

$$eqfunT := p + y^2 t T^2 + \frac{(p-1) t (T-p)^2}{p y T} + \frac{t (-y T1 + T - p)}{p y} - T \quad \mathbf{(1.1.1.2)}$$

The quadratic method above gives the following algebraic equation for T1

```
> eqT1 := 64*T1^3*t^5-27*p^3*t^5-96*T1^2*p*t^4+30*T1*p^2*t^3+p^3*t^2+
  T1^2*p*t-T1*p^2;
```

$$eqT1 := 64 T1^3 t^5 - 27 p^3 t^5 - 96 T1^2 p t^4 + 30 T1 p^2 t^3 + p^3 t^2 + T1^2 p t - T1 p^2 \quad \mathbf{(1.1.1.3)}$$

we simplify it with w=t^3 and tT1=t*T1

```
> eqtT1:=subs(t=w^(1/3),simplify(subs(T1=tT1/t,t*eqT1)));
```

$$eqtT1 := \left(-27 w^2 + w\right) p^3 + tT1 (30 w - 1) p^2 + (-96 w + 1) tT1^2 p + 64 tT1^3 w \quad \mathbf{(1.1.1.4)}$$

```
> algeqtoseries(eqtT1, w, tT1, 4);
```

$$\left[-\frac{p}{64} w^{-1} + \frac{p}{2} - 2 p w + O\left(w^3\right), p + p w - 4 p w^2 + 32 p w^3 + O\left(w^4\right), p w + 4 p w^2 + 32 p w^3 + 336 p w^4 + O\left(w^5\right)\right] \quad \mathbf{(1.1.1.5)}$$

Similarily for Tt=T(t,ty,p)

```
> eqTt := numer(factor(subs(T1 = tT1 / t, T = Tt, y = t·y, t = w^(1/3), eqfunT))); indets(%);
```

$$eqTt := w y^3 Tt^3 p - p y Tt^2 + p^2 y Tt + Tt^2 p - 2 Tt p^2 - Tt\, tT1\, y + p^3 + Tt p - p^2$$

$$\{Tt, p, tT1, w, y\} \quad \mathbf{(1.1.1.6)}$$

```
> allvalues( algeqtoseries(eqTt, y, Tt, 3, true));
```

$$\left[p - 1 - \frac{\left(p (p-1) - p^2 + tT1\right)\left(-2 p (p-1) + 2 p^2 - p - 1\right)}{p} y \quad \mathbf{(1.1.1.7)}\right.$$

$$+ \frac{1}{p}\left(-tT1 p^2 + tT1^2 p - p^2 (p-1) - (p-1) tT1 + p (p-1) - tT1^2 + p^3\right.$$

$$-p^2 + p\, tT1)\, y^2 + O(y^3)\Big],\ \Big[p + tT1\, y - \frac{-tT1\, p^2 + tT1^2\, p - tT1^2}{p}\, y^2 + O(y^3)\Big]$$

tT1 = t2T2 for p=1 (which is ok : root loop transfrom with the root loop counted twice)

> $simplify\left(subs\left(p = 1, -\frac{-p^2\, tT1 + p\, tT1^2 - tT1^2}{p}\right)\right);$

$$tT1 \quad \textbf{(1.1.1.8)}$$

> $t2T2 := factor\left(-\frac{-p^2\, tT1 + p\, tT1^2 - tT1^2}{p}\right);$

$$t2T2 := \frac{tT1\,(p^2 - p\, tT1 + tT1)}{p} \quad \textbf{(1.1.1.9)}$$

Rational parametrizations (Lemma 3, Lemma 4 amd proposition 5)

Rational partametrization for tT_1(p,t)

> $wU := \frac{U\,(U-1)\,(2\,U-1)}{2};$

$$wU := \frac{U\,(U-1)\,(2\,U-1)}{2} \quad \textbf{(1.1.2.1)}$$

> $tT1U := \frac{p\,(3\,U-1)\,U}{2\,(2\,U-1)};$

$$tT1U := \frac{p\,(3\,U-1)\,U}{4\,U-2} \quad \textbf{(1.1.2.2)}$$

> $simplify(subs(w = wU, tT1 = tT1U, eqtT1));$

$$0 \quad \textbf{(1.1.2.3)}$$

Rational parametrization for T(p,t,ty):

> $yUV := -\frac{2\,V\,(V-2+4\,U)}{8\,U^3\,p - 6\,U^2\,V - 12\,U^2\,p - 6\,U\,V^2 - V^3 + 2\,U\,V + 4\,U\,p + 2\,V^2};$

$$yUV := -\frac{2\,V\,(V-2+4\,U)}{8\,U^3\,p - 6\,U^2\,V - 12\,U^2\,p - 6\,U\,V^2 - V^3 + 2\,U\,V + 4\,U\,p + 2\,V^2} \quad \textbf{(1.1.2.4)}$$

> $TtUV := \frac{8\,U^3\,p - 6\,U^2\,V - 12\,U^2\,p - 6\,U\,V^2 - V^3 + 2\,U\,V + 4\,U\,p + 2\,V^2}{4\,(U-1)\,U\,(2\,U-1)};$

$$TtUV := \frac{8\,U^3\,p - 6\,U^2\,V - 12\,U^2\,p - 6\,U\,V^2 - V^3 + 2\,U\,V + 4\,U\,p + 2\,V^2}{4\,(U-1)\,U\,(2\,U-1)} \quad \textbf{(1.1.2.5)}$$

The equation verified by Ttp defines a unique power series in y with constant term p.

> $factor(subs(y = 0, eqTt));$

$$p\,(Tt - p + 1)\,(Tt - p) \quad \textbf{(1.1.2.6)}$$

> $factor(eqTt - subs(y = 0, eqTt));$

$$Tt\,y\,(Tt^2\,p\,w\,y^2 - Tt\,p + p^2 - tT1) \quad \textbf{(1.1.2.7)}$$

the rational parametrization verifies eqTt:

> $simplify(subs(w = wU, y = yUV, Tt = TtUV, tT1 = tT1U, eqTt));$

(1.1.2.8)

$$0 \quad \textbf{(1.1.2.8)}$$

The parametrization is good : V=0 is the only solution of y=0 and T=p, and y(V) is increasing in a neigborhood of 0:

```
> simplify(subs(V = 0, TtUV)); simplify(subs(V = 2 - 4 U, TtUV))
```

$$p$$

$$p - 1 \quad \textbf{(1.1.2.9)}$$

```
> factor(subs(V = 0, factor(diff(yUV, V))));
```

$$-\frac{1}{U\,p\,(U-1)} \quad \textbf{(1.1.2.10)}$$

Singularity of U:

```
> eqUc := factor(diff(wU, U)); solve(%);
```

$$eqUc := 3\,U^2 - 3\,U + \frac{1}{2}$$

$$\frac{1}{2} + \frac{\sqrt{3}}{6}, \frac{1}{2} - \frac{\sqrt{3}}{6} \quad \textbf{(1.1.2.11)}$$

```
> Uc := 1/2 - sqrt(3)/6; wc := simplify(subs(U = Uc, wU));
```

$$Uc := \frac{1}{2} - \frac{\sqrt{3}}{6}$$

$$wc := \frac{\sqrt{3}}{36} \quad \textbf{(1.1.2.12)}$$

```
> allvalues(algeqtoseries((wc·(1 - WW^2) - wU), WW, U, 6));
```

$$\left[\frac{1}{2} + \frac{\sqrt{3}}{3} - \frac{1}{27}\sqrt{3}\,WW^2 - \frac{4}{729}\sqrt{3}\,WW^4 + \mathrm{O}\left(WW^6\right), \frac{1}{2} - \frac{\sqrt{3}}{6} + \frac{\sqrt{2}\,WW}{6} + \frac{\sqrt{3}\,WW^2}{54} + \frac{5\sqrt{2}\,WW^3}{648} + \mathrm{O}\left(WW^{7/2}\right)\right], \left[\frac{1}{2} + \frac{\sqrt{3}}{3} - \frac{1}{27}\sqrt{3}\,WW^2 - \frac{4}{729}\sqrt{3}\,WW^4 + \mathrm{O}\left(WW^6\right), \frac{1}{2} - \frac{\sqrt{3}}{6} - \frac{\sqrt{2}\,WW}{6} + \frac{\sqrt{3}\,WW^2}{54} - \frac{5\sqrt{2}\,WW^3}{648} + \mathrm{O}\left(WW^{7/2}\right)\right] \quad \textbf{(1.1.2.13)}$$

```
> Using := 1/2 - sqrt(3)/6 - sqrt(2) WW/6 + sqrt(3) WW^2/54 - 5 sqrt(2) WW^3/648;
```

$$Using := \frac{1}{2} - \frac{\sqrt{3}}{6} - \frac{\sqrt{2}\,WW}{6} + \frac{\sqrt{3}\,WW^2}{54} - \frac{5\sqrt{2}\,WW^3}{648} \quad \textbf{(1.1.2.14)}$$

Transfer to tT1 (not needed for the moment):

```
> collect(expand(convert(simplify(series(subs(U = Using, tT1U), WW, 4)), polynom)), WW,
    factor);
```

(1.1.2.15)

$$\frac{p\sqrt{2}\,WW^3}{9} - \frac{p\sqrt{3}\,WW^2}{12} - \frac{(\sqrt{3}-2)\,p}{4} \quad \textbf{(1.1.2.15)}$$

expansion of the partition function Z=t2T2/(p^2*t^3):

$$> Ztser := collect\left(expand\left(convert\left(simplify\left(series\left(subs\left(tT1 = tT1U, U = Using, \frac{t2T2}{p^2 \cdot wU}\right), WW, 4\right)\right), polynom\right)\right), WW, factor\right);$$

$$Ztser := \frac{2\sqrt{2}\,(3p - 3 + 2\sqrt{3})\,WW^3}{3p} - \frac{\sqrt{3}\,(3p - 27 + 16\sqrt{3})\,WW^2}{4p} - \frac{3\sqrt{3}\,(-p - 7 + 4\sqrt{3})}{4p} \quad \textbf{(1.1.2.16)}$$

Critical points and poles of y(V) for t arbitrary (Lemma 4)

We can also chek that y(V) has only one pole for V>0, which will be useful later.

1) The Polynom of degree 3 giving the poles goes to -infinity when V goes to infinity and its value at V=0 is positive (U is between 0 and Uc):

$$> collect(8U^3p - 6U^2V - 12U^2p - 6UV^2 - V^3 + 2UV + 4Up + 2V^2, V, factor);$$

$$-V^3 + (-6U + 2)V^2 - 2U(3U - 1)V + 4Up(2U - 1)(U - 1) \quad \textbf{(1.1.3.1)}$$

2) The polynom is increasing at V=0:

$$> factor(subs(V = 0, diff(8U^3p - 6U^2V - 12U^2p - 6UV^2 - V^3 + 2UV + 4Up + 2V^2, V)));$$

$$-2U(3U - 1) \quad \textbf{(1.1.3.2)}$$

3) This leaves only two possibilities: a) only one pole at some V>0, b) One positive pole and two negative poles. The positive pole is between 1-2U and 2*(1-2U):

The value of the polynom at 2(1-2U) is negative:

$$> factor(subs(V = 2\cdot(1 - 2U), \textbf{(1.1.3.1)}));$$

$$4U(2U - 1)(U - 1)(p - 1) \quad \textbf{(1.1.3.3)}$$

The value of the polynom at (1-2U) is positive:

$$> factor(subs(V = 1 - 2U, \textbf{(1.1.3.1)}));$$

$$(2U - 1)(4U^2p - 2U^2 - 4Up + 4U - 1) \quad \textbf{(1.1.3.4)}$$

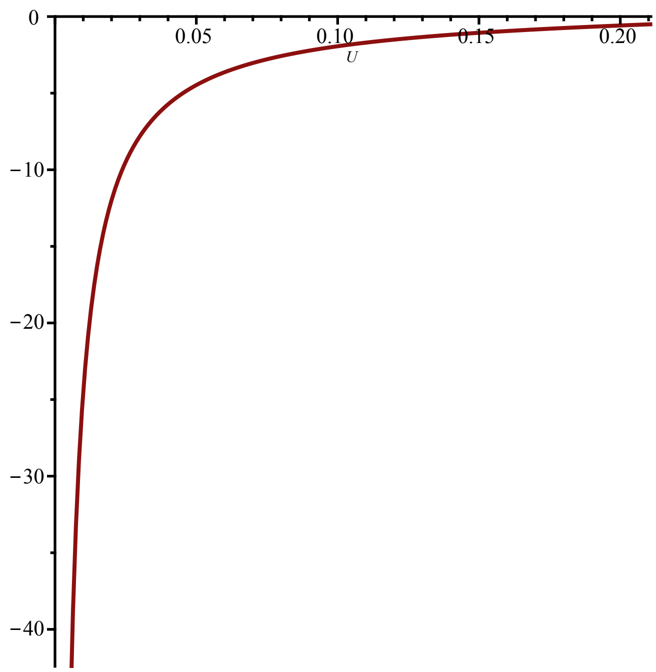
The second factor is negative for p in (0,1) and U in (0,Uc]:

$$> factor(solve(\%, p)); evalf(solve(\%)); evalf(Uc); plot\left(\frac{2U^2 - 4U + 1}{4U(2U - 1)}, U = 0..Uc\right);$$

$$\frac{2U^2 - 4U + 1}{4U(U - 1)}$$

1.707106781, 0.2928932190

0.2113248653



Now we look for the critical points in V of yUV for fixed U in (0,Uc]:

> $eqVcU := collect(numer(factor((diff(yUV, V)))), V, factor);$

$$eqVcU := -2\,V^4 + (-16\,U + 8)\,V^3 - 4\,(3\,U - 1)\,(3\,U - 2)\,V^2 - 16\,U\,p\,(2\,U - 1)\,(U - 1)\,V - 16\,U\,p\,(U - 1)\,(2\,U - 1)^2 \quad \textbf{(1.1.3.5)}$$

There is a double root only when U=Uc:

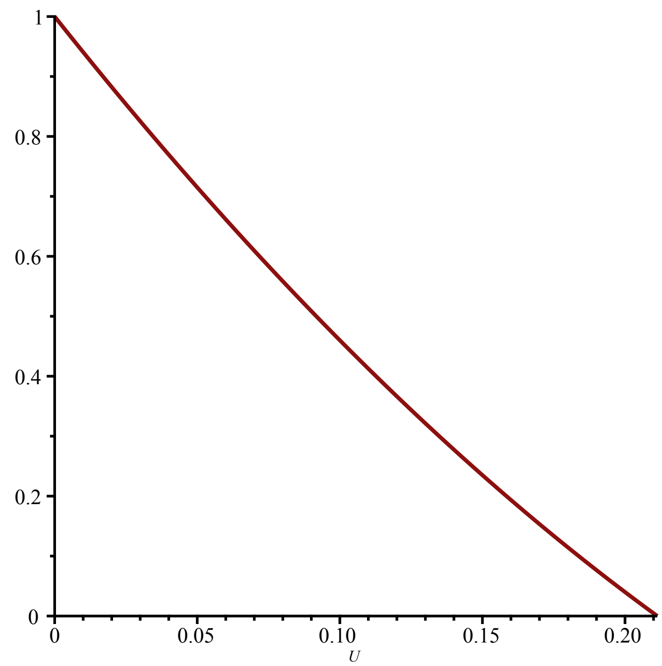
> $factor(discrim(eqVcU, V));$

$$-131072\,U^2\,p\,(p - 1)\,(2\,U - 1)^2\,(U - 1)^2\,\left(216\,U^6\,p^2 - 216\,U^6\,p - 648\,U^5\,p^2 + 729\,U^6 + 648\,U^5\,p + 702\,U^4\,p^2 - 2187\,U^5 - 702\,U^4\,p - 324\,U^3\,p^2 + 2673\,U^4 + 324\,U^3\,p + 54\,U^2\,p^2 - 1701\,U^3 - 54\,U^2\,p + 594\,U^2 - 108\,U + 8\right) \quad \textbf{(1.1.3.6)}$$

> $factor\big(discrim\big(216\,U^6\,p^2 - 216\,U^6\,p - 648\,U^5\,p^2 + 729\,U^6 + 648\,U^5\,p + 702\,U^4\,p^2 - 2187\,U^5 - 702\,U^4\,p - 324\,U^3\,p^2 + 2673\,U^4 + 324\,U^3\,p + 54\,U^2\,p^2 - 1701\,U^3 - 54\,U^2\,p + 594\,U^2 - 108\,U + 8, p\big)\big);$

$$-108\,U^2\,(2\,U - 1)^2\,(U - 1)^2\,\left(6\,U^2 - 6\,U + 1\right)\left(15\,U^2 - 15\,U + 4\right)^2 \quad \textbf{(1.1.3.7)}$$

> $plot\big(\left(6\,U^2 - 6\,U + 1\right), U = 0..Uc\big);$



Recall that this polynomial is positive at V=0: there are 2 or 4 real critical points, we will see that it is 4.

The value at 2(1-2U) is always positive:

```
> factor(subs(V = 2(1 - 2 U), eqVcU));
```

$$16\, U\, (U-1)\, (2\, U-1)^2\, (p-1) \qquad \textbf{(1.1.3.8)}$$

The value at 1-2U is negative if U <Uc, 0 if U=Uc:

```
> factor(subs(V = 1 - 2 U, eqVcU));
```

$$-2\, \left(6\, U^2-6\, U+1\right)\, (2\, U-1)^2 \qquad \textbf{(1.1.3.9)}$$

Conclusion: We have 4 critical points v1 < 0 < v2 < 1-2U < v3 < 2(1-2U) < v4, one pole between 1-2U and 2(1-2U) and possibly two negative poles. Note that this also ensures that the positive pole is between v3 and 2(1-2U).

```
> factor(subs(V = 1 - 2 U, yUV)); factor(subs(V = 2·(1 - 2 U), yUV));
```

$$\frac{2\, (2\, U-1)}{4\, U^2\, p-2\, U^2-4\, U\, p+4\, U-1}$$

$$0 \qquad \textbf{(1.1.3.10)}$$

yUV is increasing in a neighborhood of 0:

```
> factor(subs(V = 0, diff(yUV, V)));
```

$$-\frac{1}{U\, p\, (U-1)} \qquad \textbf{(1.1.3.11)}$$

For t=tc and y+(p,tc) >1:

```
> factor(subs(U = Uc, V = 1 - 2·Uc, yUV) - 1);
```

$$\frac{-2\, p+1+\sqrt{3}}{2\, p-1+\sqrt{3}} \qquad \textbf{(1.1.3.12)}$$

Critical points and poles of y(V) at t_c (proof of Theorem 1)

A more detailed look for U=Uc:

> yUcV := factor(subs(U = Uc, yUV));

$$yUcV := \frac{6\,V\left(-3\,V+2\sqrt{3}\right)}{9\,V^2\sqrt{3}-9\,V^3+6\,V\sqrt{3}+2\,p\sqrt{3}-9\,V^2-9\,V} \quad \textbf{(1.2.1)}$$

We first want to compute the 4 critical points:

> eqVcUc := factor(subs(U = Uc, eqVcU));

$$eqVcUc := -\frac{2\left(9\,V^2\sqrt{3}-9\,V^3-4\,p\sqrt{3}\right)\left(-3\,V+\sqrt{3}\right)}{27} \quad \textbf{(1.2.2)}$$

One of the critical points is 1-2Uc=sqrt(3)/3, which we alrealdy knew. We can have explicit trigonometric expressions for the three others:

> $Vcminus := -\dfrac{\mathrm{sqrt}\left(\frac{p}{3}\right)}{\cos\left(\frac{1}{3}\cdot\arccos(\mathrm{sqrt}(p))\right)}$; $Vcplusright := 2\dfrac{\mathrm{sqrt}(3)}{3} + \dfrac{\mathrm{sqrt}\left(\frac{1-p}{3}\right)}{\cos\left(\frac{1}{3}\cdot\arccos(\mathrm{sqrt}(1-p))\right)}$; $Vcplusleft := \dfrac{\mathrm{sqrt}\left(\frac{p}{3}\right)}{\cos\left(\frac{1}{3}\cdot\arccos(\mathrm{sqrt}(p))-\frac{\mathrm{Pi}}{3}\right)}$;

$$Vcminus := -\frac{\sqrt{3}\sqrt{p}}{3\cos\left(\frac{\arccos\left(\sqrt{p}\right)}{3}\right)}$$

$$Vcplusright := \frac{2\sqrt{3}}{3}+\frac{\sqrt{-3\,p+3}}{3\cos\left(\frac{\arccos\left(\sqrt{1-p}\right)}{3}\right)}$$

$$Vcplusleft := \frac{\sqrt{3}\sqrt{p}}{3\sin\left(\frac{\arccos\left(\sqrt{p}\right)}{3}+\frac{\pi}{6}\right)} \quad \textbf{(1.2.3)}$$

> simplify(subs(V = Vcminus, $9\,V^2\sqrt{3}-9\,V^3-4\,p\sqrt{3}$)); simplify(subs(V = Vcplusleft, $9\,V^2\sqrt{3}-9\,V^3-4\,p\sqrt{3}$)); simplify(subs(V = Vcplusright, $9\,V^2\sqrt{3}-9\,V^3-4\,p\sqrt{3}$)) assuming $p<1$ **and** $p>0$;

$$0$$

$$0$$

$$-\frac{1}{\cos\left(\frac{\arccos\left(\sqrt{1-p}\right)}{3}\right)^3}\left(4\,(p-1)\left(\left(\cos\left(\frac{\arccos\left(\sqrt{1-p}\right)}{3}\right)^3-\frac{3\cos\left(\frac{\arccos\left(\sqrt{1-p}\right)}{3}\right)}{4}\right)\sqrt{3}-\frac{\sqrt{-3\,p+3}}{4}\right)\right) \quad \textbf{(1.2.4)}$$

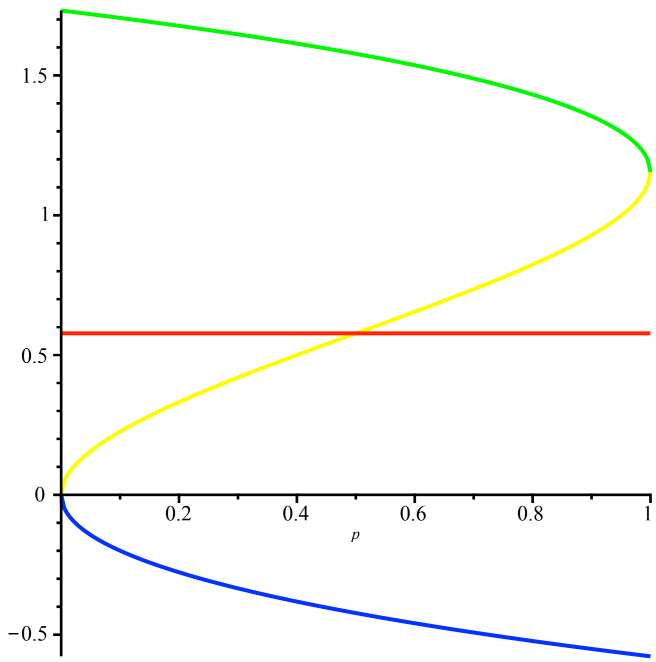
Maple does not recognize directly the trigonometric identity cos (3t) = 4 cos^3 t - 3 cot t which is weird :

```
> simplify(subs(p = 1 - p, (1.2.4)));
```

$$0 \quad \mathbf{(1.2.5)}$$

We can plot the 4 critical points of y(V):

```
> plotVcminus := plot(Vcminus, p = 0 ..1, color = blue) : plotVcplusleft := plot(Vcplusleft, p = 0
    ..1, color = yellow) : plotVcplusright := plot(Vcplusright, p = 0 ..1, color = green) :
    plotVc := plot(sqrt(3) / 3, p = 0 ..1, color = red) : plots[display]({plotVcminus,
    plotVcplusleft, plotVcplusright, plotVc});
```



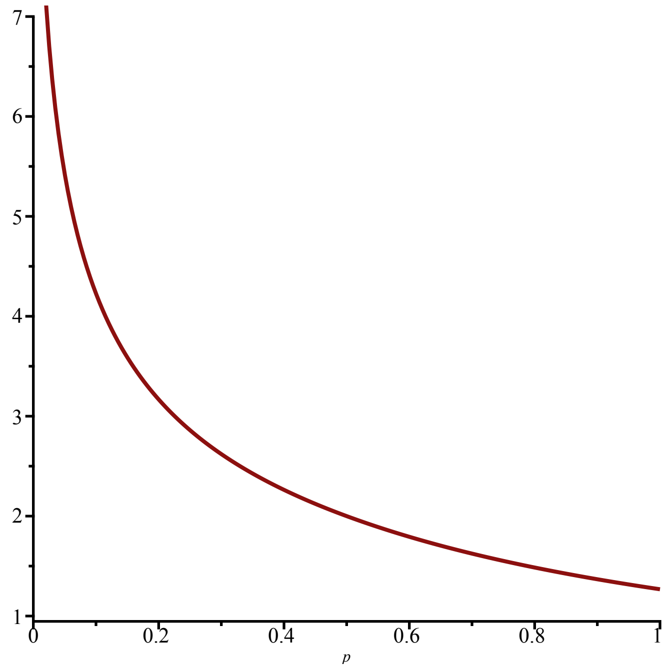
The smallest positive critical point is the yellow one (Vcplusleft) for p<1/2 and sqrt(3)/3 for p>1/2:

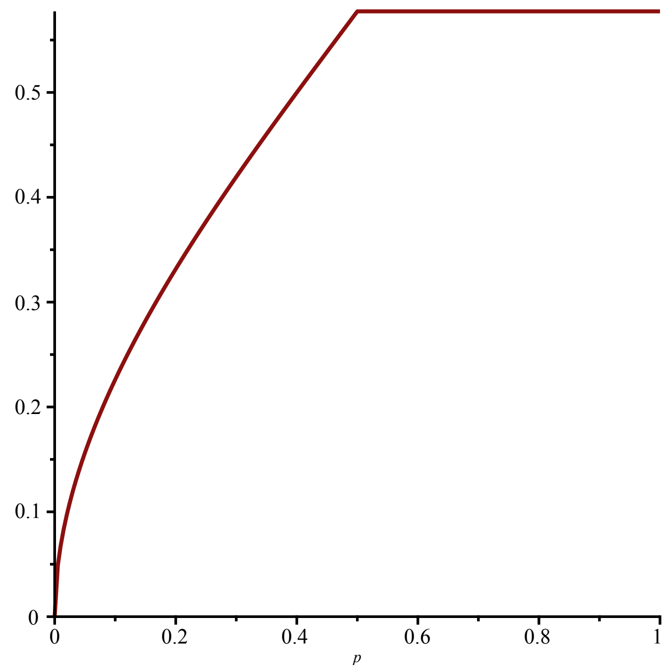
```
> Vplus := min(Vcplusleft, sqrt(3) / 3);
```

$$Vplus := \min\left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}\,\sqrt{p}}{3\sin\left(\frac{\arccos\left(\sqrt{p}\right)}{3} + \frac{\pi}{6}\right)}\right) \quad \mathbf{(1.2.6)}$$

The corresponding value for y, which is the radius of cv of T(p,t_c,t_c y), is >2 for p<1/2 and <2 for p>1/2 but always >1

```
> plot(subs(V = Vplus, yUcV), p = 0 ..1); plot(Vplus, p = 0 ..1);
```





There is only one negative critical point (Vcminus), but we have to check where it is compared to the potential negative poles of Vc. First we check for which values of p such poles exist:

> *collect*(*denom*(*yUcV*), *V*, *factor*); *factor*(*discrim*(%, *V*)); *solve*(%); *evalf*(%);

$$-9\,V^3+\left(9\sqrt{3}-9\right)V^2+\left(6\sqrt{3}-9\right)V+2\,p\sqrt{3}$$

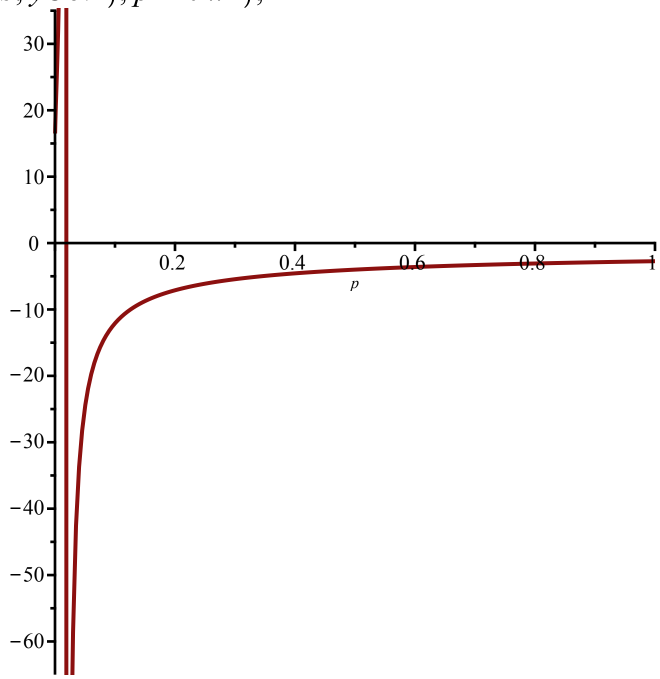
$$-729\left(2\,p-1+\sqrt{3}\right)\left(18\,p-9+5\sqrt{3}\right)$$

$$\frac{1}{2}-\frac{5\sqrt{3}}{18},\ \frac{1}{2}-\frac{\sqrt{3}}{2}$$

$$0.0188747755,\ -0.3660254040 \qquad \mathbf{(1.2.7)}$$

The are negative poles are for p<0.018 and one of them is between the negative singularity and 0. We will not consider these values of p in this paper but it might be interesting to investigate: for these values of p, y(V) has no negative singularity.

> $plot(subs(V = Vcminus, yUcV), p = 0..1);$



Perimeter asymptotics at p=1/2 and t=t_c (Lemma 12)

The singularity of V are at y+=2 and y-=-4:

> $simplify\left(subs\left(V = \frac{\mathrm{sqrt}(3)}{3}, p = \frac{1}{2}, U = Uc, yUV\right)\right); simplify\left(subs\left(V = Vcminus, p = \frac{1}{2}, U = Uc, yUV\right)\right);$

$$2$$
$$-4 \qquad \textbf{(1.3.1)}$$

The singular expansion of V at YY=1-y/2

> $algeqtoseries\left(subs\left(U = Uc, p = \frac{1}{2}, y = 2\cdot(1 - YY), numer(y - yUV)\right), YY, V, 5\right);$

$$\left[\frac{\sqrt{3}}{3} + RootOf(3\_Z^3 + 1)\, YY^{1/3} + \frac{YY}{3} + \frac{RootOf(3\_Z^3 + 1)\, YY^{4/3}}{3} + \mathrm{O}\left(YY^{5/3}\right)\right] \qquad \textbf{(1.3.2)}$$

> $Vcritser := \frac{\mathrm{sqrt}(3)}{3} - \left(\frac{1}{3}\right)^{\frac{1}{3}}\cdot YY^{\frac{1}{3}} + \frac{YY}{3} - \frac{1}{3}\cdot\left(\frac{1}{3}\right)^{\frac{1}{3}}\cdot YY^{\frac{4}{3}};$

$$Vcritser := \frac{\sqrt{3}}{3} - \frac{3^{2/3}\, YY^{1/3}}{3} + \frac{YY}{3} - \frac{3^{2/3}\, YY^{4/3}}{9} \qquad \textbf{(1.3.3)}$$

Singular expansion of T

> $Ttcritser := simplify\left(series\left(subs\left(V = \frac{\mathrm{sqrt}(3)}{3} - \left(\frac{1}{3}\right)^{\frac{1}{3}}\cdot YY^{\frac{1}{3}} + \frac{YY}{3} - \frac{1}{3}\cdot\left(\frac{1}{3}\right)^{\frac{1}{3}}\right.\right.\right.$

$$\cdot\, YY^{\frac{4}{3}}, p = \frac{1}{2}, U = Uc, TtUV\Bigg), YY, 2\Bigg)\Bigg); collect(\%, YY, factor);$$

$$Ttcritser := \mathrm{O}\left(YY^2\right) - \frac{3^{5/6}\, YY^{2/3}}{2} + 3^{1/6}\, YY^{4/3} - \frac{5\, 3^{5/6}\, YY^{5/3}}{6} + \frac{(3\, YY + 3)\sqrt{3}}{6}$$

$$-\frac{5\, 3^{5/6}\, YY^{5/3}}{6} - \frac{3^{5/6}\, YY^{2/3}}{2} + 3^{1/6}\, YY^{4/3} + \frac{\sqrt{3}\, YY}{2} + \frac{\sqrt{3}}{2} + \mathrm{O}\left(YY^2\right) \qquad \textbf{(1.3.4)}$$

The expansions for Delta and Theta are done when needed in the worksheet.

Singular expansions at t_c of the weights and the series Delta(p,z) parametrized by V (Lemma 8)

Recall the equation satisfied by Tt = T(t,ty):

> $eqTt;$

$$Tt^3\, p\, w\, y^3 - Tt^2\, p\, y + Tt\, p^2\, y + Tt^2\, p - 2\, Tt\, p^2 - Tt\, tT1\, y + p^3 + Tt\, p - p^2 \qquad \textbf{(1.4.1)}$$

We deduce from it an equation satisfied by Fptz; = p \tilde F(p,t,z):

> $eqFptz := factor\left(numer\left(subs\left(p = 1 - p, Tt = (1 - z)\cdot Fptz\,, y = \frac{1}{1 - z}\,, eqTt\right)\right)\right);$

$$eqFptz := -Fptz^3\, w\, p - Fptz^2\, p\, z^2 + Fptz^3\, w + Fptz^2\, p\, z + Fptz^2\, z^2 + 2\, Fptz\, p^2\, z - Fptz^2\, z - Fptz\, p^2 - 3\, Fptz\, p\, z - p^3 + Fptz\, p - Fptz\, tT1 + Fptz\, z + 2\, p^2 - p \qquad \textbf{(1.4.2)}$$

> $collect(eqFptz, z, factor);$

$$-Fptz^2\,(p-1)\,z^2 + Fptz\,(p-1)\,(2\,p + Fptz - 1)\,z - Fptz^3\, w\, p + Fptz^3\, w - Fptz\, p^2 - p^3 + Fptz\, p - Fptz\, tT1 + 2\, p^2 - p \qquad \textbf{(1.4.3)}$$

We isolate the term in z=0 and substract the equation satisfied by T(t,1-p,t/(1-z):

> $factor(coeff(\textbf{(1.4.3)}, z, 0) - subs(p = 1 - p, y = 1, Tt = T1minuspyis1, eqTt));$

$$-(Fptz - T1minuspyis1)\left(Fptz^2\, p\, w + Fptz\, p\, w\, T1minuspyis1 + p\, w\, T1minuspyis1^2 - Fptz^2\, w - Fptz\, w\, T1minuspyis1 - w\, T1minuspyis1^2 + p^2 - p + tT1\right) \qquad \textbf{(1.4.4)}$$

We specialize the second factor at z=0 for which $Fptz = T1minuspyis1 = T(1 - p, t, t)$:

> $factor\big(subs\big(Fptz = T1minuspyis1, Fptz^2\, p\, w + Fptz\, p\, w\, T1minuspyis1 + p\, w\, T1minuspyis1^2 - Fptz^2\, w - Fptz\, w\, T1minuspyis1 - w\, T1minuspyis1^2 + p^2 - p + tT1\big)\big);$

$$3\, p\, w\, T1minuspyis1^2 - 3\, w\, T1minuspyis1^2 + p^2 - p + tT1 \qquad \textbf{(1.4.5)}$$

It is the derivative of the agebraic equation satisfied by T:

> $simplify(subs(y = 1, p = 1 - p, -diff(eqTt, Tt)));$

$$p^2 + \left(3\, Tt^2\, w - 1\right) p - 3\, Tt^2\, w + tT1 \qquad \textbf{(1.4.6)}$$

We now do the expansions at U=Uc:

Equation y(U,V) = y(Uc,Vuc) :

> $eqyUVc := numer(factor((subs(V = Vuc, yUcV) - yUV))); indets(\%);$

$$eqyUVc := 96\sqrt{3}\,U^3\,Vuc\,p - 72\sqrt{3}\,U^2\,V\,Vuc - 144\sqrt{3}\,U^2\,Vuc\,p - 72\sqrt{3}\,U\,V^2\,Vuc + 72\sqrt{3}\,U\,V\,Vuc^2 + 72\sqrt{3}\,U\,V\,Vuc + 16\sqrt{3}\,U\,V\,p + 48\sqrt{3}\,U\,Vuc\,p + 18\,V^3\,Vuc^2 - 18\,V^2\,Vuc^3 - 54\,V^2\,Vuc^2 + 36\,V\,Vuc^3 - 18\,V^2\,Vuc + 36\,V\,Vuc^2 + 36\,V\,Vuc - 144\,U^3\,Vuc^2\,p + 108\,U^2\,V\,Vuc^2 + 216\,U^2\,Vuc^2\,p + 108\,U\,V^2\,Vuc^2 - 72\,U\,V\,Vuc^3 - 108\,U\,V\,Vuc^2 - 72\,U\,Vuc^2\,p - 72\,U\,V\,Vuc - 12\sqrt{3}\,V^3\,Vuc + 18\sqrt{3}\,V^2\,Vuc^2 + 36\sqrt{3}\,V^2\,Vuc + 4\sqrt{3}\,V^2\,p - 36\sqrt{3}\,V\,Vuc^2 - 24\sqrt{3}\,V\,Vuc - 8\sqrt{3}\,V\,p$$

$$\{U, V, Vuc, p\} \qquad \textbf{(1.4.7)}$$

We plug the development of U at Uc to get the development of V at Uc:

> Vsing := V + subs(Vuc = V, collect(convert(simplify(op(2, algeqtoseries(subs(V = Vuc + VV, collect((simplify(subs(U = Using, eqyUVc))), WW, factor)), WW, VV, 3, true))), polynom), WW, factor));

$$Vsing := V - \left(\sqrt{2}\left(1269\,V^6\sqrt{3} - 405\,V^7 - 180\,V^4\,p\sqrt{3} + 2502\,V^4\sqrt{3} - 4698\,V^5 + 1656\sqrt{3}\,V^2\,p - 792\,V^3\,p - 1791\,V^2\sqrt{3} + 171\,V^3 + 28\,p\sqrt{3} - 1848\,V\,p + 1512\,V\right)V\,WW^3\right) \Big/ \left(12\left(9\,V^2\sqrt{3} - 9\,V^3 - 4\,p\sqrt{3}\right)\left(-3\,V + \sqrt{3}\right)^5\right) + \left(\sqrt{3}\left(72\,V^4\sqrt{3} - 27\,V^5 + 204\sqrt{3}\,V^2\,p - 162\,V^3\,p - 18\,V^2\sqrt{3} - 117\,V^3 + 8\,p\sqrt{3} - 210\,V\,p + \ldots\right.\right.$$
$$\left. + \sqrt{3}\right)^3\Big) - \frac{\sqrt{2}\,V\,WW}{-3\,V + \sqrt{3}}$$

expansion of the partition function Z=t2T2/(p^2*t^3):

> Ztser;

$$\frac{2\sqrt{2}\left(3\,p - 3 + 2\sqrt{3}\right)WW^3}{3\,p} - \frac{\sqrt{3}\left(3\,p - 27 + 16\sqrt{3}\right)WW^2}{4\,p} - \frac{3\sqrt{3}\left(-p - 7 + 4\sqrt{3}\right)}{4\,p} \qquad \textbf{(1.4.9)}$$

Expansion of \tildeF (p,t,z) = 1/p * z/(1-z) * T(1-p,t,t/(1-z)) with 1/(1-z) = \hat y(1-p):

> $collect\left(convert\left(simplify\left(\frac{1}{p} \cdot series(subs(V = Vsing, U = Using, p = 1 - p, (yUV - 1) \cdot TtUV), WW, 4)\right), polynom\right), WW, factor\right);$

$$\left(2\sqrt{2}\left(9\,V^2\sqrt{3} - 9\,V^3 - 6\,V\sqrt{3} - 2\,p\sqrt{3} + 9\,V^2 + 2\sqrt{3} - 9\,V\right)V\left(-3\,V + 2\sqrt{3}\right)WW^3\right) \Big/ \left(\left(-3\,V + \sqrt{3}\right)^3\left(9\,V^2\sqrt{3} - 9\,V^3 + 4\,p\sqrt{3} - 4\sqrt{3}\right)p\right) - \left(\sqrt{3}\left(9\,V^2\sqrt{3} - 9\,V^3 - 6\,V\sqrt{3} - 2\,p\sqrt{3} + 9\,V^2 + 2\sqrt{3} - 9\,V\right)\left(2\,V\sqrt{3} - 3\,V^2 + 1\right)V\left(-3\,V \ldots\right.\right.$$
$$\left. - 4\sqrt{3}\right)p\Big) - \frac{\left(9\,V^2\sqrt{3} - 9\,V^3 - 6\,V\sqrt{3} - 2\,p\sqrt{3} + 9\,V^2 + 2\sqrt{3} - 9\,V\right)\sqrt{3}}{6\,p}$$

The series Delta of the page:

> $Deltaser := factor\left(\frac{coeff((\mathbf{1.4.10}), WW, 3)}{coeff((\mathbf{1.4.9}), WW, 3)}\right);$

$$Deltaser := \frac{3\left(9V^2\sqrt{3} - 9V^3 - 6V\sqrt{3} - 2p\sqrt{3} + 9V^2 + 2\sqrt{3} - 9V\right)V\left(-3V + 2\sqrt{3}\right)}{\left(-3V + \sqrt{3}\right)^3\left(9V^2\sqrt{3} - 9V^3 + 4p\sqrt{3} - 4\sqrt{3}\right)\left(3p - 3 + 2\sqrt{3}\right)} \quad \textbf{(1.4.11)}$$

Finite cluster probability (Proof of Theorem 1)

Formulas for cylinder generating functions

> $Wcyl := \frac{1}{2\cdot\left(\frac{1}{z1} - \frac{1}{z2}\right)^2}\cdot\left(\frac{1}{\mathrm{sqrt}\left(\left(\frac{1}{z1} - cplus\right)\cdot\left(\frac{1}{z1} - cmoins\right)\right)}\cdot\frac{1}{\mathrm{sqrt}\left(\left(\frac{1}{z2} - cplus\right)\cdot\left(\frac{1}{z2} - cmoins\right)\right)}\cdot\left(\frac{1}{z1\cdot z2} - \frac{cplus + cmoins}{2}\cdot\left(\frac{1}{z1} + \frac{1}{z2}\right) + cplus\cdot cmoins\right) - 1\right);$

$$Wcyl := \frac{\frac{\frac{1}{z1\,z2} - \frac{(cplus + cmoins)\left(\frac{1}{z1} + \frac{1}{z2}\right)}{2} + cplus\ cmoins}{\sqrt{\left(\frac{1}{z1} - cplus\right)\left(\frac{1}{z1} - cmoins\right)}\sqrt{\left(\frac{1}{z2} - cplus\right)\left(\frac{1}{z2} - cmoins\right)}} - 1}{2\left(\frac{1}{z1} - \frac{1}{z2}\right)^2} \quad \textbf{(1.5.1.1)}$$

The coefficient we need

> $simplify\left(subs\left(z2 = \frac{1}{z}, coeff(series(Wcyl, z1, 4), z1, 3)\right)\right)$ assuming $z1 > 0;$

$$-\frac{-z^2 + \left(\sqrt{(-z + cplus)(-z + cmoins)} + \frac{cplus}{2} + \frac{cmoins}{2}\right)z + \frac{(cplus - cmoins)^2}{8}}{\sqrt{(-z + cplus)(-z + cmoins)}} \quad \textbf{(1.5.1.2)}$$

And the antiderivative

> $-int((\mathbf{1.5.1.2}), z);$

$$\frac{z^2}{2} - \frac{z\sqrt{z^2 + (-cmoins - cplus)z + cplus\ cmoins}}{2} - \frac{cmoins\sqrt{z^2 + (-cmoins - cplus)z + cplus\ cmoins}}{4} - \frac{cplus\sqrt{z^2 + (-cmoins - cplus)z + cplus\ cmoins}}{4} \quad \textbf{(1.5.1.3)}$$

Computation of the integral bounds

We have to solve y(1-p,V) = y(p,V+/-(p))/(y(p,V+/-(p))-1); there is a symetry p <--> 1-p and V <--> 2sqrt(3)/3 - V in play:

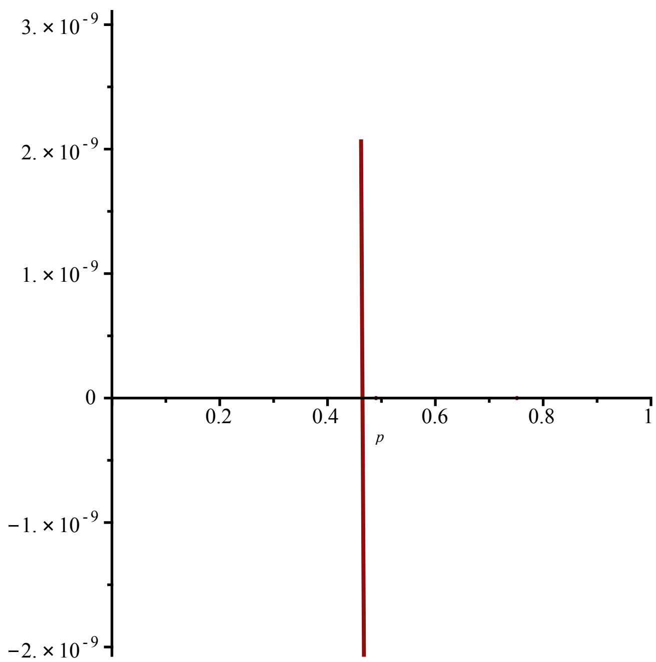
$$> \ simplify\left(subs\left(p=1-p,\ V=\frac{2\,\mathrm{sqrt}(3)}{3}-V,\ yUcV\right)-factor\left(\frac{yUcV}{yUcV-1}\right)\right);$$

$$0 \qquad \textbf{(1.5.2.1)}$$

We also know that (2sqrt(3)/3 - V-(p)) is V+right(1-p) and (2sqrt(3)/3 - V+left(p)) is V+left(1-p), therefore we want to solve y(p,V)=y(p,Vc) for Vc=V-(p), V+left(p) and sqrt(3)/3. Obviously, Vc is a double solution and we want to identify the third one. We will proceed formally since we can't get Maple to simplify the expressions:

$$> \ collect\left(simplify\left(rem\left(numer(yUcV - subs(V=Vcplusleft, yUcV)),\ (V-Vcplusleft)^2,\ V\right),\ trig\right),\ V, factor\right) \text{ assuming } p>0 \textbf{ and } p<1;\ plot(\%, p=0..1);$$

$$18\sin\left(\frac{\arccos(\sqrt{p})}{3}+\frac{\pi}{6}\right)\left(4p^{3/2}\sin\left(\frac{\arccos(\sqrt{p})}{3}+\frac{\pi}{6}\right)^3-4\sin\left(\frac{\arccos(\sqrt{p})}{3}+\frac{\pi}{6}\right)^4 p-4p^{3/2}\sin\left(\frac{\arccos(\sqrt{p})}{3}+\frac{\pi}{6}\right)+3p\sin\left(\frac{\arccos(\sqrt{p})}{3}+\frac{\pi}{6}\right)^2+p^2\right)V - 6\sqrt{3}\left(-4p^{3/2}\sin\left(\frac{\arccos(\sqrt{p})}{3}+\frac{\pi}{6}\right)^4+p^{5/2}+3p^{3/2}\sin\left(\frac{\arccos(\sqrt{p})}{3}+\frac{\pi}{6}\right)^2+4\sin\left(\frac{\arccos(\sqrt{p})}{3}+\frac{\pi}{6}\right)^3p^2-4\sin\left(\frac{\arccos(\sqrt{p})}{3}+\frac{\pi}{6}\right)p^2\right)$$



This is the equation we want to solve:

$$> \ factor(numer(yUcV - subs(V=Vc, yUcV)));$$

$$18\left(6\sqrt{3}\,V^2\,Vc+6\sqrt{3}\,V\,Vc^2-9V^2\,Vc^2-2\sqrt{3}\,V\,p-2\sqrt{3}\,Vc\,p-9\,V\,Vc+4p\right)(V-Vc) \qquad \textbf{(1.5.2.2)}$$

One of the roots of the polynom of degree 2 is Vc, we want the second one:

> $collect\left(6\sqrt{3}\,V^2\,Vc + 6\sqrt{3}\,V\,Vc^2 - 9\,V^2\,Vc^2 - 2\,p\sqrt{3}\,V - 2\sqrt{3}\,Vc\,p - 9\,V\,Vc + 4\,p, V, factor\right);$

$$3\,Vc\left(-3\,Vc + 2\sqrt{3}\right)V^2 - \sqrt{3}\left(3\,Vc\sqrt{3} - 6\,Vc^2 + 2\,p\right)V + \frac{2\sqrt{3}\left(-3\,Vc + 2\sqrt{3}\right)p}{3} \qquad \textbf{(1.5.2.3)}$$

> $factor(subs(V = Vc, \%));$

$$-\frac{\left(-9\,Vc^3 + 9\,Vc^2\sqrt{3} - 4\,p\sqrt{3}\right)\left(\sqrt{3} - 3\,Vc\right)}{3} \qquad \textbf{(1.5.2.4)}$$

The third solution is given by the ratio

> $Vcint := \frac{coeff(\textbf{(1.5.2.3)}, V, 0)}{coeff(\textbf{(1.5.2.3)}, V, 2)\cdot Vc};$

$$Vcint := \frac{2\sqrt{3}\,p}{9\,Vc^2} \qquad \textbf{(1.5.2.5)}$$

We can check that it is indeed a root if Vc is critical.

> $factor(subs(V = Vcint, \textbf{(1.5.2.3)}));$

$$-\frac{2\sqrt{3}\left(-9\,Vc^3 + 9\,Vc^2\sqrt{3} - 4\,p\sqrt{3}\right)\left(\sqrt{3} - 3\,Vc\right)p}{27\,Vc^3} \qquad \textbf{(1.5.2.6)}$$

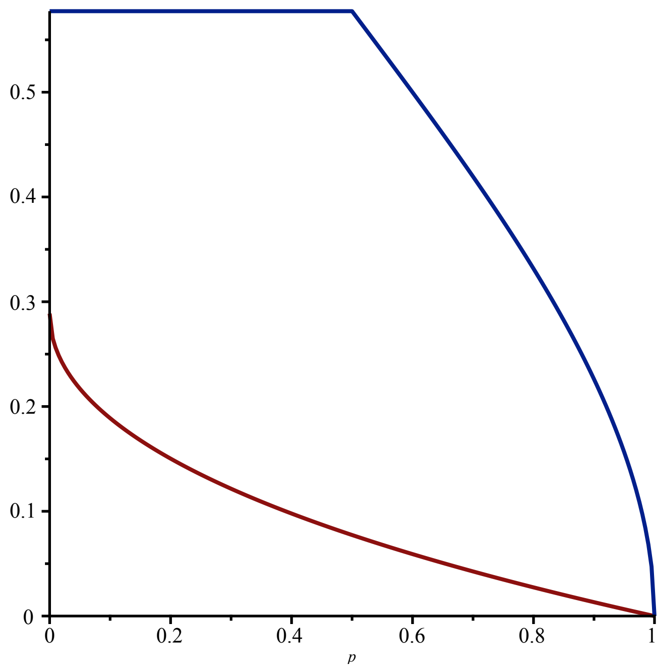
First we look at the upper bound for the integral. When there is no negative pole (so p not too close to 0), it is given by:

> $Vminusint := simplify(subs(Vc = Vcminus, Vcint), trig);$

$$Vminusint := \frac{2\sqrt{3}\,\cos\left(\frac{\arccos\left(\sqrt{p}\right)}{3}\right)^2}{3} \qquad \textbf{(1.5.2.7)}$$

We have to check that 2sqrt(3)/3 - Vminusint is smaller than Vplus(1-p):

> $plot\left(\left\{\frac{2\cdot \mathrm{sqrt}(3)}{3} - Vminusint, subs(p = 1 - p, Vplus)\right\}, p = 0..1\right);$



The lower bound is given by:

> $Vplusint := simplify(subs(Vc = Vplus, Vcint), trig);$

$$Vplusint := \frac{2\sqrt{3}\,p}{9\min\left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}\sqrt{p}}{3\sin\left(\frac{\arccos(\sqrt{p})}{3} + \frac{\pi}{6}\right)}\right)^2} \tag{1.5.2.8}$$

When p<1/2, it is given by:

> $Vplusintsubcrit := simplify\left(subs\left(Vc = \frac{\sqrt{3}\sqrt{p}}{3\sin\left(\frac{\arccos(\sqrt{p})}{3} + \frac{\pi}{6}\right)}, Vcint\right)\right);$

$$Vplusintsubcrit := \frac{2\sqrt{3}\sin\left(\frac{\arccos(\sqrt{p})}{3} + \frac{\pi}{6}\right)^2}{3} \tag{1.5.2.9}$$

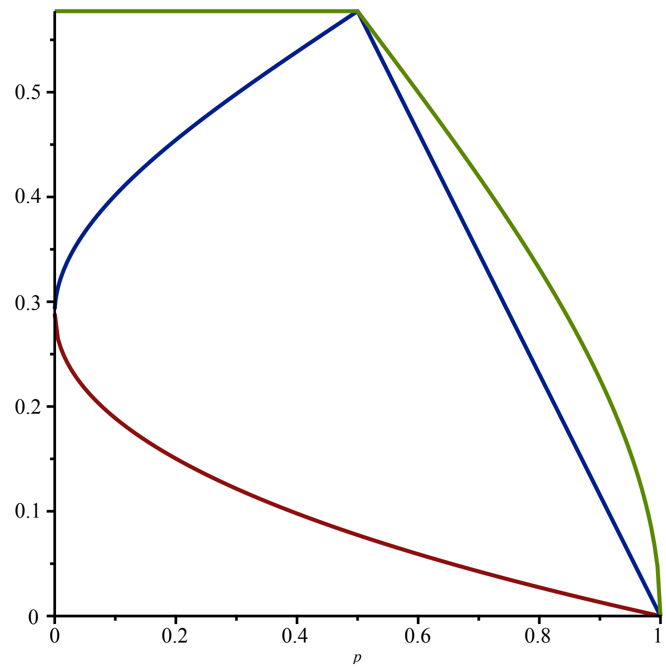
When p>1/2:

> $Vplusintsupcrit := simplify\left(subs\left(Vc = \frac{\mathrm{sqrt}(3)}{3}, Vcint\right)\right);$

$$Vplusintsupcrit := \frac{2\,p\sqrt{3}}{3} \tag{1.5.2.10}$$

And it is also smaller than Vplus:

> $plot\left(\left\{\frac{2\,\mathrm{sqrt}(3)}{3} - Vminusint, \frac{2\,\mathrm{sqrt}(3)}{3} - Vplusint, subs(p = 1 - p, Vplus)\right\}, p = 0..1\right);$



Computation of the integral

>

The root factor factorized as in the paper:

> $\frac{1}{yUcV}$; $\frac{coeff\left(numer\left(\frac{1}{yUcV}\right), V, 3\right)}{coeff\left(denom\left(\frac{1}{yUcV}\right), V, 2\right)}$;

$$\frac{9 V^2 \sqrt{3} - 9 V^3 + 6 V \sqrt{3} + 2 p \sqrt{3} - 9 V^2 - 9 V}{6 V \left(-3 V + 2 \sqrt{3}\right)}$$

$$\frac{1}{2} \tag{1.5.3.1}$$

> $rootfactor := \frac{1}{2} \cdot \mathrm{sqrt}((Vpi - V) \cdot (V - Vmi)) \cdot \frac{(V - Vp) \cdot (V - Vm)}{V \cdot \left(\frac{2 \cdot \mathrm{sqrt}(3)}{3} - V\right)}$;

$$rootfactor := \frac{\sqrt{(Vpi - V)\ (V - Vmi)}\ (V - Vp)\ (V - Vm)}{2 V \left(\frac{2\sqrt{3}}{3} - V\right)} \tag{1.5.3.2}$$

The term involving Delta:

> $deltafactor := factor\left(Deltaser \cdot subs\left(p = 1 - p, \frac{factor(diff(yUcV, V))}{yUcV \cdot (yUcV - 1)}\right)\right)$;

$$deltafactor := \frac{3}{\left(3 p - 3 + 2\sqrt{3}\right) \left(-3 V + \sqrt{3}\right)^2} \tag{1.5.3.3}$$

We saw that Vp = 1/2 + sqrt(3)/2 - 3/2*Vpi and that Vm = 1/2 + sqrt(3)/2 - 3/2*Vmi, the last factor is given by:

> $lastfactor := \frac{1}{yUcV} + \frac{1}{2} + \frac{\mathrm{sqrt}(3)}{2} - \frac{3}{4} \cdot Vmi - \frac{3}{4} \cdot Vpi;$

$$lastfactor := \frac{9 V^2 \sqrt{3} - 9 V^3 + 6 V \sqrt{3} + 2 p \sqrt{3} - 9 V^2 - 9 V}{6 V \left(-3 V + 2 \sqrt{3}\right)} + \frac{1}{2} + \frac{\sqrt{3}}{2} - \frac{3 Vmi}{4} - \frac{3 Vpi}{4} \quad \textbf{(1.5.3.4)}$$

There is also a constant in from of the intergal:

> $constfactor := rationalize\left(\frac{1}{p \cdot subs(U = Uc, wU) \cdot 2 \cdot \mathrm{Pi}}\right);$

$$constfactor := \frac{6 \sqrt{3}}{p \pi} \quad \textbf{(1.5.3.5)}$$

Maple cant compute the integral in general.
We try to compute the integral when p<1/2. In this regime Vm = sqrt(3) - 2*Vmi and Vp = sqrt(3) - 2* Vpi. (warning, this takes a long time !!! You can skip the next 2 entries and go to the supercritical range direclty.).

> $ProbaFiniteSubcrit := simplify(int(subs(Vm = \mathrm{sqrt}(3) - 2 * Vmi, Vp = \mathrm{sqrt}(3) - 2 * Vpi, constfactor \cdot rootfactor \cdot deltafactor \cdot lastfactor), V = Vpi..Vmi))$ assuming $Vpi < Vmi$ **and** $Vpi > \frac{\mathrm{sqrt}(3)}{3}$ **and** $Vmi < \frac{2\,\mathrm{sqrt}(3)}{3}$:

> $ProbaFiniteSubcritVal := simplify\left(subs\left(Vmi = \frac{2 \cdot \mathrm{sqrt}(3)}{3} \cdot \left(\cos\left(\frac{\arccos(\mathrm{sqrt}(p))}{3}\right)\right)^2, Vpi = \frac{2 \cdot \mathrm{sqrt}(3)}{3} \cdot \left(\sin\left(\frac{\arccos\left(\sqrt{p}\right)}{3} + \frac{\pi}{6}\right)\right)^2, ProbaFiniteSubcrit\right)\right)$ assuming $p > 0$ **and** $p < \frac{1}{2}$:

The expression looks awful but should simplify to 1. We do not do it since this was already known.

> $ProbaFiniteSubcritVal;$

$$\Bigg(3 \Bigg(\Bigg(\frac{1}{9}\Bigg(1024 \Bigg(\left(2 + (p-1)\sqrt{3}\right) \cos\left(\frac{2 \arccos\left(\sqrt{p}\right)}{3}\right)^6 + \Bigg(3 + \left(\frac{3p}{2} - \frac{3}{2}\right)\sqrt{3}\Bigg) \cos\left(\frac{2 \arccos\left(\sqrt{p}\right)}{3}\right)^5 + \Bigg(\Bigg(\left(3\sqrt{3} + \frac{9p}{2} - \frac{9}{2}\right) \sin\left(\frac{2 \arccos\left(\sqrt{p}\right)}{3}\right) - \frac{15}{2} + \left(\frac{15}{4} - \frac{15p}{4}\right)\sqrt{3}\Bigg) \cos\left(\frac{2 \arccos\left(\sqrt{p}\right)}{3}\right)^4 + \Bigg(\Bigg(\left(\frac{9\sqrt{3}}{2} + \frac{27p}{4} - \frac{27}{4}\right) \sin\left(\frac{2 \arccos\left(\sqrt{p}\right)}{3}\right) - \frac{53}{4} + \Bigg(-\frac{53p}{8} \quad \textbf{(1.5.3.6)}$$

$$+\frac{53}{8}\Big)\sqrt{3}\Big)\cos\left(\frac{2\arccos(\sqrt{p})}{3}\right)^3+\Big(\Big(-\frac{27p}{8}-\frac{9\sqrt{3}}{4}$$

$$+\frac{27}{8}\Big)\sin\left(\frac{2\arccos(\sqrt{p})}{3}\right)+\frac{27}{16}+\left(\frac{27p}{32}-\frac{27}{32}\right)\sqrt{3}\Big)$$

$$\cos\left(\frac{2\arccos(\sqrt{p})}{3}\right)^2+\Big(\Big(-\frac{99\sqrt{3}}{16}+\frac{297}{32}-\frac{297p}{32}\Big)\sin\left(\frac{2\arccos(\sqrt{p})}{3}\right)$$

$$+\frac{171}{16}+\left(\frac{171p}{32}-\frac{171}{32}\right)\sqrt{3}\Big)\cos\left(\frac{2\arccos(\sqrt{p})}{3}\right)+\Big(\frac{117}{32}-\frac{117p}{32}$$

$$-\frac{39\sqrt{3}}{16}\Big)\sin\left(\frac{2\arccos(\sqrt{p})}{3}\right)-\frac{377}{32}+\left(\frac{377}{64}-\frac{377p}{64}\right)\sqrt{3}\Big)$$

$$\sin\left(\frac{\arccos(\sqrt{p})}{3}\right)\sqrt{6-3\sqrt{3}\sin\left(\frac{2\arccos(\sqrt{p})}{3}\right)+3\cos\left(\frac{2\arccos(\sqrt{p})}{3}\right)}\Bigg)$$

$$+\Big(-\frac{512}{3}+\left(-\frac{256p}{3}+\frac{256}{3}\right)\sqrt{3}\Big)\cos\left(\frac{2\arccos(\sqrt{p})}{3}\right)^{10}+\Big(\Big(\frac{512}{3}$$

$$+\left(\frac{256p}{3}-\frac{256}{3}\right)\sqrt{3}\Big)\cos\left(\frac{4\arccos(\sqrt{p})}{3}\right)+\Big(-\frac{128}{3}p-\frac{256}{3}p^2$$

$$+\frac{128}{9}\Big)\sqrt{3}-\frac{256}{3}-\frac{1024p}{3}\Big)\cos\left(\frac{2\arccos(\sqrt{p})}{3}\right)^9+\Big(\Big(\Big(-\frac{256}{3}$$

$$-\frac{1024p}{3}\Big)\sqrt{3}-128p-256p^2+\frac{128}{3}\Big)\sin\left(\frac{2\arccos(\sqrt{p})}{3}\right)+\Big(\Big(-\frac{128}{9}$$

$$+\frac{128}{3}p+\frac{256}{3}p^2\Big)\sqrt{3}+\frac{1024p}{3}+\frac{256}{3}\Big)\cos\left(\frac{4\arccos(\sqrt{p})}{3}\right)+\Big(-\frac{128}{3}$$

$$+\left(\frac{64}{3}-\frac{64p}{3}\right)\sqrt{3}\Big)\cos\left(2\arccos(\sqrt{p})\right)+\left(\frac{64}{3}p+\frac{1024}{3}p^2+\frac{320}{9}\right)\sqrt{3}$$

$$+ 128 + 1280\, p \Big) \cos\left(\frac{2 \arccos(\sqrt{p}\,)}{3}\right)^{8} + \left(\left(\left(\left(\left(\frac{1024\, p}{3} + \frac{256}{3}\right)\sqrt{3} + 256\, p^{2}\right.\right.\right.\right.$$

$$+ 128\, p - \frac{128}{3}\Big) \cos\left(\frac{4 \arccos(\sqrt{p}\,)}{3}\right) + \left(-\frac{128\sqrt{3}}{3} + 64\right.$$

$$- 64\, p\Big) \cos(2 \arccos(\sqrt{p}\,)) + \left(-\frac{3328\, p}{3} + \frac{512}{3}\right)\sqrt{3} - \frac{1664}{3} + 128\, p$$

$$- 768\, p^{2}\Big) \sin\left(\frac{2 \arccos(\sqrt{p}\,)}{3}\right) + \left(\left(-\frac{256}{3}\, p - \frac{896}{3}\, p^{2} - \frac{128}{9}\right)\sqrt{3} - \frac{3584\, p}{3}\right.$$

$$- \frac{512}{3}\Big) \cos\left(\frac{4 \arccos(\sqrt{p}\,)}{3}\right) + \left(\frac{448}{3} + \left(\frac{224\, p}{3}\right.\right.$$

$$- \frac{224}{3}\Big)\sqrt{3}\Big) \cos(2 \arccos(\sqrt{p}\,)) + \left(1088\, p^{2} + \frac{7904}{9} - \frac{1120}{3}\, p\right)\sqrt{3} - 960$$

$$+ \frac{13696\, p}{3}\Big) \cos\left(\frac{2 \arccos(\sqrt{p}\,)}{3}\right)^{7} + \left(\left(\left(\left(\left(-\frac{640}{3} + \frac{3584\, p}{3}\right)\sqrt{3} + 896\, p^{2}\right.\right.\right.\right.$$

$$- 320\, p + \frac{1856}{3}\Big) \cos\left(\frac{4 \arccos(\sqrt{p}\,)}{3}\right) + \left(-\frac{448\sqrt{3}}{3} + 224\right.$$

$$- 224\, p\Big) \cos(2 \arccos(\sqrt{p}\,)) + \left(\frac{2432\, p}{3} - \frac{64}{3}\right)\sqrt{3} + 832\, p^{2} - 480\, p + 160\Big)$$

$$\sin\left(\frac{2 \arccos(\sqrt{p}\,)}{3}\right) + \left(\left(\left(\frac{1792}{3}\, p - \frac{3712}{3}\, p^{2} - \frac{8576}{9}\right)\sqrt{3} - 4864\, p\right.\right.$$

$$+ \frac{3328}{3}\Big) \cos\left(\frac{4 \arccos(\sqrt{p}\,)}{3}\right) + \left(\frac{1792}{3} + \left(\frac{896\, p}{3}\right.\right.$$

$$- \frac{896}{3}\Big)\sqrt{3}\Big) \cos(2 \arccos(\sqrt{p}\,)) + \left(-448\, p^{2} + \frac{1432}{9} + \frac{1208}{3}\, p\right)\sqrt{3}$$

$$- \frac{2176\,p}{3} - \frac{784}{3}\Bigg) \cos\left(\frac{2 \arccos(\sqrt{p})}{3}\right)^{6} + \Bigg(\Bigg(\Big(\big(-512\,p - 128\big)\sqrt{3} - 384\,p^{2}$$

$$- 192\,p + 64\Big) \cos\left(\frac{4 \arccos(\sqrt{p})}{3}\right) + \left(64\sqrt{3} - 96 + 96\,p\right) \cos\left(2 \arccos(\sqrt{p})\right)$$

$$+ \left(\frac{11776\,p}{3} - \frac{3440}{3}\right)\sqrt{3} + \frac{8104}{3} - 1720\,p + 2944\,p^{2}\Bigg) \sin\left(\frac{2 \arccos(\sqrt{p})}{3}\right)$$

$$+ \Bigg(\left(\left(\frac{1480}{3}\,p - \frac{448}{3}\,p^{2} - \frac{4120}{9}\right)\sqrt{3} - \frac{1408\,p}{3} + \frac{2576}{3}\right) \cos\left(\frac{4 \arccos(\sqrt{p})}{3}\right)$$

$$+ \left(\frac{128}{3} + \left(\frac{64\,p}{3} - \frac{64}{3}\right)\sqrt{3}\right) \cos\left(2 \arccos(\sqrt{p})\right) + \Bigg(\Bigg(-\frac{7264}{3}\,p^{2} - \frac{19616}{9}$$

$$+ \frac{4928}{3}\,p\Bigg)\sqrt{3} + 2880 - 9280\,p\Bigg) \cos\left(\frac{2 \arccos(\sqrt{p})}{3}\right)^{5} + \Bigg(\Bigg(\Bigg(\Bigg(\Bigg(\frac{3632}{3}$$

$$- \frac{12160\,p}{3}\Bigg)\sqrt{3} - 3136\,p^{2} + 2008\,p - \frac{8392}{3}\Bigg) \cos\left(\frac{4 \arccos(\sqrt{p})}{3}\right)$$

$$+ \left(\frac{1472\sqrt{3}}{3} - 736 + 736\,p\right) \cos\left(2 \arccos(\sqrt{p})\right) + \left(\frac{3904\,p}{3} - \frac{1760}{3}\right)\sqrt{3}$$

$$+ \frac{4208}{3} + 304\,p + 384\,p^{2}\Bigg) \sin\left(\frac{2 \arccos(\sqrt{p})}{3}\right) + \Bigg(\Bigg(-\frac{4736}{3}\,p + \frac{7136}{3}\,p^{2}$$

$$+ \frac{19424}{9}\Bigg)\sqrt{3} + \frac{27584\,p}{3} - \frac{8512}{3}\Bigg) \cos\left(\frac{4 \arccos(\sqrt{p})}{3}\right) + \Bigg(-\frac{3328}{3} + \Bigg($$

$$-\frac{1664\,p}{3} + \frac{1664}{3}\Bigg)\sqrt{3}\Bigg) \cos\left(2 \arccos(\sqrt{p})\right) + \Bigg(-584\,p^{2} - \frac{12074}{9}$$

$$+ \frac{134}{3}\,p\Bigg)\sqrt{3} - \frac{11968\,p}{3} + \frac{5228}{3}\Bigg) \cos\left(\frac{2 \arccos(\sqrt{p})}{3}\right)^{4} + \Bigg(\Bigg(\Bigg(\Bigg(\frac{3232}{3}$$

$$- \frac{6848\,p}{3}\Big)\sqrt{3} - 1856\,p^2 + 1904\,p - \frac{6416}{3}\Big)\cos\left(\frac{4\arccos(\sqrt{p})}{3}\right) + \Big(\frac{784\sqrt{3}}{3}$$

$$- 392 + 392\,p\Big)\cos(2\arccos(\sqrt{p})) + \Big(\Big(-\frac{8672\,p}{3} + \frac{3148}{3}\Big)\sqrt{3} - 2632\,p^2$$

$$+ 2502\,p - 2142\Big)\sin\left(\frac{2\arccos(\sqrt{p})}{3}\right) + \Big(\Big(\Big(-\frac{5126}{3}\,p + \frac{5080}{3}\,p^2$$

$$+ \frac{17066}{9}\Big)\sqrt{3} + 6208\,p - 2852\Big)\cos\left(\frac{4\arccos(\sqrt{p})}{3}\right) + \Big(-\frac{2116}{3} + \Big(-\frac{1058\,p}{3}$$

$$+ \frac{1058}{3}\Big)\sqrt{3}\Big)\cos(2\arccos(\sqrt{p})) + \Big(\frac{4816}{3}\,p^2 + \frac{3542}{3} - 1634\,p\Big)\sqrt{3}$$

$$+ \frac{14816\,p}{3} - \frac{5356}{3}\Big)\cos\left(\frac{2\arccos(\sqrt{p})}{3}\right)^3 + \Big(\Big(\Big((2368\,p - 788)\sqrt{3} + 1848\,p^2$$

$$- 1326\,p + 1750)\cos\left(\frac{4\arccos(\sqrt{p})}{3}\right) + \big(-426\,p - 284\sqrt{3}$$

$$+ 426\big)\cos(2\arccos(\sqrt{p})) + (-2064\,p + 876)\sqrt{3} - 1320\,p^2 + 858\,p - 1906\Big)$$

$$\sin\left(\frac{2\arccos(\sqrt{p})}{3}\right) + \big((-900\,p^2 + 576\,p - 828)\sqrt{3} - 3528\,p$$

$$+ 1080\big)\cos\left(\frac{4\arccos(\sqrt{p})}{3}\right) + \big(432 + (216\,p - 216)\sqrt{3}\big)\cos(2\arccos(\sqrt{p}))$$

$$+ (822\,p^2 - 585\,p + 1131)\sqrt{3} + 3696\,p - 1578\Big)\cos\left(\frac{2\arccos(\sqrt{p})}{3}\right)^2$$

$$+ \Big(\Big(\big((2632\,p - 1160)\sqrt{3} + 2172\,p^2 - 2136\,p + 2332\big)\cos\left(\frac{4\arccos(\sqrt{p})}{3}\right) + ($$

$$-444\,p - 296\sqrt{3} + 444\Big)\cos\big(2\arccos(\sqrt{p})\big) + (-80\,p + 40)\sqrt{3} + 306\,p^2 - 672\,p$$

$$-202\Big)\sin\left(\frac{2\arccos(\sqrt{p})}{3}\right) + \Big(\big((-1254\,p^2 + 1233\,p - 1347\big)\sqrt{3} - 4560\,p$$

$$+2010\Big)\cos\left(\frac{4\arccos(\sqrt{p})}{3}\right) + \left(513 + \left(\frac{513\,p}{2}\right.\right.$$

$$\left.\left.-\frac{513}{2}\right)\sqrt{3}\right)\cos\big(2\arccos(\sqrt{p})\big) + \left(-\frac{5651}{3}\,p^2 - \frac{30671}{18} + \frac{12571}{6}\,p\right)\sqrt{3}$$

$$\left.-6006\,p + \frac{7985}{3}\right)\cos\left(\frac{2\arccos(\sqrt{p})}{3}\right) + \Big(\big((672\,p - 336)\sqrt{3} + 582\,p^2$$

$$-660\,p + 646\big)\cos\left(\frac{4\arccos(\sqrt{p})}{3}\right) + 213\left(p + \frac{2\sqrt{3}}{3} - 1\right)\Big(p$$

$$\left.\left.-\frac{\cos\big(2\arccos(\sqrt{p})\big)}{2} - \frac{1}{2}\right)\right)\sin\left(\frac{2\arccos(\sqrt{p})}{3}\right) + \left(\left(\left(-\frac{3977}{3}\,p + \frac{4112}{3}\,p^2\right.\right.\right.$$

$$\left.+\frac{13027}{9}\right)\sqrt{3} + 4980\,p - \left.\frac{6446}{3}\right)\cos\left(\frac{4\arccos(\sqrt{p})}{3}\right)$$

$$\left.+\frac{1679\left(2 + (p-1)\sqrt{3}\right)\left(p - \frac{\cos\left(2\arccos(\sqrt{p})\right)}{2} - \frac{1}{2}\right)}{3}\right)$$

$$\sqrt{2\sqrt{3} - \sqrt{3}\cos\left(\frac{2\arccos(\sqrt{p})}{3}\right) + 3\sin\left(\frac{2\arccos(\sqrt{p})}{3}\right)} + \left(\left(\left(\left(\frac{8299}{6}\right.\right.\right.\right.$$

$$\left.\left.-1380\,p\right)3^{1/4} - \frac{\left(p^2 - \frac{27}{4}\,p + \frac{33173}{12}\right)3^{3/4}}{3}\right)\cos\left(\frac{2\arccos(\sqrt{p})}{3}\right)$$

$$+\left(\left(\frac{31487}{36} + 19\,p^2 - \frac{233}{12}\,p\right)3^{1/4} + 3^{3/4}\left(-\frac{7873}{18}\right.\right.$$

$$+\frac{4048\,p}{9}\Big)\Big)\sin\left(\frac{2\arccos(\sqrt{p})}{3}\right)+\Bigg(\left(\frac{235}{6}-12\,p\right)3^{1/4}$$

$$+\frac{35\left(p^2-\frac{117}{140}\,p-\frac{131}{60}\right)3^{3/4}}{3}\Bigg)\cos\left(\frac{4\arccos(\sqrt{p})}{3}\right)+\Bigg(\Big(\frac{47}{6}$$

$$-\frac{74\,p}{3}\Big)3^{1/4}-\frac{22\,3^{3/4}\left(p^2-\frac{75}{88}\,p+\frac{67}{88}\right)}{3}\Bigg)\cos\left(\frac{10\arccos(\sqrt{p})}{3}\right)+\Big(\Big($$

$$-\frac{83}{6}+26\,p\Big)3^{1/4}+7\,3^{3/4}\left(p^2-\frac{95}{84}\,p+\frac{107}{84}\right)\Big)\cos\left(\frac{14\arccos(\sqrt{p})}{3}\right)+\Bigg(\Big($$

$$-\frac{31}{6}+6\,p\Big)3^{1/4}+\frac{4\left(p^2-\frac{27}{16}\,p+\frac{113}{48}\right)3^{3/4}}{3}\Bigg)\cos\left(\frac{16\arccos(\sqrt{p})}{3}\right)$$

$$+\Bigg(\left(\frac{5}{6}-\frac{4\,p}{3}\right)3^{1/4}-\frac{\left(p^2-\frac{5}{4}\,p+\frac{19}{12}\right)3^{3/4}}{3}\Bigg)\cos\left(\frac{20\arccos(\sqrt{p})}{3}\right)$$

$$+\Bigg(\left(\frac{52}{3}\,p^2-\frac{33}{4}-\frac{221}{12}\,p\right)3^{1/4}$$

$$+\frac{62\,3^{3/4}\left(p+\frac{71}{124}\right)}{9}\Bigg)\sin\left(\frac{10\arccos(\sqrt{p})}{3}\right)+\Bigg(\left(\frac{13}{4}-\frac{5}{3}\,p^2+\frac{13}{12}\,p\right)3^{1/4}$$

$$+\frac{2\,3^{3/4}\left(p-\frac{31}{4}\right)}{9}\Bigg)\sin\left(\frac{14\arccos(\sqrt{p})}{3}\right)+\Big(\Big(-\frac{319}{36}+\frac{85}{12}\,p$$

$$-\frac{20}{3}\,p^2\Big)3^{1/4}+3^{3/4}\left(\frac{9}{2}-\frac{26\,p}{3}\right)\Big)\sin\left(\frac{16\arccos(\sqrt{p})}{3}\right)+\Bigg(\Big(\frac{1}{3}\,p^2-\frac{5}{36}$$

$$+\frac{1}{4}\,p\Big)3^{1/4}+\frac{4\,3^{3/4}\left(p+\frac{3}{8}\right)}{9}\Bigg)\sin\left(\frac{20\arccos(\sqrt{p})}{3}\right)+\Big(\Big(-\frac{137}{6}$$

$$-\frac{10\,p}{3}\Big)3^{1/4}-12\left(p^2-\frac{131}{144}\,p-\frac{535}{432}\right)3^{3/4}\Big)\cos\left(\frac{8\arccos(\sqrt{p})}{3}\right)$$

$$+\Big(\left(\frac{889}{36}+\frac{1}{3}\,p^2-\frac{13}{4}\,p\right)3^{1/4}+3^{3/4}\Big(-\frac{77}{6}$$

$$
+ \frac{100 p}{9} \Big) \Big) \sin\left(\frac{4 \arccos(\sqrt{p})}{3} \right) + \left(\left(-\frac{20}{3} p^2 - \frac{289}{36} + \frac{43}{12} p \right) 3^{1/4} \right.
$$

$$
+ 3^{3/4} \left(\frac{7}{2} - 10 p \right) \Big) \sin\left(\frac{8 \arccos(\sqrt{p})}{3} \right) + \left(\frac{3^{1/4}}{6} + \left(-\frac{1}{12} \right. \right.
$$

$$
+ \frac{p}{12} \Big) 3^{3/4} \Big) \cos\left(\frac{22 \arccos(\sqrt{p})}{3} \right) + \left(\left(-\frac{1}{12} + \frac{p}{12} \right) 3^{1/4} \right.
$$

$$
+ \frac{3^{3/4}}{18} \Big) \sin\left(\frac{22 \arccos(\sqrt{p})}{3} \right) + \left(\left(-14 + \frac{50 p}{3} \right) 3^{1/4} + 3^{3/4} \left(-\frac{2}{3} p + p^2 \right. \right.
$$

$$
+ \frac{85}{9} \Big) \Big) \cos(2 \arccos(\sqrt{p})) + \left(\left(\frac{1}{2} + \frac{28 p}{3} \right) 3^{1/4} \right.
$$

$$
+ \frac{17 \left(p^2 - \frac{77}{68} p - \frac{7}{68} \right) 3^{3/4}}{3} \Big) \cos(4 \arccos(\sqrt{p})) + \left(\left(2 - \frac{16 p}{3} \right) 3^{1/4} \right.
$$

$$
- \frac{4 \left(p^2 - \frac{3}{4} p + \frac{13}{12} \right) 3^{3/4}}{3} \Big) \cos(6 \arccos(\sqrt{p})) + \left(\left(\frac{251}{6} - 21 p^2 \right. \right.
$$

$$
+ \frac{33}{2} p \Big) 3^{1/4} + \frac{14 \left(p - \frac{65}{14} \right) 3^{3/4}}{3} \Big) \sin(2 \arccos(\sqrt{p})) + \left(\left(\frac{185}{12} + 15 p^2 \right. \right.
$$

$$
- \frac{63}{4} p \Big) 3^{1/4} + 3^{3/4} \left(-\frac{47}{6} + \frac{52 p}{3} \right) \Big) \sin(4 \arccos(\sqrt{p})) + \left(\left(-\frac{11}{3} + 3 p \right. \right.
$$

$$
- 2 p^2 \Big) 3^{1/4} + 3^{3/4} \left(-\frac{8 p}{3} + 2 \right) \Big) \sin(6 \arccos(\sqrt{p})) + \left(\frac{4106 p}{3} \right.
$$

$$
- \frac{8267}{6} \Big) 3^{1/4} - \frac{16 \left(p^2 - \frac{73}{64} p - \frac{33077}{192} \right) 3^{3/4}}{3} \Big)
$$

$$
\sqrt{6 - 3 \sqrt{3} \sin\left(\frac{2 \arccos(\sqrt{p})}{3} \right) + 3 \cos\left(\frac{2 \arccos(\sqrt{p})}{3} \right)} + \left(\left(\left(-\frac{11264}{3} \right. \right. \right.
$$

$$
- 4608 p^2 + 5632 p \Big) 3^{1/4} + 3^{3/4} \left(-\frac{13312 p}{3} + 2048 \right) \Big) \sin\left(\frac{\arccos(\sqrt{p})}{3} \right)
$$

$$+ \Bigg(\bigg(\Big(-\frac{2048}{3} + \frac{5120\, p}{3} \Big)\, 3^{1/4}$$

$$+ \frac{512\, 3^{3/4} \left(p^2 + p + \frac{10}{3}\right)}{3} \bigg) \cos\left(\frac{\arccos\left(\sqrt{p}\right)}{3}\right) + \bigg(\Big(\frac{16384}{3} + 5120\, p^2$$

$$- 5120\, p \Big)\, 3^{1/4} + 6144 \left(p - \frac{4}{9}\right) 3^{3/4} \bigg) \sqrt{1-p} + \bigg(\Big(\frac{5120}{3} - 3072\, p \Big)\, 3^{1/4}$$

$$- \frac{2560 \left(p^2 - \frac{6}{5}\, p + \frac{19}{15}\right) 3^{3/4}}{3} \bigg) \cos\left(\frac{5 \arccos\left(\sqrt{p}\right)}{3}\right) + \bigg(\Big(-\frac{6656}{3} - 2560\, p^2$$

$$+ 2048\, p \Big)\, 3^{1/4} + (-3072\, p + 1024)\, 3^{3/4} \bigg) \sin\left(\frac{5 \arccos\left(\sqrt{p}\right)}{3}\right) + \bigg(\frac{1024\, 3^{1/4}}{3}$$

$$+ \left(\frac{512\, p}{3} - \frac{512}{3} \right) 3^{3/4} \bigg) \cos\left(\frac{7 \arccos\left(\sqrt{p}\right)}{3}\right) + 512 \bigg((p-1)\, 3^{1/4}$$

$$+ \frac{2\, 3^{3/4}}{3} \bigg) \sin\left(\frac{7 \arccos\left(\sqrt{p}\right)}{3}\right) \bigg) \bigg) \Bigg) \Bigg/$$

$$\Bigg(65536 \sqrt{2\sqrt{3} - \sqrt{3} \cos\left(\frac{2 \arccos\left(\sqrt{p}\right)}{3}\right) + 3 \sin\left(\frac{2 \arccos\left(\sqrt{p}\right)}{3}\right)}$$

$$\sqrt{6 - 3\sqrt{3} \sin\left(\frac{2 \arccos\left(\sqrt{p}\right)}{3}\right) + 3 \cos\left(\frac{2 \arccos\left(\sqrt{p}\right)}{3}\right)}\; p \Bigg(\bigg(\Big(\Big(-\frac{4}{3}$$

$$+ \frac{4\, p}{3} \Big) \sqrt{3} + \frac{7}{3} + p^2 - 2\, p \Big) \bigg(\frac{145}{8192} + \cos\left(\frac{\arccos\left(\sqrt{p}\right)}{3}\right)^{12}$$

$$- \frac{9 \cos\left(\frac{\arccos\left(\sqrt{p}\right)}{3}\right)^{10}}{4} + \frac{15 \cos\left(\frac{\arccos\left(\sqrt{p}\right)}{3}\right)^{8}}{16}$$

$$+ \frac{27 \cos\left(\frac{\arccos\left(\sqrt{p}\right)}{3}\right)^{6}}{64} - \frac{57 \cos\left(\frac{\arccos\left(\sqrt{p}\right)}{3}\right)^{4}}{512}$$

$$- \frac{129 \cos\left(\frac{\arccos\left(\sqrt{p}\right)}{3}\right)^{2}}{2048} \bigg) \sin\left(\frac{\arccos\left(\sqrt{p}\right)}{3}\right) + \bigg(\Big(-\frac{7}{4} - \frac{3}{4}\, p^2 + \frac{3}{2}\, p \Big) \sqrt{3}$$

$$- 3 p + 3\Big) \cos\left(\frac{\arccos(\sqrt{p})}{3}\right)^{11} + \left(\left(\frac{63}{16} + \frac{27}{16} p^2 - \frac{27}{8} p\right)\sqrt{3} + \frac{27 p}{4}\right.$$

$$- \frac{27}{4}\Big) \cos\left(\frac{\arccos(\sqrt{p})}{3}\right)^{9} + \left(\left(-\frac{161}{64} - \frac{69}{64} p^2 + \frac{69}{32} p\right)\sqrt{3} - \frac{69 p}{16}\right.$$

$$+ \frac{69}{16}\Big) \cos\left(\frac{\arccos(\sqrt{p})}{3}\right)^{7} + \left(\left(\frac{175}{512} + \frac{75}{512} p^2 - \frac{75}{256} p\right)\sqrt{3} + \frac{75 p}{128}\right.$$

$$- \frac{75}{128}\Big) \cos\left(\frac{\arccos(\sqrt{p})}{3}\right)^{5} + \left(\left(-\frac{77}{2048} - \frac{33}{2048} p^2 + \frac{33}{1024} p\right)\sqrt{3} - \frac{33 p}{512}\right.$$

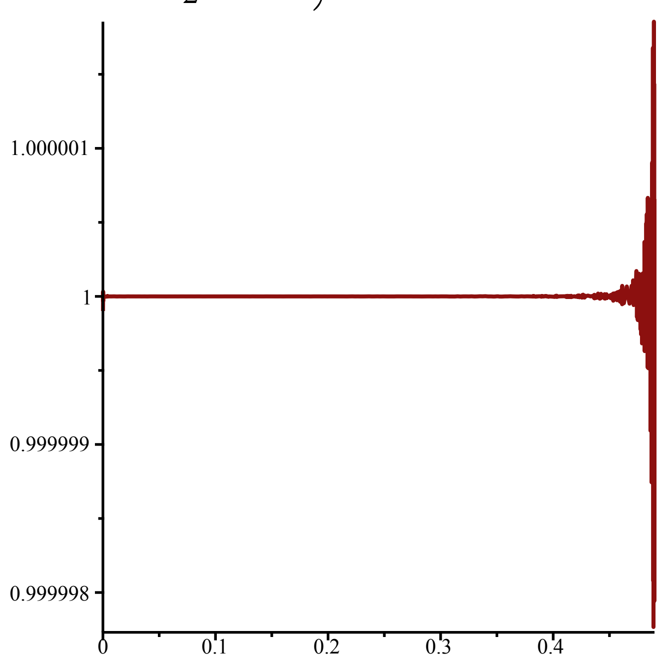
$$+ \frac{33}{512}\Big) \cos\left(\frac{\arccos(\sqrt{p})}{3}\right)^{3} + \left(\left(\frac{147}{8192} + \frac{63}{8192} p^2 - \frac{63}{4096} p\right)\sqrt{3} - \frac{63}{2048}\right.$$

$$+ \frac{63 p}{2048}\Big) \cos\left(\frac{\arccos(\sqrt{p})}{3}\right) + \left(\left(-\frac{49}{2048} + \frac{49 p}{2048}\right)\sqrt{3} + \frac{343}{8192} + \frac{147 p^2}{8192}\right.$$

$$- \frac{147 p}{4096}\Big)\sqrt{1-p} + \left(-\frac{21\sqrt{3}}{4096} + \frac{21}{2048}\right) p^{3/2} + \frac{21\sqrt{3}\, p^{5/2}}{8192}$$

$$+ \frac{49\sqrt{p}\left(\sqrt{3} - \frac{12}{7}\right)}{8192}\Bigg)\Bigg)$$

> $plot\left(ProbaFiniteSubcritVal, p = 0 .. \frac{1}{2} - 0.01\right);$



In the supercritical range p>1/2 (we will replace Vmi by its value later, it is easier for Maple):

> $ProbaFiniteSupcrit := simplify\left(int\left(subs\left(Vm = \text{sqrt}(3) - 2 * Vmi, Vp = \frac{\text{sqrt}(3)}{3}, Vpi = \frac{2 \cdot p \cdot \text{sqrt}(3)}{3}, constfactor \cdot rootfactor \cdot deltafactor \cdot lastfactor\right), V = \frac{2 \cdot p \cdot \text{sqrt}(3)}{3} .. Vmi\right)\right)$ assuming $p > \frac{1}{2}$ **and** $p < 1$ **and** $Vmi > \frac{2 \cdot p \cdot \text{sqrt}(3)}{3}$ **and** $Vmi < \frac{2 \cdot \text{sqrt}(3)}{3}$;

$$ProbaFiniteSupcrit := -\Bigg(648\Bigg(\Bigg(\sqrt{2\sqrt{3} - 3\,Vmi}\,\Bigg(\Bigg(4\,Vmi^{9/2} + \frac{64\sqrt{Vmi}}{9} + 32\,Vmi^{5/2} - \frac{32\sqrt{3}\,Vmi^{7/2}}{3} - \frac{128\sqrt{3}\,Vmi^{3/2}}{9}\Bigg)p^{3/2} + \frac{64\sqrt{p}\,(3+\sqrt{3})\,Vmi^{3/2}}{9} - 16\sqrt{p}\,(1+\sqrt{3})\,Vmi^{5/2} + \frac{16\sqrt{p}\,(3+\sqrt{3})\,Vmi^{7/2}}{3} - 2\sqrt{p}\,(1+\sqrt{3})\,Vmi^{9/2} - \frac{32\sqrt{Vmi}\,(1+\sqrt{3})\sqrt{p}}{9} + \Bigg(\Bigg(\Bigg(p - \frac{31}{2}\Bigg)Vmi^{6} + 19\,Vmi^{5} + \Bigg(\frac{47\,p}{4} - 49\Bigg)Vmi^{4} + \frac{136\,Vmi^{3}}{3} + \Bigg(\frac{46\,p}{9} - \frac{116}{9}\Bigg)Vmi^{2} + \frac{16\,Vmi}{3} - \frac{4\,p}{9}\Bigg)3^{3/4} - \frac{1}{2}\Bigg(19\Bigg(-\frac{20}{57} - \frac{9\,Vmi^{6}}{19} + \frac{12\,Vmi^{5}}{19} + \Bigg(p - \frac{527}{76}\Bigg)Vmi^{4} + \frac{144\,Vmi^{3}}{19} + \Bigg(\frac{124\,p}{57} - \frac{362}{57}\Bigg)Vmi^{2} + \frac{256\,Vmi}{57}\Bigg)Vmi\,3^{1/4}\Bigg)\Bigg)\sqrt{2}\,p\Bigg)\sqrt{1-p} \qquad \textbf{(1.5.3.7)}$$

$$- \frac{1}{9} \left(4 \left(\left(\left(\left(\frac{123\ 3^{3/4}}{2} + 108\ 3^{1/4} \right) Vmi^{3/2} + \left(-180\ 3^{3/4} - \frac{783\ 3^{1/4}}{2} \right) Vmi^{5/2} \right.\right.\right.\right.$$

$$+ \left(\frac{1629\ 3^{3/4}}{4} + 450\ 3^{1/4} \right) Vmi^{7/2} + \left(- \frac{405\ 3^{3/4}}{2} - \frac{11421\ 3^{1/4}}{16} \right) Vmi^{9/2}$$

$$+ \left(\frac{7695\ 3^{3/4}}{32} + \frac{567\ 3^{1/4}}{4} \right) Vmi^{11/2} + \left(- \frac{1053\ 3^{1/4}}{8} - \frac{27\ 3^{3/4}}{2} \right) Vmi^{13/2}$$

$$+ \frac{81\ 3^{3/4}\ Vmi^{15/2}}{8} - 9 \left(\frac{8\ 3^{3/4}}{9} + 3^{1/4} \right) \sqrt{Vmi} \Bigg) p^{3/2} + \Bigg(-12\ 3^{3/4}\ Vmi^{3/2}$$

$$+ \frac{153\ Vmi^{5/2}\ 3^{1/4}}{2} - \frac{141\ 3^{3/4}\ Vmi^{7/2}}{2} + \frac{1593\ 3^{1/4}\ Vmi^{9/2}}{16}$$

$$- \frac{189\ 3^{3/4}\ Vmi^{11/2}}{8} + 3^{1/4} \left(\frac{27\ Vmi^{13/2}}{4} + \sqrt{Vmi} \right) \Bigg) p^{5/2} + 8 \sqrt{p} \Bigg(\Bigg($$

$$- \frac{27\ 3^{1/4}}{2} - \frac{99\ 3^{3/4}}{16} \Bigg) Vmi^{3/2} + \left(\frac{315\ 3^{1/4}}{8} + \frac{45\ 3^{3/4}}{2} \right) Vmi^{5/2} + \Bigg($$

$$- \frac{225\ 3^{1/4}}{4} - \frac{1347\ 3^{3/4}}{32} \Bigg) Vmi^{7/2} + \left(\frac{2457\ 3^{1/4}}{32} + \frac{405\ 3^{3/4}}{16} \right) Vmi^{9/2} + \Bigg($$

$$- \frac{567\ 3^{1/4}}{32} - \frac{6939\ 3^{3/4}}{256} \Bigg) Vmi^{11/2} + \left(\frac{999\ 3^{1/4}}{64} + \frac{27\ 3^{3/4}}{16} \right) Vmi^{13/2}$$

$$- \frac{81\, 3^{3/4}\, Vmi^{15/2}}{64} + \sqrt{Vmi}\,\left(3^{1/4} + 3^{3/4}\right)\Big)\Big)\sqrt{2}\Big)\Big)\sqrt{2p-1}\,\sqrt{-\sqrt{3} + 3\, Vmi}$$

$$- \frac{1}{9}\Bigg(64\sqrt{2\sqrt{3} - 3\, Vmi}\,\sqrt{1-p}\,\Bigg(\Bigg(\Bigg(\left(\frac{23\, 3^{3/4}}{2} + 24\, 3^{1/4}\right) Vmi^{3/2} + \Bigg($$

$$-39\, 3^{3/4} - \frac{267\, 3^{1/4}}{4}\Bigg) Vmi^{5/2} + \left(\frac{141\, 3^{3/4}}{2} + 99\, 3^{1/4}\right) Vmi^{7/2} + \Bigg(-\frac{369\, 3^{3/4}}{8}$$

$$- \frac{4221\, 3^{1/4}}{32}\Bigg) Vmi^{9/2} + \left(\frac{1557\, 3^{3/4}}{32} + \frac{135\, 3^{1/4}}{4}\right) Vmi^{11/2} + \Bigg(-\frac{945\, 3^{1/4}}{32}$$

$$- \frac{27\, 3^{3/4}}{8}\Bigg) Vmi^{13/2} + \frac{81\, 3^{3/4}\, Vmi^{15/2}}{32} - \frac{5\sqrt{Vmi}\left(\frac{4\, 3^{3/4}}{5} + 3^{1/4}\right)}{2}\Bigg) p^{3/2}$$

$$+ \Bigg(-4\, 3^{3/4}\, Vmi^{3/2} + \frac{39\, Vmi^{5/2}\, 3^{1/4}}{2} - \frac{33\, 3^{3/4}\, Vmi^{7/2}}{2} + \frac{369\, 3^{1/4}\, Vmi^{9/2}}{16}$$

$$- \frac{45\, 3^{3/4}\, Vmi^{11/2}}{8} + 3^{1/4}\left(\frac{27\, Vmi^{13/2}}{16} + \sqrt{Vmi}\right)\Bigg) p^{5/2} + \sqrt{p}\,\Bigg(\Bigg(-\frac{19\, 3^{3/4}}{4}$$

$$- 12\, 3^{1/4}\Bigg) Vmi^{3/2} + \left(\frac{39\, 3^{3/4}}{2} + \frac{57\, 3^{1/4}}{2}\right) Vmi^{5/2} + \Bigg(-\frac{249\, 3^{3/4}}{8}$$

$$- \frac{99\, 3^{1/4}}{2}\Bigg) Vmi^{7/2} + \left(\frac{369\, 3^{3/4}}{16} + \frac{963\, 3^{1/4}}{16}\right) Vmi^{9/2} + \Bigg(-\frac{1467\, 3^{3/4}}{64}$$

$$- \frac{135\, 3^{1/4}}{8}\Bigg) Vmi^{11/2} + \left(\frac{27\, 3^{3/4}}{16} + \frac{459\, 3^{1/4}}{32}\right) Vmi^{13/2} - \frac{81\, 3^{3/4}\, Vmi^{15/2}}{64}$$

$$+ \sqrt{Vmi}\,\left(3^{1/4} + 3^{3/4}\right)\Bigg)\Bigg)\Bigg)\Bigg)\Bigg) \Bigg/ \Bigg(\left(2\sqrt{3} - 3\, Vmi\right)^{9/2}\sqrt{Vmi}\,\sqrt{1-p}\,\sqrt{-\sqrt{3} + 3\, Vmi}\,\sqrt{2p-1}\; p^{3/2}\left(48p - 48 + 32\sqrt{3}\right)\Bigg)$$

We replace Vmi by its value:

$$> ProbaFiniteSupcritVal := simplify\left(subs\left(Vmi = \frac{2\cdot \mathrm{sqrt}(3)}{3}\cdot\left(\cos\left(\frac{\arccos(\mathrm{sqrt}(p))}{3}\right)\right)^2, ProbaFiniteSupcrit\right)\right) \text{ assuming } p > \frac{1}{2} \textbf{ and } p < 1;$$

$$
\begin{aligned}
&\mathit{ProbaFiniteSupcritVal} := -\Bigg(2\Bigg(\sqrt{2p-1}\,\Bigg(3\sqrt{3}\,\left(\sqrt{p}-p^{3/2}\right)\cos\left(\frac{\arccos(\sqrt{p})}{3}\right)^{7} \qquad \textbf{(1.5.3.8)}\\
&+3\sqrt{3}\sqrt{1-p}\,\sin\left(\frac{\arccos(\sqrt{p})}{3}\right)\cos\left(\frac{\arccos(\sqrt{p})}{3}\right)^{6} p+\Bigg(\Bigg(6\\
&+\frac{15\sqrt{3}}{2}\Bigg)p^{3/2}-\sqrt{3}\,p^{5/2}+\sqrt{p}\left(-6-\frac{13\sqrt{3}}{2}\right)\Bigg)\cos\left(\frac{\arccos(\sqrt{p})}{3}\right)^{5}\\
&+\sqrt{1-p}\,\sin\left(\frac{\arccos(\sqrt{p})}{3}\right)\left(-6+\left(p-\frac{7}{2}\right)\sqrt{3}\right)p\cos\left(\frac{\arccos(\sqrt{p})}{3}\right)^{4}\\
&+\Bigg(\left(-\frac{15}{2}-\frac{87\sqrt{3}}{16}\right)p^{3/2}+\frac{5\sqrt{3}\,p^{5/2}}{4}+\sqrt{p}\,\Bigg(\frac{15}{2}\\
&+\frac{67\sqrt{3}}{16}\Bigg)\Bigg)\cos\left(\frac{\arccos(\sqrt{p})}{3}\right)^{3}\\
&-\frac{3\sqrt{1-p}\,\sin\left(\frac{\arccos(\sqrt{p})}{3}\right)\left(-6+\left(p-\frac{5}{4}\right)\sqrt{3}\right)p\cos\left(\frac{\arccos(\sqrt{p})}{3}\right)^{2}}{4}\\
&+\Bigg(-\frac{\sqrt{1-p}\,\left(-2p^{3/2}\sqrt{3}+\sqrt{p}\,(3+\sqrt{3})\right)\sin\left(\frac{\arccos(\sqrt{p})}{3}\right)}{2}+\Bigg(\frac{9\sqrt{3}}{16}\\
&+\frac{3}{2}\Bigg)p^{3/2}-\frac{\sqrt{3}\,p^{5/2}}{16}-\frac{\sqrt{p}\,(3+\sqrt{3})}{2}\Bigg)\cos\left(\frac{\arccos(\sqrt{p})}{3}\right)\\
&-\frac{3\sqrt{3}\sqrt{1-p}\,\sin\left(\frac{\arccos(\sqrt{p})}{3}\right)p^{2}}{16}\Bigg)\sqrt{2\cos\left(\frac{\arccos(\sqrt{p})}{3}\right)^{2}-1}\\
&+6\sqrt{1-p}\,\Bigg(\cos\left(\frac{\arccos(\sqrt{p})}{3}\right)^{2}\\
&-\frac{1}{2}\Bigg)^{2}\sin\left(\frac{\arccos(\sqrt{p})}{3}\right)\cos\left(\frac{\arccos(\sqrt{p})}{3}\right)\Bigg(\sqrt{3}\,\Big(\sqrt{p}
\end{aligned}
$$

$$- 2 p^{3/2}\Big) \cos\left(\frac{\arccos(\sqrt{p})}{3}\right)^2 + \left(\frac{5\sqrt{3}}{3} + 4\right) p^{3/2} - \frac{2\sqrt{3}\, p^{5/2}}{3}$$

$$- \frac{2\sqrt{p}\,(3+\sqrt{3})}{3}\Bigg)\Bigg)\sqrt{3}\left(\cos\left(\frac{\arccos(\sqrt{p})}{3}\right) + 1\right)^4 \left(\cos\left(\frac{\arccos(\sqrt{p})}{3}\right)\right.$$

$$\left.- 1\right)^4\Bigg) \Bigg/ \Bigg(3\sqrt{1-p}\, p^{3/2}\sqrt{2p-1}\sqrt{2\cos\left(\frac{\arccos(\sqrt{p})}{3}\right)^2 - 1}\,\Big(3p - 3$$

$$+ 2\sqrt{3}\Big) \sin\left(\frac{\arccos(\sqrt{p})}{3}\right)^9 \cos\left(\frac{\arccos(\sqrt{p})}{3}\right)\Bigg)$$

> ProbaPercoSupcVal1 := simplify(1 - ProbaFiniteSupcritVal, trig); plot(%, p = 1/2 ..1);

$$ProbaPercoSupcVal1 := \Bigg(16\Bigg(\sqrt{2p-1}\Bigg(\big(3\sqrt{p} - 3p^{3/2}\big)\cos\left(\frac{\arccos(\sqrt{p})}{3}\right)^7$$

$$+ 3\cos\left(\frac{\arccos(\sqrt{p})}{3}\right)^6 \sqrt{1-p}\,\sin\left(\frac{\arccos(\sqrt{p})}{3}\right) p + \Bigg(\left(\left(\frac{15}{2} + 2\sqrt{3}\right)p^{3/2} - p^{5/2}\right.$$

$$\left. + \left(-\frac{13}{2} - 2\sqrt{3}\right)\sqrt{p}\right)\cos\left(\frac{\arccos(\sqrt{p})}{3}\right)^5 + \sqrt{1-p}\,\sin\left(\frac{\arccos(\sqrt{p})}{3}\right) p\,\Big(p - \frac{7}{2}$$

$$- 2\sqrt{3}\Big)\cos\left(\frac{\arccos(\sqrt{p})}{3}\right)^4 + \Bigg(\left(\left(-\frac{87}{16} - \frac{5\sqrt{3}}{2}\right)p^{3/2} + \frac{5p^{5/2}}{4} + \Big(\frac{67}{16}\right.$$

$$\left. + \frac{5\sqrt{3}}{2}\Big)\sqrt{p}\right)\cos\left(\frac{\arccos(\sqrt{p})}{3}\right)^3$$

$$- \frac{3\sqrt{1-p}\,\sin\left(\frac{\arccos(\sqrt{p})}{3}\right) p\left(p - \frac{5}{4} - 2\sqrt{3}\right)\cos\left(\frac{\arccos(\sqrt{p})}{3}\right)^2}{4} + \Bigg($$

$$-\frac{\sqrt{1-p}\left(\left(-2\sqrt{3}+1\right)p^{3/2} - 3p^{5/2} + \left(1+\sqrt{3}\right)\sqrt{p}\right)\sin\left(\frac{\arccos(\sqrt{p})}{3}\right)}{2} + \Bigg(\frac{\sqrt{3}}{2}$$

$$+ \frac{9}{16}\Bigg)p^{3/2} - \frac{p^{5/2}}{16} + \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\right)\sqrt{p}\Bigg)\cos\left(\frac{\arccos(\sqrt{p})}{3}\right)$$

$$- \frac{3\sqrt{1-p}\,\sin\left(\frac{\arccos\left(\sqrt{p}\right)}{3}\right)p^{2}}{16}\Bigg)\sqrt{2\cos\left(\frac{\arccos\left(\sqrt{p}\right)}{3}\right)^{2}-1}$$

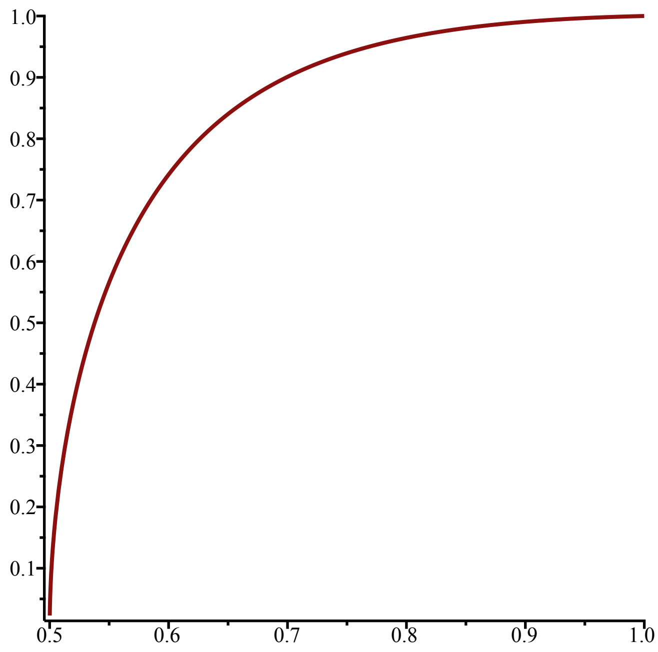
$$+6\sqrt{1-p}\left(\cos\left(\frac{\arccos\left(\sqrt{p}\right)}{3}\right)^{2}-\frac{1}{2}\right)^{2}\sin\left(\frac{\arccos\left(\sqrt{p}\right)}{3}\right)\Bigg(\Big(\big(\sqrt{p}$$

$$-2p^{3/2}\big)\cos\left(\frac{\arccos\left(\sqrt{p}\right)}{3}\right)^{2}+\left(\frac{4\sqrt{3}}{3}+\frac{5}{3}\right)p^{3/2}-\frac{2p^{5/2}}{3}+\Big(-\frac{2\sqrt{3}}{3}$$

$$-\frac{2}{3}\Big)\sqrt{p}\Big)\cos\left(\frac{\arccos\left(\sqrt{p}\right)}{3}\right)\Bigg)\left(\cos\left(\frac{\arccos\left(\sqrt{p}\right)}{3}\right)+1\right)^{4}\Big(\cos\left(\frac{\arccos\left(\sqrt{p}\right)}{3}\right)$$

$$-1\Big)^{4}\Bigg)\Bigg/\Bigg(\sqrt{1-p}\,p^{3/2}\sqrt{2p-1}\sqrt{2\cos\left(\frac{\arccos\left(\sqrt{p}\right)}{3}\right)^{2}-1}\,\big(24p-24$$

$$+16\sqrt{3}\big)\sin\left(\frac{\arccos\left(\sqrt{p}\right)}{3}\right)^{9}\cos\left(\frac{\arccos\left(\sqrt{p}\right)}{3}\right)\Bigg)$$



We will try to simplify the expression. If you have an nicer way to do it, please tell me !

> $ProbaPercoSupcVal2 := factor\left(rationalize\left(simplify\left(expand\left(simplify\left(subs\left(Vmi = \frac{2\cdot \mathrm{sqrt}(3)}{3}\cdot(\cos(\mathrm{theta}))^{2}, p = (\cos(3\cdot\mathrm{theta}))^{2}, 1 - ProbaFiniteSupcrit\right), trig\right)\right)\right)\right)\right)$ assuming theta > 0 **and** theta $< \frac{\mathrm{Pi}}{12}$;

$$ProbaPercoSupcVal2 := -\Big(2\Big(-16\cos(\theta)^{8}\sqrt{-1+2\cos(\theta)^{2}} \qquad (1.5.3.9)$$

$$+ 32 \cos(\theta)^6 \sqrt{-1 + 2\cos(\theta)^2}$$
$$+ 16 \cos(\theta)^6 \sqrt{\left(-1 + 2\cos(\theta)^2\right)\left(16\cos(\theta)^4 - 16\cos(\theta)^2 + 1\right)}$$
$$- 24 \cos(\theta)^4 \sqrt{-1 + 2\cos(\theta)^2}$$
$$- 24 \cos(\theta)^4 \sqrt{\left(-1 + 2\cos(\theta)^2\right)\left(16\cos(\theta)^4 - 16\cos(\theta)^2 + 1\right)}$$
$$+ 2\sqrt{3}\cos(\theta)^2 \sqrt{-1 + 2\cos(\theta)^2} + 8\cos(\theta)^2 \sqrt{-1 + 2\cos(\theta)^2}$$
$$+ 9\cos(\theta)^2 \sqrt{\left(-1 + 2\cos(\theta)^2\right)\left(16\cos(\theta)^4 - 16\cos(\theta)^2 + 1\right)}$$
$$- \sqrt{3}\sqrt{-1 + 2\cos(\theta)^2} - \sqrt{-1 + 2\cos(\theta)^2}\Big)$$
$$\sqrt{\left(-1 + 2\cos(\theta)^2\right)\left(16\cos(\theta)^4 - 16\cos(\theta)^2 + 1\right)}\ (\cos(\theta) + 1)^5 \big(\cos(\theta)$$
$$- 1\big)^5\Big) \Big/ \Big(\left(27\cos(\theta)^2 - 72\cos(\theta)^4 + 48\cos(\theta)^6 + 2\sqrt{3} - 3\right)\big(-2\cos(\theta)$$
$$+ \sqrt{3}\big)^2 \cos(\theta)^2 \left(2\cos(\theta) + \sqrt{3}\right)^2 \sin(\theta)^{10}\Big)$$

> *ProbaPercoSupcVal3* := *simplify*(*expand*(**(1.5.3.9)**), *trig*);

$$ProbaPercoSupcVal3 := -\Bigg(4\bigg(\sqrt{-48\cos(\theta)^4 + 18\cos(\theta)^2 - 1 + 32\cos(\theta)^6}\ \bigg(\sqrt{3} \qquad \textbf{(1.5.3.10)}$$
$$- 8\left(\cos(\theta)^2 - \frac{1}{2}\right)^3\bigg)\sqrt{-1 + 2\cos(\theta)^2} + 256\cos(\theta)^{10} - 640\cos(\theta)^8$$
$$+ 544\cos(\theta)^6 - 168\cos(\theta)^4 + 9\cos(\theta)^2\bigg)\ (\cos(\theta) + 1)^5\ \big(\cos(\theta)$$
$$- 1\big)^5 \left(\cos(\theta)^2 - \frac{1}{2}\right)\Bigg) \Bigg/ \Big(\big(27\cos(\theta)^2 - 72\cos(\theta)^4 + 48\cos(\theta)^6 + 2\sqrt{3}$$
$$- 3\big)\left(4\cos(\theta)^2 - 3\right)^2 \cos(\theta)^2 \sin(\theta)^{10}\Big)$$

The terms to the power 5 simplify with the sinus^10 and a minus sign. For the rest several factors simplify with cos(6theta) = 2p-1.

> *combine*$\left(27\cos(\theta)^2 - 72\cos(\theta)^4 + 48\cos(\theta)^6 + 2\sqrt{3} - 3\right)$;

$$-\frac{3}{2} + 2\sqrt{3} + \frac{3\cos(6\theta)}{2} \qquad \textbf{(1.5.3.11)}$$

> *combine*$\left(\left(4\cos(\theta)^2 - 3\right)^2 \cos(\theta)^2\right)$;

$$\frac{1}{2} + \frac{\cos(6\theta)}{2} \qquad \textbf{(1.5.3.12)}$$

> *combine*$\left(32\cos(\theta)^6 - 48\cos(\theta)^4 + 18\cos(\theta)^2 - 1\right)$;

$$\cos(6\theta) \qquad \textbf{(1.5.3.13)}$$

> *factor*$\left(256\cos(\theta)^{10} - 640\cos(\theta)^8 + 544\cos(\theta)^6 - 168\cos(\theta)^4 + 9\cos(\theta)^2\right)$;

(1.5.3.14)

$$\cos(\theta)^2\left(16\cos(\theta)^4-16\cos(\theta)^2+1\right)\left(4\cos(\theta)^2-3\right)^2 \quad \textbf{(1.5.3.14)}$$

> $combine\left(\cos(\theta)^2\ \left(4\cos(\theta)^2-3\right)^2\right)$

$$\frac{1}{2}+\frac{\cos(6\,\theta)}{2} \quad \textbf{(1.5.3.15)}$$

> $combine\left(\left(16\cos(\theta)^4-16\cos(\theta)^2+1\right)\cdot\left(2\cos(\theta)^2-1\right)\right);$

$$\cos(6\,\theta) \quad \textbf{(1.5.3.16)}$$

The final expression for the probability that the root cluster is infinite:

> $ProbaPercoSupcVal4 := \left(2\left(\sqrt{2\,p-1}\left(\sqrt{3}-\left(2\cdot\cos(\theta)^2-1\right)^3\right)\cdot\left(2\cdot\cos(\theta)^2-1\right)\cdot\sqrt{-1+2\cos(\theta)^2}+(2\,p-1)\cdot\left(\frac{1}{2}+\frac{2\,p-1}{2}\right)\right)\right)\Big/\left(\left(-\frac{3}{2}+2\sqrt{3}+\frac{3\cdot(2\,p-1)}{2}\right)\cdot\left(\frac{1}{2}+\frac{2\,p-1}{2}\right)\right);$

$$ProbaPercoSupcVal4 := \frac{2\left(\sqrt{2\,p-1}\left(\sqrt{3}-\left(-1+2\cos(\theta)^2\right)^3\right)\left(-1+2\cos(\theta)^2\right)^{3/2}+(2\,p-1)\,p\right)}{\left(3\,p-3+2\sqrt{3}\right)p} \quad \textbf{(1.5.3.17)}$$

Again, Maple does not simplify ...

> $simplify\left(subs\left(\text{theta}=\frac{\arccos(\text{sqrt}(p))}{3}, ProbaPercoSupcVal4\right)-ProbaPercoSupcVal1\right)$ assuming $p>\frac{1}{2}$ **and** $p<1; plot(\%, p=0..1);$

$$-\left(16\left(\sqrt{2\,p-1}\left(\left(3\sqrt{p}-3\,p^{3/2}\right)\cos\left(\frac{\arccos(\sqrt{p})}{3}\right)^7+3\cos\left(\frac{\arccos(\sqrt{p})}{3}\right)^6\sqrt{1-p}\,\sin\left(\frac{\arccos(\sqrt{p})}{3}\right)p+\left(\left(\frac{15}{2}+2\sqrt{3}\right)p^{3/2}-p^{5/2}+\left(-\frac{13}{2}-2\sqrt{3}\right)\sqrt{p}\right)\cos\left(\frac{\arccos(\sqrt{p})}{3}\right)^5+\sqrt{1-p}\,\sin\left(\frac{\arccos(\sqrt{p})}{3}\right)p\left(p-\frac{7}{2}-2\sqrt{3}\right)\cos\left(\frac{\arccos(\sqrt{p})}{3}\right)^4+\left(\left(-\frac{87}{16}-\frac{5\sqrt{3}}{2}\right)p^{3/2}+\frac{5\,p^{5/2}}{4}+\left(\frac{67}{16}+\frac{5\sqrt{3}}{2}\right)\sqrt{p}\right)\cos\left(\frac{\arccos(\sqrt{p})}{3}\right)^3\right.\right.\right.$$

$$- \frac{3\sqrt{1-p}\,\sin\left(\frac{\arccos\left(\sqrt{p}\right)}{3}\right) p \left(p - \frac{5}{4} - 2\sqrt{3}\right)\cos\left(\frac{\arccos\left(\sqrt{p}\right)}{3}\right)^{2}}{4} + \Bigg($$

$$-\frac{\sqrt{1-p}\left(\left(-2\sqrt{3}-1\right)p^{3/2} + p^{5/2} + \left(1+\sqrt{3}\right)\sqrt{p}\right)\sin\left(\frac{\arccos\left(\sqrt{p}\right)}{3}\right)}{2} + \left(\frac{\sqrt{3}}{2}\right.$$

$$\left. + \frac{9}{16}\right) p^{3/2} - \frac{p^{5/2}}{16} + \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\right)\sqrt{p}\Bigg)\cos\left(\frac{\arccos\left(\sqrt{p}\right)}{3}\right)$$

$$- \frac{3\sqrt{1-p}\,\sin\left(\frac{\arccos\left(\sqrt{p}\right)}{3}\right) p^{2}}{16}\Bigg)\sqrt{2\cos\left(\frac{\arccos\left(\sqrt{p}\right)}{3}\right)^{2} - 1}$$

$$- 32\sqrt{1-p}\,\sin\left(\frac{\arccos\left(\sqrt{p}\right)}{3}\right)\left(\cos\left(\frac{\arccos\left(\sqrt{p}\right)}{3}\right)^{2}\right.$$

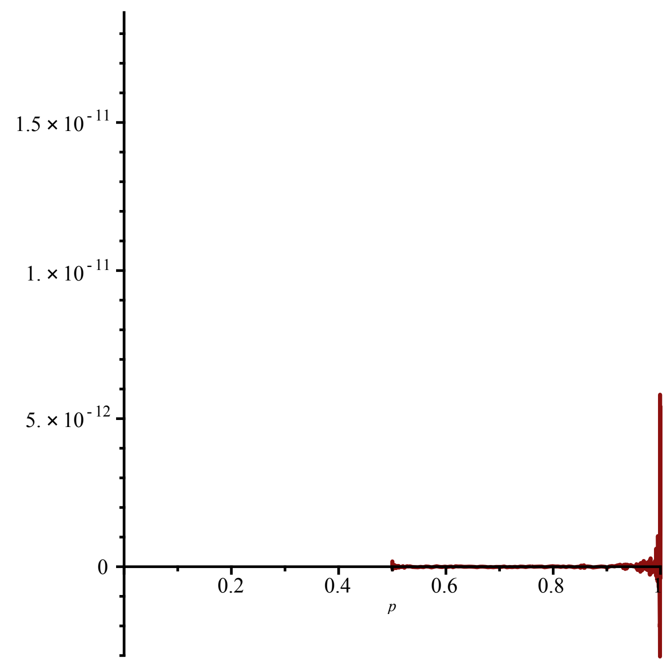
$$\left.-\frac{1}{2}\right)^{2}\cos\left(\frac{\arccos\left(\sqrt{p}\right)}{3}\right)\Bigg(\left(\sqrt{p} - 2p^{3/2}\right)\cos\left(\frac{\arccos\left(\sqrt{p}\right)}{3}\right)^{6} + \left(-\frac{3\sqrt{p}}{2}\right.$$

$$\left.+ 3p^{3/2}\right)\cos\left(\frac{\arccos\left(\sqrt{p}\right)}{3}\right)^{4} + \left(\frac{9\sqrt{p}}{16} - \frac{9p^{3/2}}{8}\right)\cos\left(\frac{\arccos\left(\sqrt{p}\right)}{3}\right)^{2} - \frac{p^{3/2}}{16}$$

$$+ \frac{p^{5/2}}{8}\Bigg)\Bigg)\left(\cos\left(\frac{\arccos\left(\sqrt{p}\right)}{3}\right) + 1\right)^{4}\left(\cos\left(\frac{\arccos\left(\sqrt{p}\right)}{3}\right) - 1\right)^{4}\Bigg) \Big/$$

$$\Bigg(\sqrt{1-p}\,p^{3/2}\sqrt{2p-1}\sqrt{2\cos\left(\frac{\arccos\left(\sqrt{p}\right)}{3}\right)^{2} - 1}\,\Big(24p - 24$$

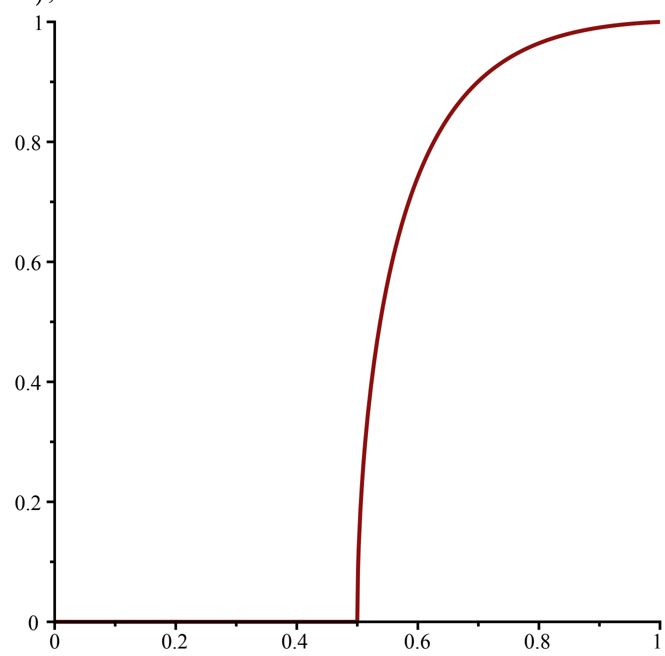
$$+ 16\sqrt{3}\Big)\sin\left(\frac{\arccos\left(\sqrt{p}\right)}{3}\right)^{9}\cos\left(\frac{\arccos\left(\sqrt{p}\right)}{3}\right)\Bigg)$$



Plot of the probability displayed in the article

> $probaplot := piecewise\left(\frac{1}{2} < p, subs\left(\text{theta} = \frac{\text{arccos}(\text{sqrt}(p))}{3}, ProbaPercoSupcVal4\right)\right):$

> $plot(probaplot, p = 0..1);$



The symptotic expansion in pp=p-1/2:

> $simplify\left(expand\left(rationalize\left(convert\left(simplify\left(series\left(subs\left(\text{theta} = \frac{\text{arccos}(\text{sqrt}(p))}{3}, p = \frac{1}{2} + pp, ProbaPercoSupcVal4\right), pp, 1\right)\right), polynom\right)\right)\right)\right);$

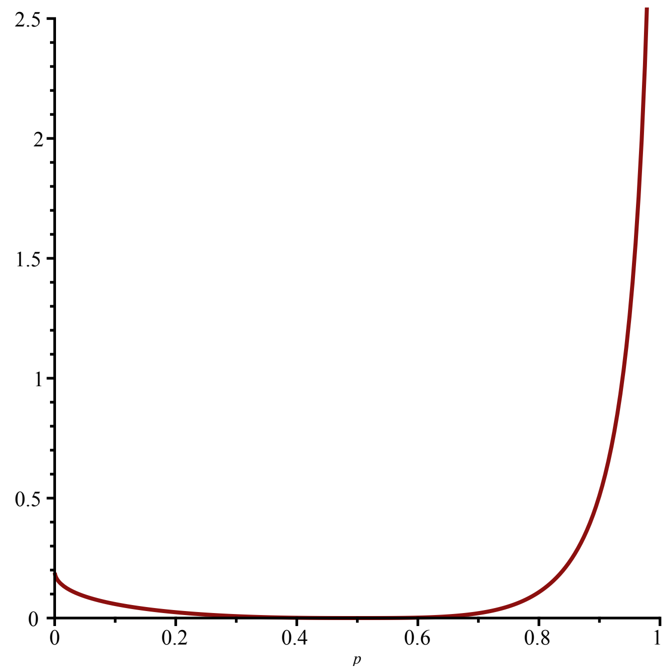
$$\frac{5\, 3^{1/4} \sqrt{pp}\, \left(3 + 4\sqrt{3}\right)}{26} \quad \textbf{(1.5.3.18)}$$

BDFG functions (Lemmas 9 and 10)

Lemma 9

The condition on y_+(p) and y_+(1-p) for a contour integral representation of f^bullet and f^diamond in Lemma 7

> $plot\left(subs(p = 1 - p, subs(V = Vplus, yUcV)) - \left(1 - \frac{1}{subs(V = Vplus, yUcV)}\right)^{-1}, p = 0..1\right);$



Lemma 10

The singularity of T are at y+=2 and y-=-4:

> $simplify\left(subs\left(V = \frac{\text{sqrt}(3)}{3}, p = \frac{1}{2}, U = Uc, yUV\right)\right); simplify\left(subs\left(V = Vcminus, p = \frac{1}{2}, U = Uc, yUV\right)\right);$

$$2$$
$$-4 \tag{1.6.2.1}$$

The values of z^+ and z^diamond:

> $zd := simplify\left(\frac{1}{2} \cdot \frac{\left(\frac{1}{2} - \frac{1}{4}\right)}{\text{sqrt}\left(\frac{wc}{2}\right)}\right); zp := simplify\left(\left(\frac{1}{4} \cdot \frac{\left(\frac{1}{2} + \frac{1}{4}\right)}{\text{sqrt}\left(\frac{wc}{2}\right)}\right)^2\right);$

$$zd := \frac{3^{3/4}\sqrt{2}}{4}$$

(1.6.2.2)

$$zp := \frac{27\sqrt{3}}{32} \tag{1.6.2.2}$$

Asymptotic expansion for T at y=2. First we compute an expansion of V(y), then we inject in the expression for T:

> $algeqtoseries\left(subs\left(U = Uc, p = \frac{1}{2}, y = 2\cdot(1 - YY), numer(y - yUV)\right), YY, V, 5\right);$

$$\left[\frac{\sqrt{3}}{3} + RootOf\left(3\_Z^3 + 1\right) YY^{1/3} + \frac{YY}{3} + \frac{RootOf\left(3\_Z^3 + 1\right) YY^{4/3}}{3} + \mathrm{O}\left(YY^{5/3}\right)\right] \tag{1.6.2.3}$$

> $simplify\left(series\left(subs\left(V = \frac{\mathrm{sqrt}(3)}{3} - \left(\frac{1}{3}\right)^{\frac{1}{3}}\cdot YY^{\frac{1}{3}} + \frac{YY}{3} - \frac{1}{3}\cdot\left(\frac{1}{3}\right)^{\frac{1}{3}}\cdot YY^{\frac{4}{3}}, p = \frac{1}{2}, U = Uc, TtUV\right), YY, 2\right)\right); collect(\%, YY, factor);$

$$\mathrm{O}\left(YY^2\right) - \frac{3^{5/6}\, YY^{2/3}}{2} + 3^{1/6}\, YY^{4/3} - \frac{5\, 3^{5/6}\, YY^{5/3}}{6} + \frac{(3\, YY + 3)\sqrt{3}}{6}$$

$$-\frac{5\, 3^{5/6}\, YY^{5/3}}{6} - \frac{3^{5/6}\, YY^{2/3}}{2} + 3^{1/6}\, YY^{4/3} + \frac{\sqrt{3}\, YY}{2} + \frac{\sqrt{3}}{2} + \mathrm{O}\left(YY^2\right) \tag{1.6.2.4}$$

The hypergeometric functions and their developments

> $simplify\left(\frac{1}{\mathrm{Pi}}\cdot int\left(\frac{1}{\mathrm{sqrt}(x\cdot(1-x))}\cdot(1 - u\cdot x)^{\frac{2}{3}}, x = 0..1\right)\right)$ assuming $|u| < 1$; $map\left(simplify, series\left(subs\left(u = 1 - uu, \mathrm{hypergeom}\left(\left[-\frac{2}{3}, \frac{1}{2}\right], [1], u\right)\right), uu = 0, 2\right)\right);$

$$\mathrm{hypergeom}\left(\left[-\frac{2}{3}, \frac{1}{2}\right], [1], u\right)$$

$$\frac{\pi\, 2^{2/3}\sqrt{3}}{6\,\Gamma\left(\frac{2}{3}\right)^3} + \frac{\pi\, 2^{2/3}\sqrt{3}\, uu}{3\,\Gamma\left(\frac{2}{3}\right)^3} - \frac{18\, 2^{1/3}\,\Gamma\left(\frac{2}{3}\right)^3 uu^{7/6}}{7\,\pi^2} + \mathrm{O}\left(uu^2\right) \tag{1.6.2.5}$$

> $simplify\left(\frac{1}{\mathrm{Pi}}\cdot int\left(\frac{1}{\mathrm{sqrt}(x\cdot(1-x))}\cdot(1 - u\cdot x)^{\frac{4}{3}}, x = 0..1\right)\right)$ assuming $|u| < 1$; $map\left(simplify, series\left(subs\left(u = 1 - uu, \mathrm{hypergeom}\left(\left[-\frac{4}{3}, \frac{1}{2}\right], [1], u\right)\right), uu = 0, 2\right)\right);$

$$\mathrm{hypergeom}\left(\left[-\frac{4}{3}, \frac{1}{2}\right], [1], u\right)$$

(1.6.2.6)

$$\frac{45\,2^{1/3}\,\Gamma\left(\frac{2}{3}\right)^3}{32\,\pi^2} + \frac{9\,2^{1/3}\,\Gamma\left(\frac{2}{3}\right)^3 uu}{8\,\pi^2} + \frac{32\,\pi\,2^{2/3}\sqrt{3}\,uu^{11/6}}{165\,\Gamma\left(\frac{2}{3}\right)^3} + \mathrm{O}\left(uu^2\right) \quad \textbf{(1.6.2.6)}$$

> $simplify\left(\frac{1}{\mathrm{Pi}} \cdot int\left(\frac{1}{\mathrm{sqrt}(x\cdot(1-x))}\cdot(1-u\cdot x), x=0..1\right)\right)$ assuming $|u| < 1$;

$$-\frac{u}{2}+1 \quad \textbf{(1.6.2.7)}$$

Asymptotic expansion of phi:

> $\mathrm{phi} := \dfrac{\dfrac{1-\mathrm{sqrt}\left(\frac{wc}{2}\right)\cdot(z2-2\,\mathrm{sqrt}(z1))}{1-\mathrm{sqrt}\left(\frac{wc}{2}\right)\cdot(z2+2\,\mathrm{sqrt}(z1))}-1}{1-2\cdot\mathrm{sqrt}\left(\frac{wc}{2}\right)\cdot(z2-2\,\mathrm{sqrt}(z1))}$:

value at (zp,zd):

> $simplify(subs(z1=zp, z2=zd, \mathrm{phi}))$;

$$1 \quad \textbf{(1.6.2.8)}$$

derivatives at (zp,zd):

> $simplify(subs(z1=zp, z2=zd, simplify(diff(\mathrm{phi}, z1))))$; $simplify\left(subs\left(z1=zp, z2=zd, simplify\left(\frac{diff(\mathrm{phi}, z2)}{\mathrm{sqrt}(zp)}\right)\right)\right)$;

$$\frac{20\sqrt{3}}{81}$$

$$\frac{20\sqrt{3}}{81} \quad \textbf{(1.6.2.9)}$$

The factor in front of the functions I_\alpha:

> $simplify\left(1-\frac{1}{2\cdot\left(1-\mathrm{sqrt}\left(\frac{wc}{2}\right)\cdot(zd-2\,\mathrm{sqrt}(zp))\right)}\right)$;

$$\frac{3}{5} \quad \textbf{(1.6.2.10)}$$

value of kappa^diamond and kappa^+

> $kappa\_d := simplify\left(\frac{18\,2^{1/3}\,\Gamma\left(\frac{2}{3}\right)^3}{7\,\pi^2}\cdot\left(\frac{3}{5}\right)^{\frac{2}{3}}\cdot\left(\frac{20\sqrt{3}}{81}\right)^{\frac{7}{6}}\cdot\mathrm{sqrt}(2\cdot wc)\cdot\frac{3^{\frac{5}{6}}}{2}\right)$;

$$kappa\_d := \frac{4\,2^{1/6}\,\Gamma\left(\frac{2}{3}\right)^3 3^{2/3}\sqrt{5}}{63\,\pi^2} \quad \textbf{(1.6.2.11)}$$

$$> \ kappa\_p := simplify\left(\frac{18\,2^{1/3}\,\Gamma\left(\frac{2}{3}\right)^3}{7\,\pi^2}\cdot\left(\frac{3}{5}\right)^{\frac{2}{3}}\cdot\left(\frac{20\sqrt{3}}{81}\right)^{\frac{7}{6}}\cdot\frac{1-2\cdot\mathrm{sqrt}\left(\frac{wc}{2}\right)\cdot zd}{2\cdot wc\cdot(zp)^2}\cdot\frac{3^{\frac{5}{6}}}{2}\right);$$

$$kappa\_p := \frac{2048\,2^{2/3}\,\Gamma\left(\frac{2}{3}\right)^3\,3^{11/12}\,\sqrt{5}}{5103\,\pi^2} \quad \textbf{(1.6.2.12)}$$

$$> \ simplify\left(\frac{kappa\_p}{kappa\_d}\right);$$

$$\frac{512\sqrt{2}\,3^{1/4}}{81} \quad \textbf{(1.6.2.13)}$$

Volume and perimeter exponents (proof of Theorem 2)

The factor Delta(z)/z in the integral in terms of V((1-z)^(-1)) meaning

$$> \ factor\left(subs\left(p=\frac{1}{2}, Deltaser\cdot\left(1-\frac{1}{yUcV}\right)^{-1}\right)\right);$$

$$-\frac{36\left(3+4\sqrt{3}\right)V^2\left(-3V+2\sqrt{3}\right)^2}{13\left(-3V+3+\sqrt{3}\right)\left(-3V-3+\sqrt{3}\right)\left(-3V+\sqrt{3}\right)^4} \quad \textbf{(1.7.1)}$$

$$> \ simplify\left(series\left(subs\left(V=\frac{\mathrm{sqrt}(3)}{3}-\left(\frac{1}{3}\right)^{\frac{1}{3}}\cdot YY^{\frac{1}{3}}+\frac{YY}{3}-\left(\frac{1}{3}\right)^{\frac{1}{3}}\cdot YY^{\frac{4}{3}}, p=\frac{1}{2}, Deltaser\cdot\left(1-\frac{1}{yUcV}\right)^{-1}\right), YY, 2\right)\right); collect(\%, YY, factor);$$

$$\frac{16+\left(-81+108\sqrt{3}\right)\mathrm{O}\left(YY^{1/3}\right)+\frac{-4\,3^{2/3}\,YY^{2/3}+(-48\,YY+12)\,3^{1/3}}{YY^{4/3}}}{-81+108\sqrt{3}}$$

$$\mathrm{O}\left(YY^{1/3}\right)+\frac{64\sqrt{3}}{1053}+\frac{16}{351}+\frac{-\frac{64\,3^{5/6}}{351}-\frac{16\,3^{1/3}}{117}}{YY^{1/3}}+\frac{-\frac{16\,3^{1/6}}{351}-\frac{4\,3^{2/3}}{351}}{YY^{2/3}}+\frac{\frac{16\,3^{5/6}}{351}+\frac{4\,3^{1/3}}{117}}{YY^{4/3}} \quad \textbf{(1.7.2)}$$

In the previous display YY = 1 - y/2 = 1 - 1/(2(1-z)) :

$$> \ series\left(1-\frac{1}{2\cdot(1-z)}, z=\frac{1}{2}, 2\right);$$

$$(-2)\left(z-\frac{1}{2}\right)+\mathrm{O}\left(\left(z-\frac{1}{2}\right)^2\right) \tag{1.7.3}$$

Expansion of the hypergeometric function

> $series\left(\mathrm{hypergeom}\left(\left[\frac{4}{3},\frac{3}{2}\right],[3],z\right),z=1,2\right);$

$$\frac{12\sqrt{\pi}}{\Gamma\left(\frac{5}{6}\right)\Gamma\left(\frac{2}{3}\right)}-\frac{36\,\Gamma\left(\frac{5}{6}\right)\sqrt{3}\,\Gamma\left(\frac{2}{3}\right)(-1)^{1/6}(z-1)^{1/6}}{\pi^{3/2}}-\frac{144\sqrt{\pi}\,(z-1)}{5\,\Gamma\left(\frac{5}{6}\right)\Gamma\left(\frac{2}{3}\right)}+\frac{540\,\Gamma\left(\frac{5}{6}\right)\sqrt{3}\,\Gamma\left(\frac{2}{3}\right)(-1)^{1/6}(z-1)^{7/6}}{7\,\pi^{3/2}}+\mathrm{O}\left((z-1)^2\right) \tag{1.7.4}$$

There is a Beta prefactor in Euler's integral representation of the hypergeometric function:

> $\mathrm{Beta}\left(\frac{3}{2},\frac{3}{2}\right);$

$$\frac{\pi}{8} \tag{1.7.5}$$

> $kappaprime := simplify\left(\frac{36\,\Gamma\left(\frac{5}{6}\right)\sqrt{3}\,\Gamma\left(\frac{2}{3}\right)}{\pi^{3/2}}\cdot\left(\frac{4\,3^{5/6}}{351}+\frac{3^{1/3}}{117}\right)\cdot\left(\frac{3}{4}-\frac{1}{8}\right)\cdot\frac{1}{16}\cdot\left(\frac{1}{\mathrm{sqrt}\left(\frac{wc}{2}\right)\cdot 2}+\frac{1}{\mathrm{sqrt}\left(\frac{wc}{2}\right)\cdot 4}\right)^2\cdot\left(1+\frac{2}{4}\right)^{-\frac{4}{3}}\cdot\left(\frac{2\cdot\mathrm{sqrt}\left(\frac{wc}{2}\right)}{1+\frac{2}{4}}\cdot\frac{1}{\mathrm{sqrt}(zp)}\right)^{\frac{1}{6}}\cdot\left(\mathrm{sqrt}(zp)\cdot kappa\_d+zp\cdot kappa\_p\right)^{-\frac{1}{7}}\right);$

$$kappaprime := \frac{9\,137^{6/7}\,7^{1/7}\,\Gamma\left(\frac{2}{3}\right)^{18/7}\,3^{17/21}\,5^{13/14}\left(\frac{4\,3^{5/6}}{3}+3^{1/3}\right)2^{3/14}}{56992\,\pi^{12/7}} \tag{1.7.6}$$

the constant in the expansion for the tail probability of the volume of the cluster

> $\mathrm{kappa} := simplify\left(\frac{kappaprime}{\mathrm{GAMMA}\left(\frac{8}{7}\right)}\right); evalf(\%);$

$$\kappa := \frac{63\, 137^{6/7}\, 7^{1/7}\, \Gamma\left(\frac{2}{3}\right)^{18/7} 3^{17/21}\, 5^{13/14} \left(\frac{4\, 3^{5/6}}{3} + 3^{1/3}\right) 2^{3/14}}{56992\, \pi^{12/7}\, \Gamma\left(\frac{1}{7}\right)}$$

$$0.2782825983 \qquad (1.7.7)$$

The constant in the expansion of the law of the perimeter of the cluste

$$> \mathit{kappaprime} := \mathit{simplify}\left(\frac{-8}{\mathrm{GAMMA}\left(\frac{4}{3}\right)} \cdot \frac{\left(\frac{8\, 3^{5/6}}{351} + \frac{2\, 3^{1/3}}{117}\right) \cdot 3^{\frac{5}{6}}}{\mathrm{GAMMA}\left(-\frac{2}{3}\right)}\right); \mathit{evalf}(\%);$$

$$\mathit{kappaprime} := \frac{8\, 3^{5/6} \left(4\, 3^{5/6} + 3\, 3^{1/3}\right) \Gamma\left(\frac{2}{3}\right)^2}{117\, \pi^2}$$

$$0.4543907536 \qquad (1.7.8)$$

>