

Electron. J. Probab. **19** (2014), no. 79, 1–27.
ISSN: 1083-6489 DOI: 10.1214/EJP.v19-2675

Percolation on uniform infinite planar maps

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Abstract

We construct the uniform infinite planar map (UIPM), obtained as the $n \rightarrow \infty$ local limit of planar maps with n edges, chosen uniformly at random. We then describe how the UIPM can be sampled using a “peeling” process, in a similar way as for uniform triangulations. This process allows us to prove that for bond and site percolation on the UIPM, the percolation thresholds are $p_c^{\text{bond}} = 1/2$ and $p_c^{\text{site}} = 2/3$ respectively. This method also works for other classes of random infinite planar maps, and we show in particular that for bond percolation on the uniform infinite planar quadrangulation, the percolation threshold is $p_c^{\text{bond}} = 1/3$.

Keywords: random map ; UIPQ ; percolation threshold ; peeling process.

AMS MSC 2010: 05C80 ; 82B43.

Submitted to EJP on March 12, 2013, final version accepted on August 10, 2014.

Supersedes arXiv:1302.2851v1.

1 Introduction

1.1 Background and motivations

A lot of progress has been made in the past decade toward the understanding of statistical physics models in dimension 2. All these models, when examined at their critical point, share a strong property of conformal invariance, a property which has been established for a number of them, in the scaling limit. Without aiming at exhaustivity, let us mention the Loop-Erased Random Walk [25], site percolation on the triangular lattice [32], the Ising model of ferromagnetism [14] and its dual FK-Ising representation [33]. This property leads to a precise description of geometric objects in terms of the Schramm-Loewner Evolution (SLE) processes introduced in [31], and subsequently studied in a number of papers – let us mention the groundbreaking works [30, 23, 24], to name but a few.

For percolation in particular, this led to the derivation of the so-called “arm exponents”, that describe the probability of observing disjoint long-range paths: for instance, at criticality, the probability for a given vertex to be connected to distance n follows a power law: it decays like $n^{-\alpha'_1+o(1)}$ as $n \rightarrow \infty$, with $\alpha'_1 = \frac{5}{48}$. Combining this new understanding with Kesten’s scaling relations [19], one can then describe the

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behavior of percolation not only at criticality, but also near criticality, i.e. through its phase transition. Let us mention in particular that the density of the infinite connected component decays as $\theta(p) = (p - p_c)^{\beta+o(1)}$ as $p \searrow p_c$, with $\beta = \frac{5}{36}$ [34].

Such exponents had however been predicted much earlier by powerful but non-rigorous methods, such as quantum gravity. Let us mention in particular the paper [1], where arm exponents in their own right were first considered and derived. Random graphs have been extensively used in the statistical physics literature, with a view to analyzing random spatial processes such as percolation or the Ising model (see for instance [9] and [10]). Studying these models in random geometries can provide a useful insight on their behavior on Euclidean lattices such as \mathbb{Z}^2 or the triangular lattice. Once derived the critical exponents in the random graph setting, the Knizhnik-Polyakov-Zamolodchikov (KPZ) formula [20] predicts what the values of these exponents are for (regular) Euclidean lattices.

Let us now make a bit more precise what is meant by random geometries. In the following, we consider proper embeddings of finite connected graphs in the sphere \mathbb{S}^2 , where loops and multiple edges are allowed. A *finite planar map* is then an equivalence class of such embeddings with respect to orientation-preserving homeomorphisms of the sphere. A planar map is *rooted* if it has furthermore a distinguished oriented edge $\vec{e} = (v_0, v_1)$, which is then called the *root edge* (v_0 being the root vertex). Faces of the map are the connected components of the complement of the union of its edges, and a map is a *p-angulation* if all its faces have degree p . In particular, when $p = 3$ (resp. 4), we obtain triangulations (resp. quadrangulations).

The set of vertices of a given map will always be equipped with the graph distance. From this point of view, a random planar map can be considered as a random discrete metric space, giving a precise mathematical framework for two-dimensional quantum gravity. In particular, it is believed that random planar maps provide a good approximation for continuous random surfaces. Recently, Le Gall [26] and Miermont [29] proved that random planar p -angulations (for $p = 3$ or $p \geq 4$ even) properly rescaled converge towards a universal random surface, called the Brownian Map, in analogy with the fact that the Brownian motion arises as the scaling limit of discrete random walks.

In this paper, rather than dealing with continuous scaling limits, we consider *local limits* of random maps as introduced in [7], which is a natural way to construct random infinite planar graphs. We define the distance d as: for every pair of finite rooted maps \mathbf{m}, \mathbf{m}' ,

$$d(\mathbf{m}, \mathbf{m}') = (1 + \sup \{r : B_r(\mathbf{m}) = B_r(\mathbf{m}')\})^{-1}$$

where, for $r \geq 1$, $B_r(\mathbf{m})$ is the planar map consisting of all edges of \mathbf{m} that have at least one vertex at distance strictly smaller than r from the root (and $\sup \emptyset = 0$ by convention). We denote by (\mathcal{M}, d) the completion of the space of all finite rooted maps with respect to d . Elements of \mathcal{M} that are not finite maps are called *infinite maps*. Note that one can extend the function defined for finite maps $\mathbf{m} \mapsto B_r(\mathbf{m})$ to a continuous function B_r on \mathcal{M} . The ball $B_r(\mathbf{m})$ can be interpreted in a natural way as the union of the edges of \mathbf{m} that have a vertex at distance strictly smaller than r from the root.

In a pioneering work [5], Angel and Schramm constructed the uniform infinite planar triangulation (UIPT) as the local limit of uniformly distributed large triangulations. Shortly after, Krikun [22] defined similarly the uniform infinite planar quadrangulation (UIPQ): if \mathbf{q}_n is distributed according to the uniform measure on the set of all rooted quadrangulations with n faces, then it is proved in [22] that the distribution of \mathbf{q}_n converges weakly to a probability measure τ in the set of all probability measures on infinite quadrangulations: the measure τ is the law of the UIPQ. Both the UIPT and the UIPQ have been the focus of numerous works in recent years such as [3, 13, 15, 21, 27, 28],

but it is fair to say that they are not yet fully understood. Statistical mechanics models on planar maps are also starting to attract attention: see for example the recent proof of recurrence for the simple random walk on the UIPQ [17], or the study of Ising and Potts models from a combinatorial point of view [8, 11]. In this paper, we study in more detail independent percolation on the UIPQ and the UIPM.

1.2 Organization of the paper and main results

In Section 2, we remind several important properties of planar quadrangulations that will be instrumental in the present paper. We also describe a natural bijection between quadrangulations and planar maps. This bijection allows one to use properties for quadrangulations in order to study planar maps. In particular, it provides an easy way to construct the uniform infinite planar map (UIPM) from the uniform infinite planar quadrangulation (UIPQ). This bijection also behaves nicely through restrictions. In particular, uniform infinite planar p -angulations could also be constructed in this way.

We then describe in Section 3 a “peeling process” similar to the process introduced by Angel in [3] for triangulations. This process offers a useful description of the usual exploration process, that follows the interface between black (occupied) and white (vacant) sites, as a simple Markov chain for which the transition probabilities are known rather explicitly.

In [3], Angel used this description to study site percolation on the uniform infinite planar triangulation (UIPT). The usual planar triangular lattice has a “self-matching” property that suggests that for site percolation on this lattice, one has $p_c = 1/2$, which is a celebrated result of Kesten [18]. The UIPT is “stochastically” self-matching, and it also holds in this case that $p_c = 1/2$, in both annealed and quenched environments. Similarly, \mathbb{Z}^2 has a self-duality property that implies that $p_c = 1/2$ for bond percolation on this lattice (strictly speaking, this is the actual result proved in [18]). The UIPM happens to be “stochastically” self-dual too, and in Section 5 we use the peeling process to prove that $p_c^{\text{bond}} = 1/2$ a.s. in this case. Before that, we derive the site percolation threshold on the UIPM in Section 4, where we show that $p_c^{\text{site}} = 2/3$ a.s. In the last part of Section 5, we explain how the method allows one to compute bond percolation thresholds for other classes of random infinite planar maps. In particular, we show that for bond percolation on the UIPQ, $p_c^{\text{bond}} = 1/3$ a.s. The main result of our paper is thus the following.

Theorem 1.1. *For site and bond percolation on the UIPM, one has, respectively, $p_c^{\text{site}} = 2/3$ and $p_c^{\text{bond}} = 1/2$ almost surely. For bond percolation on the UIPQ, one has $p_c^{\text{bond}} = 1/3$ almost surely.*

To our knowledge, all the percolation thresholds known explicitly in the Euclidean case are either $1/2$, or can be related to $1/2$ in some way (e.g. via a star-triangle transformation). Here, the values $1/3$ and $2/3$ show up from the computations, but there does not seem to be any direct heuristic explanation for them.

We would also like to mention that the value $1/3$ for bond percolation on the UIPQ echoes a similar result found in [4]. In this paper, Angel and Curien study bond percolation on uniform infinite quadrangulations of the half-plane (that is, with an infinite boundary), and prove that the threshold is also $1/3$. Their proof relies on the peeling process and is similar to ours, the main difference being that dealing with full plane quadrangulations adds some technical difficulty. Indeed, the Markov chain studied in Section 5 becomes, in their paper, a random walk for half-plane quadrangulations, with

simpler transition probabilities. To the best of our knowledge, there is no simple argument to deduce percolation thresholds for the full plane from half-plane thresholds (or the other way around).

2 Main tools

2.1 Quadrangulations and planar maps

Recall that a finite planar map is a *quadrangulation* if all its faces have degree 4, that is 4 adjacent edges. Note that the underlying graph of a quadrangulation is bipartite. A planar map is a *quadrangulation with a boundary* or with *holes* if all its faces have degree 4, except for a number of distinguished faces which can be arbitrary even-sided polygons (we assume these boundaries to be simple, i.e. the polygons are not “folded”). In the case when there is only one hole, of perimeter $2p$, we obtain what is called a quadrangulation of the $2p$ -gon. For every integer $n \geq 0$, we denote by \mathcal{Q}_n the set of all rooted quadrangulations with n faces. We also denote by \mathcal{Q}_n^p the set of all quadrangulations of the $2p$ -gon with n inner faces, such that the external face contains the root edge and lies on the right-hand side of it. The completion of the set \mathcal{Q}_f of all rooted finite quadrangulations for the distance d (defined in the introduction) is denoted by \mathcal{Q} ; it is a subset of \mathcal{M} . Elements of $\mathcal{Q}_\infty = \mathcal{Q} \setminus \mathcal{Q}_f$ are called infinite rooted quadrangulations. Similarly, we denote the set of finite (resp. infinite) quadrangulations of the $2p$ -gon by \mathcal{Q}_f^p (resp. \mathcal{Q}_∞^p). We refer to [15] for more details.

For our purpose, when dealing with quadrangulations, it turns out to be more convenient to work with faces rather than with edges, which leads us to introduce the new distance d^* on \mathcal{M} defined by

$$d^*(\mathbf{m}, \mathbf{m}') = (1 + \sup \{r : B_r^*(\mathbf{m}) = B_r^*(\mathbf{m}')\})^{-1}$$

for all rooted maps \mathbf{m}, \mathbf{m}' , where, for $r \geq 1$, we denote by $B_r^*(\mathbf{m})$ the planar map obtained as the union of all faces of \mathbf{m} that have at least one vertex at distance strictly smaller than r from the root. Note that if \mathbf{m} is a quadrangulation, then $B_r^*(\mathbf{m})$ is a quadrangulation with (possibly adjacent) holes.

Let us stress that if maps with faces of arbitrarily large degrees are considered, then the distances d and d^* give rise to two different topologies. However, when restricted to quadrangulations, the two distances are equivalent (more generally, this holds true for closed sets of maps with faces of bounded degree). Indeed, for every $\mathbf{q} \in \mathcal{Q}$ and $r \geq 1$, one has

$$B_r(\mathbf{q}) \subset B_r^*(\mathbf{q}) \subset B_{r+2}(\mathbf{q}),$$

where $\mathbf{m} \subset \mathbf{m}'$ means that the edge set of \mathbf{m} is included in the edge set of \mathbf{m}' (we will use this notation throughout the paper). Therefore, \mathcal{Q} can also be seen as the completion of \mathcal{Q}_f for the distance d^* .

If we are given a quadrangulation with holes, it is natural to construct a full quadrangulation of the sphere by filling its holes of degree $2p$ with quadrangulations of the $2p$ -gon. However, one has to make sure that filling the holes with different quadrangulations leads to different maps. This can be ensured by dealing only with *rigid* quadrangulations, as in [5]: we say that a rooted quadrangulation with holes \mathbf{q} is rigid if no quadrangulation of the sphere includes two different copies of \mathbf{q} with coinciding roots. As stated in [6], an easy adaptation of Lemma 4.8 of [5] yields that any rooted quadrangulation with holes is rigid.

2.2 UIPQ and UIPM

As we already mentioned, the law of the UIPQ can be constructed as the weak local limit of uniform measures on large quadrangulations. Recently, Curien and Miermont [16] constructed similar measures for quadrangulations with a boundary. More precisely, if \mathbf{q}_n^p is distributed according to the uniform measure on \mathcal{Q}_n^p , then the distribution of \mathbf{q}_n^p converges weakly to a probability measure τ^p in the set of all probability measures on (\mathcal{Q}^p, d^*) : the measure τ^p is the law of the uniform infinite planar quadrangulation of the $2p$ -gon.

There is a natural bijection between rooted quadrangulations and rooted planar maps, which we now describe (see Figure 1). Starting from a quadrangulation, its bipartite structure allows one to divide its set of vertices into two sets: *circle-vertices* are the vertices which are at an even distance from the root vertex (including the root vertex itself), and *square-vertices* are the vertices at an odd distance. Now, draw an edge between any two circle-vertices on the same face: we produce in this way a planar map with n edges, rooted at the edge corresponding to the face (in the initial quadrangulation) which is on the left hand-side of the root edge. Making explicit the reverse map is straightforward: it suffices to add one square-vertex on each face, and connect it to all vertices of this face. This bijection is used throughout the paper.

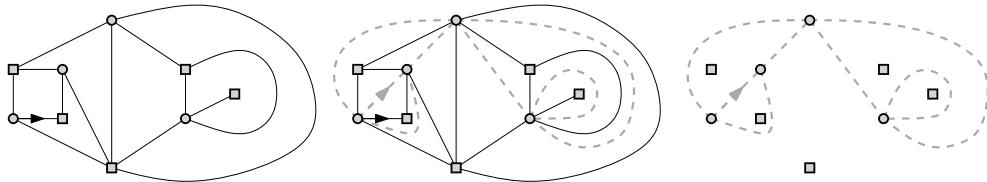


Figure 1: The bijection between quadrangulations and planar maps.

This bijection maps the uniform measure on rooted quadrangulations with n faces to the uniform measure on rooted planar maps with n edges. Therefore, it can be used to define a (random) uniform infinite planar map (UIPM), whose law is just the weak limit of the uniform measure on rooted planar maps with n edges for the distance d . Indeed, it is easy to check that this bijection is continuous for the topologies considered (note that we could possibly use the distance d^* to construct the UIPM, but this approach would require controlling the degree of faces, which makes it slightly less direct).

Note also that this bijection maps the circle-vertices onto the vertices of the final map, while the square-vertices are mapped to faces. The dual graph of the random map can thus be obtained by simply choosing to draw edges between square-vertices, instead of between circle-vertices. This also corresponds to re-rooting the original quadrangulation by reversing orientation of the root edge.

The planar map so obtained is thus “stochastically” self-dual (because the uniform infinite quadrangulation is invariant under the previous re-rooting, or because the dual of a random uniform planar map with n edges has the same law), which seems to indicate that the bond percolation threshold on this map is $p_c = 1/2$, as in the case of \mathbb{Z}^2 (see [18]) which is “truly” self-dual.

2.3 Counting quadrangulations

In this short section, we collect some enumeration results for quadrangulations that are instrumental for our purpose. We refer the reader to [12] for proofs.

If we denote by $a_{n,p}$ the number of quadrangulations of the $2p$ -gon with n internal faces rooted on the boundary face, one has (see [12], equation (2.11)):

$$a_{n,p} = 3^{n-p} \frac{(3p)!}{p!(2p-1)!} \frac{(2n+p-1)!}{(n-p+1)!(n+2p)!}. \quad (2.1)$$

Actually, the exact value of $a_{n,p}$ is not needed, but only its asymptotic behavior in n for p fixed and the values of the generating series in n at their convergence radius. That is, we need the value of C_p defined by

$$a_{n,p} \underset{n \rightarrow \infty}{\sim} C_p 12^n n^{-5/2}, \quad (2.2)$$

$$\text{which is } C_p = \frac{1}{2\sqrt{\pi}} \left(\frac{2}{3}\right)^p \frac{(3p)!}{p!(2p-1)!}. \quad (2.3)$$

And we need the value of Z_p defined by the value of the generating series

$$Z_p(t) := \sum_{n \geq 0} a_{n,p} t^n$$

at $1/12$, which is the convergence radius of this series for each fixed p . This value is

$$Z_p := Z_p(1/12) = 2 \left(\frac{2}{3}\right)^p \frac{(3p-3)!}{p!(2p-1)!}. \quad (2.4)$$

Following [5, 6], we define the free distribution on rooted quadrangulations of a $2p$ -gon as the probability measure μ^p that assigns the weight

$$\mu^p(\mathbf{q}) = \frac{12^{-n}}{Z_p(1/12)}$$

to each quadrangulation \mathbf{q} of the $2p$ -gon having n internal faces and rooted on its boundary face.

2.4 Spatial Markov property for the UIPQ

We now state the spatial Markov property of the UIPQ, grouping into a unique lemma all the properties that are needed.

Lemma 2.1. *Let us denote by \mathbf{q}_∞ the UIPQ, and let \mathbf{q} be a rigid quadrangulation with n internal faces and k boundary faces, with perimeters $2p_1, \dots, 2p_k$.*

(i) *One has*

$$\tau(\mathbf{q} \subset \mathbf{q}_\infty) = \frac{12^{-n}}{C_1} \left(\prod_{i=1}^k Z_{p_i} \right) \sum_{i=1}^k \frac{C_{p_i}}{Z_{p_i}}. \quad (2.5)$$

When $\mathbf{q} \subset \mathbf{q}_\infty$ holds, let us denote by \mathbf{q}_i the component of the UIPQ in the i -th face.

(ii) *Almost surely, only one of these components is infinite: the probability that it is \mathbf{q}_j is given by the j -th term in the previous sum, i.e.*

$$\tau(\mathbf{q} \subset \mathbf{q}_\infty, \mathbf{q}_j \text{ is infinite}) = \frac{12^{-n}}{C_1} C_{p_j} \left(\prod_{\substack{i=1 \\ i \neq j}}^k Z_{p_i} \right). \quad (2.6)$$

(iii) *If we condition on the event that $\{\mathbf{q} \subset \mathbf{q}_\infty\}$, and that the external faces of \mathbf{q} all contain finitely many vertices of \mathbf{q}_∞ , except (possibly) the j -th one, then*

- the quadrangulations $(q_i)_{1 \leq i \leq k}$ are independent,
- q_j has the same distribution as the UIPQ of the $2p_j$ -gon,
- and for $i \neq j$, q_i is distributed as the free quadrangulation of a $2p_i$ -gon.

This spatial Markov property is proved in [5] for uniform triangulations, and a strictly identical proof applies in our setting of quadrangulations.

3 Peeling process for quadrangulations

We now describe the peeling process, a growth process that can be used to sample planar maps. It has first been used in physics [2] to derive heuristics for the scaling limit of 2-dimensional quantum gravity. Later, Angel [3] adapted this process to triangulations, and used it to study volume growth and site percolation on the UIPT. Benjamini and Curien [6] adapted this process to quadrangulations in order to prove that the simple random walk on the UIPQ is subdiffusive. We will make extensive use of this process to study both site and bond percolation on the UIPM associated with the UIPQ.

Let q_∞ be the UIPQ. The peeling process is a sequence $(q_n)_{n \geq 0}$ of (finite) random quadrangulations with simple boundary, such that:

- q_0 is the root edge of q_∞ and one has $q_0 \subset q_1 \subset \dots \subset q_n \subset \dots \subset q_\infty$.
- Let \mathcal{F}_n be the filtration generated by q_0, q_1, \dots, q_n . Then conditionally on \mathcal{F}_n , the part of q_∞ that has not been discovered yet, that is $q_\infty \setminus q_n$, is a UIPQ of the $|\partial q_n|$ -gon.

Let us now describe the conditional distribution of q_{n+1} knowing \mathcal{F}_n , and write down explicit transition probabilities. First, we have to choose an oriented edge e on ∂q_n . Any choice, deterministic or random, is acceptable as long as it depends only on \mathcal{F}_n , and q_n lies on the right hand side of e . The map $q_\infty \setminus q_n$ rooted at e is a UIPQ of the $|\partial q_n|$ -gon. Let $p = |\partial q_n|/2$, and denote the vertices of ∂q_n by x_1, \dots, x_{2p} so that $e = (x_{2p}, x_1)$ (see Figure 2). Now, let us reveal the face of $q_\infty \setminus q_n$ containing e . Following the orientation given by e , we denote the vertices of this face by (x_{2p}, x_1, y_0, y_1) . Four cases may occur, depending on whether y_0 and / or y_1 belong to ∂q_n : we now describe q_{n+1} in each case, and give the corresponding probability.

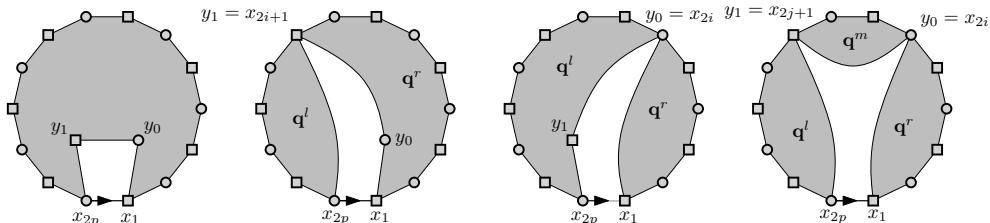


Figure 2: Discovering a new face during the peeling process. Note that y_1 can coincide with x_1 , and y_0 can coincide with x_{2p} – in the second and third cases, respectively.

- (1) $y_0, y_1 \notin \partial q_n$ (Figure 2, left). In this case, we set q_{n+1} to be the union of q_n and the face discovered. Therefore, $q_\infty \setminus q_{n+1}$ is a quadrangulation of a $2(p+1)$ -gon, and the spatial Markov property ensures that conditionally on this event and \mathcal{F}_n , the map $q_\infty \setminus q_{n+1}$ is a UIPQ of the $2(p+1)$ -gon. Hence, conditionally on \mathcal{F}_n , this event has probability

$$\tau(y_0, y_1 \notin \partial q_n | \mathcal{F}_n) = \tau^p(y_0, y_1 \notin \partial q^p) = \lim_{N \rightarrow \infty} \frac{a_{N-1, p+1}}{a_{N, p}} = \frac{C_{p+1}}{12C_p}$$

(using (2.2)).

- (2) $y_0 \notin \partial \mathbf{q}_n$ and $y_1 = x_{2i+1}$ with $0 \leq i \leq p-1$ (Figure 2, middle left). In this case, the new face divides the remaining part of \mathbf{q}_∞ into two separate quadrangulations: \mathbf{q}_n^r with perimeter $2(i+1)$ and \mathbf{q}_n^l with perimeter $2(p-i)$. Conditionally on this event and \mathcal{F}_n , exactly one of these two quadrangulations is infinite. If it is \mathbf{q}_n^r , the spatial Markov property ensures that it is a UIPQ of the $2(i+1)$ -gon, while \mathbf{q}_n^l is independent of \mathbf{q}_n^r and is a free quadrangulation of the $2(p-i)$ -gon. We set \mathbf{q}_{n+1} to be the union of \mathbf{q}_n , the face discovered, and \mathbf{q}_n^l , so that $\mathbf{q}_\infty \setminus \mathbf{q}_{n+1} = \mathbf{q}_n^r$. Using (2.6), the probability of this event is given by

$$\tau(y_0 \notin \partial \mathbf{q}_n, y_1 = x_{2i+1}, \mathbf{q}_n^r \text{ infinite } | \mathcal{F}_n) = \frac{Z_{p-i} C_{i+1}}{12C_p}.$$

If \mathbf{q}_n^l is infinite, the situation is similar, and we set \mathbf{q}_{n+1} to be the union of \mathbf{q}_n , the face discovered, and \mathbf{q}_n^r , so that $\mathbf{q}_\infty \setminus \mathbf{q}_{n+1} = \mathbf{q}_n^l$. The corresponding probability is

$$\tau(y_0 \notin \partial \mathbf{q}_n, y_1 = x_{2i+1}, \mathbf{q}_n^l \text{ infinite } | \mathcal{F}_n) = \frac{C_{p-i} Z_{i+1}}{12C_p}.$$

- (3) $y_1 \notin \partial \mathbf{q}_n$ and $y_0 = x_{2i}$ with $1 \leq i \leq p$ (Figure 2, middle right). The situation is similar to the second case, and conditionally on this event and \mathcal{F}_n , either $\mathbf{q}_n^r = \mathbf{q}_\infty \setminus \mathbf{q}_{n+1}$ is a UIPQ of the $2i$ -gon, or $\mathbf{q}_n^l = \mathbf{q}_\infty \setminus \mathbf{q}_{n+1}$ is a UIPQ of the $2(p+1-i)$ -gon. The respective probabilities are:

$$\begin{aligned} \tau(y_1 \notin \partial \mathbf{q}_n, y_0 = x_{2i}, \mathbf{q}_n^r \text{ infinite } | \mathcal{F}_n) &= \frac{Z_{p+1-i} C_i}{12C_p}, \\ \tau(y_1 \notin \partial \mathbf{q}_n, y_0 = x_{2i}, \mathbf{q}_n^l \text{ infinite } | \mathcal{F}_n) &= \frac{C_{p+1-i} Z_i}{12C_p}. \end{aligned}$$

- (4) $y_0 = x_{2i}$ and $y_1 = x_{2j+1}$ with $1 \leq i \leq j \leq p-1$ (Figure 2, right). In this case, the new face divides the remaining part of \mathbf{q}_∞ into three separate quadrangulations: \mathbf{q}_n^r with perimeter $2i$, \mathbf{q}_n^m with perimeter $2(j-i+1)$, and \mathbf{q}_n^l with perimeter $2(p-j)$. Here again, the spatial Markov property ensures that conditionally on the corresponding event and \mathcal{F}_n , exactly one of these quadrangulations is infinite, and we set \mathbf{q}_{n+1} to be the union of \mathbf{q}_n , the face discovered, and the other two finite quadrangulations. The corresponding probabilities are given by:

$$\begin{aligned} \tau(y_0 = x_{2i}, y_1 = x_{2j+1}, \mathbf{q}_n^r \text{ infinite } | \mathcal{F}_n) &= \frac{Z_{p-j} Z_{j-i+1} C_i}{12C_p}, \\ \tau(y_0 = x_{2i}, y_1 = x_{2j+1}, \mathbf{q}_n^m \text{ infinite } | \mathcal{F}_n) &= \frac{Z_{p-j} C_{j-i+1} Z_i}{12C_p}, \\ \tau(y_0 = x_{2i}, y_1 = x_{2j+1}, \mathbf{q}_n^l \text{ infinite } | \mathcal{F}_n) &= \frac{C_{p-j} Z_{j-i+1} Z_i}{12C_p}. \end{aligned}$$

Let us insist on the fact that the peeling procedure that we just described, and its transition probabilities, do not depend on the choice of the edge e , provided that at each step n , this choice depends only on \mathcal{F}_n . This will allow us to study both site and bond percolation on the UIPM by following the percolation interface along the way. This peeling procedure also allowed Benjamini and Curien [6] to study the simple random walk on the UIPQ.

A more straightforward yet very useful consequence of this fact is that the sequence $(|\partial \mathbf{q}_n|, |\mathbf{q}_n|)_{n \geq 0}$ is a homogeneous Markov chain whose transition probabilities

do not depend on the particular peeling process performed. For instance, let us write $|\partial \mathbf{q}_{n+1}| = |\partial \mathbf{q}_n| + 2X_n$ for every $n \geq 0$. Then one has, for this increment X_n , using the transition probabilities of the peeling process that we derived explicitly,

$$P(X_n = 1 | |\partial \mathbf{q}_n| = 2p) = \frac{C_{p+1}}{12C_p} \quad (3.1)$$

(corresponding to case (1) above), and for every $k = 0, \dots, p-1$,

$$P(X_n = -k | |\partial \mathbf{q}_n| = 2p) = 4 \frac{C_{p-k} Z_{k+1}}{12C_p} + 3 \frac{C_{p-k}}{12C_p} \sum_{i=1}^k Z_i Z_{k+1-i} \quad (3.2)$$

(combining cases (2) and (3) for the first term, and (4) for the second term). Of particular interest is the following asymptotics proven in Theorem 5 of [6]:

Lemma 3.1. *If $\mathbf{q}_0, \mathbf{q}_1, \dots, \mathbf{q}_n, \dots$ is generated by a peeling procedure of the UIPQ, then one has*

$$\begin{aligned} |\partial \mathbf{q}_n| &\approx n^{2/3}, \\ |\mathbf{q}_n| &\approx n^{4/3}, \end{aligned}$$

where, if $(Y_n)_{n \geq 0}$ is a random process, $Y_n \approx n^\alpha$ means that for some constant $\kappa > 0$, $Y_n n^{-\alpha} \log^\kappa(n) \rightarrow \infty$ and $Y_n n^{-\alpha} \log^{-\kappa}(n) \rightarrow 0$ almost surely.

This property is proved in [6] by using geometric properties of the UIPQ, without appealing to the peeling process directly. However, it should also be possible to prove these asymptotics by using the explicit transition probabilities for the peeling process, and the enumeration results of Section 2.3. An easy consequence of Lemma 3.1 – actually, only the fact that $|\partial \mathbf{q}_n| \rightarrow \infty$ a.s. – that will be useful for our purpose is the following:

Corollary 3.2. *Let $\mathbf{q}_0, \mathbf{q}_1, \dots, \mathbf{q}_n, \dots$ be generated by a peeling procedure of the UIPQ, and set $|\partial \mathbf{q}_{n+1}| = |\partial \mathbf{q}_n| + 2X_n$ for every $n \geq 0$. Then one has*

$$E[X_n | \mathcal{F}_n] \xrightarrow{n \rightarrow \infty} 0.$$

Proof. For $p > 0$ and $0 \leq k \leq p-1$, one can easily derive from (3.2):

$$P(X_n = -k | |\partial \mathbf{q}_n| = 2p) = \frac{\left(p - \frac{1}{2}\right)_k (p-1)_k}{\left(p - \frac{1}{3}\right)_k \left(p - \frac{2}{3}\right)_k} q_k$$

where $(x)_k = x(x-1) \cdots (x-k+1)$ and

$$q_k = \frac{1}{3} Z_{k+1} \left(\frac{2}{9}\right)^k + \frac{1}{4} \sum_{i=1}^k Z_i Z_{k+1-i} \left(\frac{2}{9}\right)^k. \quad (3.3)$$

Therefore, the probabilities $P(X_n = -k | |\partial \mathbf{q}_n| = 2p)$ are increasing in p and converge to q_k . Let us denote by X a random variable with law given by

$$P(X = -k) = q_k, \text{ for } k \geq 0,$$

$$P(X = 1) = \lim_{p \rightarrow \infty} P(X_n = 1 | |\partial \mathbf{q}_n| = 2p) = 3/8.$$

Since $|\partial \mathbf{q}_n| \rightarrow \infty$ almost surely as n grows, an argument of dominated convergence shows that $E[X_n | \mathcal{F}_n]$ converges to $E[X]$.

Now, let us show that $E[X] = 0$. To this aim, we introduce the series

$$Z(x) = \sum_{k \geq 1} Z_k x^k,$$

with convergence radius $2/9$. The series $Z'(x)$ corresponds to the generating series of ternary trees, and classical arguments (see [12], (5.29)) yield

$$\begin{aligned} Z(x) &= -\frac{2}{3} + \frac{2}{3} {}_2F_1\left(-\frac{2}{3}, -\frac{1}{3}; \frac{1}{2}; \frac{9x}{2}\right), \\ \text{and } Z'(x) &= 4\sqrt{\frac{2}{9x}} \sin\left(\frac{1}{3} \arcsin\left(\sqrt{\frac{9x}{2}}\right)\right). \end{aligned}$$

This allows one to compute the generating function of the numbers q_k from (3.3):

$$\sum_{k \geq 0} q_k x^k = \frac{9}{2x} \left(\frac{Z(2x/9)}{3} + \frac{Z(2x/9)^2}{4} \right).$$

With the value $Z(2/9) = 1/3$, this implies readily that $\sum_{k \geq 0} q_k = 5/8$ and the probabilities in the definition of X add up to 1. From here, the value $Z'(2/9) = 2$ and basic computations give

$$\sum_{k \geq 1} k q_k = \frac{1}{3} Z'\left(\frac{2}{9}\right) + \frac{1}{4} (Z^2)' \left(\frac{2}{9}\right) - \frac{3}{2} Z\left(\frac{2}{9}\right) - \frac{9}{8} Z\left(\frac{2}{9}\right)^2 = \frac{3}{8}.$$

□

To conclude this section, let us stress that the peeling procedure for the UIPQ also provides a sampling of the UIPM. Indeed, consider $(\mathbf{q}_n)_{n \geq 0}$ a peeling-generated sequence for the UIPQ. For every $n \geq 0$, we can associate to \mathbf{q}_n , which is a quadrangulation with a boundary, a map \mathbf{m}_n by a slight modification of the bijection of Section 2.2: there is an edge of \mathbf{m}_n inside each face of \mathbf{q}_n except for the boundary face. We obtain in this way an increasing sequence of maps, which are all submaps of \mathbf{m}_∞ .

Note that different quadrangulations \mathbf{q}_n may produce the same map \mathbf{m}_n . In fact there is more information on the UIPM in \mathbf{q}_n than in \mathbf{m}_n , since \mathbf{q}_n also gives information on the faces of \mathbf{m}_∞ . Indeed, let us consider two edges of \mathbf{m}_n . Considering only \mathbf{m}_n , it is not possible to say if the two edges are part of the same face in \mathbf{m}_∞ . However, this information is available in \mathbf{q}_n : the two edges belong to the same face of \mathbf{m}_∞ iff their associated quadrangles share a common square-vertex in \mathbf{q}_n . This is not problematic for our purpose, since we are not interested in the sequence $(\mathbf{m}_n)_{n \geq 0}$ by itself.

4 Site percolation on the UIPM

In this section, we consider Bernoulli site percolation on the UIPM: the vertices are colored, independently of each other, *black* with probability q , and *white* with probability $(1-q)$. We prove the first part of Theorem 1.1: for site percolation on the UIPM, the percolation threshold is almost surely

$$p_c^{\text{site}} = 2/3.$$

4.1 Exploration process

Consider m_∞ the UIPM, and q_∞ the associated UIPQ. Suppose that each vertex of m_∞ is colored independently at random, black with probability q and white with probability $(1 - q)$ (in q_∞ , this corresponds to a coloring of circle-vertices only). We are interested in percolation of the origin, i.e. the existence of an infinite black connected component containing the origin.

We also assume for simplicity that the root vertex of m_∞ – which is also the root vertex of q_∞ – is colored black. We can sample percolation on the UIPM simultaneously with a peeling process of the UIPQ: each time a new vertex of the UIPM is added, we color it randomly, independently of all previous steps. Note that if at some step n , all the vertices of the UIPM that are on the boundary ∂q_n are white, then these vertices separate from infinity (in m_∞) the root vertex, which therefore does not percolate (for black sites).

Now, recall that we can choose where the next quadrangle is revealed at each step of the peeling process. In particular, we can let this choice depend on the percolation configuration sampled so far. On the one hand, if all the vertices of the UIPM that are on the boundary ∂q_n have the same color, then we can make an arbitrary choice. On the other hand, if there are white and black vertices on ∂q_n , then we can ensure that ∂q_n remains divided in two arcs: one arc with black vertices only, and the other one with white vertices only. If we then follow the orientation of the boundary, there is a unique choice of three consecutive vertices x_{2p} , x_1 , and x_2 , where x_{2p} and x_2 are black and white respectively, and x_1 is a square-vertex between them. We then reveal the quadrangle on the left side of the oriented edge (x_{2p}, x_1) (see Figure 3).

If this rule is followed, it is easy to see that all black vertices on ∂q_n belong to the percolation cluster containing the root vertex of m_∞ , as long as the boundary does not become totally white, which corresponds to detecting a white circuit. However, note that white vertices of ∂q_n do not necessarily belong to the same white cluster, so black and white sites do not play symmetric roles in this process: one cannot simply use the symmetry $q \leftrightarrow 1 - q$. The connectedness of white sites corresponds to “*-connectedness”, as it is usually called for percolation on planar graphs such as \mathbb{Z}^2 .

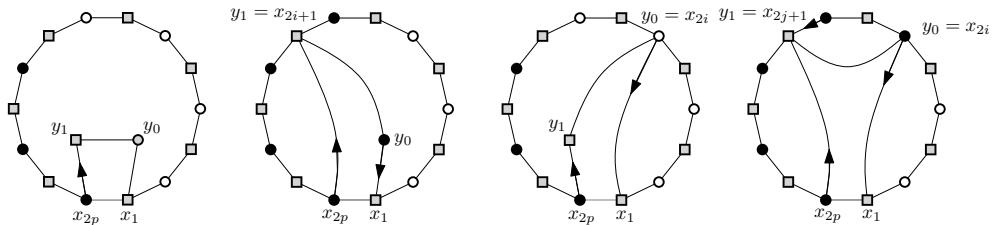


Figure 3: This figure shows how percolation is sampled during the peeling process. The arrows on each figure indicate the possible reroottings for the next peeling step. Note that in the middle right case, if $y_0 = x_{2i}$ is white and if the quadrangulation on the right is infinite, then a circuit of white vertices separates the root vertex from infinity, so percolation does not occur.

Let us denote by B_n the number of black vertices on ∂q_n , W_n the number of white vertices, and by \mathcal{F}_n the filtration generated by q_0, q_1, \dots, q_n and their coloring. Recall that X_n denotes the increment size of the boundary length conditionally on \mathcal{F}_n , and that its distribution is given by (3.1), (3.2). We now give the explicit transition probabilities of B_n conditionally on \mathcal{F}_n . In order to simplify notation, we write $|\partial q_n| = 2p$.

- (1) When $X_n = 1$, the face discovered has two new vertices, among them one belonging to the UIPM, that gets color black or white (see Figure 3, left for an illustration). Therefore,

$$B_{n+1} = \begin{cases} B_n + 1 & \text{with probability } q \frac{C_{p+1}}{12C_p}, \\ B_n & \text{with probability } (1-q) \frac{C_{p+1}}{12C_p}. \end{cases}$$

We now consider the event $X_n = -k \leq 0$ for some $k \in \{0, \dots, p-1\}$, that is, some vertices are removed from $\partial \mathbf{q}_n$. Let us discuss the different cases that may occur, according to Section 3.

- (2) $y_0 \notin \partial \mathbf{q}_n$ and $y_1 \in \partial \mathbf{q}_n$ (Figure 3, middle left). The vertex y_0 belongs to the unexplored part of the UIPM, and it is colored black or white (with the corresponding probabilities), independently of previously chosen colors.

On the one hand, if the quadrangulation \mathbf{q}_n^l is infinite (this corresponds to $i = k$ on Figure 3, i.e. $y_1 = x_{2k+1}$), then black vertices are removed if and only if $p-k < B_n$, and in this case $\partial \mathbf{q}_{n+1}$ has no white vertices. If $p-k \geq B_n$, then no black vertex is removed and $B_{n+1} = B_n$. Hence, $B_{n+1} = \min(B_n, p-k)$ in this case.

On the other hand, if \mathbf{q}_n^r is infinite (this corresponds to $i = p-k-1$, i.e. $y_1 = x_{2(p-k)-1}$), then $|\partial \mathbf{q}_n^l| = 2(k+1)$ and the number of black vertices removed is $\min(B_n, k+1)$. In addition, one black vertex is added with probability q . This gives $B_{n+1} = \max(B_n - k - 1, 0) + 1$ with probability q , and $B_{n+1} = \max(B_n - k - 1, 0)$ with probability $(1-q)$.

- (3) $y_0 \in \partial \mathbf{q}_n$ and $y_1 \notin \partial \mathbf{q}_n$ (Figure 3, middle right). The situation is very similar to (2), except that no new colored vertex is added. If the quadrangulation \mathbf{q}_n^l is infinite (so that $i = k+1$), then one has $B_{n+1} = \min(B_n, p-k)$, and if \mathbf{q}_n^r is infinite (so that $i = p-k$), then $B_{n+1} = \max(B_n - k, 0)$.

- (4) $y_0, y_1 \in \partial \mathbf{q}_n$ (Figure 3, right). If \mathbf{q}_n^r is infinite (so that $i = p-k$), the situation is identical to the corresponding case in (3) and $B_{n+1} = \max(B_n - k, 0)$, while if \mathbf{q}_n^l is infinite (so that $j = k$), the situation is identical to the corresponding case in (2) and $B_{n+1} = \min(B_n, p-k)$.

Finally, if \mathbf{q}_n^m is infinite, then there is $1 \leq m \leq k$ such that $|\partial \mathbf{q}_n^l| = 2m$ (this corresponds to $j = p-m$ and $i = k-m+1$), and $B_{n+1} = \max(B_n - m, 0)$.

For each of the previous cases, the corresponding probabilities have been determined in Section 3. We deduce that conditionally on $|\partial \mathbf{q}_n| = 2p$, and when $X_n = -k$:

$$B_{n+1} = \begin{cases} \min(B_n, p-k) & \text{w. p. } 2 \frac{C_{p-k} Z_{k+1}}{12C_p} + \frac{C_{p-k}}{12C_p} \sum_{i=1}^k Z_i Z_{k+1-i}, \\ \max(B_n - k, 0) & \text{w. p. } \frac{C_{p-k} Z_{k+1}}{12C_p} + \frac{C_{p-k}}{12C_p} \sum_{i=1}^k Z_i Z_{k+1-i}, \\ \max(B_n - k - 1, 0) + 1 & \text{w. p. } q \frac{C_{p-k} Z_{k+1}}{12C_p}, \\ \max(B_n - k - 1, 0) & \text{w. p. } (1-q) \frac{C_{p-k} Z_{k+1}}{12C_p}, \\ \max(B_n - i, 0) & \text{w. p. } \frac{C_{p-k}}{12C_p} Z_i Z_{k+1-i} \text{ for } 1 \leq i \leq k. \end{cases}$$

4.2 Derivation of p_c^{site}

We now show that $p_c^{\text{site}} = 2/3$ a.s. We first prove that black vertices do not percolate when $q < 2/3$, and then that they percolate when $q > 2/3$. We denote by C_∞ the event that the root vertex is in an infinite black cluster.

Let us first consider $q < 2/3$. We start by noting that

$$P(C_\infty \cap \{B_n = 1 \text{ infinitely often}\}) = 0, \quad (4.1)$$

which follows from the observation that

$$P(C_\infty | B_n = 1) \leq 1 - c$$

for some universal constant $c > 0$. Indeed, if $B_n = 1$ and $X_n \leq -1$, then black vertices disappear on the next step with probability at least $1/2$. Hence, $B_{n+1} = 0$ with probability at least

$$\frac{1}{2}P(X_n \leq -1) = \frac{1}{2}\left(1 - \frac{4}{9} - \frac{3}{8} + o(1)\right)$$

(using the distribution of X). This implies that

$$P(C_\infty \cap \{B_n = 1 \text{ at least } k \text{ times}\}) \leq (1 - c)^k,$$

by conditioning on the first k such times, and (4.1) follows readily.

We will now assume that $P(C_\infty) > 0$. As we have just observed, we can suppose that a.s., $B_n \geq 2$ for n large enough. We introduce a modified Markov chain (B'_n) obtained by "simplifying" (B_n) , in particular by allowing it to take negative values (and coupled in a natural way). More precisely, we consider the chain with the following transition probabilities, conditionally on $|\partial q_n| = 2p$:

$$B'_{n+1} = \begin{cases} B'_n + 1 & \text{with probability } q \frac{C_{p+1}}{12C_p}, \\ B'_n & \text{with probability } (1 - q) \frac{C_{p+1}}{12C_p} \end{cases}$$

(corresponding to $X_n = 1$), and

$$B'_{n+1} = \begin{cases} B'_n & \text{w. p. } 2 \frac{C_{p-k} Z_{k+1}}{12C_p} + \frac{C_{p-k}}{12C_p} \sum_{i=1}^k Z_i Z_{k+1-i}, \\ B'_n - k & \text{w. p. } (1 + q) \frac{C_{p-k} Z_{k+1}}{12C_p} + \frac{C_{p-k}}{12C_p} \sum_{i=1}^k Z_i Z_{k+1-i}, \\ B'_n - k - 1 & \text{w. p. } (1 - q) \frac{C_{p-k} Z_{k+1}}{12C_p}, \\ B'_n - i & \text{w. p. } \frac{C_{p-k}}{12C_p} Z_i Z_{k+1-i} \text{ for } 1 \leq i \leq k \end{cases}$$

for every $k = 0, \dots, p-1$ (corresponding to $X_n = -k$).

Now, let us note that the increment $(B'_{n+1} - B'_n)$ is equal to the increment $(B_{n+1} - B_n)$ except in the following three cases.

- $B_{n+1} = \min(B_n, p - k)$ and $B_n > p - k$: in this case,

$$B_{n+1} - B_n = \min(B_n, p - k) - B_n = (p - k) - B_n < 0 = B'_{n+1} - B'_n.$$

- $B_{n+1} = \max(B_n - k - 1, 0) + 1$ and $B_n - k - 1 < 0$: in this case, $B_{n+1} = 1$, which is ruled out by (4.1) (for n large enough).
- In each of the remaining three sub-cases, when $B_n - k < 0$, $B_n - k - 1 < 0$, or $B_n - i < 0$ (resp.): this means that the number of black vertices gets negative, so that percolation does not occur.

Therefore, conditionally on C_∞ , one has $B_n \leq B'_n + O(1)$. We will see that almost surely, $B'_n \rightarrow -\infty$, and therefore there exists n such that $B_n = 0$. This will imply that the probability that percolation occurs is 0. One has:

$$\begin{aligned}
 & E[B'_{n+1} - B'_n | |\partial \mathbf{q}_n| = 2p] \\
 &= qP(X_n = 1 | |\partial \mathbf{q}_n| = 2p) - \sum_{k=0}^{p-1} k \left(2 \frac{C_{p-k} Z_{k+1}}{12C_p} + \frac{C_{p-k}}{12C_p} \sum_{i=1}^k Z_i Z_{k+1-i} \right) \\
 &\quad - (1-q) \sum_{k=0}^{p-1} \frac{C_{p-k} Z_{k+1}}{12C_p} - \sum_{k=1}^{p-1} \sum_{i=1}^k i \frac{C_{p-k}}{12C_p} Z_i Z_{k+1-i} \\
 &= \left(q - \frac{1}{2} \right) P(X_n = 1 | |\partial \mathbf{q}_n| = 2p) + \frac{1}{2} E[X_n | |\partial \mathbf{q}_n| = 2p] \\
 &\quad - (1-q) \sum_{k=0}^{p-1} \frac{C_{p-k} Z_{k+1}}{12C_p} + \sum_{k=1}^{p-1} \sum_{i=1}^k \left(\frac{k}{2} - i \right) \frac{C_{p-k}}{12C_p} Z_i Z_{k+1-i} \\
 &= \left(q - \frac{1}{2} \right) \frac{C_{p+1}}{12C_p} + \frac{1}{2} E[X_n | |\partial \mathbf{q}_n| = 2p] \\
 &\quad - (1-q) \sum_{k=0}^{p-1} \frac{C_{p-k} Z_{k+1}}{12C_p} - \frac{1}{2} \sum_{k=1}^{p-1} \sum_{i=1}^k \frac{C_{p-k}}{12C_p} Z_i Z_{k+1-i}.
 \end{aligned}$$

Corollary 3.2, and the computations performed in its proof, ensure that a.s.

$$E[B'_{n+1} - B'_n | \mathcal{F}_n] \longrightarrow \left(q - \frac{1}{2} \right) \frac{3}{8} - (1-q) \frac{1}{8} - \frac{1}{2} \frac{1}{24} = \frac{q}{2} - \frac{1}{3}$$

as $n \rightarrow \infty$. Therefore, $E[B'_{n+1} - B'_n | \mathcal{F}_n]$ is negative and bounded away from 0 for n large enough. This suffices to prove that $B'_n \rightarrow -\infty$ almost surely, and percolation does not occur.

Now, let us take a value $q > 2/3$. As mentioned earlier, one cannot simply exchange the roles of black and white sites to prove that W_n stays small, and that consequently black vertices percolate. However, using $X_n = (W_{n+1} - W_n) + (B_{n+1} - B_n)$, we can obtain: conditionally on $|\partial \mathbf{q}_n| = 2p$,

$$W_{n+1} = \begin{cases} W_n + 1 & \text{w. p. } (1-q) \frac{C_{p+1}}{12C_p}, \\ W_n & \text{w. p. } q \frac{C_{p+1}}{12C_p}, \end{cases}$$

and on the event $X_n = -k$ (for $k \in \{0, \dots, p-1\}$),

$$W_{n+1} = \begin{cases} \max(W_n - k, 0) & \text{w. p. } 2 \frac{C_{p-k} Z_{k+1}}{12C_p} + \frac{C_{p-k}}{12C_p} \sum_{i=1}^k Z_i Z_{k+1-i}, \\ \min(W_n, p-k) & \text{w. p. } \frac{C_{p-k} Z_{k+1}}{12C_p} + \frac{C_{p-k}}{12C_p} \sum_{i=1}^k Z_i Z_{k+1-i}, \\ \min(W_n, p-k-1) & \text{w. p. } q \frac{C_{p-k} Z_{k+1}}{12C_p}, \\ \min(W_n, p-k-1) + 1 & \text{w. p. } (1-q) \frac{C_{p-k} Z_{k+1}}{12C_p}, \\ \max(W_n - (i-1), 0) & \text{w. p. } \frac{C_{p-k}}{12C_p} Z_i Z_{k+1-i} \text{ for } 1 \leq i \leq k. \end{cases}$$

In a similar way as for B'_n , we consider the process W'_n , coupled with W_n and with increments given conditionally on $|\partial \mathbf{q}_n| = 2p$ by:

$$W'_{n+1} = \begin{cases} W'_n + 1 & \text{w. p. } (1-q) \frac{C_{p+1}}{12C_p}, \\ W'_n & \text{w. p. } q \frac{C_{p+1}}{12C_p} \end{cases}$$

(corresponding to $X_n = 1$), and for every $k \in \{0, \dots, p-1\}$,

$$W'_{n+1} = \begin{cases} W'_n + 1 & \text{w. p. } (1-q) \frac{C_{p-k} Z_{k+1}}{12C_p}, \\ W'_n & \text{w. p. } (1+q) \frac{C_{p-k} Z_{k+1}}{12C_p} + \frac{C_{p-k}}{12C_p} \sum_{i=1}^k Z_i Z_{k+1-i}, \\ W'_n - k & \text{w. p. } 2 \frac{C_{p-k} Z_{k+1}}{12C_p} + \frac{C_{p-k}}{12C_p} \sum_{i=1}^k Z_i Z_{k+1-i}, \\ W'_n - (i-1) & \text{w. p. } \frac{C_{p-k}}{12C_p} Z_i Z_{k+1-i} \text{ for } 1 \leq i \leq k \end{cases}$$

(corresponding to $X_n = -k$). Then, conditionally on the event $\{W_n > 0, \forall n \geq 0\}$, the increments of W'_n are bigger than the increments of W_n . As $n \rightarrow \infty$, one has

$$E[W'_{n+1} - W'_n | \mathcal{F}_n] \longrightarrow \frac{1}{3} - \frac{q}{2} := -\alpha < 0,$$

from which one can easily deduce that a.s. $W_n = O(\ln n)$: we now provide an explicit proof for the sake of completeness.

If we write $\Delta_n = W'_{n+1} - W'_n$, we obtain, for N large enough: for all $n \geq N$,

$$E[\Delta_n | \mathcal{F}_n] \leq -\frac{\alpha}{2}.$$

We first claim that it implies: there exists $\lambda > 0$ small enough so that for all $n \geq N$,

$$E[e^{\lambda \Delta_n} | \mathcal{F}_{n-1}] \leq 1. \quad (4.2)$$

In order to prove (4.2), let us start by noting that

$$E[|\Delta_n|^{5/4} | \mathcal{F}_{n-1}] \leq M \quad (4.3)$$

for some universal constant M : this follows from the fact that for any fixed k , the probabilities $P(X_n = -k | |\partial \mathbf{q}_n| = 2p)$ are increasing in p and converge to q_k , which is of order $q_k \sim ck^{-5/2}$ – using (2.4) and (3.3). We can then use that for some constants $C_1, C_2 > 0$,

$$e^x \leq 1 + C_1 x + C_2 |x|^{5/4}$$

for all $x \leq 1$: since $\Delta_n \leq 1$ by definition, we obtain

$$E[e^{\lambda \Delta_n} | \mathcal{F}_{n-1}] \leq 1 + C_1 \lambda E[\Delta_n | \mathcal{F}_{n-1}] + C_2 |\lambda|^{5/4} E[|\Delta_n|^{5/4} | \mathcal{F}_{n-1}],$$

which (using (4.3)) is at most 1 for λ small enough.

Now, for any fixed constant $C > 0$, we can write: for all $n \geq m \geq N$,

$$P\left(\sum_{l=m}^n \Delta_l > C \ln n | \mathcal{F}_m\right) \leq e^{-\lambda C \ln n} E\left[\exp\left(\lambda \sum_{l=m}^n \Delta_l\right) | \mathcal{F}_m\right],$$

and

$$E\left[\exp\left(\lambda \sum_{l=m}^n \Delta_l\right) | \mathcal{F}_m\right] = E\left[\exp\left(\lambda \sum_{l=m}^{n-1} \Delta_l\right) E[e^{\lambda \Delta_n} | \mathcal{F}_{n-1}] | \mathcal{F}_m\right] \leq E\left[\exp\left(\lambda \sum_{l=m}^{n-1} \Delta_l\right) | \mathcal{F}_m\right],$$

by using (4.2). By iterating this reasoning, we find

$$P\left(\sum_{l=m}^n \Delta_l > C \ln n | \mathcal{F}_m\right) \leq e^{-\lambda C \ln n},$$

which allows one to conclude, by using a Borel Cantelli argument (choosing a large enough C in the beginning). Since $B_n + W_n = |\partial \mathbf{q}_n| \approx n^{2/3}$, we deduce that $B_n \approx n^{2/3}$: in particular, black vertices percolate.

5 Bond percolation on the UIPM

In this section, we study bond percolation, instead of site percolation, on the UIPM: each edge is open with probability q , and closed with probability $(1-q)$, independently of other edges. We prove the second part of Theorem 1.1: the corresponding percolation threshold is almost surely

$$p_c^{\text{bond}} = 1/2.$$

5.1 Exploration process

In this section, we describe how to sample bond percolation on the UIPM simultaneously with a peeling process of the UIPQ. This is similar to the exploration process for site percolation described in Section 4.1, but small adaptations are needed for the process to actually follow the boundary of the percolation cluster of the root vertex. We will assume for simplicity that the root edge of m_∞ is open.

Let us consider the UIPM m_∞ , and q_∞ the associated UIPQ. Let us denote by q_0, q_1, \dots, q_n the peeling process for q_∞ , and m_0, m_1, \dots, m_n the associated submaps of m_∞ . Each time a new face of q_∞ is discovered, the corresponding edge of m_∞ is opened with probability q , and closed with probability $(1-q)$ independently of all previous steps. The percolation interfaces between open and closed edges can be viewed as a random tiling of q_∞ , as illustrated in Figure 4.

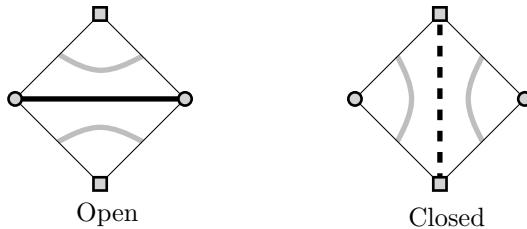


Figure 4: The exploration process can be seen as a random tiling of the quadrangles that are successively discovered.

It is possible to adapt the peeling process in order to follow percolation interfaces. Let m_n^0 denote the set of vertices connected to the root vertex of m_n by open paths lying in m_n : this is the part of the cluster of the root m_∞^0 discovered before time n with the peeling procedure. The choice of the next quadrangle to reveal is very similar to what we did for site percolation. Recall that on the quadrangulation, circle-vertices belong to the associated map, while square-vertices lie on the dual of this map. On the one hand, if all circle-vertices of ∂q_n belong to m_n^0 , or if, on the contrary, no circle-vertex of ∂q_n belongs to m_n^0 , then we can make an arbitrary choice for the next step. On the other hand, if some, but not all, circle-vertices of ∂q_n belong to m_n^0 , then we can find three vertices x_{2p}, x_1, x_2 (in this order) such that x_{2p} belongs to m_n^0 , but not x_2 (see Figure 5): we reveal the quadrangle on the left side of the edge $(x_{2p}x_1)$. Provided that this procedure is followed during the peeling process, then the vertices of $\partial q_n \cap m_n^0$ form an arc of ∂q_n .

Now, let A_n denote the number of vertices of ∂q_n that belong to m_n^0 . If there exists n such that $A_n = 0$, then the root vertex does not percolate, and $A_k = 0$ for all $k \geq n$. On the other hand, if (A_n) is unbounded, then percolation does occur. Let \mathcal{F}_n denote the filtration generated by q_0, \dots, q_n and bond percolation on them. Let $n > 0$, and suppose that $A_n > 0$. Following a similar strategy as for site percolation, we give explicit transition probabilities for A_n conditionally on \mathcal{F}_n . Recall that X_n denotes the

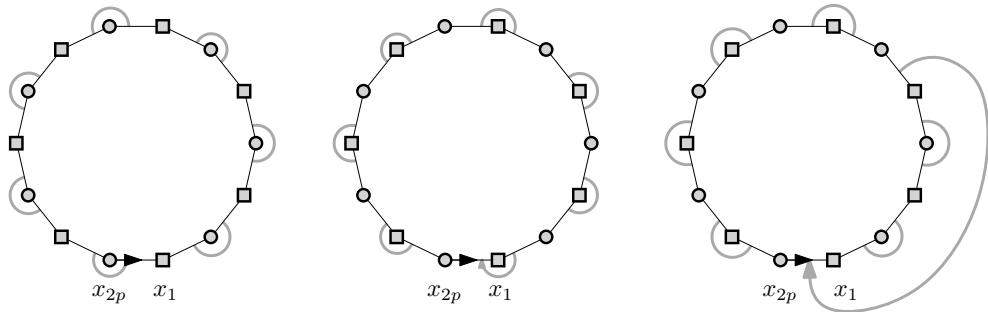


Figure 5: Configurations obtained by following the exploration process during the peeling procedure. *Left:* No vertex of the boundary belongs to the explored part of the cluster of the root, so that percolation does not occur. This corresponds to $A_n = 0$. *Middle:* All vertices of the boundary belong to the explored part of the cluster of the root. This corresponds to $A_n = p$. *Right:* The vertices on the left belong to the explored part of the cluster of the root, whereas the vertices on the right do not belong to the discovered part of the cluster of the root. This corresponds to $0 < A_n < p$.

increment size for the boundary length conditionally on \mathcal{F}_n , and that its distribution is given by (3.1), (3.2). Let us also set $|\partial \mathbf{q}_n| = 2p$, as before.

- (1) When $X_n = 1$, let us denote by (x_{2p}, x_1, y_0, y_1) the face discovered: it has two new vertices, and a new edge (x_{2p}, y_0) of the UIPM. With probability q , the new edge is open and there is an open path joining y_0 to the root vertex in \mathbf{m}_{n+1} . With probability $(1-q)$, this edge is closed and y_0 does not belong to the part discovered of the cluster of the root (note that y_0 may still belong to the cluster of the root, if some of the edges that connect it to the root have not yet been discovered). This yields

$$A_{n+1} = \begin{cases} A_n + 1 & \text{with probability } q \frac{C_{p+1}}{12C_p}, \\ A_n & \text{with probability } (1-q) \frac{C_{p+1}}{12C_p} \end{cases}$$

(see Figure 6 for an illustration).

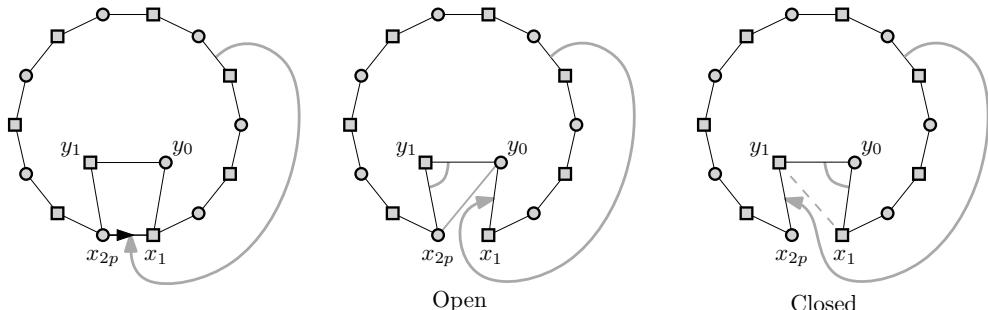


Figure 6: Evolution of the exploration process in case (1), when two new vertices are discovered.

Suppose now that $X_n = -k$. We discuss the different cases that appeared in Section 3.

- (2) $y_0 \notin \partial q_n$ and $y_1 \in \partial q_n$ (see Figure 7). The situation is somewhat similar to site percolation, except that the vertices of $\partial q_n \cap m_n$ that do not belong to m_n^0 may still be connected to it by not-yet-discovered open edges. We claim that, except for y_0 , a vertex of $\partial q_{n+1} \cap m_\infty$ belongs to m_{n+1}^0 if and only if it belongs to m_n^0 . Indeed, the two parts q_n^l and q_n^r can only be connected by the new edge or by vertices of m_n , therefore, filling the finite part with a mix of open and closed edges will not change whether vertices on the boundary of the infinite one belong or not to m_n^0 . This gives the same transitions as for site percolation:

$$A_{n+1} = \begin{cases} \min(A_n, p - k) & \text{(if } q_n^l \text{ is infinite),} \\ \max(A_n - k - 1, 0) + 1 & \text{with probability } q \text{ (if } q_n^r \text{ is infinite),} \\ \max(A_n - k - 1, 0) & \text{with probability } (1 - q) \text{ (if } q_n^r \text{ is infinite).} \end{cases}$$

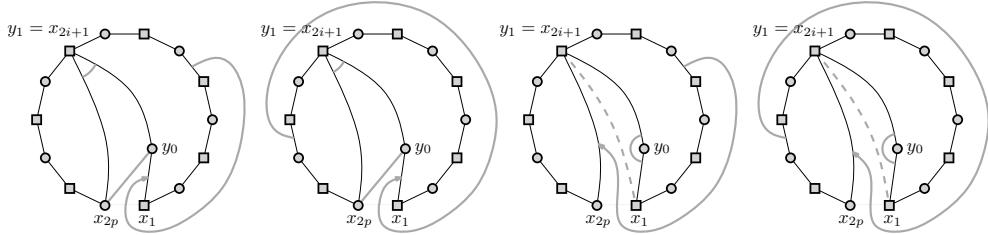


Figure 7: Evolution of the exploration process in case (2): on the first two pictures, the new edge is open, while on the two other ones, it is closed. One has $i = k$ when q_n^l is infinite, and $i = p - k - 1$ when q_n^r is infinite.

- (3) $y_0 \in \partial q_n$ and $y_1 \notin \partial q_n$. Here the situation is similar, except for a notable difference when $y_0 \notin m_n^0$. Indeed, in this case one can have $y_0 \in m_{n+1}^0$ even if the new bond is closed. This happens when q_n^r is infinite: filling e.g. q_n^l with open edges connects y_0 to the root vertex by a path of open edges belonging to q_{n+1} as long as there is at least one vertex of ∂q_n that belongs to m_n^0 . On the other hand, filling q_n^l with closed edges leaves y_0 disconnected from the root vertex in q_{n+1} , and percolation does not occur (see Figure 8 for an illustration). The corresponding probabilities depend on q_n^l , but their exact values will not be needed. Note that if q_n^l is infinite, then y_0 stays disconnected from the root vertex in q_{n+1} if $y_0 \notin m_n^0$.

The transitions are thus given by:

$$A_{n+1} = \begin{cases} \min(A_n, p - k) & \text{with probability } (1 - q) \text{ (if } q_n^l \text{ is infinite),} \\ \min(A_n + 1, p - k) & \text{with probability } q \text{ (if } q_n^l \text{ is infinite),} \\ A_n - k & \text{if } A_n - k > 0 \text{ (if } q_n^r \text{ is infinite),} \\ 0 & \text{with probability } > 0 \text{ if } A_n - k \leq 0 \text{ (if } q_n^r \text{ is infinite),} \\ 1 & \text{with probability } > 0 \text{ if } A_n - k \leq 0 \text{ (if } q_n^r \text{ is infinite).} \end{cases}$$

- (4) $y_0, y_1 \in \partial q_n$. If q_n^l is infinite, then the situation is simple and a vertex of ∂q_{n+1} belongs to m_{n+1}^0 iff it belongs to m_n^0 . We thus obtain $A_{n+1} = \min(A_n, p - k)$ in this case.

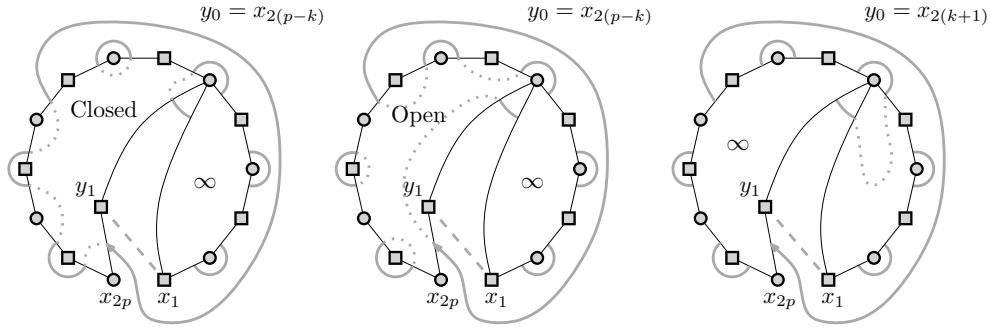


Figure 8: Evolution of the exploration process in case (3), when $y_0 \notin \mathbf{m}_n^0$. Left: $y_0 \notin \mathbf{m}_{n+1}^0$ and percolation does not occur. Middle: $y_0 \in \mathbf{m}_{n+1}^0$. Right: $y_0 \notin \mathbf{m}_{n+1}^0$.

If \mathbf{q}_n^r is infinite, then the situation is identical to case (3) (when \mathbf{q}_n^r is infinite), which gives

$$A_{n+1} = \begin{cases} A_n - k & \text{if } A_n - k > 0, \\ 0 & \text{with probability } > 0 \text{ if } A_n - k \leq 0, \\ 1 & \text{with probability } > 0 \text{ if } A_n - k \leq 0. \end{cases}$$

Finally, when \mathbf{q}_n^m is infinite, let us write $y_0 = x_{2i}$ ($1 \leq i \leq k$). If $p - i \leq A_n - 1$, then every vertex of $\partial\mathbf{q}_n^m \cap \mathbf{m}_\infty$ is in \mathbf{m}_n^0 . In this case we have $A_{n+1} = p - k$. Suppose now that $p - i \geq A_n$. The $(k - i + 1)$ circle-vertices of $\partial\mathbf{q}_n^l$ are not in $\partial\mathbf{q}_{n+1}$, which means that $\max(A_n - k + i - 1, 0)$ vertices in $\partial\mathbf{q}_{n+1}$ belong to \mathbf{m}_n^0 . These vertices also belong to \mathbf{m}_{n+1}^0 , and in addition, the vertex y_0 belongs to \mathbf{m}_{n+1}^0 iff the new edge is open. To sum up, the transitions in this final situation are:

$$A_{n+1} = \begin{cases} p - k & \text{if } A_n > p - i, \\ \max(A_n - k + i - 1, 0) + 1 & \text{with probability } q \text{ if } A_n \leq p - i, \\ \max(A_n - k + i - 1, 0) & \text{with probability } (1 - q) \text{ if } A_n \leq p - i. \end{cases}$$

5.2 Derivation of p_c^{bond}

Suppose now $q < 1/2$, and consider the modified Markov chain (A'_n) with conditional transition probabilities given \mathcal{F}_n : if $X_n = 1$,

$$A'_{n+1} = \begin{cases} A'_n + 1 & \text{with probability } q \frac{C_{p+1}}{12C_p}, \\ A'_n & \text{with probability } (1 - q) \frac{C_{p+1}}{12C_p}. \end{cases} \quad (5.1)$$

If $X_n = -k$, we set

$$A'_{n+1} = \begin{cases} A'_n & \text{with probability } (2 - q) \frac{C_{p-k}Z_{k+1}}{12C_p} + \frac{C_{p-k}}{12C_p} \sum_{i=1}^k Z_i Z_{k+1-i}, \\ A'_n + 1 & \text{with probability } q \frac{C_{p-k}Z_{k+1}}{12C_p} \end{cases} \quad (5.2)$$

(corresponding to \mathbf{q}_n^l infinite),

$$A'_{n+1} = \begin{cases} A'_n - k & \text{with probability } (1 + q) \frac{C_{p-k}Z_{k+1}}{12C_p} + \frac{C_{p-k}}{12C_p} \sum_{i=1}^k Z_i Z_{k+1-i}, \\ A'_n - k - 1 & \text{with probability } (1 - q) \frac{C_{p-k}Z_{k+1}}{12C_p} \end{cases} \quad (5.3)$$

(corresponding to \mathbf{q}_n^r infinite), and

$$A'_{n+1} = \begin{cases} A'_n - k + i & \text{with probability } q \frac{C_{p-k}}{12C_p} Z_i Z_{k+1-i} \text{ for } 1 \leq i \leq k, \\ A'_n - k + i - 1 & \text{with probability } (1-q) \frac{C_{p-k}}{12C_p} Z_i Z_{k+1-i} \text{ for } 1 \leq i \leq k \end{cases} \quad (5.4)$$

(corresponding to \mathbf{q}_n^m infinite).

In a similar way as for site percolation, we can write

$$\begin{aligned} E[A'_{n+1} - A'_n | |\partial\mathbf{q}_n| = 2p] &= qP(X_n = 1 | |\partial\mathbf{q}_n| = 2p) - \sum_{k=0}^{p-1} k \left(\frac{1}{2} P(X_n = -k | |\partial\mathbf{q}_n| = 2p) - \frac{1}{2} \sum_{i=1}^k \frac{C_{p-k}}{12C_p} Z_i Z_{k+1-i} \right) \\ &\quad - (1-q) \sum_{k=0}^{p-1} \frac{C_{p-k} Z_{k+1}}{12C_p} + q \sum_{k=0}^{p-1} \frac{C_{p-k} Z_{k+1}}{12C_p} \\ &\quad + \sum_{k=1}^{p-1} \sum_{i=1}^k (-k+i) \frac{C_{p-k}}{12C_p} Z_i Z_{k+1-i} - (1-q) \sum_{k=1}^{p-1} \sum_{i=1}^k \frac{C_{p-k}}{12C_p} Z_i Z_{k+1-i} \\ &= \left(q - \frac{1}{2} \right) P(X_n = 1 | |\partial\mathbf{q}_n| = 2p) + \frac{1}{2} E[X_n | |\partial\mathbf{q}_n| = 2p] \\ &\quad + (2q-1) \sum_{k=0}^{p-1} \frac{C_{p-k} Z_{k+1}}{12C_p} + \sum_{k=1}^{p-1} \sum_{i=1}^k \left(-\frac{k}{2} + i \right) \frac{C_{p-k}}{12C_p} Z_i Z_{k+1-i} \\ &\quad + (q-1) \sum_{k=0}^{p-1} \sum_{i=1}^k \frac{C_{p-k}}{12C_p} Z_i Z_{k+1-i} \\ &= \left(q - \frac{1}{2} \right) P(X_n = 1 | |\partial\mathbf{q}_n| = 2p) + \frac{1}{2} E[X_n | |\partial\mathbf{q}_n| = 2p] \\ &\quad + (2q-1) \sum_{k=0}^{p-1} \frac{C_{p-k} Z_{k+1}}{12C_p} + \left(q - \frac{1}{2} \right) \sum_{k=0}^{p-1} \sum_{i=1}^k \frac{C_{p-k}}{12C_p} Z_i Z_{k+1-i}, \end{aligned}$$

which is negative and stays bounded away from 0 as $n \rightarrow \infty$. Using a domination of A_n by A'_n as we did for site percolation, we deduce that percolation does not occur a.s., and A_n “stays small”. Here we can then use directly a symmetry argument, and deduce that $p_c = 1/2$ a.s.

5.3 Bond percolation on quadrangulations

In this last part, we would like to mention that the previous reasoning can easily be adapted to study bond percolation on various classes of maps, in particular on p -angulations, as soon as one has counting formulas such as (2.1) at one's disposal.

For example, the previous peeling process can be used for bond percolation on the UIPQ: we now describe explicitly the exploration process in this case. We consider percolation with parameter q , and will follow the boundary of a cluster of open edges by exploring only the neighboring closed edges, and leaving “undetermined” the remaining ones. More precisely, conditionally on $|\partial\mathbf{q}_n| = 2p$, the boundary $\partial\mathbf{q}_n$ will consist in this case of a certain number A_n of vertices of the UIPQ connected to the root edge by open edges belonging to $\partial\mathbf{q}_n$ (that is, these vertices belong to \mathbf{q}_n^0 , the set of vertices connected to the root vertex of \mathbf{q}_n by open paths lying in \mathbf{q}_n , as in Section 5.1). When $A_n > 0$, the A_n vertices are connected by $(A_n - 1)$ open edges followed by 1 closed edge, and $U_n = 2p - A_n$ undetermined edges. Note that U_n also counts the number of “free” vertices that can get connected in a later step to the open cluster that we are following.

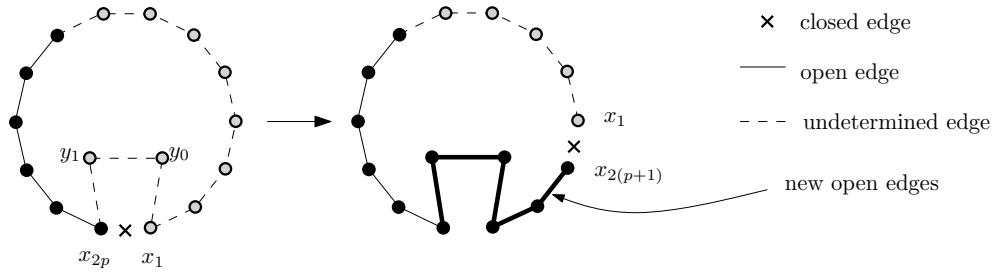


Figure 9: Exploration process for bond percolation on the UIPQ. At each step, we explore iteratively the available “undetermined” edges until we find a closed one. The remaining boundary edges are then left undetermined.

If we reach $A_n = 0$, we stop the process: in this case, we know that the root cluster has been fully explored and contains only finitely many edges, so that percolation does not occur.

At each step, we reveal a quadrangle (x_{2p}, x_1, y_0, y_1) as before, lying on the left hand-side of the unique closed edge $e = (x_{2p}, x_1)$, and we explore successively the undetermined edges following x_{2p} on ∂q_{n+1} , until we find a closed one (or no undetermined edge remains, in which case we can consider the edge explored last to be closed without any loss of generality). A certain number of new vertices get connected in this way, which follows a geometric distribution with parameter q , truncated by the number of free vertices N : let us introduce the notation $G_q(N)$ for such a distribution (i.e. $P(G_q(N) = k) = q^k(1 - q)$ for $0 \leq k < N$, and $= q^N$ for $k = N$).

- (1) When $y_0, y_1 \notin \partial q_n$, i.e. $X_n = 1$, we simply have $U_n + 2 = 2p - A_n + 2$ free vertices at our disposal. This yields

$$A_{n+1} = A_n \stackrel{(\perp\!\!\!\perp)}{+} G_q(2p - A_n + 2)$$

(see Figure 9 for an illustration).

Let us now assume that $X_n = -k$.

- (2) In the case when $y_0 \notin \partial q_n$ and $y_1 \in \partial q_n$, we obtain:

- if q_n^l is infinite,

$$A_{n+1} = \begin{cases} A_n \stackrel{(\perp\!\!\!\perp)}{+} G_q(2(p - k) - A_n) & (\text{if } A_n < 2(p - k)), \\ 2(p - k) & (\text{if } A_n \geq 2(p - k)), \end{cases}$$

- if q_n^r is infinite,

$$A_{n+1} = \begin{cases} [A_n - (2k + 1)] \stackrel{(\perp\!\!\!\perp)}{+} G_q(2p - A_n + 1) & (\text{if } A_n \geq 2k + 2), \\ 0 & \text{w.p.} > 0 \text{ (if } A_n < 2k + 2\text{),} \\ 1 \stackrel{(\perp\!\!\!\perp)}{+} G_q(2(p - k) - 1) & \text{w.p.} > 0 \text{ (if } A_n < 2k + 2\text{).} \end{cases}$$

(3) In the case when $y_0 \in \partial \mathbf{q}_n$ and $y_1 \notin \partial \mathbf{q}_n$, we obtain:

- if \mathbf{q}_n^l is infinite,

$$A_{n+1} = \begin{cases} A_n +^{(\pm)} G_q(2(p-k) - A_n) & \text{(if } A_n < 2(p-k) - 1\text{),} \\ 2(p-k) - 1 +^{(\pm)} G_q(1) & \text{(if } A_n \geq 2(p-k) - 1\text{),} \end{cases}$$

- if \mathbf{q}_n^r is infinite,

$$A_{n+1} = \begin{cases} [A_n - 2k] +^{(\pm)} G_q(2p - A_n) & \text{(if } A_n \geq 2k + 1\text{),} \\ 0 & \text{w.p. } > 0 \text{ (if } A_n < 2k + 1\text{),} \\ 1 +^{(\pm)} G_q(2(p-k) - 1) & \text{w.p. } > 0 \text{ (if } A_n < 2k + 1\text{).} \end{cases}$$

(4) In the case when $y_0, y_1 \in \partial \mathbf{q}_n$, we obtain transitions of the same type as in case (2) (when \mathbf{q}_n^l is infinite) or in case (3) (when \mathbf{q}_n^r is infinite) – note however that the corresponding (unevaluated) probabilities differ. Finally, if \mathbf{q}_n^m is infinite, let us write $y_0 = x_{2i}$ ($1 \leq i \leq k$). Then

$$A_{n+1} = \begin{cases} 2(p-k) & \text{(if } A_n > 2(p-i)\text{),} \\ [A_n - 2(k-i) - 1] +^{(\pm)} G_q(2(p-i) + 1 - A_n) & \text{(if } 2(k-i) + 2 \leq A_n \leq 2(p-i)\text{),} \\ 0 & \text{w.p. } > 0 \text{ (if } A_n < 2(k-i) + 2\text{),} \\ 1 +^{(\pm)} G_q(2(p-k) - 1) & \text{w.p. } > 0 \text{ (if } A_n < 2(k-i) + 2\text{).} \end{cases}$$

We can prove, in the same way as for site and bond percolation on the UIPM, that if percolation occurs, then we fall only finitely many times into one of the cases when one returns to either 0 or 1 before exploring undetermined edges. This is because if at a certain time n , we reach either 0 or 1 (plus an independent geometric random variable), then $A_{n+1} = 0$ with a probability at least c , for some universal constant $c > 0$ (and in this case, percolation does not occur).

We now prove that $q_c = 1/3$. Let us first assume $q < 1/3$, and dominate A_n by A'_n obtained by replacing all truncated geometric distributions by non-truncated ones (denoted by \mathcal{G}_q in the following), and allowing it to take negative values as before. We first have, when $X_n = 1$,

$$A'_{n+1} = A'_n +^{(\pm)} \mathcal{G}_q \quad \text{with probability } \frac{C_{p+1}}{12C_p}. \quad (5.5)$$

If $X_n = -k$, we set

$$A'_{n+1} = A'_n +^{(\pm)} \mathcal{G}_q \quad \text{with probability } 2 \frac{C_{p-k} Z_{k+1}}{12C_p} + \frac{C_{p-k}}{12C_p} \sum_{i=1}^k Z_i Z_{k+1-i} \quad (5.6)$$

(corresponding to \mathbf{q}_n^l infinite),

$$A'_{n+1} = \begin{cases} A'_n - 2k +^{(\pm)} \mathcal{G}_q & \text{with probability } \frac{C_{p-k} Z_{k+1}}{12C_p} + \frac{C_{p-k}}{12C_p} \sum_{i=1}^k Z_i Z_{k+1-i}, \\ A'_n - (2k+1) +^{(\pm)} \mathcal{G}_q & \text{with probability } \frac{C_{p-k} Z_{k+1}}{12C_p} \end{cases} \quad (5.7)$$

(corresponding to \mathbf{q}_n^r infinite), and

$$A'_{n+1} = A'_n - 2(k-i) - 1 \stackrel{(\perp)}{=} G_q \text{ with probability } \frac{C_{p-k}}{12C_p} Z_i Z_{k+1-i} \text{ for } 1 \leq i \leq k \quad (5.8)$$

(corresponding to \mathbf{q}_n^m infinite). We can then write

$$\begin{aligned} E[A'_{n+1} - A'_n | |\partial\mathbf{q}_n| = 2p] &= \frac{q}{1-q} \left(\frac{C_{p+1}}{12C_p} + \sum_{k=0}^{p-1} \left(2 \frac{C_{p-k} Z_{k+1}}{12C_p} + \sum_{i=1}^k \frac{C_{p-k}}{12C_p} Z_i Z_{k+1-i} \right) \right) \\ &\quad + \sum_{k=0}^{p-1} \left(-(2k+1) + \frac{q}{1-q} \right) \frac{C_{p-k} Z_{k+1}}{12C_p} \\ &\quad + \sum_{k=0}^{p-1} \left(-2k + \frac{q}{1-q} \right) \left(\frac{C_{p-k} Z_{k+1}}{12C_p} + \sum_{i=1}^k \frac{C_{p-k}}{12C_p} Z_i Z_{k+1-i} \right) \\ &\quad + \sum_{k=1}^{p-1} \sum_{i=1}^k \left(-2(k-i) - 1 + \frac{q}{1-q} \right) \frac{C_{p-k}}{12C_p} Z_i Z_{k+1-i} \\ &= \frac{q}{1-q} + \sum_{k=0}^{p-1} -k \left(4 \frac{C_{p-k} Z_{k+1}}{12C_p} + 3 \sum_{i=1}^k \frac{C_{p-k}}{12C_p} Z_i Z_{k+1-i} \right) - \sum_{k=0}^{p-1} \frac{C_{p-k} Z_{k+1}}{12C_p} \\ &\quad + \sum_{k=1}^{p-1} \sum_{i=1}^k (2i - (k+1)) \frac{C_{p-k}}{12C_p} Z_i Z_{k+1-i} \\ &= \frac{q}{1-q} + E[X_n | |\partial\mathbf{q}_n| = 2p] - \frac{C_{p+1}}{12C_p} - \sum_{k=0}^{p-1} \frac{C_{p-k} Z_{k+1}}{12C_p} \\ &\xrightarrow[p \rightarrow \infty]{} \frac{q}{1-q} - \frac{1}{2}, \end{aligned}$$

which is negative and bounded away from 0. This implies that percolation does not occur for $q < 1/3$.

To prove that percolation occurs for $q > 1/3$, we can, in the same way as in Section 4.2, compute the law of U_n using the fact that $2X_n = (A_{n+1} - A_n) + (U_{n+1} - U_n)$. Here we will need to carry on with the exploration process when A_n hits 0, that is $U_n = |\partial\mathbf{q}_n|$. In this case, we simply choose one boundary edge arbitrarily, and continue. This yields, conditionally on $|\partial\mathbf{q}_n| = 2p$:

(1) When $y_0, y_1 \notin \partial\mathbf{q}_n$:

$$U_{n+1} = U_n + 2 \stackrel{(\perp)}{=} G_q(U_n + 2).$$

(2) When $y_0 \notin \partial\mathbf{q}_n$ and $y_1 \in \partial\mathbf{q}_n$:

- if \mathbf{q}_n^l is infinite,

$$U_{n+1} = \begin{cases} U_n - 2k \stackrel{(\perp)}{=} G_q(U_n - 2k) & (\text{if } 2k < U_n), \\ 0 & (\text{if } 2k \geq U_n), \end{cases}$$

- if \mathbf{q}_n^r is infinite,

$$U_{n+1} = \begin{cases} [U_n + 1] \stackrel{(\perp)}{=} G_q(U_n + 1) & (\text{if } 2(p-k) \geq U_n + 2), \\ 2(p-k) & \text{w.p.} > 0 \text{ (if } 2(p-k) < U_n + 2\text{)}, \\ 2(p-k) - 1 \stackrel{(\perp)}{=} G_q(2(p-k) - 1) & \text{w.p.} > 0 \text{ (if } 2(p-k) < U_n + 2\text{)}. \end{cases}$$

(3) When $y_0 \in \partial \mathbf{q}_n$ and $y_1 \notin \partial \mathbf{q}_n$:

- if \mathbf{q}_n^l is infinite,

$$U_{n+1} = \begin{cases} U_n - 2k \stackrel{(\text{II})}{=} G_q(U_n - 2k) & (\text{if } 2k + 1 < U_n), \\ 1 \stackrel{(\text{II})}{=} G_q(1) & (\text{if } 2k + 1 \geq U_n), \end{cases}$$

- if \mathbf{q}_n^r is infinite,

$$U_{n+1} = \begin{cases} U_n \stackrel{(\text{II})}{=} G_q(U_n) & (\text{if } 2(p-k) - 1 \geq U_n), \\ 2(p-k) & \text{w.p.} > 0 \text{ (if } 2(p-k) - 1 < U_n\text{)}, \\ 2(p-k) - 1 \stackrel{(\text{II})}{=} G_q(2(p-k) - 1) & \text{w.p.} > 0 \text{ (if } 2(p-k) - 1 < U_n\text{)}. \end{cases}$$

(4) When $y_0, y_1 \in \partial \mathbf{q}_n$, we obtain transitions of the same type as in case (2) when \mathbf{q}_n^l is infinite, and as in case (3) when \mathbf{q}_n^r is infinite. Finally, if \mathbf{q}_n^m is infinite, we write $y_0 = x_{2i}$ ($1 \leq i \leq k$), and

$$U_{n+1} = \begin{cases} 0 & (\text{if } 2i > U_n), \\ [U_n - 2i + 1] \\ \stackrel{(\text{II})}{=} G_q(U_n - 2i + 1) & (\text{if } 2i \leq U_n \leq 2(p-k+i) - 2), \\ 2(p-k) & \text{w.p.} > 0 \text{ (if } U_n > 2(p-k+i) - 2\text{)}, \\ 2(p-k) - 1 \stackrel{(\text{II})}{=} G_q(2(p-k) - 1) & \text{w.p.} > 0 \text{ (if } U_n > 2(p-k+i) - 2\text{)}. \end{cases}$$

From here, computations are the same as with A_n , except that we have to take extra care of the truncated geometric random variables. We consider the process (U'_n) coupled with (U_n) , and with increments given conditionally on $|\partial \mathbf{q}_n| = 2p$ by

$$U'_{n+1} = \begin{cases} U'_n + 2 \stackrel{(\text{II})}{=} G_q(U_n + 2) & \text{w.p. } \frac{C_{p+1}}{12C_p}, \\ U'_n - 2k \stackrel{(\text{II})}{=} G_q(U_n - 2k) & \text{w.p. } 2 \frac{C_{p-k}Z_{k+1}}{12C_p} + \sum_{i=1}^k \frac{C_{p-k}}{12C_p} Z_i Z_{k+1-i}, \\ U'_n + 1 \stackrel{(\text{II})}{=} G_q(U_n + 1) & \text{w.p. } \frac{C_{p-k}Z_{k+1}}{12C_p}, \\ U'_n \stackrel{(\text{II})}{=} G_q(U_n) & \text{w.p. } \frac{C_{p-k}Z_{k+1}}{12C_p} + \sum_{i=1}^k \frac{C_{p-k}}{12C_p} Z_i Z_{k+1-i}, \\ U'_n - 2i + 1 \stackrel{(\text{II})}{=} G_q(U_n - 2i + 1) & \text{w.p. } \frac{C_{p-k}}{12C_p} Z_i Z_{k+1-i} \text{ for } 1 \leq i \leq k \end{cases}$$

(we adopt the convention that for $N \leq 0$, $G_q(N) \equiv 0$). Let us fix $q > 1/3$, and $\varepsilon > 0$. We can write

$$\begin{aligned} E[U'_{n+1} - U'_n | |\partial \mathbf{q}_n| = 2p] &= 2 \frac{C_{p+1}}{12C_p} - \sum_{k=0}^{p-1} k \left(4 \frac{C_{p-k}Z_{k+1}}{12C_p} + 3 \sum_{i=1}^k \frac{C_{p-k}}{12C_p} Z_i Z_{k+1-i} \right) \\ &\quad + \sum_{k=1}^{p-1} \sum_{i=1}^k (k+1-2i) \frac{C_{p-k}}{12C_p} Z_i Z_{k+1-i} + \sum_{k=0}^{p-1} \frac{C_{p-k}Z_{k+1}}{12C_p} \\ &\quad - E[G_q(V_n) | |\partial \mathbf{q}_n| = 2p] \\ &= \frac{C_{p+1}}{12C_p} + E[X_n | |\partial \mathbf{q}_n| = 2p] + \sum_{k=0}^{p-1} \frac{C_{p-k}Z_{k+1}}{12C_p} - E[G_q(V_n) | |\partial \mathbf{q}_n| = 2p], \end{aligned}$$

where the random variable V_n encodes the various truncations. We can then estimate the last term by choosing $C > 0$ large enough such that $E[G_q(C)] > \frac{q}{1-q} - \varepsilon$ and for all $p \geq 1$, $P(X_n < -C ||\partial \mathbf{q}_n| = 2p) < \varepsilon$: on the event $\{U_n > 3C\}$, we note that $V_n > C$ if $X_n \geq -C$, which implies

$$E[G_q(V_n) ||\partial \mathbf{q}_n| = 2p] \geq \left(\frac{q}{1-q} - \varepsilon \right) P(X_n \geq -C ||\partial \mathbf{q}_n| = 2p) \geq \left(\frac{q}{1-q} - \varepsilon \right) (1 - \varepsilon).$$

Hence,

$$\begin{aligned} E[U'_{n+1} - U'_n ||\partial \mathbf{q}_n| = 2p] \\ \leq \frac{C_{p+1}}{12C_p} + E[X_n ||\partial \mathbf{q}_n| = 2p] + \sum_{k=0}^{p-1} \frac{C_{p-k} Z_{k+1}}{12C_p} - \left(\frac{q}{1-q} - \varepsilon \right) (1 - \varepsilon) \\ \xrightarrow[p \rightarrow \infty]{} \frac{1}{2} - \left(\frac{q}{1-q} - \varepsilon \right) (1 - \varepsilon). \end{aligned}$$

This shows that for $q > 1/3$, there exists a constant $\alpha > 0$ (depending only on q) such that: for n large enough,

$$E[U'_{n+1} - U'_n ||\partial \mathbf{q}_n| = 2p] \leq -\alpha < 0.$$

We now have to be a bit careful in order to conclude. We first note that $U'_{n+1} - U'_n \geq U_{n+1} - U_n$, except in two cases: when $U_{n+1} = 0$, and when $U_{n+1} = 2(p-k)$ (i.e. $A_{n+1} = 0$), which is the case that we want to avoid. We need to show that this latter case occurs only finitely many times: let us argue by contradiction, and introduce $T_1 < T_2 < \dots$ the corresponding infinite sequence of stopping times. Using the fact that the increments $U'_n - U'_m$ stay small when positive (of order $O(\log n)$), and that at a given time T_k , the boundary has size U_{T_k} , we can show that $|\partial \mathbf{q}_n|$ stays of order $O(\log n)$, contradicting the fact that it grows like $n^{2/3}$. Hence, $A_n > 0$ for n large enough, so percolation occurs.

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Acknowledgments. The two authors would like to thank the anonymous referee for a very careful reading of this paper, and insightful comments. Part of this work was done while P.N. was affiliated with the Courant Institute (New York University), when it was supported in part by the NSF grants OISE-0730136 and DMS-1007626. L.M. would also like to thank the ForschungsInstitut für Mathematik at ETH for its hospitality.

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