Depth First Exploration of a Configuration Model

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Abstract

We introduce an algorithm that constructs a random uniform graph with prescribed degree sequence together with a depth first exploration of it. In the so-called supercritical regime where the graph contains a giant component, we prove that the renormalized contour process of the Depth First Search Tree has a deterministic limiting profile that we identify. The proof goes through a detailed analysis of the evolution of the empirical degree distribution of unexplored vertices. This evolution is driven by an infinite system of differential equations which has a unique and explicit solution. As a byproduct, we deduce the existence of a macroscopic simple path and get a lower bound on its length.

Keywords. Configuration Model, Depth First Search Algorithm, Differential Equation Method.

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1 Introduction

Historically, the configuration model was introduced by Bollobás in [3] as a random uniform multigraph among all the multigraphs with N vertices and prescribed degree sequence $d_1, \dots, d_N$. It turns out that this model shares a lot of features with the Erdős-Rényi random graph. In particular it exhibits a phase transition for the existence of a unique macroscopic connected component. This phase transition, as well as the size of this so-called giant component, was studied in detail in [11, 12, 10]. The proof of these results relies on the analysis of a construction algorithm which takes as input a collection of N vertices having respectively $d_1, \dots, d_N$ half-edges coming out of them, and returns as output a random uniform multigraph, by connecting step by step the half-edges. The way [11, 12, 10, 5] connect these half-edges is as follows: at a given step in this algorithm, a uniform half-edge of the growing cluster is connected to a uniform not yet connected half-edge.

In this paper, we introduce a construction algorithm which, in addition of constructing the configuration model, provides an exploration of it. This exploration corresponds to the Depth First Search algorithm which is roughly a nearest neighbor walk on the vertices that greedily tries to go as deep as possible in the graph. The output of the Depth First Search Algorithm is a spanning planar rooted tree for each connected component of the graph, whose height provides a lower bound on the length of the largest simple path in the corresponding component.

A simlilar exploration has been successfuly used by Aldous [2] for the Erdős-Rényi model in the critical window where the connected componenents are of polynomial size. The structure of the graph in this window was further studied in [1]. For the configuration model, a similar critical window was also identified and studied. See [9, 13, 5, 6].

The purpose of this article is to study this algorithm on a supercritical configuration model and in particular the limiting shape of the contour process of the tree associated to the Depth First exploration of the giant component. Unlike in the previous construction of [11, 12, 10, 5], where the authors only studied the evolution of the empirical distribution of the degree of the unexplored vertices, we have to deal with the empirical distribution of the degree of the unexplored vertices in the graph that they induce inside the final graph. The analysis of this evolution is much more delicate and is in fact the heart of our work, this is the content of Theorem 1. This is also in contrast with the critical window where the induced degree are stationary during the exploration.

It turns out that a step by step analysis of the construction does not work. Still, it is possible to track, at some ladder times, the evolution of the degrees of the unexplored vertices in the graph they induce. In this time scale, using a generalization of the celebrated differential equation method of Wormald [14] provided in the appendix, we are able to show that the evolution of the empirical degree distribution of the unexplored vertices has a fluid limit which is driven by an infinite system of differential equations. This system as such cannot be handled. We have to introduce a time change which, surprisingly, corresponds to the proportion of explored vertices, in term of the construction algorithm. Another surprise is that the resulting new system of differential equations admits an explicit solution through the generating series they form. In order to apply Wormald's method, we need to establish the uniqueness of this solution. This task, presented in Section 6.2, is also intricate and is based on the knowledge of the explicit solution mentioned above.

Combining Theorem 1 with an analysis of the ladder times, we prove that the renormalized contour process of the spanning tree of the Depth First Search algorithm converges towards a deterministic profile for which we give an explicit parametric representation. This is the object of Theorem 2. A direct consequence is a lower bound on the length of the longest simple path in a supercritical model, see Corollary 1. To the best of our knowledge, this lower bound seems to be the best available for a generic initial degree distribution. We refer here to the work [8], where the authors establish a lower bound on the longest induced path in a configuration model with bounded degree with a bound that becomes microscopic as the largest degree tends to infinity. We do not believe that our bound is sharp. The question of the length of the longest simple path in a configuration model is actually still open in generic cases. To the best of our knowledge, the only solved cases are d-regular random graphs that are known to be (almost) Hamiltonian [4]. However, a main advantage of our bound is that it is given by an explicit contruction in linear time, which is not the case for the regular graphs setting.

Let us mention that the ingredient of ladder times, used in the proof of Theorem 2, was already present in the context of Erdös-Rényi graphs in [7]. The novelty and core of the present article is the analysis of the empirical degree distribution of the unexplored vertices at the ladder times, which was straightforward in the case of Erdös-Rényi graphs as it is in that case, along the construction, a Binomial with decreasing parameter.

In order to illustrate our results, we provide explicit computations together with simulations in the setting where the initial degree distribution follows respectively a Poisson law (recovering results of [7] in the Erdős-Rényi setting), a Dirac mass at $d \geq 3$ (corresponding to d-regular random graphs) and a Geometric law. We also discuss briefly the heavy tailed case which also falls into the scope of our results.

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2 Definition of the DFS exploration and main results

2.1 The Depth First Search algorithm

Consider a multigraph $\mathrm{G} = (\mathrm{V}, \mathrm{E})$. The DFS exploration of G is the following algorithm.

For every step n we consider the following objects, defined by induction.

- A_n, the active vertices, is an ordered list of elements of V.
- S_n, the sleeping vertices, is a subset of V. This subset will never contain a vertex of A_n.
- R_n, the retired vertices, is another subset of V composed of all the vertices that are neither in A_n nor S_n.

At time $n = 0$, choose a vertex v uniformly at random. Set:

$$\begin{cases} A_0 &= \left((v, d_v^{(N)})\right), \\ S_0 &= \mathrm{V}_N \setminus \{v\}, \\ R_0 &= \emptyset. \end{cases}$$

Suppose that A_n, S_n and R_n are constructed. Three cases are possible.

1. If $A_n = \emptyset$, the algorithm has just finished exploring a connected component of G. In that case, we pick a vertex v_{n+1} uniformly at random inside S_n and set:

$$\begin{cases} A_{n+1} &= (v_{n+1}), \\ S_{n+1} &= S_n \setminus \{v_{n+1}\}, \\ R_{n+1} &= R_n. \end{cases}$$

2. If $A_n \neq \emptyset$ and if its last element u has a neighbor v in S_n, the DFS goes to v and we set:

$$\begin{cases} A_{n+1} &= A_n + v \\ S_{n+1} &= S_n \setminus \{v\}, \\ R_{n+1} &= R_n. \end{cases}$$

3. If $A_n \neq \emptyset$ and if its last element u has no neighbor in S_n, the DFS backtracks and we set:

$$\begin{cases} A_{n+1} &= A_n - u, \\ S_{n+1} &= S_n, \\ R_{n+1} &= R_n \cup \{u\}. \end{cases}$$

This algorithm explores the whole graph and provides a spanning tree of each connected component as well as a contour process of this tree. In Section 4, we will provide an algorithm that construct simultaneously a random graph and a DFS on it.

The algorithm finishes after $2N$ steps. For every $0 \leq n \leq 2N$, we set $X_n = |A_n|$. This walk is the contour process associated to the spanning forest of the DFS. Notice that $X_n = 0$ when the process starts the exploration of a new connected component. Therefore, each excursion of (X_n) corresponds to a connected component of $\mathscr{C}(\boldsymbol{d}^{(N)})$.

2.2 The Configuration model

We now turn to the definition of the configuration model.

Definition 1. *Let $\mathbf{d} = (d_1, \ldots, d_N) \in \mathbb{Z}_+^N$. Let $\mathscr{C}(\mathbf{d})$ be a random graph whose law is uniform among all multigraphs with degree sequence $\mathbf{d}$ if $d_1 + \cdots d_N$ is even, and $(d_1, d_2, \ldots, d_N + 1)$ otherwise.*

We will study sequence of configuration models whose associated sequence of empirical degree distribution converges towards a given probability measure.

Definition 2. *Let $\boldsymbol{\pi}$ be a probability distribution on $\mathbb{Z}_+$. Let $(\mathbf{d}^{(N)})_{N\geq 1}$ be a sequence of random variables defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that, for every $N \geq 1$, $\mathbf{d}^{(N)} = (d_1^{(N)}, \dots, d_N^{(N)}) \in \mathbb{Z}_+^N$. We say that $(\mathscr{C}(\mathbf{d}^{(N)}))_{N\geq 1}$ has asymptotic degree distribution $\boldsymbol{\pi}$ if*

$$\forall k \geq 0, \quad \frac{1}{N}\sum_{i=1}^{N} \mathbf{1}_{\{d_i^{(N)}=k\}} \underset{N\to+\infty}{\longrightarrow} \boldsymbol{\pi}(\{k\}).$$

As observed in [11], the configuration model exhibits a phase transition for the existence of a unique macroscopic connected component. In this article, we will restrict our attention to supercritical configuration models, that is where this giant component exists.

Definition 3. *Let $\boldsymbol{\pi}$ be a probability distribution on $\mathbb{Z}_+$ such that $\sum_{k\geq 0} \boldsymbol{\pi}(\{k\})k^2 < \infty$ and denote by $f_{\boldsymbol{\pi}}$ its generating function. Let $\hat{\boldsymbol{\pi}}$ be the probability distribution having generating function*

$$\widehat{f_{\boldsymbol{\pi}}}(s) := f_{\hat{\boldsymbol{\pi}}}(s) = \frac{f'_{\boldsymbol{\pi}}(s)}{f'_{\boldsymbol{\pi}}(1)}.$$

We say that $\boldsymbol{\pi}$ is supercritical if $\widehat{f_{\boldsymbol{\pi}}}'(1) > 1$ in which case we define $\rho_{\boldsymbol{\pi}}$ as the smallest positive solution of the equation

$$1 - \rho_{\boldsymbol{\pi}} = \widehat{f_{\boldsymbol{\pi}}}(1 - \rho_{\boldsymbol{\pi}}).$$

Finally, we set

$$\xi_{\boldsymbol{\pi}} := 1 - f_{\boldsymbol{\pi}}(1 - \rho_{\boldsymbol{\pi}}).$$

The number $\rho_{\boldsymbol{\pi}}$ is the probability that a Galton-Watson tree with distribution $\widehat{\boldsymbol{\pi}}$ is infinite, whereas the number $\xi_{\boldsymbol{\pi}}$ is the survival probability of a tree where the root has degree distribution $\boldsymbol{\pi}$ and individuals of the next generations have a number of children distributed according to $\widehat{\boldsymbol{\pi}}$. In this article, we study sequence of configuration models $\mathcal{C}\left(\mathbf{d}^{(N)}\right)$ whose asymptotic degree distribution is a supercritical probability measure $\boldsymbol{\pi}$.

As shown in [11, 12, 10, 5], in this context, denoting by $\mathrm{C}_1^{(N)}, \mathrm{C}_2^{(N)}, \dots$ the sequence of connected components of $\mathcal{C}(\mathbf{d}^{(N)})$ ordered by decreasing number of vertices,

- with high probability, $\lim\limits_{N\to+\infty} \frac{|\mathrm{C}_1^{(N)}|}{N} = \rho_\pi$,
- with high probability, $|\mathrm{C}_2^{(N)}| = \mathcal{O}(\log(N))$.

We finally make the two following technical assumptions:

- The following convergence holds:

$$\lim_{N\to+\infty} \mathbb{E}\left[\frac{{d_1^{(N)}}^2 + \cdots + {d_N^{(N)}}^2}{N}\right] = \sum_{k\geq 0} k^2 \pi(\{k\}). \quad \textbf{(A1)}$$

- There exists $\gamma > 2$ such that, for some $\lambda > 0$:

$$\mathbb{P}\left(\max\left\{d_1^{(N)}, \dots, d_N^{(N)}\right\} \geq N^{1/\gamma}\right) = \mathcal{O}\left(N^{-1-\lambda}\right). \quad \textbf{(A2)}$$

2.3 Main results

We now state our first result. Define $\alpha \geq 0$. Consider the graph induced by the sleeping vertices after having explored $\lfloor \alpha N \rfloor$ vertices when performing the DFS algorithm on a configuration model. It is clear that this induced graph is also a configuration model. The purpose of the following theorem is to identify its asymptotic degree distribution. It turns out this distribution only depends on α and on the initial degree distribution $\boldsymbol{\pi}$.

Theorem 1. *Let $\boldsymbol{\pi}$ be a probability measure on $\mathbb{Z}_+$ with generating series f and $(\mathscr{C}(\mathbf{d}^{(N)}))_{N\geq 1}$ be a configuration model with supercritical asymptotic degree distribution $\boldsymbol{\pi}$. Assume* **(A1)** *and* **(A2)***.*

Let α_c be the smallest positive solution of the equation

$$\frac{f''_{\boldsymbol{\pi}}\left(f_{\boldsymbol{\pi}}^{-1}(1-\alpha)\right)}{f'_{\boldsymbol{\pi}}(1)} = 1.$$

For every $\alpha \in [0, \alpha_c]$, let $\boldsymbol{\pi}_\alpha$ be the probability distribution on $\mathbb{Z}_+$ with generating series

$$g(\alpha, s) = \frac{1}{1-\alpha} f_{\boldsymbol{\pi}}\left(f_{\boldsymbol{\pi}}^{-1}(1-\alpha) - (1-s)\frac{f'_{\boldsymbol{\pi}}(f_{\boldsymbol{\pi}}^{-1}(1-\alpha))}{f'_{\boldsymbol{\pi}}(1)}\right).$$

Then, for every $\alpha \in [0, \alpha_c]$, denoting $\tau^{(N)}(\alpha) = \inf\left\{k \geq 1 : |S_k^{(N)}| \leq (1-\alpha)N\right\}$, the random graphs $S^{(N)}_{\tau^{(N)}(\alpha)}$ are configuration models with asymptotic degree distribution $\boldsymbol{\pi}_\alpha$.

Remark 1. *We consider α up to some constant α_c, which correspond to the time where so many vertices have been visited that the remaining graph of sleeping vertices is subcritical.*

The second result concerns the asymptotic profile of the planar rooted tree constructed by the DFS on a configuration model. This profile is defined by the current height of the walker inside the current tree constructed by the algorithm. It turns out that at time n this current height X_n coincides with $|A_n| - 1$.

Theorem 2. *Under the assumptions of Theorem 1, the following limit holds in probability for the topology of uniform convergence:*

$$\forall t \in [0,2], \quad \lim_{N\to\infty} \frac{X_{\lceil tN \rceil}}{N} = h(t),$$

where the function h is continuous on $[0,2]$, null on the interval $[2\xi_{\boldsymbol{\pi}}, 2]$ and defined hereafter on the interval $[0, 2\xi_{\boldsymbol{\pi}}]$.

There exists an implicit function $\alpha(\rho)$ defined on $[0, \rho_{\boldsymbol{\pi}}]$ such that $1-\rho = \widehat{g}(\alpha(\rho), 1-\rho)$ where, for any $\alpha \in [0, \alpha_c]$, the function $s \mapsto \widehat{g}(\alpha, s)$ is the size biased version of $s \mapsto g(\alpha, s)$ defined in Theorem 1. The graph $(t, h(t))_{t\in[0,2\xi_{\boldsymbol{\pi}}]}$ can be divided into a first increasing part and a second decreasing part. These parts are respectively parametrized for $\rho \in [0, \rho_{\boldsymbol{\pi}}]$ by :

$$\begin{cases} x^{\uparrow}(\rho) & := (2-\rho)\,\alpha(\rho) - \int_\rho^{\rho_{\boldsymbol{\pi}}} \alpha(u)\mathrm{d}u, \\ y^{\uparrow}(\rho) & := \rho\,\alpha(\rho) + \int_\rho^{\rho_{\boldsymbol{\pi}}} \alpha(u)\mathrm{d}u, \end{cases}$$

for the increasing part and

$$\begin{cases} x^{\downarrow}(\rho) := x^{\uparrow}(\rho) + 2\,(1-\alpha(\rho))\left(1 - g\big(\alpha(\rho), 1-\rho\big)\right), \\ y^{\downarrow}(\rho) := y^{\uparrow}(\rho), \end{cases}$$

for the decreasing part.

A direct consequence of this result in the following.

Corollary 1. *Let $\mathcal{H}_N$ be the length of the longest simple path in a configuration model of size N with asymptotic distribution $\boldsymbol{\pi}$ satisfying hypothesis of Theorem 1. Then, with the notations of Theorem 2, in probability,*

$$\liminf_{N\to+\infty} \mathcal{H}_N \geq h(0) = \int_0^{\rho_{\boldsymbol{\pi}}} \alpha(u)\mathrm{d}u.$$

Remark 2. *Note that the formulas in Theorems 1 and 2 have a meaning when $\boldsymbol{\pi}$ has a first moment. Therefore, it is natural to expect that the restriction on the tail of $\boldsymbol{\pi}$ is only technical.*

3 Examples

In this section we provide explicit formulations of Theorems 1 and 2 for particular choices of the initial probability distribution $\boldsymbol{\pi}$.

3.1 Poisson distribution

Since the Erdős-Rényi model on N vertices with probability of connexion c/N is contiguous to the configuration model on N vertices with sequence of degree $D_1^{(N)}, \dots, D_N^{(N)}$ that are i.i.d. with Poisson law of parameter c, we can recover the result of Enriquez, Faraud and Ménard [7]. Indeed, in the Erdős-Rényi case, after having explored a proportion α of vertices, the graph induced by the unexplored vertices is an Erdős-Rényi random graph with $(1-\alpha)N$ vertices and parameter c/N, hence its asymptotic distribution is a Poisson with parameter $(1-\alpha)c$. This is in accordance with our Theorem 1 since in that case, denoting $f(s) = \exp(c(s-1))$ the generating series of the Poisson law with parameter c,

$$\begin{aligned} g(\alpha, s) &= \frac{1}{1-\alpha} f\left(f^{-1}(1-\alpha) - (1-s)\frac{f'(f^{-1}(1-\alpha))}{f'(1)}\right) \\ &= \frac{1}{1-\alpha} \exp\left(c\left(f^{-1}(1-\alpha) - (1-s)\frac{f'(f^{-1}(1-\alpha))}{f'(1)} - 1\right)\right) \\ &= \frac{1}{1-\alpha} \exp\left(c\left(1 + \frac{\log(1-\alpha)}{c} - (1-s)\frac{cf(f^{-1}(1-\alpha))}{c} - 1\right)\right) \\ &= \frac{1}{1-\alpha} \exp\left(c\left(1 + \frac{\log(1-\alpha)}{c} - (1-s)(1-\alpha) - 1\right)\right) \\ &= \exp\left(c(1-\alpha)(s-1)\right). \end{aligned}$$

Using the formulas of Theorem 2, we obtain the same equations as in [7] for the limiting profile of the DFS spanning tree.

3.2 d-Regular and Binomials distributions

Let $d \geq 3$. Since the results of Theorem 1 and 2 hold with probability tending to 1, we can obtain results on d-regular uniform random graphs by applying them to the contiguous model which consists in choosing $\boldsymbol{\pi} = \delta_d$. By Theorem 1, the degree distribution $\boldsymbol{\pi}_\alpha$ has generating function

$$\begin{aligned} g(\alpha, s) &= \frac{1}{1-\alpha}\left((1-\alpha)^{1/d} - (1-s)\frac{d(1-\alpha)^{(d-1)/d}}{d}\right)^d \\ &= \left(1 + (s-1)(1-\alpha)^{\frac{d-2}{d}}\right)^d. \end{aligned} \tag{1}$$

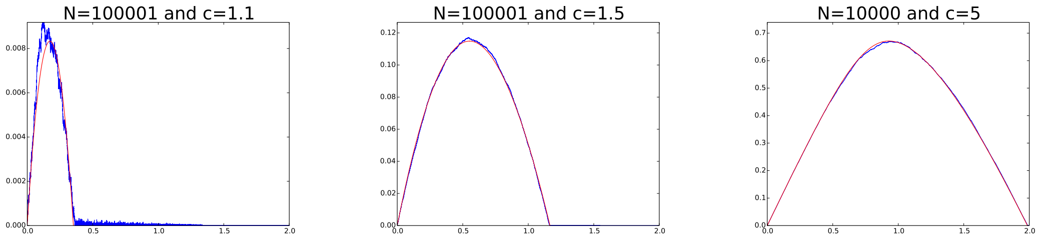

Figure 1: Simulations of $(X_{\lceil tN \rceil}/N)_{t\in[0,2]}$ (blue) and the limiting shape (red) for various values of N and c. Notice that when c is close to 1, we have to take N very large for the walk to be close to its limit.

Hence, $\boldsymbol{\pi}_\alpha$ is a binomial distribution $\mathrm{Bin}\left(d, (1-\alpha)^{\frac{d-2}{d}}\right)$. From (1), we get $\hat{g}(\alpha, s) = (1 + (s-1)(1-\alpha)^{(d-2)/d})^{d-1}$. Solving the equation in α, $1-\rho = \hat{g}(\alpha, 1-\rho)$ gives, we obtain:

$$\alpha(\rho) = 1 - \left(\frac{1-(1-\rho)^{\frac{1}{d-1}}}{\rho}\right)^{\frac{d}{d-2}}.$$

From this, we deduce a parametrization of the limiting profile in terms of hypergeometric functions. In particular, the height of the limiting profile of the DFS spanning tree is given by

$$H_{\max}(d) = 1 - \int_0^1 \left(\frac{1-x^{\frac{1}{d-1}}}{1-x}\right)^{\frac{d}{d-2}} \mathrm{d}x.$$

When $\boldsymbol{\pi}$ has binomial distribution with parameters d and p, $\boldsymbol{\pi}_\alpha$ is a binomial distribution.

$$\boldsymbol{\pi}_\alpha = \mathrm{Bin}\left(d, p(1-\alpha)^{\frac{d-2}{d}}\right).$$

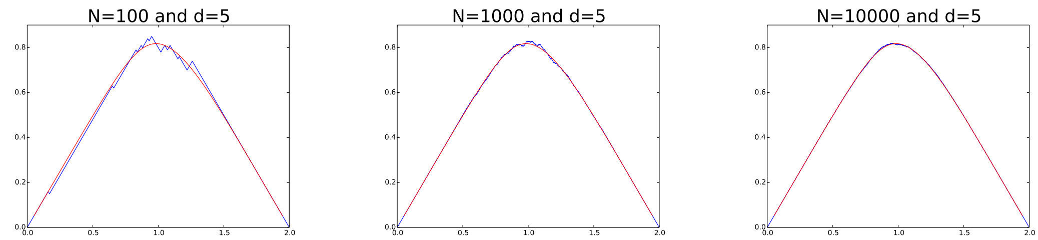

Figure 2: Simulations of $(X_{\lceil tN \rceil}/N)_{t\in[0,2]}$ (blue) and the limiting shape (red) for 5-regular graphs of various sizes.

3.3 Geometric distribution

Let $p > 0$ and suppose that the initial distribution $\boldsymbol{\pi}$ is a geometric distribution starting at 0 with parameter p. The generating series of $\boldsymbol{\pi}$ is $f(s) = \frac{p}{1-(1-p)s}$. We assume $p < 2/3$ so that the configuration model with asymptotic degree distribution $\boldsymbol{\pi}$ has a giant component. Then, by Theorem 1, the distribution $\boldsymbol{\pi}_\alpha$ has generating series

$$g(\alpha, s) = \frac{p(\alpha)}{1-(1-p(\alpha))s},$$

where $p(\alpha) = \frac{p}{p+(1-p)(1-\alpha)^3}$. Hence, $\boldsymbol{\pi}_\alpha$ is a geometric distribution that starts at 0 with parameter $p(\alpha)$. The generating series of $\hat{\boldsymbol{\pi}}_\alpha$ is $\hat{g}(\alpha, s) = \left(\frac{p(\alpha)}{1-(1-p(\alpha))s}\right)^2$. Therefore, the solution in α of $1-\rho = \hat{g}(\alpha, 1-\rho)$ is

$$\alpha(\rho) = 1 - \left(\frac{p}{1-p}\right)^{1/3}\left(\frac{1}{1-\rho+\sqrt{1-\rho}}\right)^{1/3}.$$

In particular, the height of the limiting profile of the DFS spanning tree is given by:

$$H_{\max}(p) = \rho_{\boldsymbol{\pi}} - \left(\frac{p}{1-p}\right)^{1/3}\int_0^{\rho_{\boldsymbol{\pi}}}\left(\frac{1}{x+\sqrt{x}}\right)^{1/3}\mathrm{d}x,$$

where $\rho_{\boldsymbol{\pi}}$ is given by:

$$\rho_{\boldsymbol{\pi}} = \frac{1}{2}\left(\frac{1-3p}{1-p} + \sqrt{\frac{1+3p}{1-p}}\right).$$

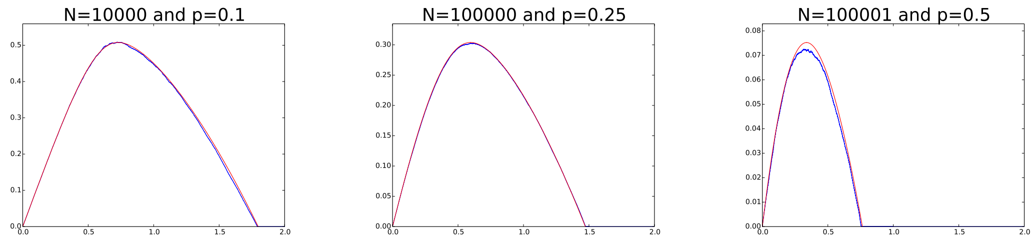

Figure 3: Simulations of $(X_{\lceil tN \rceil}/N)_{t\in[0,2]}$ (blue) and the limiting shape (red) for random graphs with geometric degrees with various perimeters.

3.4 Heavy tailed distribution

When $\boldsymbol{\pi}$ is a power law distribution of parameter $\gamma > 2$, that is when $\boldsymbol{\pi}(\{k, k+1, \ldots\}) \sim C/k^\gamma$ for a constant C, only the first $\lfloor \gamma \rfloor$ moments of $\boldsymbol{\pi}$ are finite. Let $\alpha \in (0, \alpha_c)$. Then, for all $n \geq 0$, the n-th factorial moment of $\boldsymbol{\pi}_\alpha$ is equal to

$$\begin{aligned}\boldsymbol{\pi}_\alpha(x^n) &= \frac{\partial^n}{\partial s^n}\bigg|_{s=1} g(\alpha, s) \\ &= \left(\frac{f'_{\boldsymbol{\pi}}\left(f_{\boldsymbol{\pi}}^{-1}(1-\alpha)\right)}{f'_{\boldsymbol{\pi}}(1)}\right)^n \frac{f^{(n)}\left(f^{-1}(1-\alpha)\right)}{1-\alpha}.\end{aligned}$$

Therefore, after visiting a proportion εN of the vertices along the DFS, the asymptotic distribution of the degrees of the graph induced by the unexplored vertices is not a power law and has moments of all order. This remarkable phenomenon could be explained by the fact that vertices of high degree are visited in a microscopic time. We believe that a precise study of this case could be of independent interest.

4 Constructing while exploring

Let $(\mathbf{d}^{(N)})_{N\geq 1}$ be a sequence of degree sequences of increasing length satisfying the assumptions of Theorem 1. For a fixed $N \geq 1$, we use the sequence $\mathbf{d}^{(N)} = (d_1^{(N)}, \dots, d_N^{(N)})$ to construct a configuration model $\mathscr{C}(\mathbf{d}^{(N)})$ with vertex set $\mathrm{V}_N = \{1, \dots, N\}$. More precisely, we simultaneously build the graph and its DFS exploration. This will be done in a similar way as for the DFS defined in Section 2.1, while revealing as little information on the unexplored part of the graph as possible. For every step n we consider the following objects, defined by induction.

- A_n, the active vertices, is an ordered list of pairs $(v, \mathbf{m}_v)$ where v is a vertex of V_N and $\mathbf{m}_v$ is the list of vertices corresponding to the vertices that will be **m**atched to v during the rest of the exploration.
- S_n, the sleeping vertices, is a subset of V_N. This subset will never contain a vertex of A_n.
- R_n, the retired vertices, is another subset of V_N composed of all the vertices that are neither in A_n nor S_n.

At time $n = 0$, choose a vertex v uniformly at random and pair each of its $d_v^{(N)}$ half edges to a half edge of the graph. This gives an unordered set of vertices that will be matched to v at one point of the exploration. We denote by $\mathbf{m}_v$ this set with a uniform order. Set:

$$\begin{cases} A_0 &= ((v, \mathbf{m}_v)), \\ S_0 &= \mathrm{V}_N \setminus \{v\}, \\ R_0 &= \emptyset. \end{cases}$$

Suppose that A_n, S_n and R_n are constructed. Three cases are possible.

1. If $A_n = \emptyset$, the algorithm has just finished exploring and building a connected component of $\mathscr{C}(\boldsymbol{d}^{(N)})$. In that case, we pick a vertex v_{n+1} uniformly at random inside S_n and we pair each of its $d_{v_{n+1}}^{(N)}$ half edges to a uniform half edge belonging to a vertex of S_n. We denote by $\mathbf{m}_{v_{n+1}}$ the set of these paired vertices which are different from v_{n+1} (corresponding to loops in the graph), ordered uniformly and set:

$$\begin{cases} A_{n+1} &= \left(v_{n+1}, \mathbf{m}_{v_{n+1}}\right), \\ S_{n+1} &= S_n \setminus \{v_{n+1}\}, \\ R_{n+1} &= R_n. \end{cases}$$

2. If $A_n \neq \emptyset$ and if its last element $(u, \mathbf{m}_u)$ is such that $\mathbf{m}_u = \emptyset$, the DFS backtracks and we set:

$$\begin{cases} A_{n+1} &= A_n - (u, \mathbf{m}_u), \\ S_{n+1} &= S_n, \\ R_{n+1} &= R_n \cup \{u\}. \end{cases}$$

3. If $A_n \neq \emptyset$ and if its last element $(u, \mathbf{m}_u)$ is such that $\mathbf{m}_u \neq \emptyset$, the algorithm goes to the first vertex of $\mathbf{m}_u$, say v_{n+1}. By construction, this vertex always belongs to S_n. We first update A_n into A'_n by withdrawing each occurrence of v_{n+1} in the lists $\mathbf{m}_x$ for $x \in A_n$. The half edges of v_{n+1} that have not been matched up to now are uniformly matched with half edges of S_n

that have not yet been matched. We order the set of corresponding vertices that v_{n+1} itself uniformly and denote $\mathbf{m}_{v_{n+1}}$ this list. We finally set

$$\begin{cases} A_{n+1} &= A'_n + (v_{n+1}, \mathbf{m}_{v_{n+1}}) \\ S_{n+1} &= S_n \setminus \{v_{n+1}\}, \\ R_{n+1} &= R_n. \end{cases}$$

Since each matching of half-edges in the algorithm is uniform, it indeed constructs a random graph $\mathscr{C}(\boldsymbol{d}^{(N)})$. Moreover, as advertised at the end of Section 2.1, this algorithm simultaneously constructs the DFS on this random graph as each of the three cases are in correspondence to the same three cases in the definition of the DFS given in Section 2.1.

From this construction, it is clear that for every n, the graph induced by S_n in the whole graph is a configuration model. Moreover, for each vertex v of S_n, its degree in this induced graph is given by its initial degree $d_v^{(N)}$ minus the number of times that v appears in the lists $\mathbf{m}_x$ for $x \in A_n$.

In order to prove Theorem 1, we will first analyse the part of the algorithm corresponding to the increasing part of the limiting profile. This has the same law as the increasing part of the process $(X_n)_{0 \le n \le 2N}$. During this first phase, at each time, the graph induced by the sleeping vertices, which we will call the remaining graph, is a supercritical configuration model. We will see in Section 4.1 that there is a sequence of random times where the DFS discovers a vertex belonging to what will turn out to be the giant component of the remaining graph. We will call these times ladder times and study in detail the law of the remaining graph at these times in Section 4.2.

4.1 Ladder times

Fix $\delta \in (0,1)$. Let $T_0 = 0$ and define, for $k \in \{0, \dots, K\}$,

$$T_{k+1} := \min\left\{ i > T_k,\ X_i = k+1 \text{ and } \forall i \le j \le i + N^\delta,\ X_j \ge k+1 \right\},$$

where K is the last index for which this definition makes sense (i.e. the set for which the min is taken is not empty). Of course, this sequence of times will only be useful to analyse the DFS on $\mathscr{C}(\mathbf{d}^{(N)})$ when K is of macroscopic order, which is indeed the case with high probability under the assumptions of Theorem 1.

For all $k \in \{0, \dots, K\}$, let $\mathscr{S}_k$ be the graph induced by the vertices of $S_{T_k - 1}$ in the graph constructed by the algorithm of the previous section. We also denote by v_k the last vertex of A_{T_k}. The graphs $\mathscr{S}_k$ and S_{T_k} have the same vertex set except for v_k which belongs to $\mathscr{S}_k$ but not to S_{T_k}. See Figure 4 for an illustration of these definitions. We chose to emphasize $\mathscr{S}_k$ because the structural changes between two such consecutive graphs will be easier to track.

Fix $k < K$. From the definition of the times T_k and T_{k+1}, we can deduce that v_{k+1} and v_k are neighbors in $\mathscr{S}_k$. Between the times $n = T_k$ and $n = T_{k+1}$ the process $X_n = |A_n|$ stays above k and is equal to k at time $T_{k+1} - 1$. Each excursion of X_n strictly above k between T_k and $T_{k+1} - 1$ corresponds to the exploration of a different connected component of $\mathscr{S}_k \setminus \{v_k\}$ and we have

$$T_{k+1} - T_k = 1 + 2 \times (\text{ number of vertices in } \mathscr{S}_k \setminus \mathscr{S}_{k+1} \ - 1).$$

In addition, the definition of the ladder times implies that these connected components have sizes smaller than N^δ.

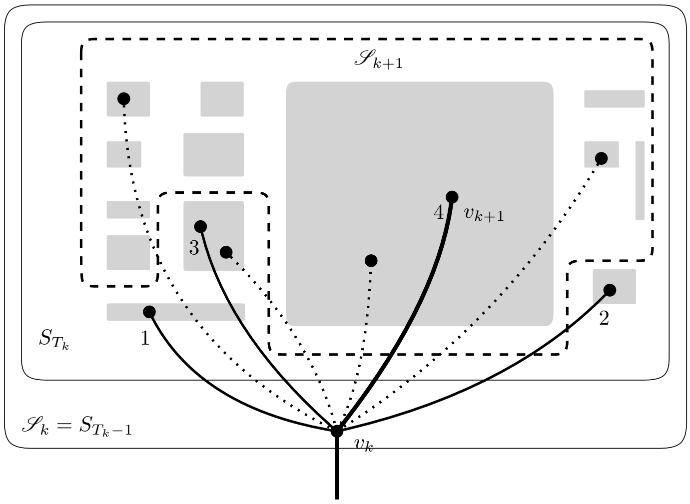

Figure 4: Structure of the remaining graph at a ladder time. The first half edges of v_k are numbered according to their matching order during the construction. Here, the last matched half edge is in bold and connect v_k to v_{k+1}. The remaing half edges of v_k are represented with dotted line and matched to unexplored vertices.

For every $\alpha \in [0,1]$, let $D_\alpha^{(N)}$ be the degree of a uniform vertex in $S_{\alpha N}$. Let $\varepsilon > 0$ and define

$$\alpha_\varepsilon^{(N)} = \sup\left\{\alpha \in [0,1] : \forall \beta \in [0,\alpha], \frac{\mathbb{E}[D_\alpha^{(N)}(D_\alpha^{(N)} - 1)]}{\mathbb{E}[D_\alpha^{(N)}]} > 1 + \varepsilon\right\}.$$

Note that $\alpha_0^{(N)}$ corresponds to the proportion of explored vertices such that $S_{\alpha_0 N}$ is critical. For $\varepsilon > 0$, as long as we have not explored a fraction $\alpha_\varepsilon^{(N)}$ of vertices, the subgraphs S_n are all supercritical. For $0 < \delta < 1/2$, let $\mathbf{G}_\varepsilon = \mathbf{G}_\varepsilon^{(N)}(\gamma, \delta)$ be the event that for all $n \in [0, \alpha_\varepsilon^{(N)} N]$

- the maximum degree of a vertex in S_0, hence in S_n, is at most $N^{1/\gamma}$;
- there is at least one connected component with size greater than $N^{1-\delta}$ in S_n;
- there is no connected component of size between N^δ and $N^{1-\delta}$ in S_n.

Under the assumptions of Theorem 1 we have for every $\lambda > 0$,

$$\mathbb{P}(\mathbf{G}_\varepsilon) = 1 - \mathcal{O}(N^{-\lambda}). \tag{2}$$

The event $\mathbf{G}_\varepsilon$ will be instrumental in the analysis of the DFS and the times T_k because, on this event, if $T_k < \alpha_\varepsilon^{(N)} N$, then the graph $S_{T_k} = \mathscr{S}_k \setminus \{v_k\}$ has a connected component of size larger than $N^{1-\delta}$ and, in $\mathscr{S}_k$, v_k has a neighbor in this giant component. Indeed, if every neighbor of v_k in S_{T_k} belonged to a small component, the size of the connected component of v_k in $\mathscr{S}_k$ would be at most $N^{1/\gamma} N^\delta \ll N^{1-\delta}$. On the other hand, we know that this component has size larger than N^δ meaning that, on $\mathbf{G}_\varepsilon$, it is in fact larger than $N^{1-\delta}$ leading to a contradiction. By induction, this means that on $\mathbf{G}_\varepsilon$ and if $T_k < \alpha_\varepsilon^{(N)} N$, then $k < K$.

4.2 Analysis of the graphs $\mathscr{S}_k$

Let $N_i(k)$ be the number of vertices of degree i in $\mathscr{S}_k$. The graph $\mathscr{S}_k$ has the law of a configuration model with vertex degrees given by the sequence $(N_i(k))_{i\geq 0}$. Recalling that $\mathbf{m}_{v_k}$ denotes the list of neighbors of of v_k in $\mathscr{S}_k$ (self-loops not included), the evolution of N_i is given by:

$$N_i(k+1) - N_i(k) = -\, V_i(\mathscr{S}_k \setminus \mathscr{S}_{k+1}) \tag{3}$$

$$+ \sum_{v \in \mathbf{m}_{v_k} \cap \mathscr{S}_{k+1}} \left(-\mathbf{1}_{\deg_{\mathscr{S}_k}(v)=i} + \mathbf{1}_{\deg_{\mathscr{S}_k}(v)=i+1} \right), \tag{4}$$

where $V_i(S)$ stands for the number of vertices with degree i in the graph S and, if H is a subgraph of S, $S \setminus H$ is the subgraph of S induced by its vertices that do not belong to H. Indeed, the first contribution corresponds to the complete removal of vertices belonging to $\mathscr{S}_k$ but not $\mathscr{S}_{k+1}$. The second contribution corresponds to edges of $\mathscr{S}_k$ connecting v_k and vertices of $\mathscr{S}_{k+1}$. Figure 4 gives an illustration of this situation. In this figure, the contribution 3 comes from the connected components of the vertices attaches to the half edges of v_k numbered 1, 2 and 3. The contribution 4 comes from v_{k+1} and the vertices matched to dotted half edges.

A fundamental step in understanding the behaviour of the exploration process is to identify the asymptotic behaviour of the variables T_k and $N_i(k)$ for large N. This is the object of Theorem 3. To state this, we first introduce some technical notation.

Let $(z_i)_{i\geq 0} \in \mathbb{R}^{\mathbb{Z}_+}$ be such that $\sum_{i\geq 0} z_i \leq 1$ and $\sum_{k\geq 0} iz_i < \infty$. for any $i \geq 0$ let $\hat{z}_i = (i+1)z_i / \sum_j jz_j$ and define:

$$\begin{cases} g_{(z_i)_{i\geq 0}}(s) &= \sum_{i\geq 0} \frac{z_i}{\sum_{l\geq 0} z_l} s^i \\ \hat{g}_{(z_i)_{i\geq 0}}(s) &= \sum_{i\geq 0} \hat{z}_i s^i = \frac{g'_{(z_i)_{i\geq 0}}(s)}{g'_{(z_i)_{i\geq 0}}(1)} \end{cases} \tag{5}$$

respectively the generating series associated to $(z_k)_{k\geq 0}$ and its sized biased version. Let also $\rho_{(z_i)_{i\geq 0}}$ be the largest solution in $[0,1]$ of

$$1 - s = \hat{g}_{(z_i)_{i\geq 0}}(1-s). \tag{6}$$

Remark 3. *Since $\hat{g}$ is the generating function of a probability distribution on the integers, it is convex on $[0,1]$. Therefore, Equation (6) has a positive solution in $(0,1]$ if and only if $\hat{g}'(1) > 1$, which is equivalent to $\frac{\sum_{l\geq 1}(l-1)lz_l}{\sum_{l\geq 1} lz_l} > 1$.*

We also define the following functions:

$$f(z_0, z_1, \dots) = \frac{2 - \rho_{(z_i)_{i\geq 0}}}{\rho_{(z_i)_{i\geq 0}}} \tag{7}$$

$$f_i(z_0, z_1, \dots) = -\frac{1}{\rho_{(z_j)_{j\geq 0}}} \frac{iz_i}{\sum_{j\geq 0} jz_j} + \frac{1}{\rho_{(z_j)_{j\geq 0}}} \left(1 - \frac{\sum_{j\geq 0}(j-1)jz_j}{\sum_{n\geq 0} jz_j} \right) \left(\frac{iz_i}{\sum_{j\geq 0} jz_j} - \frac{(i+1)z_{i+1}}{\sum_{j\geq 0} jz_j} \right). \tag{8}$$

The asymptotic behaviour of the variables T_k and $N_i(k)$ will be driven by the solution of an infinite system of differential equations whose uniqueness and existence is provided by the following lemma, whose proof is postponed to Section 6.2.

Lemma 1. *Let $\boldsymbol{\pi} = (\boldsymbol{\pi}_i)_{i\geq 0} \in [0,1]^{\mathbb{N}}$ such that $\sum_{i\geq 0} \boldsymbol{\pi}_i = 1$. Then, the following system of differential equations has a unique solution which is well defined on $[0, t_{\max})$ for some $t_{\max} > 0$:*

$$\begin{cases} \frac{\mathrm{d}z_i}{\mathrm{d}t} &= f_i(z_0, z_1, \ldots); \\ z_i(0) &= \boldsymbol{\pi}_i. \end{cases} \tag{S}$$

We are now ready to state the main result of this section.

Theorem 3. *With high probability, for all $k \leq \alpha_\varepsilon^{(N)} N$:*

$$T_k = N z\left(\frac{k}{N}\right) + o(N)$$
$$N_i(k) = N z_i\left(\frac{k}{N}\right) + o(N),$$

where $(z_0, z_1, \ldots)$ is the unique solution of (S) and z is the unique solution of $\frac{\mathrm{d}z}{\mathrm{d}t} = f(z_0, z_1, \ldots)$ with initial condition given by $z(0) = 0$.

Proof. Our main tool to prove this result is Corollary 2, which is stated and proved in the Appendix. This corollary is a version of a result of Wormald [14] tailored for our purpose. To apply this result we need to check the following two points:

1. There exists $0 < \beta < 1/2$ such that with high probability for all $k \leq \alpha_\varepsilon^{(N)} N$,

$$|T_{k+1} - T_k| \leq N^\beta \text{ and for all } k \geq 0,\ |N_i(k+1) - N_i(k)| \leq N^\beta.$$

2. We denote by $(\mathcal{F}_k)_{k\geq 0}$ the canonical filtration associated to the sequence $((N_i(k))_{i\geq 0})_{k\geq 0}$. There exists $\lambda > 0$ such that for every k and n,

$$\mathbb{E}[T_{k+1} - T_k \,|\, \mathcal{F}_k] = f\left(\frac{N_0(k)}{N}, \frac{N_1(k)}{N}, \ldots\right) + O\left(N^{-\lambda}\right),$$
$$\mathbb{E}[N_i(k+1) - N_i(k) \,|\, \mathcal{F}_k] = f_i\left(\frac{N_0(k)}{N}, \frac{N_1(k)}{N}, \ldots\right) + O\left(N^{-\lambda}\right).$$

The first identity is a consequence of Equation (2) with $\delta < 1/2 - 1/\gamma$. Indeed on the event $\mathbf{G}_\varepsilon$ the vertices v_k have degree at most $N^{1/\gamma}$ and therefore $T_{k+1} - T_k \leq 1 + 2N^{1/\gamma}N^\delta \ll N^\beta$ for some $\beta < 1/2$. Since $|N_i(k+1) - N_i(k)| \leq (T_{k+1} - T_k)/2$ the second inequality is trivial.

In order to establish the second identity, we need to analyse the structure of $\mathscr{S}_k$ and the contributions (3) and (4). To this end, we will study the random variable $\mathbb{e}_k$ that counts the number of excursions strictly above k of the walker (X_n) coding the DFS between the times T_k and $T_{k+1} - 1$ (in Figure 4, $\mathbb{e}_k = 3$). In particular, the expectation of $\mathbb{e}_k$conditional on $\mathcal{F}_k$ is well defined on the event $\mathbf{G}_\varepsilon$.

If we disconnect the edges joining the $\mathbb{e}_k$ first children of v_k in the tree constructed by the DFS, the remaining connected components in $\mathscr{S}_k$ of these children have size smaller than N^δ. This motivates the following notation:

- for every $i \geq 0$, let $\mathbf{Ext}_i^k$ (resp. $\mathbf{Surv}_i^k$) be the set of half-edges $e \in \mathscr{S}_k$ connected to a vertex w of degree i (in $\mathscr{S}_k$) such that the the connected component of w after removing this half-edge has size smaller than N^δ (resp. larger than N^δ);

- let $\mathbf{Ext}^k$ (resp. $\mathbf{Surv}^k$) be the set of half-edges $e \in \mathscr{S}_k$ connected to a vertex w such that the connected component of w after removing this half-edge has size smaller than N^δ (resp. larger than N^δ). Note that $\mathbf{Ext}^k = \sqcup_{j\geq 0}\mathbf{Ext}^k_j$ and $\mathbf{Surv}^k = \sqcup_{j\geq 0}\mathbf{Surv}^k_j$.

Recall that on $\mathbf{G}_\varepsilon$, for all $k \leq \alpha_\varepsilon^{(N)} N$, v_k has a neighbor in $\mathscr{S}_k$ that belongs to a connected component of $\mathscr{S}_k$ with more than N^δ vertices. This means that for every such k, with probability $1-\mathcal{O}(N^{-1-\lambda})$, the random variable $\mathbb{e}_k$ is the number of half edges of $\mathbf{Ext}^k$ attached to v_k before attaching a half edge of $\mathbf{Surv}^k$ during the DFS. In order to compute its expectation, we first condition on $\{\deg_{\mathscr{S}_k}(v_k) = d\}$, with $d > 0$ fixed.

Conditional on the event $\{\deg_{\mathscr{S}_k}(v_k) = d\} \cap \{\mathbb{e}_k < \deg_{\mathscr{S}_k}(v_k)\}$, the law of $(\mathscr{S}_k, v_k)$ is the law of a rooted configuration model $\mathbf{C}^d_{\mathbf{N}(k)}$ with root degree d and degree sequence $\mathbf{N}(k) := (N_i(k))_{i\geq 0}$, conditioned on the root to have one of its half-edges paired to an element of $\mathbf{Surv}(\mathbf{C}^d_{\mathbf{N}(k)})$. We define the new random variable $\tilde{\mathbb{e}}_k$ as the number of half edges of the root paired to an element of $\mathbf{Ext}(\mathbf{C}^d_{\mathbf{N}(k)})$ before pairing a half edge to an element of $\mathbf{Surv}(\mathbf{C}^d_{\mathbf{N}(k)})$ when doing successive uniform matching in the configuration model (with the convention $\tilde{\mathbb{e}}_k = d$ if the root has no half-edged paired to an element of $\mathbf{Surv}(\mathbf{C}^d_{\mathbf{N}(k)})$). We have the following equality for all j:

$$\mathbb{P}\big(\mathbb{e}_k = j \,\big|\, \mathcal{F}_k \text{ and } \deg_{\mathscr{S}_k}(v_k) = d\big) = \mathbb{P}\big(\tilde{\mathbb{e}}_k = j \,\big|\, \tilde{\mathbb{e}}_k < d\big) + \mathcal{O}(N^{-1-\lambda}).$$

Let

$$\tilde{\rho}_k := \frac{|\mathbf{Surv}(\mathbf{C}^d_{\mathbf{N}(k)})|}{2|E(\mathbf{C}^d_{\mathbf{N}(k)})|} = 1 - \frac{|\mathbf{Ext}(\mathbf{C}^d_{\mathbf{N}(k)})|}{2|E(\mathbf{C}^d_{\mathbf{N}(k)})|},$$

the proportion of half-edges in $\mathbf{Surv}(\mathbf{C}^d_{\mathbf{N}(k)})$ (resp. $\mathbf{Ext}(\mathbf{C}^d_{\mathbf{N}(k)})$). This proportion is close to a constant ρ_k that we now define with the help of additional notation. Recalling (5), let

$$\begin{array}{lcl} p_i = p_i(k) = \frac{N_i(k)}{\sum_{j\geq 0} N_j(k)} & ; & g_k = g_{(p_j)_{j\geq 0}} \\ \hat{p}_i = \hat{p}_i(k) = \frac{(i+1)p_{i+1}(k)}{\sum_{j\geq 0} j p_j(k)} & ; & \hat{g}_k = \hat{g}_{(p_j)_{j\geq 0}} = g_{(\hat{p}_j)_{j\geq 0}} \end{array},$$

and let $\rho_k = \rho_{(p_j(k))_{j\geq 0}}$ be the largest solution in $[0,1]$ of $1-s = \hat{g}_k(1-s)$. We have the following lemma, whose proof is postponed to Section 6.1.

Lemma 2. *For all $0 \leq k \leq \alpha_\varepsilon^{(N)} N$, there exists $\lambda > 0$ and $\eta > 0$ such that, conditionally on $\mathcal{F}_k$, uniformly in k,*

$$\begin{cases} \mathbb{P}\left(\left|\frac{|\mathbf{Ext}_i(\mathbf{C}^d_{\mathbf{N}(k)})|}{2|E(\mathbf{C}^d_{\mathbf{N}(k)})|} - \frac{ip_i}{g_k'(1)}(1-\rho_k)^{i-1}\right| \geq N^{-\lambda}\right) = \mathcal{O}\left(N^{-1-\lambda}\right), \\ \mathbb{P}\left(\left|\frac{|\mathbf{Surv}_i(\mathbf{C}^d_{\mathbf{N}(k)})|}{2|E(\mathbf{C}^d_{\mathbf{N}(k)})|} - \frac{ip_i}{g_k'(1)}(1-(1-\rho_k)^{i-1})\right| \geq N^{-\lambda}\right) = \mathcal{O}\left(N^{-1-\lambda}\right), \\ \mathbb{P}\left(\left|\frac{|\mathbf{Ext}(\mathbf{C}^d_{\mathbf{N}(k)})|}{2|E(\mathbf{C}^d_{\mathbf{N}(k)})|} - (1-\rho_k)\right| \geq N^{-\lambda}\right) = \mathcal{O}\left(N^{-1-\lambda}\right), \\ \mathbb{P}\left(\left|\frac{|\mathbf{Surv}(\mathbf{C}^d_{\mathbf{N}(k)})|}{2|E(\mathbf{C}^d_{\mathbf{N}(k)})|} - \rho_k\right| \geq N^{-\lambda}\right) = \mathcal{O}\left(N^{-1-\lambda}\right). \end{cases}$$

Using this lemma, we obtain:

$$\mathbb{P}\big(\mathbb{e}_k = j \,\big|\, \mathcal{F}_k \text{ and } \deg_{\mathscr{S}_k}(v_k) = d\big) = \frac{\mathbb{P}\big(\{\tilde{\mathbb{e}}_k = j\} \cap \{\tilde{\mathbb{e}}_k < d\}\} \cap \{|\tilde{\rho}_k - \rho_k| \leq \mathcal{O}(N^{-\lambda})\}\big)}{\mathbb{P}\big(\tilde{\mathbb{e}}_k < d \cap \{|\tilde{\rho}_k - \rho_k| \leq \mathcal{O}(N^{-\lambda})\}\big)} + \mathcal{O}\left(N^{-1-\lambda}\right). \quad (9)$$

Fix $j < d$. To estimate the probabilities of (9), we successively match the half edges $c_1, \ldots, c_{j+1}$ of the root uniformly among the half edges of $\mathbf{C}^d_{\mathbf{N}(k)}$. Notice that if none of these half edges are matched together, this is equivalent to an urn model without replacement. At each of these steps, the proportion of available half edges of $\mathbf{Ext}(\mathbf{C}^d_{\mathbf{N}(k)})$ diminishes and is therefore between $1 - \widetilde{\rho}_k - \frac{d}{2|E(\mathbf{C}^d_{\mathbf{N}(k)})|}$ and $1 - \widetilde{\rho}_k$. Recalling that $|E(\mathbf{C}^d_{\mathbf{N}(k)})|$ is uniformly of order N, we can write for every $j < d$

$$\frac{\left(1-\rho_k - C\frac{d}{N} + \mathcal{O}\left(N^{-\lambda}\right)\right)^j \left(\rho_k + \mathcal{O}\left(N^{-\lambda}\right)\right)}{1 - \left(1-\rho_k - C\frac{d}{N} + \mathcal{O}\left(N^{-\lambda}\right)\right)^d} + \mathcal{O}\left(N^{-1-\lambda}\right)$$
$$\leq \mathbb{P}\big(\mathbb{e}_k = j \,\big|\, \mathcal{F}_k \text{ and } \deg_{\mathscr{S}_k}(v_k) = d\big) \leq$$
$$\frac{\left(1-\rho_k + \mathcal{O}\left(N^{-\lambda}\right)\right)^j \left(\rho_k + C\frac{d}{N} + \mathcal{O}\left(N^{-\lambda}\right)\right)}{1 - \left(1-\rho_k + \mathcal{O}\left(N^{-\lambda}\right)\right)^d} + \mathcal{O}\left(N^{-1-\lambda}\right).$$

where C is a constant and the error terms $\mathcal{O}(N^{-\lambda})$ are the same everywhere and uniform in d. This easily translates into

$$\mathbb{P}\big(\mathbb{e}_k = j \,\big|\, \mathcal{F}_k \text{ and } \deg(v_k) = d\big)$$
$$= \frac{(1-\rho_k)^j \rho_k}{1-(1-\rho_k)^d}\left(1 + \mathcal{O}\left(d^2 N^{-1} + dN^{-\lambda}\right)\right)\mathbf{1}_{\{j<d\}} + \mathcal{O}\left(N^{-1-\lambda}\right),$$

where, once again, the error terms are uniform. We can now compute the conditional expectation of $\mathbb{e}_k$:

$$\mathbb{E}\left[\mathbb{e}_k \middle| \mathcal{F}_k, \deg_{\mathscr{S}_k}(v_k) = d\right]$$
$$= \frac{1-\rho_k}{\rho_k\left(1-(1-\rho_k)^d\right)}\left(-d\rho_k(1-\rho_k)^{d-1} + 1 - (1-\rho_k)^d\right)\left(1 + \mathcal{O}\left(d^2N^{-1} + dN^{-\lambda}\right)\right) + \mathcal{O}\left(N^{-\lambda}\right),$$

where the last error term comes from the fact the $\mathbb{e}_k$ is smaller that $\mathcal{O}(N)$ by definition.

To finally compute the expectation of $\mathbb{e}_k$, we want to sum the above equality with respect to the law of $\deg_{\mathscr{S}_k}(v_k)$. By construction, in $\mathscr{S}_{k-1}$, the vertex v_k is attached to v_{k-1} by a half edge of $\mathbf{Surv}^{k-1}$ chosen uniformly. Therefore, by Lemma 2, the law of the degree of v_k in $\mathscr{S}_k$ is given by

$$\mathbb{P}(\deg_{\mathscr{S}_k}(v_k) = d \,|\, \mathcal{F}_k) = \frac{(d+1)p_{d+1}(k-1)}{\rho_{k-1}g'_{k-1}(1)}\left(1-(1-\rho_{k-1})^d\right)(1 + \mathcal{O}(N^{-\lambda})),$$

where the error term is uniform in d and k. We can replace $k-1$ by k in the above probabilities at the cost of a factor $1 + \mathcal{O}(N^{-\lambda})$ which is uniform in k and d. Indeed, on $\mathbf{G}_\varepsilon$, the difference between $\mathscr{S}_{k-1}$ and $\mathscr{S}_k$ consists of at most $N^{1/\gamma}$ components of size at most N^δ and we have $p_d(k-1) = p_d(k)\left(1 + \mathcal{O}(N^{1/\gamma+\delta-1})\right)$ uniformly in k and d. The difference between ρ_{k-1} and ρ_k is then of the same order by a Taylor expansion. Therefore

$$\mathbb{P}(\deg_{\mathscr{S}_k}(v_k) = d \,|\, \mathcal{F}_k) = \frac{(d+1)p_{d+1}(k)}{\rho_k g'_k(1)}\left(1-(1-\rho_k)^d\right)(1 + \mathcal{O}(N^{-\lambda})), \tag{10}$$

and we get:

$$
\begin{aligned}
&\mathbb{E}\left[\mathbb{e}_k | \mathcal{F}_k\right] \\
&= \frac{(1-\rho_k)}{g_k'(1)\rho_k^2} \sum_{d\geq 0} (d+1) p_{d+1}(k) \left(-d\rho_k (1-\rho_k)^{d-1} + 1 - (1-\rho_k)^d \right) \left(1 + \mathcal{O}\left(d^2 N^{-1} + dN^{-\lambda}\right)\right) + \mathcal{O}\left(N^{-\lambda}\right) \\
&= \frac{(1-\rho_n)}{g_n'(1)\rho_n^2} \left(g_n'(1) - \rho_n g_n''(1-\rho_n) - g_n'(1-\rho_n) \right) + \mathcal{O}(N^{\frac{1}{\gamma}-1}) \cdot \mathcal{O}\left(\sum_{d\geq 0} d^2 p_d(k) \right) + \mathcal{O}(N^{-\lambda}).
\end{aligned}
$$

Notice that the error $\mathcal{O}(N^{-\lambda})$ is uniform in k and d. Let us prove that $\sum_{d\geq 0} d^2 p_d(k)$ is of order 1. First note that it is of the same order as $\frac{1}{N}\sum_{d\geq 0} d^2 N_d(k)$, where we recall that $N_d(k)$ is the number of vertices of degree d in $\mathscr{S}_k$. Indeed the number of vertices of $\mathscr{S}_k$ is of order N. Denoting by $N_{\geq d}(k)$ the number of vertices of degree larger than d in $\mathscr{S}_k$, it holds that $N_{\geq d}(k) \geq N_{\geq d}(k+1)$ from the definition of the algorithm. This monotonicity implies that

$$
\frac{1}{N} \sum_{d\geq 0} d^2 N_d(k) \leq \sum_{d\geq 0} d^2 \frac{N_d(0)}{N},
$$

where the right-hand side converges towards a finite limit by our assumptions on the initial degree sequence of the model. Therefore

$$
\begin{aligned}
\mathbb{E}\left[\mathbb{e}_k | \mathcal{F}_k\right] &= \frac{(1-\rho_k)}{g_k'(1)\rho_k^2} \left(g_k'(1) - \rho_k g_n''(1-\rho_k) - g_k'(1-\rho_k) \right) + \mathcal{O}(N^{-\lambda}) \\
&= \frac{1-\rho_k}{\rho_k} \left(1 - \hat{g}_k'(1-\rho_k)\right) + \mathcal{O}(N^{-\lambda}), \qquad (11)
\end{aligned}
$$

where we used $1-\rho_k = \hat{g}_k(1-\rho_k) = g_k'(1-\rho_k)/g_k'(1)$.

Now that we know more about the random variable $\mathbb{e}_k$, we can study more in depth the time difference between two consecutive ladder times.

With high probability, the first $\mathbb{e}_k$ neighbours of v_k in the tree constructed by the DFS all belong to distinct connected components of $\mathscr{S}_k \backslash \{v_k\}$. We denote these components by $W^{(1)}, \ldots, W^{(\mathbb{e}_k)}$. Notice that by Lemma 2, for all $i \geq 0$, the ratio $|\mathbf{Ext}_i^k|/|\mathbf{Ext}^k|$ concentrates around $ip_i(k)(1-\rho_k)^{i-1}/g_k'(1)$. Therefore, conditional on $\mathbb{e}_k$, with probability $1-\mathcal{O}(N^{-\lambda})$, the size of these components can be coupled with the size of $\mathbb{e}_k$ i.i.d. Galton-Watson trees independent of $\mathbb{e}_k$ and whose reproduction laws have generating series given by $\tilde{g}_k(s) := \hat{g}_k((1-\rho_k)s)/(1-\rho_k)$. Therefore, the expected size of a component is given by:

$$
\mathbb{E}\left[\left|W^{(1)}\right| \,\middle|\, \mathcal{F}_k\right] = \frac{1}{1-\tilde{g}_k'(1)} + \mathcal{O}(N^{-\lambda}) = \frac{1}{1-\hat{g}_k'(1-\rho_k)} + \mathcal{O}(N^{-\lambda}),
$$

and we obtain, using Equation (11):

$$
\begin{aligned}
\mathbb{E}\left[T_{k+1} - T_k \,\middle|\, \mathcal{F}_k\right] &= 1 + 2 \times \mathbb{E}\left[\sum_{p=1}^{\mathbb{e}_k} \left|W^{(i)}\right| \,\middle|\, \mathcal{F}_k\right] \\
&= 1 + 2\left(\frac{1-\rho_k}{\rho_k}\left(1-\hat{g}_k'(1-\rho_k)\right) + \mathcal{O}(N^{-\lambda})\right)\left(\frac{1}{1-\hat{g}_k'(1-\rho_k)} + \mathcal{O}(N^{-\lambda})\right) \\
&= \frac{2-\rho_k}{\rho_k} + \mathcal{O}\left(N^{-\lambda}\right) \\
&= f\left(\frac{N_0(k)}{N}, \frac{N_1(k)}{N}, \ldots\right) + \mathcal{O}(N^{-\lambda}) \qquad (12)
\end{aligned}
$$

which is the desired result for the evolution of (T_k).

We now turn to the evolution of the $(N_i(k))$ which follows from the analysis of the expectation of the terms (3) and (4). The term (3) accounts for the vertices of degree i in the graph $\mathscr{S}_k \setminus \mathscr{S}_{k+1}$. Among these vertices, the vertex v_k has a special role because it is conditioned to be matched to an element of $\mathbf{Surv}^k$. Therefore we write

$$V_i(\mathscr{S}_k \setminus \mathscr{S}_{k+1}) = \mathbf{1}_{\{\deg_{\mathscr{S}_k}(v_k)=i\}} + \sum_{j=1}^{\mathbb{e}_k} \sum_{v \in W^{(j)}} \mathbf{1}_{\{\deg_{\mathscr{S}_k}(v)=i\}}.$$

We first compute the expectation of the sum in the right hand side of the previous equation. The connected components $W^{(1)}, \ldots, W^{(\mathbb{e}_k)}$ are well approximated by independent Galton-Watson trees with offspring distribution given by $\hat{g}_n$, conditioned on extinction. Let C_i be the number of individuals that have $i-1$ children in such a tree. These individuals all have degree i in $\mathscr{S}_k$ and contribute to the sum. The quantity C_i satisfies the following recursion established by summing over the possible number of children of the root:

$$\mathbb{E}[C_i] = \mathbb{E}\left[\sum_{l \geq 0} \hat{p}_l (1-\rho_k)^l \left(l C_i + \delta_{l=i-1}\right)\right] = \mathbb{E}[C_i] \hat{g}_k'(1-\rho_k) + \hat{p}_{i-1}(1-\rho_k)^{i-1},$$

which leads to

$$\mathbb{E}[C_i] = \frac{\hat{p}_{i-1}(1-\rho_k)^{i-1}}{1-\hat{g}_k'(1-\rho_k)}. \tag{13}$$

Therefore, multiplying (11) and (13), we obtain

$$\mathbb{E}\left[V_i(\mathscr{S}_k \setminus \mathscr{S}_{k+1}) \,\middle|\, \mathcal{F}_k\right] = \mathbb{P}\left(\deg_{\mathscr{S}_k}(v_k) = i \,\middle|\, \mathcal{F}_k\right) + \frac{\hat{p}_{i-1}}{\rho_k}(1-\rho_k)^{i-1} + \mathcal{O}\left(N^{-\lambda}\right). \tag{14}$$

Note that the sum over i of these terms gives the total number of vertices in the connected components associated to the first $\mathbb{e}_k$ children of v_k: $(1-\rho_n)/\rho_n + o(1)$. This is in agreement with Equation (12).

For the last term (4), we use the fact that, with probability $1-\mathcal{O}(N^{-\lambda})$, the elements of $\mathbf{m}_{v_k}$ that belong to $\mathscr{S}_{k+1}$ are distinct. One of these elements is v_{k+1} and has a special role, while all the others correspond to a uniform matching to a half edge of a vertex of $\mathscr{S}_{k+1} \setminus \{v_{k+1}\}$ and therefore have degree i with probability $\hat{p}_{i-1}$. Note that there are $\deg_{\mathscr{S}_k}(v_k) - \mathbb{e}_k - 1$ terms in the sum (4) when excluding v_{k+1}. We have:

$$\begin{aligned}
&\mathbb{E}\left[\sum_{v \in \mathbf{m}_{v_k} \cap \mathscr{S}_{k+1}} \left(-\mathbf{1}_{\deg_{\mathscr{S}_k}(v)=i} + \mathbf{1}_{\deg_{\mathscr{S}_k}(v)=i+1}\right)\right] \\
&\quad = -\mathbb{P}\left(\deg_{\mathscr{S}_k}(v_{k+1}) = i \,\middle|\, \mathcal{F}_k\right) + \mathbb{P}\left(\deg_{\mathscr{S}_k}(v_{k+1}) = i+1 \,\middle|\, \mathcal{F}_k\right) \\
&\qquad + \mathbb{E}\left[\deg(v_k) - \mathbb{e}_k - 1 \,\middle|\, \mathcal{F}_k\right](-\hat{p}_{i-1} + \hat{p}_i) + \mathcal{O}(N^{-\lambda}) \\
&\quad = -\mathbb{P}\left(\deg_{\mathscr{S}_k}(v_{k+1}) = i \,\middle|\, \mathcal{F}_k\right) + \mathbb{P}\left(\deg_{\mathscr{S}_k}(v_{k+1}) = i+1 \,\middle|\, \mathcal{F}_k\right) \\
&\qquad + \frac{1}{\rho_k}\left(\hat{g}_k'(1) - (1-\rho_k)\hat{g}_k'(1-\rho_k) - (1-\rho_k)(1-\hat{g}_k'(1-\rho_k)) - \rho_k\right)(-\hat{p}_{i-1} + \hat{p}_i) + \mathcal{O}(N^{-\lambda}) \\
&\quad = -\mathbb{P}\left(\deg_{\mathscr{S}_k}(v_{k+1}) = i \,\middle|\, \mathcal{F}_k\right) + \mathbb{P}\left(\deg_{\mathscr{S}_k}(v_{k+1}) = i+1 \,\middle|\, \mathcal{F}_k\right) \\
&\qquad + \frac{1}{\rho_k}\left(1 - \hat{g}_k'(1)\right)(\hat{p}_{i-1} - \hat{p}_i) + \mathcal{O}(N^{-\lambda}).
\end{aligned} \tag{15}$$

Hence, summing (14) and (15), we obtain the total contribution of (3) and (4):

$$\begin{aligned}\mathbb{E}\left[N_i(k+1)-N_i(k)\,\middle|\,\mathcal{F}_k\right] = &-\mathbb{P}\left(\deg_{\mathscr{S}_k}(v_k)=i\,\middle|\,\mathcal{F}_k\right)-\mathbb{P}\left(\deg_{\mathscr{S}_k}(v_{k+1})=i\,\middle|\,\mathcal{F}_k\right)\\ &+\mathbb{P}\left(\deg_{\mathscr{S}_k}(v_{k+1})=i+1\,\middle|\,\mathcal{F}_k\right)\\ &-\frac{\hat{p}_{i-1}}{\rho_k}(1-\rho_k)^{i-1}+\frac{1}{\rho_k}\left(1-\hat{g}_k'(1)\right)(\hat{p}_{i-1}-\hat{p}_i)+\mathcal{O}(N^{-\lambda}).\end{aligned}$$

Recall that the conditional law of $\deg_{\mathscr{S}}(v_k)$ is given by equation (10). Similar arguments than those used to compute it lead to

$$\mathbb{P}\left(\deg_{\mathscr{S}_k}(v_{k+1})=i\,\middle|\,\mathcal{F}_k\right)=\mathbb{P}\left(\deg_{\mathscr{S}_k}(v_k)=i-1\,\middle|\,\mathcal{F}_k\right)+\mathcal{O}(N^{-\lambda}).$$

Therefore we have

$$\begin{aligned}\mathbb{E}\left[N_i(k+1)-N_i(k)\,\middle|\,\mathcal{F}_k\right] &= -\frac{\hat{p}_{i-1}}{\rho_k}\left(1-(1-\rho_k)^{i-1}\right)-\frac{\hat{p}_{i-1}}{\rho_k}(1-\rho_k)^{i-1}\\ &\quad+\frac{1}{\rho_k}\left(1-\hat{g}_k'(1)\right)(\hat{p}_{i-1}-\hat{p}_i)+\mathcal{O}(N^{-\lambda})\\ &= -\frac{\hat{p}_{i-1}}{\rho_k}+\frac{1}{\rho_k}\left(1-\hat{g}_k'(1)\right)(\hat{p}_{i-1}-\hat{p}_i)+\mathcal{O}(N^{-\lambda})\\ &= f_i\left(\frac{N_0(k)}{N},\frac{N_1(k)}{N},\dots\right)+O\left(N^{-\lambda}\right).\end{aligned}$$

This ends the proof of Theorem 3. □

5 Proofs of the main results

5.1 Proof of Theorem 1

The time variable in Theorem 1 is the proportion of vertices explored by the DFS whereas in Theorem 3 it is the index of the ladder times T_k. Therefore, to prove Theorem 1, a first step is to study the asymptotic proportion of explored vertices at time T_k. For all $N\geq 1$ and all $1\leq k\leq \alpha_\varepsilon^{(N)}N$, this proportion is given by $\omega(T_k):=\frac{k+T_k}{2N}$. Therefore, by Theorem 3, this proportion satisfies

$$\omega(T_k)=\widetilde{z}\left(\frac{k}{N}\right)+o(1),\qquad\text{with}\quad \widetilde{z}(t)=\frac{1}{2}\left(t+z\left(t\right)\right). \tag{16}$$

Fix $0\leq\alpha\leq\alpha_\varepsilon^{(N)}$ and recall the definition of $\tau^{(N)}(\alpha)$ given in Theorem 1. At time $T_{N\widetilde{z}^{-1}(\alpha)}$, by Equation (16), the number of explored vertices is $\alpha N+o(N)$. Therefore $\tau^{(N)}(\alpha)=T_{N\widetilde{z}^{-1}(\alpha)}+o(1)$. Hence, for all $i\geq 0$,

$$\begin{aligned}N_i(\tau^{(N)}(\alpha)) &= N_i\left(T_{N\widetilde{z}^{-1}(\alpha)}+o(1)\right)\\ &= Nz_i\left(\widetilde{z}^{-1}(\alpha)\right)+o(N).\end{aligned}$$

It is easy to check that the sequence of functions $(z_i\circ\widetilde{z}^{-1})_{i\geq 0}$ is solution of the system (S') of Lemma 5. The generating function $g(\alpha,s)$ of Theorem 1 is given by

$$g(\alpha,s)=\frac{1}{1-\alpha}\sum_{i\geq 0}z_i\circ\widetilde{z}^{-1}(\alpha)s^i,$$

which is the desired result by Equation (25) and Proposition 1.

5.2 Proof of Theorem 2

Let $N \geq 1$. By definition, for all $1 \leq k \leq K^{(N)}$, the contour process of the tree constructed by the DFS algorithm at time T_k is located at point (T_k, k). Furthermore, by Theorem 3,

$$(T_k, k) = N\left(z\left(\frac{k}{N}\right) + o(1), \frac{k}{N}\right).$$

Note that $|T_{k+1} - T_k| = o(N)$ and that, between two consecutive T_k's, the contour process cannot fluctuate by more than $o(N)$. Hence, after normalization by N, the limiting contour process converges to the curve $(z(t), t)$ where t ranges from 0 to $t_{\max} = \sup\{t > 0, z'(t) < +\infty\}$. Recall that by the definition of z in Theorem 3 and Equation (7), $z'(t) = (2 - \rho_{(z_i(t))_{i\geq 0}})/\rho_{(z_i(t))_{i\geq 0}}$. Hence, if we parametrize $(z(t), t)$ in terms of $\rho = \rho_{(z_i(t))_{i\geq 0}}$, the curve can be written $(x(\rho), y(\rho))$ where the functions x and y satisfy

$$\frac{x'(\rho)}{y'(\rho)} = \frac{2-\rho}{\rho}.$$

Note that when t ranges from 0 to $t_{\max}$, the parameter ρ decreases from $\rho_{\boldsymbol{\pi}}$ to 0. In order to get a second equation connecting x' and y', we go back to the discrete process and observe that the number of explored vertices at time T_k is given by $(k + T_k)/2$. This comes from a general fact about contour processes of trees. Using notations of Theorem 1, let $\hat{g}(\alpha, \cdot)$ be the size-biased version of $g(\alpha, \cdot)$. For all $\rho \in (0, \rho_{\boldsymbol{\pi}}]$, let $\alpha(\rho)$ be the unique solution of $1 - \rho = \hat{g}(\alpha(\rho), 1 - \rho)$. After renormalizing by N, we get that:

$$\frac{x(\rho) + y(\rho)}{2} = \alpha(\rho).$$

This yields the following system of equations:

$$\begin{cases} \frac{x'(\rho)}{y'(\rho)} = \frac{2-\rho}{\rho} \\ \frac{x'(\rho)+y'(\rho)}{2} = \alpha'(\rho). \end{cases}$$

Therefore,

$$\begin{cases} x'(\rho) = (2-\rho)\alpha'(\rho) \\ y'(\rho) = \rho\alpha'(\rho). \end{cases}$$

Integrating by part, this gives the formulas for $x^{\uparrow}$ and $y^{\uparrow}$ of Theorem 2. Fix $\rho \in (0, \rho_{\boldsymbol{\pi}}]$. Then, the asymptotic profile of the decreasing phase of the DFS is obtained by translating horizontally each point $(x^{\uparrow}(\rho), y^{\uparrow}(\rho))$ of the ascending phase to the right by twice the asymptotic proportion of the giant component of the remaining graph of parameter ρ, which is $2(1 - g(\alpha(\rho), 1 - \rho))$.

6 Technical lemmas

6.1 Asymptotic densities in a configuration model

In this section we establish Lemma 2. The proofs of each of the four estimates follow the same scheme, therefore we only focus on the proof the last one, namely that there exists $\lambda > 0$ such that:

$$\mathbb{P}\left(\left|\frac{|\mathbf{Surv}(\mathbf{C}^d_{\mathbf{N}(k)})|}{2|E(\mathbf{C}^d_{\mathbf{N}(k)})|} - \rho_k\right| \geq N^{-\lambda}\right) = \mathcal{O}\left(N^{-1-\lambda}\right).$$

First, notice that for the values of k that we consider and under our assumptions **A1** and **A2**, the number of edges and vertices of the graphs $\mathbf{C}^d_{\mathbf{N}(k)}$ are, with probability $\mathcal{O}\left(N^{-1-\lambda}\right)$, all of order N. Therefore, it is enough to prove the following bound:

$$\mathbb{P}\left(\left|\frac{|\mathbf{Surv}(\mathbf{C}^d_{\mathbf{N}(k)})|}{2|E(\mathbf{C}^d_{\mathbf{N}(k)})|} - \rho_k\right| \geq |E(\mathbf{C}^d_{\mathbf{N}(k)})|^{-\lambda}\right) = \mathcal{O}\left(|E(\mathbf{C}^d_{\mathbf{N}(k)})|^{-1-\lambda}\right).$$

This is a direct consequence of the two following Lemmas. The first one is a general concentration result for configuration graphs.

Lemma 3. *Fix $\gamma > 2$ and $n \geq 1$. Let $\mathbf{d} = (d_1, \ldots, d_n)$ be such that $\max\{d_1, \ldots, d_n\} \leq n^{1/\gamma}$. Fix also $\delta \in (0, 1/2)$ and recall that, for a graph G, $\mathbf{Surv}(G)$ denotes the set of half edges of G attached to a vertex v such that the connected component of v after removing this half edge has at least n^δ vertices. Let $m = \sum_i d_i$ the number of half edges of a configuration graph $\mathcal{C}(\mathbf{d})$, then, for any $\delta' \geq \delta$ one has*

$$\mathbb{P}\left(\left|\frac{|\mathbf{Surv}(\mathcal{C}(\mathbf{d}))|}{m} - \frac{\mathbb{E}\left(|\mathbf{Surv}(\mathcal{C}(\mathbf{d}))|\right)}{m}\right| \geq \frac{n^{\delta'+\frac{1}{\gamma}}}{2\sqrt{m}}\right) \leq C\exp\left(-Cn^{2(\delta'-\delta)}\right).$$

The second Lemma consists in an estimation of the expectation of $|\mathbf{Surv}(\mathcal{C}(\mathbf{d}^{(n)}))|$ for a sequence of configuration models that satisfy the assumptions of Theorem 1.

Lemma 4. *Let $(\mathcal{C}(\mathbf{d}^{(n)}))_{n\geq 1}$ be a sequence of configuration models with asymptotic degree distribution $\boldsymbol{\pi}$. We suppose that $\boldsymbol{\pi}$ is supercritical in the sense of Definition 3 and that the sequence $\mathbf{d}^{(n)}$ satisfies assumption* **A1***. Moreover, suppose that for all $n \geq 1$, $\max\{d_1, \ldots, d_n\} \leq n^{1/\gamma}$.*

For all $n \geq 1$, let g_n be the generating series associated to the empirical distribution of the degree sequence $\mathbf{d}^{(n)}$. Let ρ_n be the smallest positive solution of the equation $\hat{g}_n(1-x) = 1-x$. Then, for n sufficiently large:

$$\frac{\mathbb{E}\left[|\mathbf{Surv}(\mathcal{C}(\mathbf{d}^{(n)}))|\right]}{2g'_n(1)} = \rho_n + \mathcal{O}\left(n^{2\delta+\frac{1}{\gamma}-1}\right).$$

Proof of Lemma 3. In order to prove Lemma 3, it is sufficient to check that the application $\mathbf{Surv}(\cdot)$ is Lipschitz in the following sense. We say that two configuration models are related by a switching if they differ by exactly two pairs of matched half-edges (see Figure 5). Then, we claim that $\mathbf{Surv}(\cdot)$ is such that, for any two graphs G_1 and G_2 differing by a switching:

$$||\mathbf{Surv}(\mathrm{G}_1)| - |\mathbf{Surv}(\mathrm{G}_2)|| \leq 8n^{\delta+\frac{1}{\gamma}}. \tag{17}$$

Using a result of Bollobàs and Riordan [5, Lemma 8], this regularity implies the following concentration inequality:

$$\mathbb{P}\left(\left||\mathbf{Surv}(\mathbf{C}^d_{\mathbf{N}(k)})| - \mathbb{E}[|\mathbf{Surv}(\mathbf{C}^d_{\mathbf{N}(k)})|]\right| \geq t\right) \leq 2\exp\left(\frac{-t^2}{Cn^{2\delta+\frac{2}{\gamma}}m}\right), \tag{18}$$

By taking $t = n^{\delta'+\frac{1}{\gamma}}m^{\frac{1}{2}}$ in (18), we obtain Lemma 3.

It remains to prove inequality (17). To pass from G_1 to G_2, one has to delete two edges in G_1 and then add two other edges. Therefore, it suffices to study the effect of adding an edge e on a graph G having maximal degree $n^{1/\gamma}$.

Let u and v be the extremities of e. Let us define two partial order associated respectively to u and v among the half-edges of $\mathbf{Ext}(\mathrm{G}) = \mathbf{Surv}(\mathrm{G})^c$. We say that:

Figure 5: Switching two edges in a graph.

- $e_1 \preceq_u e_2$ if all the paths connecting e_2 to u contain e_1,
- $e_1 \preceq_v e_2$ if all the paths connecting e_2 to v contain e_1.

Let f_u (resp. f_v) be a maximal element for the partial order $\preceq_u$ (resp. $\preceq_v$), and denote by $\mathscr{C}_{f_u}$ (resp. $\mathscr{C}_{f_v}$) the connected component of the extremity of f_u (resp. f_v) after the removal of f_u (resp. f_v) in G. Then, by maximality, the set of extremities of half-edges that change their status from $\mathbf{Ext}(\mathrm{G})$ to $\mathbf{Surv}(\mathrm{G})$ after the adding of e is included in $\mathscr{C}_{f_u} \cup \mathscr{C}_{f_v}$. See Figure 6 for an illustration. Since

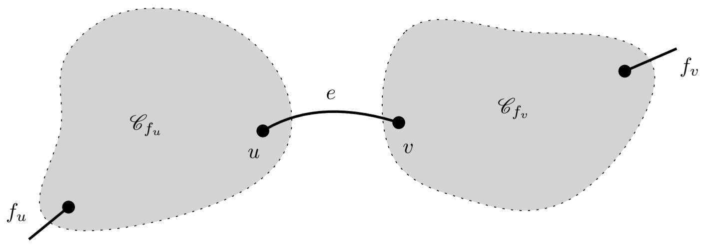

Figure 6: Effect of the edge e.

f_u (resp. f_v) was in $\mathbf{Ext}(\mathrm{G})$, the number of vertices in $\mathscr{C}_{f_u}$ (resp. $\mathscr{C}_{f_v}$) is at most n^δ. Since the maximal degree of a vertex in G is $n^{1/\gamma}$, we deduce that:

$$||\mathbf{Surv}_n(\mathrm{G})| - |\mathbf{Surv}_n(\mathrm{G} \cup e)|| \leq 2n^{\delta + \frac{1}{\gamma}}.$$

This implies (17) and Lemma 3. □

Proof of Lemma 4. Fix $n \geq 1$. Let e be a uniformly chosen half-edge in $\mathcal{C}(\mathbf{d}^{(n)})$ and let v be the extremity of e. We denote $\mathscr{C}_v$ the connected component of v inside $\mathcal{C}(\mathbf{d}^{(n)})$) after removing e. Then, since $\mathbb{E}\left[|\mathbf{Surv}(\mathcal{C}(\mathbf{d}^{(n)}))|\right] = 2g_n'(1)\mathbb{P}\left(e \in \mathbf{Surv}(\mathcal{C}(\mathbf{d}^{(n)}))\right)$, it is sufficient to prove that

$$\mathbb{P}\left(|\mathscr{C}_v| \geq n^\delta\right) = \rho_n + \mathcal{O}\left(n^{2\delta + \frac{2}{\gamma} - 1}\right). \tag{19}$$

Let $(d_i^\uparrow)_{1 \leq i \leq n}$ and $(d_i^\downarrow)_{1 \leq i \leq n}$ respectively denote the increasing and decreasing reordering of the degree sequence $(d_i)_{1 \leq i \leq n}$:

$$d_1^\uparrow \leq \cdots \leq d_n^\uparrow \qquad \text{and} \qquad d_1^\downarrow \geq \cdots \geq d_n^\downarrow.$$

In order to prove (19), we will use a coupling argument. More precisely, we first introduce two Galton-Watson trees:

- $\mathscr{T}^-$ with reproducing law: $q_i^- := \frac{(i+1)|\{j \geq \lceil n^\delta \rceil, d_j^\downarrow = i+1\}|}{\sum_{j \geq \lceil n^\delta \rceil}(j+1)d_j^\downarrow}$,

- $\mathscr{T}^+$ with reproducing law: $q_i^+ := \frac{(i+1)|\{j\geq\lceil n^\delta\rceil, d_j^\uparrow=i+1\}|}{\sum_{j\geq\lceil n^\delta\rceil}(j+1)d_j^\uparrow}$.

We also let E be the event where, in the $\lfloor n^\delta \rfloor$ first steps of the exploration of $\mathscr{C}_v$, no loop is discovered. Then, the following inequalities hold:

$$(1-\mathbb{P}(E))\,\mathbb{P}(|\mathscr{T}^-|\geq n^\delta) \leq \mathbb{P}\left(|\mathscr{C}_v|\geq n^\delta\right) \leq \mathbb{P}(|\mathscr{T}^+|\geq n^\delta). \tag{20}$$

Now, we prove that:

$$\begin{cases}\mathbb{P}(|\mathscr{T}^-|\geq n^\delta) = \rho_n + \mathcal{O}(n^{\delta+\frac{1}{\gamma}-1}),\\ \mathbb{P}(|\mathscr{T}^+|\geq n^\delta) = \rho_n + \mathcal{O}(n^{\delta+\frac{1}{\gamma}-1}).\end{cases} \tag{21}$$

Since the proofs of these two bounds are similar, we only focus on the second one. Let $g_n^+(s) = \sum_{k\geq 0} q_k^+ s^k$ be the generating series of $(q_k^+)_{k\geq 0}$. Let ρ_n^+ be the smallest positive solution of $g_n^+(1-x) = 1-x$. Then, there exists some constant $c>0$ such that

$$\begin{aligned}\mathbb{P}(|\mathscr{T}^+|\geq n^\delta) &= \mathbb{P}(|\mathscr{T}^+| = +\infty) + \mathbb{P}(n^\delta \leq |\mathscr{T}^+| < +\infty)\\ &= \rho_n^+ + o\left(\frac{1}{n}\right).\end{aligned} \tag{22}$$

The difference between ρ_n^+ and ρ_n can be written as follows:

$$\begin{aligned}\rho_n^+ - \rho_n &= g_n^+(1-\rho_n^+) - g_n(1-\rho_n)\\ &= g_n(1-\rho_n^+) - g_n(1-\rho_n) + g_n^+(1-\rho_n^+) - g_n(1-\rho_n^+)\\ &= g_n'(1-\rho_n)(\rho_n-\rho_n^+) + o\left(\rho_n^+-\rho_n\right) + g_n^+(1-\rho_n^+) - g_n(1-\rho_n^+),\end{aligned} \tag{23}$$

where in the last equality, we used a Taylor expansion. From the definition of $(q_k^+)_{k\geq 0}$, for all $k\geq 0$, it holds that:

$$q_k^+ = p_k + \mathcal{O}\left(\frac{n^{\delta+\frac{1}{\gamma}}}{n}\right),$$

where the error term is uniform in k. In particular, this implies that $g_n^+(1-\rho_n^+) - g_n(1-\rho_n^+)$ is of order $n^{\delta+\frac{1}{\gamma}-1}$. Injecting this into (23), we get

$$\left(1-g_n'(1-\rho_n)+o(1)\right)\left(\rho_n^+-\rho_n\right) = \mathcal{O}\left(n^{\delta+\frac{1}{\gamma}-1}\right).$$

By the assumptions of Lemma 4, ρ_n converges to the fixed point of $g_{\boldsymbol{\pi}}$, which is bounded away from 0. Therefore, for large enough n , $g_n'(1-\rho_n)$ is bounded away from 1. Hence

$$|\rho_n^+-\rho_n| = \mathcal{O}\left(n^{\delta+\frac{1}{\gamma}-1}\right).$$

Together with (22), this implies (21).

It remains to estimate the probability of the event E. During the first $\lfloor n^\delta\rfloor$ steps of the exploration of $\mathscr{C}_v$, the number of half-edges of the explored cluster is at most $n^\delta \times n^{1/\gamma}$. Hence, the probability of creating a loop at each of these steps is of order $n^{\delta+\frac{1}{\gamma}-1}$. Therefore, by the union bound:

$$\mathbb{P}(E) = \mathcal{O}\left(n^{2\delta+\frac{1}{\gamma}-1}\right). \tag{24}$$

Gathering (20), (21) and (24), we get (19) and therefore Lemma 4. □

6.2 An infinite system of differential equations

The aim of this section is to prove Lemma 1. In the following, we fix a probability distribution $\boldsymbol{\pi} = (\boldsymbol{\pi}_i)_{i\geq 0}$, supercritical in the sense of Definition 3.

First, we prove that the problem can be reduced to the study of another differential system.

Lemma 5. *If the following system has a unique solution well defined on some maximal interval $[0, t'_{\max})$ for some $t'_{\max} > 0$:*

$$\begin{cases} \frac{\mathrm{d}\zeta_i}{\mathrm{d}t} &= -\frac{i\zeta_i}{\sum_{j\geq 0} j\zeta_j} + \frac{1}{\sum_{j\geq 0} j\zeta_j}\left(1 - \frac{\sum_{j\geq 0}(j-1)j\zeta_j}{\sum_{n\geq 0} j\zeta_j}\right)(i\zeta_i - (i+1)\zeta_{i+1}) \\ \zeta_i(0) &= \boldsymbol{\pi}_i, \end{cases} \tag{S'}$$

then the system (S) has a unique solution well defined on a maximal interval $[0, t_{\max})$ for some $t_{\max} > 0$.

Proof. Suppose that (S') has a unique solution $(\zeta_i)_{i\geq 0}$. Let ϕ be the unique function defined by

$$\begin{cases} \phi'(t)\rho_{(\zeta_i(t))_{i\geq 0}} = 1, \\ \phi(0) = 0. \end{cases}$$

Then, for all $i \geq 0$, $(\zeta_i \circ \phi)'(t) = \frac{1}{\rho_{(\zeta_i(t))_{i\geq 0}}} \times \rho_{(\zeta_i(t))_{i\geq 0}} f_i(\zeta_0(t), \zeta_1(t), \ldots) = f_i(\zeta_0(t), \zeta_1(t), \ldots)$ which proves that $(\zeta_i \circ \phi)_{i\geq 0}$ is a solution of the system (S).

Let $(z_i)_{i\geq 0}$ be a solution of (S). Then, for all $t \geq 0$ where it is well defined,

$$\sum_{i\geq 0} z_i(t) = 1 - \int_0^t \frac{1}{\rho_{(z_i(t))_{i\geq 0}}}\mathrm{d}u =: 1 - \psi(t).$$

Then, $(z_i \circ \psi^{-1})_{i\geq 0}$ is a solution of (S'). Therefore, since $(\zeta_i)_{i\geq 0}$ is unique, $(z_i \circ \psi^{-1} \circ \phi)_{i\geq 0} = (\zeta_i \circ \phi)_{i\geq 0}$ is also solution of (S). In particular, it implies that

$$\frac{-1}{\rho_{(z_i\circ\psi^{-1}\circ\phi(t))_{i\geq 0}}} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i\geq 0} z_i \circ \psi^{-1} \circ \phi\right)(t) = \left(\psi^{-1} \circ \phi\right)'(t) \times \frac{-1}{\rho_{(z_i\circ\psi^{-1}\circ\phi(t))_{i\geq 0}}}.$$

Therefore, $\psi = \phi$ only depends on $(\zeta_i)_{i\geq 0}$, yielding the uniqueness of the solution. □

We now exhibit a solution of (S'). Let $f_{\boldsymbol{\pi}}(s) = \sum_{i\geq 0} \boldsymbol{\pi}_i s^i$ be the generating series associated to $\boldsymbol{\pi}$. Define $t'_{\max}$ as the unique root between 0 and $1 - \boldsymbol{\pi}_0$ of the equation

$$\frac{f''_{\boldsymbol{\pi}}\left(f_{\boldsymbol{\pi}}^{-1}(1-t)\right)}{f'_{\boldsymbol{\pi}}(1)} = 1.$$

For all $0 \leq t \leq t'_{\max}$ and $0 \leq s \leq 1$, let

$$f(t,s) := f_{\boldsymbol{\pi}}\left(f_{\boldsymbol{\pi}}^{-1}(1-t) - (1-s)\frac{f'_{\boldsymbol{\pi}}(f_{\boldsymbol{\pi}}^{-1}(1-t))}{f'_{\boldsymbol{\pi}}(1)}\right). \tag{25}$$

Note that this restriction to the interval $[0, t'_{\max})$ will have a crucial role in the analytic proof of the uniqueness of the solution. Moreover, from a probabilistic point of view, it corresponds to the range of times where $\frac{1}{1-t} f_{\boldsymbol{\pi}}(t,s)$ is the generating series of a supercritical probability law.

Proposition 1. *For all $0 \le t \le t'_{\max}$ and $i \ge 0$, let $\zeta_i(t) := [s^i]f(t,s)$ be the s^i coefficient of $f(t,s)$. Then, $(\zeta_i)_{i\ge 0}$ is a solution of (S').*

Proof. It can be easily verified that $f(t,s)$ satisfies the following equation:

$$\frac{\partial f}{\partial t}(t,s) = \frac{\frac{\partial f}{\partial s}(t,s)}{\frac{\partial f}{\partial s}(t,1)}\left((1-s)\frac{\frac{\partial^2 f}{\partial s^2}(t,1)}{\frac{\partial f}{\partial s}(t,1)} - 1\right).$$

By extracting the s^i coefficient we get that

$$\frac{\mathrm{d}\zeta_i}{\mathrm{d}t} = -\frac{i\zeta_i}{\sum_{j\ge 0} j\zeta_j} + \frac{1}{\sum_{j\ge 0} j\zeta_j}\left(1 - \frac{\sum_{j\ge 0}(j-1)j\zeta_j}{\sum_{n\ge 0} j\zeta_j}\right)(i\zeta_i - (i+1)\zeta_{i+1}),$$

which ends the proof the proposition. □

It remains to prove the uniqueness of the solution that we found. Let $(\zeta_i)_{i\ge 0}$ be a solution of (S'). We will prove that $\sum_{i\ge 0}\zeta_i(t)s^i = f(t,s)$, which implies that for all $i \ge 0$, the function ζ_i is the s^i coefficient of $f(t,s)$.

Remark 4. *Notice that when π has bounded support, we only have to deal with a finite number of differential equations and the uniqueness follows merely from the Cauchy-Lipschitz Theorem.*

We introduce the following quantities:

$$E(t) := \sum_{i\ge 0} i\zeta_i(t) \qquad \text{and} \qquad Z(t) := \int_0^t \left(\frac{E'}{2\sqrt{E}} + \frac{1}{\sqrt{E}}\right)(u)\mathrm{d}u.$$

Lemma 6. *For all $0 \le t \le t'_{\max}$:*

1. $\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i\ge 0}\zeta_i\right)(t) = -1$;
2. $E'(t) = -2\frac{\sum_{i\ge 1} i(i-1)\zeta_i(t)}{E(t)}$.

In particular, $\sum_{i\ge 0}\zeta_i(t) = 1 - t$.

Proof. The first point is obtained by summing the equations of (S'). Let us prove the second point. We omit the reference on t for clarity.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i\ge 1} i\zeta_i\right) &= \frac{-1}{\left(\sum\limits_{i\ge 1} i\zeta_i\right)^2}\Bigg[\left(\sum_{i\ge 1} i\zeta_i\right)\left(\sum_{i\ge 1} i^2\zeta_i\right) \\ &\qquad - \left(\sum_{i\ge 1} i\zeta_i - \sum_{i\ge 1} i(i-1)\zeta_i\right)\left(\sum_{i\ge 1} i^2\zeta_i - \sum_{i\ge 1} i(i-1)\zeta_i\right)\Bigg] \\ &= \frac{-1}{\sum\limits_{i\ge 1} i\zeta_i}\left(2\sum_{i\ge 0} i^2\zeta_i - 2\sum_{i\ge 0} i\zeta_i\right) \\ &= -2\frac{\sum_{i\ge 0} i(i-1)\zeta_i}{\sum_{i\ge 0} i\zeta_i}.\end{aligned}$$

□

By Lemma 6, the system (S') can be rewritten:

$$\frac{\mathrm{d}\zeta_i}{\mathrm{d}t} = \frac{i}{2}\frac{E'}{E}\zeta_i - (i+1)\zeta_{i+1}\left(\frac{E'}{2E} + \frac{1}{E}\right).$$

We are going to compare ζ_i with a truncated version of it. Let $\varepsilon > 0$ and let

$$\Delta := \Delta(\varepsilon) = \left\lfloor \sqrt{\frac{\sum_{k\geq 0} \boldsymbol{\pi}(\{k\})k^2}{\varepsilon}} \right\rfloor. \tag{26}$$

Note that, by Markov's inequality, $\sum_{i\geq\Delta} \boldsymbol{\pi}(\{i\}) \leq \varepsilon$. Let $(\zeta_i^{(\Delta)})_{0\leq i\leq\Delta}$ be *the* solution of the following *finite* system of differential equations:

$$\begin{cases} \frac{\mathrm{d}\zeta_\Delta^{(\Delta)}}{\mathrm{d}t} &= \frac{i}{2}\frac{E'}{E}\zeta_\Delta^{(\Delta)}; \\ \frac{\mathrm{d}\zeta_i^{(\Delta)}}{\mathrm{d}t} &= \frac{i}{2}\frac{E'}{E}\zeta_i^{(\Delta)} - (i+1)\zeta_{i+1}^{(\Delta)}\left(\frac{E'}{2E} + \frac{1}{E}\right); \\ \zeta_i^{(\Delta)}(0) &= \boldsymbol{\pi}_i. \end{cases}$$

It turns out that the generating function of the $\zeta_i^{(\Delta)}$ has a simple expression in term of the functions E and Z.

Lemma 7. *Let $f_\Delta(s) := \sum_{0\leq i\leq\Delta} \boldsymbol{\pi}_i s^i$ be the truncated version of $f_{\boldsymbol{\pi}}$. Then, for all $0 \leq t \leq t'_{\max}$ and $0 \leq s \leq 1$:*

$$\sum_{0\leq i\leq\Delta} \zeta_i^{(\Delta)}(t)s^i = f_\Delta\left(\frac{s\sqrt{E(t)} - Z(t)}{\sqrt{E(0)}}\right). \tag{27}$$

Moreover, the initial solution is close to its truncated version.

Lemma 8. *For all $0 \leq t \leq t'_{\max}$ and all $0 \leq i \leq \Delta$, $\zeta_i^{(\Delta)}(t) \leq \zeta_i(t) \leq \zeta_i^{(\Delta)} + 2\varepsilon$.*

We postpone the proofs of Lemmas 7 and 8 at the end of this section and explain now how it leads to the uniqueness part of Lemma 1.

By Lemma 8,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{0\leq i\leq\Delta} \zeta_i^{(\Delta)}\right)(t) = -\frac{\sum_{0\leq i\leq\Delta} i\zeta_i^{(\Delta)}(t)}{\sum_{i\geq 0} i\zeta_i(t)} \geq -1.$$

By our choice of Δ, $\sum_{0\leq i\leq\Delta} \zeta_i^{(\Delta)}(0) = \sum_{0\leq i\leq\Delta} \boldsymbol{\pi}_i \geq 1 - \varepsilon$. Therefore:

$$1 - t - \varepsilon \leq \sum_{0\leq i\leq\Delta} \zeta_i^{(\Delta)}(t) \leq \sum_{0\leq i\leq\Delta} \zeta_i(t) = 1 - t.$$

Evaluating (27) at $s = 1$ gives:

$$1 - t - \varepsilon \leq f_\Delta\left(\frac{\sqrt{E(t)} - Z(t)}{\sqrt{E(0)}}\right) \leq 1 - t.$$

Recalling the definition of Δ in (26) and letting ε converge to zero, we get that

$$f_{\boldsymbol{\pi}}\left(\frac{\sqrt{E(t)} - Z(t)}{\sqrt{E(0)}}\right) = 1 - t.$$

We now take the inverse of $f_{\boldsymbol{\pi}}$ and differentiate in t to obtain

$$\begin{cases} \sqrt{\frac{E(t)}{E(0)}} &= \frac{f'_{\boldsymbol{\pi}}(f_{\boldsymbol{\pi}}^{-1}(1-t))}{f'_{\boldsymbol{\pi}}(1)}, \\ \frac{Z(t)}{\sqrt{E(0)}} &= \frac{f'_{\boldsymbol{\pi}}(f_{\boldsymbol{\pi}}^{-1}(1-t))}{f'_{\boldsymbol{\pi}}(1)} - f_{\boldsymbol{\pi}}^{-1}(1-t). \end{cases}$$

By re-injecting in (27), we proved that, for all $t \geq 0$ and $0 \leq s \leq 1$,

$$\sum_{0 \leq i \leq \Delta} \zeta_i^{(\Delta)}(t) s^i = f_\Delta \left(f_{\boldsymbol{\pi}}^{-1}(1-t) - (1-s) \frac{f'_{\boldsymbol{\pi}}(f_{\boldsymbol{\pi}}^{-1}(1-t))}{f'_{\boldsymbol{\pi}}(1)} \right).$$

It is now easy to conclude since, by Lemma 8 and our choice of Δ:

$$\begin{aligned} f(t,s) = \lim_{\varepsilon \to 0} \sum_{0 \leq i \leq \Delta} \zeta_i^{(\Delta)}(t) s^i \leq \sum_{i \geq 0} \zeta_i(t) s^i = \sum_{0 \leq i \leq \Delta} \zeta_i(t) s^i + \sum_{i > \Delta} \zeta_i(t) s^i \\ \leq \lim_{\varepsilon \to 0} \left(\sum_{0 \leq i \leq \Delta} \zeta_i^{(\Delta)}(t) s^i + 2\Delta\varepsilon + \varepsilon \right) = f(t,s). \end{aligned}$$

This ends the proof of Lemma 1. We now turn to the proofs of Lemmas 7 and 8.

Proof of Lemma 7. We first prove by an induction from Δ to 0 that for all $0 \leq i \leq \Delta$,

$$\zeta_i^{(\Delta)}(t) = \sum_{k=i}^{\Delta} c_k \binom{n}{k} (-Z)^{k-i} E^{i/2},$$

where $c_k = \boldsymbol{\pi}_k E(0)^{-k/2}$.

The initialization is straightforward because the function $\zeta_\Delta^{(\Delta)}(t) = c_\Delta E(t)^{\Delta/2}$ is indeed the solution of $y' = \frac{\Delta}{2} \frac{E'}{E} y$ with initial condition $y(0) = \boldsymbol{\pi}_\Delta$.

Suppose that the property holds at $i+1$ for some $0 \leq i \leq \Delta - 1$. Since

$$\frac{\mathrm{d}\zeta_i^{(\Delta)}}{\mathrm{d}t} = \frac{i}{2} \frac{E'}{E} \zeta_i^{(\Delta)} - (i+1) \zeta_{i+1}^{(\Delta)} \left(\frac{E'}{2E} + \frac{1}{E} \right)$$

and $\zeta_i^{(\Delta)}(0) = \boldsymbol{\pi}_i$,

$$\zeta_i^{(\Delta)}(t) = \boldsymbol{\pi}_i \left(\frac{E(t)}{E(0)} \right)^{i/2} - \int_0^t (i+1) \zeta_{i+1}^{(\Delta)}(u) \left(\frac{E'(u)}{2E(u)} + \frac{1}{E(u)} \right) \left(\frac{E(t)}{E(u)} \right)^{-i/2} \mathrm{d}u.$$

We now use the induction hypothesis to obtain:

$$\begin{aligned}
\zeta_i^{(\Delta)}(t) &= \boldsymbol{\pi}_i \left(\frac{E(t)}{E(0)}\right)^{i/2} - \int_0^t (i+1) \sum_{k=i+1}^{\Delta} c_k \binom{k}{k-i-1} (-Z(u))^{k-i-1} \\
&\qquad\qquad \times E^{(i+1)/2}(u) \left(\frac{E'(u)}{2E(u)} + \frac{1}{E(u)}\right) \left(\frac{E(t)}{E(u)}\right)^{i/2} \mathrm{d}u \\
&= \boldsymbol{\pi}_i \left(\frac{E(t)}{E(0)}\right)^{i/2} - E(t)^{i/2} \int_0^t (i+1) \sum_{k=i+1}^{\Delta} c_k \binom{k}{k-i-1} (-Z(u))^{k-i-1} \\
&\qquad\qquad \times \left(\frac{E'(u)}{2\sqrt{E(u)}} + \frac{1}{\sqrt{E(u)}}\right) \mathrm{d}u \\
&= \boldsymbol{\pi}_i \left(\frac{E(t)}{E(0)}\right)^{i/2} - E(t)^{i/2} \int_0^t (i+1) \sum_{k=i+1}^{\Delta} c_k \binom{k}{k-i-1} (-Z(u))^{k-i-1} Z'(u) \mathrm{d}u \\
&= c_i E(t)^{i/2} - \sum_{k=i+1}^{\Delta} c_k \frac{i+1}{k-i} \binom{k}{k-i-1} (-Z(t))^{k-i} E(t)^{i/2} \\
&= \sum_{k=i}^{\Delta} c_k \binom{k}{k-i} (-Z(t))^{k-i} E(t)^{i/2}.
\end{aligned}$$

This ends the proof by induction. It is now easy to conclude:

$$\begin{aligned}
\sum_{i=0}^{\Delta} \zeta_i^{(\Delta)}(t) s^i &= \sum_{i=0}^{\Delta} \sum_{k=i}^{\Delta} c_k \binom{k}{k-i} (-Z(t))^{k-i} E(t)^{i/2} s^i \\
&= \sum_{k=0}^{\Delta} \boldsymbol{\pi}_k E(0)^{-k/2} \sum_{i=0}^{k} \binom{k}{k-i} (-Z(t))^{k-i} \left(s\sqrt{E(t)}\right)^i \\
&= \sum_{k=0}^{\Delta} \boldsymbol{\pi}_k \left(\frac{s\sqrt{E(t)} - Z(t)}{\sqrt{E(0)}}\right)^k \\
&= f_\Delta \left(\frac{s\sqrt{E(t)} - Z(t)}{\sqrt{E(0)}}\right).
\end{aligned}$$

□

Proof of Lemma 8. We first prove the lower bound by an induction from Δ to 0. It is important to notice that for all $0 \leq t \leq t'_{\max}$,

$$-\left(\frac{E'}{2E} + \frac{1}{E}\right) = \frac{1}{\sum_{j\geq 0} j\zeta_j} \left(\frac{\sum_{j\geq 0}(j-1)j\zeta_j}{\sum_{n\geq 0} j\zeta_j} - 1\right) \geq 0.$$

Therefore, the lower bound holds for $i = \Delta$ since

$$\frac{\mathrm{d}}{\mathrm{d}t}\zeta_\Delta^{(\Delta)} = \frac{\Delta}{2}\frac{E'}{E}\zeta_\Delta^{(\Delta)} \leq \frac{\Delta}{2}\frac{E'}{E}\zeta_\Delta^{(\Delta)} - (\Delta+1)\zeta_{\Delta+1}\left(\frac{E'}{2E} + \frac{1}{E}\right).$$

Indeed, by Gronwall's Lemma, it implies that $\zeta_\Delta^{(\Delta)}$ is upper-bounded by the solution of the differential equation $y' = \frac{\Delta}{2}\frac{E'}{E}y - (\Delta+1)\zeta_{\Delta+1}\left(\frac{E'}{2E} + \frac{1}{E}\right)$, which is nothing but ζ_Δ.

Assume the lower bound holds for an index $1 \leq i \leq \Delta$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\zeta_{i-1}^{(\Delta)} = \frac{i-1}{2}\frac{E'}{E}\zeta_{i-1}^{(\Delta)} - i\zeta_i^{(\Delta)}\left(\frac{E'}{2E}+\frac{1}{E}\right)$$
$$\leq \frac{i-1}{2}\frac{E'}{E}\zeta_{i-1}^{(\Delta)} - i\zeta_i\left(\frac{E'}{2E}+\frac{1}{E}\right).$$

Again, by Gronwall's Lemma, it implies that $\zeta_{i-1}^{(\Delta)} \leq \zeta_{i-1}$.

The proof of the upper bound can be obtained by contradiction. Indeed, suppose that there exists a time $t \in (0, t'_{\max})$ and an index $0 \leq i \leq \Delta$ such that $\zeta_i(t) > \zeta_i^{(\Delta)}(t) + 2\varepsilon$. Then, using the lower bound we have just obtained,

$$1 - t = \sum_{i\geq 0}\zeta_i(t) \geq \sum_{0\leq i\leq\Delta}\zeta_i(t)$$
$$> \sum_{0\leq i\leq\Delta}\zeta_i^{(\Delta)}(t) + 2\varepsilon$$
$$\geq 1 - \varepsilon - t + 2\varepsilon = 1 - t + \varepsilon.$$

□

7 Appendix : a Theorem by Wormald

Theorem 4. *For all $N \geq 1$, let $Y(i) = Y^{(N)}(i)$ be a Markov chain with respect to a filtration $\{\mathcal{F}_i\}_{i\geq 1}$. Suppose that*

- *$Y(0)/N$ converges towards $z(0)$ in probability;*
- *$|Y(i+1) - Y(i)| \leq N^{\beta}$;*
- *$\mathbb{E}\left[Y(i+1) - Y(i)\,\middle|\,\mathcal{F}_i\right] = f\left(\frac{i}{N}, \frac{Y(i)}{N}\right) + O\left(N^{-\lambda}\right)$,*

where $0 < \beta < 1/2$, $\lambda > 0$ and f is a Lipschitz function. Then, the differential equation

$$z'(t) = f(t, z(t)),$$

has a unique solution z with given initial condition $z(0)$ and $Y(\lfloor tN \rfloor)/N$ converges in probability towards z for the topology of uniform convergence.

Proof. By regularity of the solution of the differential equation with respect to the initial condition, it suffices to treat the case where $Y(0)/N \equiv z(0)$. Let $1 < \varepsilon < \frac{1-\beta}{\beta}$ which exists by our hypothesis on β. Let $w = N^{(1+\varepsilon)\beta}$ and fix $\alpha \in \left(\frac{1+\varepsilon}{2}\beta, \varepsilon\beta\right)$. We prove by induction the following property for all $0 \leq i \leq N/w$:

$$\mathbb{P}\left(|Y(iw) - z(iw/N)N| > i\left(N^{\alpha+\beta} + N^{(1+\varepsilon)\beta-\lambda} + N^{2(1+\varepsilon)\beta-2}\right)\right)$$
$$\leq i\exp\left(-\frac{1}{2}N^{2\alpha-(1+\varepsilon)\beta}\right). \quad (28)$$

Note that the lower bound in the probability tends to zero for all $i \leq N/w$ and that the probability tends to zero by our hypothesis.

The initialization is satisfied by the choice we made for $Y(0)$.

Suppose that the property is verified for $0 \leq i \leq N/w - 1$. Rewriting

$$\begin{aligned} |Y((i+1)w) - z((i+1)w/N)N| = \ & |Y((i+1)w) - Y(iw) - wf(iw/N, Y(iw)/N)| && (29) \\ & + |Y(iw) - z(iw/N)N| && (30) \\ & + |z((i+1)w/N)N - z(iw/N)N - wf(iw/N, Y(iw)/N)| && (31) \end{aligned}$$

and using that (31) is bounded by $(w/N)^2 = N^{2(1+\varepsilon)\beta-2}$ by Taylor expansion:

$$\begin{aligned} &\mathbb{P}\left(|Y((i+1)w) - z((i+1)w/N)N| > (i+1)\left(N^{\alpha+\beta} + N^{(1+\varepsilon)\beta-\lambda} + N^{2(1+\varepsilon)\beta-2}\right)\right) \\ &\quad \leq \mathbb{P}\left(|Y(iw) - z(iw/N)N| > i\left(N^{\alpha+\beta} + N^{(1+\varepsilon)\beta-\lambda} + N^{2(1+\varepsilon)\beta-2}\right)\right) \\ &\quad + \mathbb{P}\left(|Y((i+1)w) - Y(iw) - wf(iw/N, Y(iw)/N)| > \left(N^{\alpha+\beta} + N^{(1+\varepsilon)\beta-\lambda}\right)\right). \end{aligned}$$

The first term of the previous upper bound is given by our induction hypothesis. The second term can be treated with the Azuma-Hoeffding inequality since $\left(Y((i+1)w) - Y(iw) - \mathbb{E}[Y((i+1)w) - Y(iw) \,|\, \mathcal{F}_i]\right)_{i \geq 0}$ is a martingale with increments bounded by N^β. □

Using Equation (28), we can even state a version of this result for a polynomial number of Markov Chains driven by an infinite number of differential equations, which is what is needed for our work.

Corollary 2. *Let $a > 0$. For all $N \geq 1$ and all $1 \leq k \leq N^a$, let $Y_k(i) = Y_k^{(N)}(i)$ be a Markov chain with respect to a filtration $\{\mathcal{F}_i\}_{i \geq 1}$. Suppose that, for all $k \geq 1$, there exists a function f_k such that:*

- *$Y_k(0)/N$ converges towards $z_k(0)$ in probability;*
- *$|Y_k(i+1) - Y_k(i)| \leq N^\beta$;*
- *$\mathbb{E}\left[Y_k(i+1) - Y_k(i) \,\middle|\, \mathcal{F}_i\right] = f_k\left(\frac{i}{N}, \frac{(Y_k(i))_{1 \leq k \leq N^a}}{N}\right) + O\left(N^{-\lambda}\right)$,*

where $0 < \beta < 1/2$, $\lambda > 0$. Suppose that the following infinite system of differential equations with initial conditions $(z_k(0))_{k \geq 1}$ has a unique solution $(z_k)_{k \geq 1}$:

$$\forall k \geq 1, \quad z_k'(t) = f_k(t, (z_k(t))_{k \geq 1}).$$

Then, for all $k \geq 1$, $Y_k(\lfloor tN \rfloor)/N$ converges in probability towards z_k for the topology of uniform convergence.

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