Introduction to algebraic operads

Thibaut Mazuir

Humboldt Universität zu Berlin
Wintersemester 2022/2023

Contents

CHAPTER 1

Homological algebra

We fix a field $\mathbb{K}$ for the rest of these notes.

1. Categories

1.1. Categories and functors.

DEFINITION 1.1. *A* category $\mathtt{C}$ *corresponds to the following data:*

(1) A class of objects $\mathrm{Ob}(\mathtt{C})$.
(2) For each pair of objects $X, Y \in \mathrm{Ob}(\mathtt{C})$ a set $\mathtt{C}(X, Y)$, called the set of morphisms *from X to Y. It is also denoted* $\mathrm{Hom}_{\mathtt{C}}(X, Y)$.
(3) For each object X of $\mathtt{C}$, an element $\mathrm{id}_X \in \mathtt{C}(X, X)$ called the identity *of X.*
(4) For each triple of objects X, Y and Z of $\mathtt{C}$ of a map $\circ : \mathtt{C}(Y, Z) \times \mathtt{C}(X, Y) \to \mathtt{C}(X, Z)$ called the composition.

These data are such that for every $f \in \mathtt{C}(X, Y)$, $g \in \mathtt{C}(Y, Z)$ and $h \in \mathtt{C}(Z, T)$ we have that $(h \circ g) \circ f = h \circ (g \circ f)$ and for very $f \in \mathtt{C}(X, Y)$ we have that $\mathrm{id}_Y \circ f = f = f \circ \mathrm{id}_X$.

A category is said to be *small* if $\mathrm{Ob}(\mathtt{C})$ is a set. A morphism $f \in \mathtt{C}(X, Y)$ is said to be an *isomorphism* if there exists a morphism $g \in \mathtt{C}(Y, X)$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. It will be denoted as $f : X \xrightarrow{\sim} Y$.

EXAMPLE 1.2. (i) The category $\mathtt{Set}$ whose objects are sets and morphisms the maps between them.
(ii) The category $\mathtt{Vect}$ whose objects are $\mathbb{K}$-vector spaces and morphisms the linear maps between them.

Both of these categories are not small.

DEFINITION 1.3. *(i) Given a category $\mathtt{C}$, the* opposite category $\mathtt{C}^{op}$ *is the category whose objects are the objects of $\mathtt{C}$, whose morphisms sets are $\mathtt{C}^{op}(X, Y) := \mathtt{C}(Y, X)$ and whose composition is defined as $g \circ_{\mathtt{C}^{op}} f := f \circ_{\mathtt{C}} g$.*
(ii) Given $\mathtt{C}$ and $\mathtt{D}$ two categories, the product category $\mathtt{C} \times \mathtt{D}$ *is defined to be the category whose objects are pairs of objects $X \times Y := (X, Y)$ for $X \in \mathrm{Ob}(\mathtt{C})$ and $Y \in \mathrm{Ob}(\mathtt{D})$, whose sets of morphisms are defined to be*

$$\mathrm{Hom}_{\mathtt{C}\times\mathtt{D}}(X_1 \times Y_1, X_2 \times Y_2) := \mathrm{Hom}_{\mathtt{C}}(X_1, X_2) \times \mathrm{Hom}_{\mathtt{D}}(Y_1, Y_2)$$

and whose composition is defined as $(f_2 \times g_2) \circ (f_1 \times g_1) := (f_2 \circ_{\mathtt{C}} f_1) \times (g_2 \circ_{\mathtt{D}} g_1)$ with identities $\mathrm{id}_{X \times Y} := \mathrm{id}_X \times \mathrm{id}_Y$.

Definition 1.4. *Given* $\mathtt{C}$ *and* $\mathtt{D}$ *two categories, a* functor $\mathcal{F} : \mathtt{C} \to \mathtt{D}$ *associates to every object* $X \in \mathtt{C}$ *an object* $\mathcal{F}(X) \in \mathtt{D}$ *and to every morphism* $f \in \mathtt{C}(X,Y)$ *a morphism* $\mathcal{F}(f) \in \mathtt{D}(\mathcal{F}(X), \mathcal{F}(Y))$ *such that* $\mathcal{F}(\mathrm{id}_X) = \mathrm{id}_{\mathcal{F}(X)}$ *and such that* $\mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f)$.

The class of functors between two categories $\mathtt{C}$ and $\mathtt{D}$ will be denoted $\mathrm{Fun}(\mathtt{C},\mathtt{D})$. If the categories $\mathtt{C}$ and $\mathtt{D}$ are small then $\mathrm{Fun}(\mathtt{C},\mathtt{D})$ is a set.

Example 1.5.
(i) The *forgetful functor* $\mathtt{Vect} \to \mathtt{Set}$ mapping a vector space to its underlying set by forgetting its vector space structure.
(ii) For a category $\mathtt{C}$ and an object $C \in \mathtt{C}$, the functor $\mathtt{C}(C,\cdot) : \mathtt{C} \to \mathtt{Set}$ associating to each object $D \in \mathtt{C}$ the set $\mathtt{C}(C,D)$.
(iii) The tensor product functor $- \otimes - : \mathtt{Vect} \times \mathtt{Vect} \to \mathtt{Vect}$.

A functor from a product category $\mathtt{C}_1 \times \mathtt{C}_2 \to \mathtt{D}$ will be called a *bifunctor.*

Definition 1.6. *The category* $\mathtt{Cat}$ *is defined to be the category whose objects are small categories and morphisms the functors between them, where two functors* $\mathcal{F} : \mathtt{C}_1 \to \mathtt{C}_2$ *and* $\mathcal{G} : \mathtt{C}_2 \to \mathtt{C}_3$ *are composed as* $\mathcal{G} \circ \mathcal{F}(X) := \mathcal{G}(\mathcal{F}(X))$ *and* $\mathcal{G} \circ \mathcal{F}(f) := \mathcal{G}(\mathcal{F}(f))$.

Definition 1.7. *Given* $\mathtt{C}$ *and* $\mathtt{D}$ *two categories and* $\mathcal{F}, \mathcal{G} : \mathtt{C} \to \mathtt{D}$ *two functors, a* natural transformation $\eta : \mathcal{F} \Rightarrow \mathcal{G}$ *is defined to be the datum of a morphism* $\eta_X \in \mathtt{D}(\mathcal{F}(X), \mathcal{G}(X))$ *for every* $X \in \mathtt{C}$ *such that for every* $f \in \mathtt{C}(X,Y)$*, the following diagram commutes*

$$\begin{tikzcd} \mathcal{F}(X) \arrow[r, "\mathcal{F}(f)"] \arrow[d, "\eta_X"] & \mathcal{F}(Y) \arrow[d, "\eta_Y"] \\ \mathcal{G}(X) \arrow[r, "\mathcal{G}(f)"] & \mathcal{G}(Y) \end{tikzcd} .$$

We will denote $\mathrm{Nat}(\mathcal{F}, \mathcal{G})$ the class of natural transformations between two functors $\mathcal{F}, \mathcal{G} : \mathtt{C} \to \mathtt{D}$. It is a set if the categories $\mathtt{C}$ and $\mathtt{D}$ are small.

Example 1.8. The abelianization $G^{ab} := G/[G,G]$ of a group G defines a functor $-^{ab} : \mathtt{Gr} \to \mathtt{Gr}$ where $\mathtt{Gr}$ denotes the category whose objects are groups and morphisms group morphisms. The collection of maps $\pi_G : G \to G^{ab}$ then defines a natural transformation $\pi : \mathrm{id}_{\mathtt{Gr}} \Rightarrow -^{ab}$.

The *identity natural transformation* $\mathrm{id}_{\mathcal{F}} : \mathcal{F} \Rightarrow \mathcal{F}$ of a functor $\mathcal{F}$ is defined for every $X \in \mathtt{C}$ as $(\mathrm{id}_{\mathcal{F}})_X = \mathrm{id}_{\mathcal{F}(X)}$. Two natural transformations $\eta : \mathcal{F}_1 \Rightarrow \mathcal{F}_2$ and $\gamma : \mathcal{F}_2 \Rightarrow \mathcal{F}_3$ can moreover be composed as $\gamma \circ \eta : \mathcal{F}_1 \Rightarrow \mathcal{F}_3$ by setting $(\gamma \circ \eta)_X := \gamma_X \circ \eta_X$.

1.2. Equivalences and adjoints.

Definition 1.9. *A natural transformation* $\eta : \mathcal{F} \Rightarrow \mathcal{G}$ *is said to be a* natural equivalence *if each morphism* η_X *is an isomorphism. In that case, we say that* $\mathcal{F}$ *and* $\mathcal{G}$ *are* naturally equivalent *and denote* $\mathcal{F} \simeq \mathcal{G}$.

For a category $\mathtt{C}$, a functor $\mathcal{F} : \mathtt{C} \to \mathtt{Set}$ is said to be *representable* if there exists an object $C \in \mathtt{C}$ such that $\mathcal{F} \simeq \mathtt{C}(C,\cdot)$.

Exercise 1.10. Prove that the morphisms $\tau_X := \eta_X^{-1}$ define a natural equivalence $\tau : \mathscr{G} \Rightarrow \mathscr{F}$ such that $\tau \circ \eta = \mathrm{id}_{\mathscr{F}}$ and that $\eta \circ \tau = \mathrm{id}_{\mathscr{G}}$.

DEFINITION 1.11. *An* equivalence of categories *is a pair of functors* $\mathscr{F} : \mathtt{C} \rightleftarrows \mathtt{D} : \mathscr{G}$ *such that* $\mathscr{G} \circ \mathscr{F} \simeq \mathrm{id}_{\mathtt{C}}$ *and* $\mathscr{F} \circ \mathscr{G} \simeq \mathrm{id}_{\mathtt{D}}$.

Exercise 1.12. Prove that a functor $\mathscr{F}$ defines an equivalence of categories if and only if for every objects $X, Y \in \mathtt{C}$, the map $\mathscr{F} : \mathtt{C}(X,Y) \to \mathtt{D}(\mathscr{F}(X), \mathscr{F}(Y))$ is bijective (the functor $\mathscr{F}$ is *fully faithful*) and for every object $D \in \mathtt{D}$ there exists an object $C \in \mathtt{C}$ and an isomorphism $\mathscr{F}(C) \to D$ (the functor $\mathscr{F}$ is *essentially surjective*).

DEFINITION 1.13. *Two functors* $\mathscr{F} : \mathtt{C} \leftrightarrows \mathtt{D} : \mathscr{G}$ *are said to be* adjoints *if there exists natural bijections*

$$\phi_{X,Y} : \mathtt{D}(\mathscr{F}(X), Y) \xrightarrow{\sim} \mathtt{C}(X, \mathscr{G}(Y))$$

where $X \in \mathtt{C}$ *and* $Y \in \mathtt{D}$. *The functors* $\mathscr{F}$ *and* $\mathscr{G}$ *are respectively the* left and right adjoints *and are denoted* $\mathscr{F} \dashv \mathscr{G}$.

In other words, two functors $\mathscr{F} : \mathtt{C} \leftrightarrows \mathtt{D} : \mathscr{G}$ are adjoints if the two functors

$$\mathtt{D}(\mathscr{F}(\cdot), \cdot), \mathtt{C}(\cdot, \mathscr{G}(\cdot)) : \mathtt{C}^{op} \times \mathtt{D} \to \mathtt{Set}$$

are naturally equivalent.

EXAMPLE 1.14. Let W be a vector space. The functors $-\otimes W : \mathtt{Vect} \to \mathtt{Vect}$ and $\mathrm{Hom}_{\mathtt{Vect}}(W, \cdot) : \mathtt{Vect} \to \mathtt{Vect}$ are adjoints: for every vector spaces V_1 and V_2 we have that

$$\mathrm{Hom}_{\mathtt{Vect}}(V_1 \otimes W, V_2) = \mathrm{Hom}_{\mathtt{Vect}}(V_1, \mathrm{Hom}_{\mathtt{Vect}}(W, V_2)) \ .$$

1.3. Monoidal categories.

1.3.1. *Symmetric monoidal categories.*

DEFINITION 1.15. *A* monoidal category *is a category* $\mathtt{C}$ *endowed with the following data:*

(1) A bifunctor $\boxtimes : \mathtt{C} \times \mathtt{C} \to \mathtt{C}$ *and an object* I *of* $\mathtt{C}$.
(2) A natural equivalence $\alpha : \boxtimes \circ (\boxtimes \times \mathrm{id}_{\mathtt{C}}) \simeq \boxtimes \circ (\mathrm{id}_{\mathtt{C}} \times \boxtimes)$ *called* the associator*, such that for every* $A, B, C, D \in \mathtt{C}$ *the following diagram commutes:*

$$\begin{tikzcd}
((A \boxtimes B) \boxtimes C) \boxtimes D \arrow[r, "\alpha_{A,B,C} \boxtimes \mathrm{id}_D"] \arrow[d, "\alpha_{A \boxtimes B, C, D}"] & (A \boxtimes (B \boxtimes C)) \boxtimes D \arrow[r, "\alpha_{A, B\boxtimes C, D}"] & A \boxtimes ((B \boxtimes C) \boxtimes D) \arrow[d, "\mathrm{id}_A \boxtimes \alpha_{B,C,D}"] \\
(A \boxtimes B) \boxtimes (C \boxtimes D) \arrow[rr, "\alpha_{A,B,C\boxtimes D}"] & & A \boxtimes (B \boxtimes (C \boxtimes D))
\end{tikzcd} .$$

(3) A natural equivalence $\lambda : I \boxtimes \mathrm{id}_{\mathtt{C}} \simeq \mathrm{id}_{\mathtt{C}}$ *and a natural equivalence* $\rho : \mathrm{id}_{\mathtt{C}} \boxtimes I \simeq \mathrm{id}_{\mathtt{C}}$ *such that*

$$\rho_I = \lambda_I : I \boxtimes I \xrightarrow{\sim} I$$

and such that the following diagram commutes for every $A, B \in \mathtt{C}$

$$\begin{tikzcd}
(A \boxtimes I) \boxtimes B \arrow[r, "\alpha_{A,I,B}"] \arrow[dr, "\rho_A \boxtimes \mathrm{id}_B"'] & A \boxtimes (I \boxtimes B) \arrow[d, "\mathrm{id}_A \boxtimes \lambda_B"] \\
& A \boxtimes B
\end{tikzcd} .$$

A monoidal category is said to be *strict* if the natural transformations α, λ and ρ are identities.

MacLane's coherence theorem states that the commutativity of the diagram for the associator implies that given two bracketings of $A_1 \boxtimes \cdots \boxtimes A_n$, two sequences of morphisms made of iterations of the associator from one bracketing to the other have equal composition. In other words, two bracketings are naturally equivalent through a unique natural equivalence made of iterations of the associator.

Definition 1.16. *A monoidal category* $\mathtt{C}$ *is said to be* symmetric *if there exists a natural equivalence*

$$\sigma_{A,B} : A \boxtimes B \xrightarrow{\sim} B \boxtimes A$$

called the braiding, *such that* $\sigma_{B,A}\sigma_{A,B} = \mathrm{id}_{A\boxtimes B}$ *and such that the following diagram commutes*

$$\begin{tikzcd}
(A \boxtimes B) \boxtimes C \arrow[r, "\alpha_{A,B,C}"] \arrow[d, "\sigma_{A,B}\boxtimes \mathrm{id}_C"] & A \boxtimes (B \boxtimes C) \arrow[r, "\sigma_{A,B\boxtimes C}"] & (B \boxtimes C) \boxtimes A \arrow[d, "\alpha_{B,C,A}"] \\
(B \boxtimes A) \boxtimes C \arrow[r, "\alpha_{B,A,C}"] & B \boxtimes (A \boxtimes C) \arrow[r, "\mathrm{id}_B \boxtimes \sigma_{A,C}"] & B \boxtimes (C \boxtimes A)
\end{tikzcd} .$$

A counterpart of MacLane's coherence theorem taking the natural transformation σ into account also holds in the case of symmetric monoidal categories.

Example 1.17. The category `Set` endowed with the cartesian product and the category `Vect` endowed with the tensor product $-\otimes_{\mathbb{K}}-$ are symmetric monoidal categories.

1.3.2. *Closed symmetric monoidal categories.*

Definition 1.18. *A symmetric monoidal category* $\mathtt{C}$ *is said to be* closed *if for every* $Y \in \mathtt{C}$ *the functor* $-\boxtimes Y : \mathtt{C} \to \mathtt{C}$ *admits a right adjoint denoted* $\underline{\mathrm{Hom}}_{\mathtt{C}}(Y, \cdot) : C \to C$ *such that the bijections*

$$\mathtt{C}(X \otimes Y, Z) \simeq \mathtt{C}(X, \underline{\mathrm{Hom}}_{\mathtt{C}}(Y, Z))$$

are natural in X, Y *and* Z.

In a closed symmetric monoidal category, the set (X, Y) is called *the external hom* while the object $\underline{\mathrm{Hom}}_{\mathtt{C}}(X, Y)$ is called the *internal hom.*

Example 1.19. (i) The symmetric monoidal categories `Set` and `Vect` are closed with internal homs their external homs.

(ii) The symmetric monoidal category gr `Vect` and `Vect` are closed, as proven in Exercise sheet 3.

1.3.3. *Monoids.*

Definition 1.20. *A* monoid M *in a monoidal category* $\mathtt{C}$ *is an object* $M \in \mathtt{C}$ *together with:*

(1) A morphism $\mu : M \boxtimes M \to M$ *called the* multiplication, *which is* associative *i.e. makes the following diagram commute*

$$\begin{tikzcd}
(M \boxtimes M) \boxtimes M \arrow[r, "\alpha_{M,M,M}"] \arrow[d, "\mu\boxtimes \mathrm{id}_M"] & M \boxtimes (M \boxtimes M) \arrow[r, "\mathrm{id}_M \boxtimes \mu"] & M \boxtimes M \arrow[d, "\mu"] \\
M \boxtimes M \arrow[rr, "\mu"] & & M
\end{tikzcd} .$$

(2) A morphism $\eta : I \to M$ called the unit, *which makes the following diagram commute*

$$\begin{tikzcd} I \boxtimes M \arrow[r, "\eta\boxtimes\mathrm{id}_M"] \arrow[dr, "\lambda_M"'] & M \boxtimes M \arrow[d, "\mu"] & M \boxtimes I \arrow[l, "\mathrm{id}_M\boxtimes\eta"'] \arrow[dl, "\rho_M"] \\ & M & \end{tikzcd}.$$

If $\mathtt{C}$ *is symmetric, a monoid is moreover said to be* commutative *if* $\mu = \mu\sigma_{A,A}$.

Example 1.21. A monoid in `Set` is a standard monoid while a monoid in `Vect` is a standard unital associative algebra (see Section 1.1.1 of Chapter 2).

Exercise 1.22. Prove that the unit of a monoid is unique.

1.3.4. *Lax monoidal functors.*

Definition 1.23. *A* lax monoidal functor $\mathscr{F}$ *between two monoidal categories* $\mathtt{C}$ *and* $\mathtt{D}$ *is a functor* $\mathscr{F} : \mathtt{C} \to \mathtt{D}$ *together with*

(1) a natural transformation $\phi_{A,B} : \mathscr{F}(A) \boxtimes_{\mathtt{D}} \mathscr{F}(B) \to \mathscr{F}(A \boxtimes_{\mathtt{C}} B)$,
(2) and a morphism $\psi : I_{\mathtt{D}} \to \mathscr{F}(I_{\mathtt{C}})$

which are such that for every $A, B, C \in \mathtt{C}$ *the following three diagrams commute*

$$\begin{tikzcd} (\mathscr{F}(A) \boxtimes_{\mathtt{D}} \mathscr{F}(B)) \boxtimes_{\mathtt{D}} \mathscr{F}(C) \arrow[r, "\phi_{A,B}\boxtimes_{\mathtt{D}}\mathrm{id}_{\mathscr{F}(C)}"] \arrow[d, "\alpha^{\mathtt{D}}_{\mathscr{F}(A),\mathscr{F}(B),\mathscr{F}(C)}"] & \mathscr{F}(A \boxtimes_{\mathtt{C}} B) \boxtimes_{\mathtt{D}} \mathscr{F}(C) \arrow[r, "\phi_{A\boxtimes_{\mathtt{C}}B,C}"] & \mathscr{F}((A \boxtimes_{\mathtt{C}} B) \boxtimes_{\mathtt{C}} C) \arrow[d, "\mathscr{F}\left(\alpha^{\mathtt{C}}_{A,B,C}\right)"] \\ \mathscr{F}(A) \boxtimes_{\mathtt{D}} (\mathscr{F}(B) \boxtimes_{\mathtt{D}} \mathscr{F}(C)) \arrow[r, "\mathrm{id}_{\mathscr{F}(A)}\boxtimes_{\mathtt{D}}\phi_{B,C}"'] & \mathscr{F}(A) \boxtimes_{\mathtt{D}} \mathscr{F}(B \boxtimes_{\mathtt{C}} C) \arrow[r, "\phi_{A,B\boxtimes_{\mathtt{C}}C}"'] & \mathscr{F}(A \boxtimes_{\mathtt{C}} (B \boxtimes_{\mathtt{C}} C)) \end{tikzcd},$$

$$\begin{tikzcd} I_{\mathtt{D}} \boxtimes_{\mathtt{D}} \mathscr{F}(A) \arrow[r, "\psi\boxtimes_{\mathtt{D}}\mathrm{id}_{\mathscr{F}(A)}"] \arrow[d, "\lambda^{\mathtt{D}}_{\mathscr{F}(A)}"] & \mathscr{F}(I_{\mathtt{C}}) \boxtimes_{\mathtt{D}} \mathscr{F}(A) \arrow[d, "\phi_{I_{\mathtt{C}},A}"] \\ \mathscr{F}(A) & \mathscr{F}(I \boxtimes_{\mathtt{C}} A) \arrow[l, "\mathscr{F}(\lambda^{\mathtt{C}}_A)"] \end{tikzcd},$$

$$\begin{tikzcd} \mathscr{F}(A) \boxtimes_{\mathtt{D}} I_{\mathtt{D}} \arrow[r, "\mathrm{id}_{\mathscr{F}(A)}\boxtimes_{\mathtt{D}}\psi"] \arrow[d, "\rho^{\mathtt{D}}_{\mathscr{F}(A)}"] & \mathscr{F}(A) \boxtimes_{\mathtt{D}} \mathscr{F}(I_{\mathtt{C}}) \arrow[d, "\phi_{A,I_{\mathtt{C}}}"] \\ \mathscr{F}(A) & \mathscr{F}(A \boxtimes_{\mathtt{C}} I) \arrow[l, "\mathscr{F}(\rho^{\mathtt{C}}_A)"] \end{tikzcd}.$$

A lax monoidal functor is said to be *strong* if ϕ and ψ are isomorphisms and is *strict* if they are identities.

Example 1.24. The forgetful functor `Vect` $\to$ `Set` is lax monoidal but not strong monoidal.

Exercise 1.25. Prove that the image of a monoid under a lax monoidal functor is again a monoid.

Definition 1.26. *Let* $\mathtt{C}$ *and* $\mathtt{D}$ *be two symmetric monoidal categories. A functor* $\mathscr{F} : \mathtt{C} \to \mathtt{D}$ *is said to be* lax symmetric monoidal *if it is lax monoidal and the following diagram commutes for every* $A, B \in \mathtt{C}$

$$\begin{tikzcd} \mathscr{F}(A) \boxtimes_{\mathtt{D}} \mathscr{F}(B) \arrow[r, "s^{\mathtt{D}}_{\mathscr{F}(A),\mathscr{F}(B)}"] \arrow[d, "\phi_{A,B}"] & \mathscr{F}(B) \boxtimes_{\mathtt{D}} \mathscr{F}(A) \arrow[d, "\phi_{B,A}"] \\ \mathscr{F}(A \boxtimes_{\mathtt{C}} B) \arrow[r, "\mathscr{F}(s^{\mathtt{C}}_{A,B})"] & \mathscr{F}(B \boxtimes_{\mathtt{C}} A) \end{tikzcd} .$$

Example 1.27. (i) The singular chains functor $C_*(-) : \mathtt{Top} \to \mathtt{dg\ Vect}$ is lax symmetric monoidal.
(ii) The homology functor $H_*(-) : \mathtt{dg\ Vect} \to \mathtt{gr\ Vect}$ is strong symmetric monoidal.
(iii) The free vector space functor $\mathbb{K}[-] : \mathtt{Set} \to \mathtt{Vect}$ is strong symmetric monoidal.

2. Homological algebra

2.1. Chain and cochain complexes.

2.1.1. *(Co)chain complexes and (co)homology.*

Definition 2.1. *A* chain complex *corresponds to the data of a vector space* C_n *for every* $n \in \mathbb{Z}$ *together with linear maps*

$$\cdots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots ,$$

which are such that $\partial_n \circ \partial_{n+1} = 0$. *The collection of these maps is called a* differential.

Definition 2.2. *A* cochain complex *corresponds to the data of a vector space* C^n *for every* $n \in \mathbb{Z}$ *together with linear maps*

$$\cdots \to C^{n-1} \xrightarrow{\partial^{n-1}} C^n \xrightarrow{\partial^n} C^{n+1} \to \cdots ,$$

which are such that $\partial^n \circ \partial^{n-1} = 0$.

Chain complexes are usually denoted as C_* while cochain complexes are denoted as C^*. Dualizing a chain complex (C_n, ∂_n) moreover gives in particular a cochain complex $(C_n^\vee, \partial_n^\vee)$.

Definition 2.3. *(i) The* homology *of a chain complex* C_* *is defined to be the collection of vector spaces* $H_n(C) := \mathrm{Ker}(\partial_n)/\mathrm{Im}(\partial_{n+1})$.
(ii) The cohomology *of a cochain complex* C^* *is defined to be the collection of vector spaces* $H^n(C) := \mathrm{Ker}(\partial^n)/\mathrm{Im}(\partial^{n-1})$.

A (co)chain complex is said to be *acyclic* if its (co)homology is null.

2.1.2. *Chain maps and homotopies.*

Definition 2.4. *A* chain map *between two chain complexes* $f_* : C_* \to D_*$ *is defined to be a collection of maps* $f_n : C_n \to D_n$ *such that for all* $n \in \mathbb{Z}$, $f_n\partial_n^C = \partial_n^D f_n$. *A chain map between cochain complexes is defined similarly.*

A chain map $f_* : C_* \to D_*$ is usually represented as

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r] \arrow[d, "f_{n+1}"] & C_n \arrow[r] \arrow[d, "f_n"] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & D_{n+1} \arrow[r] & D_n \arrow[r] & D_{n-1} \arrow[r] & \cdots .
\end{tikzcd}$$

Definition 2.5. *The* composition *of two chain maps $f_* : A_* \to B_*$ and $g_* : B_* \to C_*$ is the chain map $(g \circ f)_* : A_* \to C_*$ defined as $(g \circ f)_n := g_n \circ f_n$.*

Proposition 2.6. *A chain map $f_* : C_* \to D_*$ induces well-defined maps $H_n(C) \to H_n(D)$.*

Definition 2.7. *A chain map f_* for which all maps induced in homology are isomorphisms is called a* quasi-isomorphism.

Definition 2.8. *Two chain maps $f_*, g_* : C_* \to D_*$ are said to be* homotopic *and denoted $f_* \simeq g_*$, if there exists a collection of linear maps $h_n : C_n \to D_{n+1}$ such that for all $n \in \mathbb{Z}$*

$$\partial^D_{n+1} h_n + h_{n-1} \partial^C_n = g_n - f_n \ .$$

The collection of maps h_n is then called a (chain) homotopy *between f_* and g_*.*

Proposition 2.9. *Two homotopic chain maps induce the same map in homology.*

Definition 2.10. *A chain map $f_* : C_* \to D_*$ is said to be a* homotopy equivalence *if there exists a chain map $g_* : D_* \to C_*$ such that $g_* \circ f_* \simeq \mathrm{id}_{C_*}$ and $f_* \circ g_* \simeq \mathrm{id}_{D_*}$.*

Following Proposition 2.9, a chain equivalence is in particular a quasi-isomorphism.

2.2. The differential graded viewpoint.

Definition 2.11. *(i) A* graded vector space *is a vector space V together with a direct sum decomposition $V = \oplus_{n \in \mathbb{Z}} V_n$.*

(ii) A linear map of degree r *between two graded vector spaces V and W is a linear map $f : V \to W$ such that $f(V_n) \subset W_{n+r}$ for all $n \in \mathbb{Z}$.*

An element x of a dg module C_* is said to have *degree n* if $x \in C_n$. Its degree will then be written $|x| := n$.

Lemma 2.12. *(i) The datum of a chain complex is equivalent to the data of a graded vector space C together with a map $\partial : C \to C$ of degree -1 such that $\partial \circ \partial = 0$.*

(ii) The datum of cochain complex is equivalent to the data of a graded vector space C together with a map $\partial : C \to C$ of degree $+1$ such that $\partial \circ \partial = 0$.

A (co)chain complex seen from the viewpoint of Lemma 2.12 will be refered to as a *differential graded vector space*, or *dg vector space.* It will be said to be *homologically graded* if the differential has degree -1 and *cohomologically graded* if the differential has degree $+1$.

Lemma 2.13. *The datum of a chain map $f_* : C_* \to D_*$ is equivalent to the datum of a linear map $f : C \to D$ of degree 0 such that $\partial_D f = f \partial_C$.*

Definition 2.14. *Given C_* and D_* two dg vector spaces, their* tensor product *is defined to be the dg vector space whose degree n part is $(C \otimes D)_n := \oplus_{p+q=n} C_p \otimes D_q$ and whose differential is defined as*

$$\partial_{C\otimes D}(c \otimes d) = \partial_C c \otimes d + (-1)^{|c|} c \otimes \partial_D d \ .$$

Exercise 2.15. Prove that the formula of Definition 2.14 defines indeed a differential.

Definition 2.16. *The* suspension sC *of a dg vector space C is the dg vector space whose degree n part is $(sC)_n := C_{n-1}$ and whose differential is $\partial_{sC} := -\partial_C$.*

2.3. Exact sequences.

Definition 2.17. *A sequence of linear maps $f_n : A_n \to A_{n-1}$ is said to be* exact *if for every n, $f_n \circ f_{n-1} = 0$ and $\mathrm{Im}(f_{n-1}) = \mathrm{Ker}(f_n)$.*

In particular, a (co)chain complex is exact if and only if its (co)homology is null.

Definition 2.18. *A* short sequence *of chain maps between chain complexes*

$$0 \to A_* \xrightarrow{f} B_* \xrightarrow{g} C_* \to 0$$

is said to be exact *if for every $n \in \mathbb{Z}$ the sequence of linear maps*

$$0 \to A_n \xrightarrow{f_n} B_n \xrightarrow{g_n} C_n \to 0$$

is exact, i.e. if f_n is injective, g_n is surjective and $\mathrm{Ker}(g_n) = \mathrm{Im}(f_n)$.

Lemma 2.19. *A commutative diagram of the form*

$$\begin{tikzcd} & A_1 \arrow[r, "s_A"] \arrow[d, "f_1"] & A_2 \arrow[r, "t_A"] \arrow[d, "f_2"] & A_3 \arrow[r] \arrow[d, "f_3"] & 0 \\ 0 \arrow[r] & B_1 \arrow[r, "s_B"] & B_2 \arrow[r, "t_B"] & B_3 & \end{tikzcd}$$

whose rows are exact induces a short exact sequence

$$\mathrm{Ker}(f_1) \to \mathrm{Ker}(f_2) \to \mathrm{Ker}(f_3) \to B_1/\mathrm{Im}(f_1) \to B_2/\mathrm{Im}(f_2) \to B_3/\mathrm{Im}(f_3) \ .$$

This result is called the *snake lemma.* The map $\mathrm{Ker}(f_3) \to B_1/\mathrm{Im}(f_1)$ is moreover called the *connecting morphism.*

Theorem 1. *A* short exact sequence *of chain complexes*

$$0 \to A_* \xrightarrow{f} B_* \xrightarrow{g} C_* \to 0$$

induces a long exact sequence *of linear maps*

$$\cdots \to H_n(A) \to H_n(B) \to H_n(C) \to H_{n-1}(A) \to H_{n-1}(B) \to H_{n-1}(C) \to \cdots \ .$$

This long exact sequence is moreover natural in the short exact sequence.

2.4. Koszul conventions. Consider two maps $f : A_* \to B_*$ and $g : C_* \to D_*$ of respective degree $|f|$ and $|g|$. In the rest of this course, we will denote $f \otimes g : A_* \otimes C_* \to B_* \otimes D_*$ the map of degree $|f| + |g|$ defined as

$$(f \otimes g)(a \otimes b) = (-1)^{|g||a|} f(a) \otimes g(b) \ .$$

This convention is called the ***Koszul sign convention***. The motto is that "moving a in front of g adds the sign $(-1)^{|g||a|}$".

Example 2.20. Under the Koszul sign convention, we have in particular that

$$(f_2 \otimes g_2) \circ (f_1 \otimes g_1) = (-1)^{|f_1||g_2|} (f_2 \otimes f_1) \circ (g_2 \otimes g_1)$$

and that $\partial_{C \otimes D} = \partial_C \otimes \mathrm{id}_D + \mathrm{id}_C \otimes \partial_D$.

CHAPTER 2

Standard algebraic structures

We work with homologically graded dg vector spaces in this chapter. We will denote `dg Vect` the category of dg vector spaces with chain maps between them, and `gr Vect` the category of graded vector spaces with linear maps of degree 0 between them.

1. Associative algebras and coalgebras

1.1. Associative algebras.

1.1.1. *Definitions.*

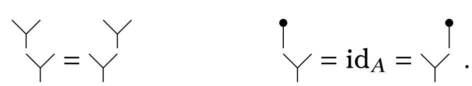
Definition 1.1. *(i) Let A be a vector space. An* associative algebra *structure on A corresponds to the datum of a map $\mu : A \otimes A \to A$ such that $\mu(\mu \otimes \mathrm{id}) = \mu(\mathrm{id} \otimes \mu)$.*
(ii) A morphism of algebras *is defined to be a linear map $f : A \to B$ such that $f\mu_A = \mu_B(f \otimes f)$.*

This definition is equivalent to the standard axiomatic definition of an algebra, whose multiplication is defined to be $a \cdot b := \mu(a, b)$. The field $\mathbb{K}$ is moreover naturally a $\mathbb{K}$-algebra.

Definition 1.2. *(i) An algebra A is said to be* unital *if there exists a morphism of algebras $u : \mathbb{K} \to A$ such that $\mu(u, \cdot) = \mathrm{id} = \mu(\cdot, u)$.*
(ii) A morphism of unital algebras *is a morphism $f : A \to B$ between unital algebras such that $fu_A = u_B$.*

Setting $u(1_{\mathbb{K}}) = 1_A$, we recover the usual axiom for a unital algebra. A morphism of unital algebras is then simply a morphism of algebras $f : A \to B$ such that $f(1_A) = 1_B$.

Representing the multiplication μ and the unit map u respectively as $\curlyvee$ and $\bullet$, the axioms for a unital algebra read as



This viewpoint will be used systematically in the rest of this course.

We denote `As-alg` the category of associative algebras with morphisms of algebras between them, and `uAs-alg` the category of unital associative algebras with morphisms of unital algebras between them.

Definition 1.3. *A unital algebra A is said to be* augmented *if there exists a morphism of unital algebras $\varepsilon : A \to \mathbb{K}$. This morphism is then called an* augmentation *of A.*

1.1.2. *Free associative algebra.* Let V be a vector space. Given $v_1, \ldots, v_n \in V$, we will denote $v_1 \cdots v_n := v_1 \otimes \cdots \otimes v_n \in V^{\otimes n}$ in the rest of this section.

Definition 1.4. *The* free tensor (unital) algebra *on V is defined to be the vector space*

$$T(V) := \mathbb{K} \oplus V \oplus V^{\otimes 2} \oplus \cdots \oplus V^{\otimes n} \oplus \cdots$$

endowed with the concatenation multiplication

$$v_1 \cdots v_n \otimes v_{n+1} \cdots v_{n+m} \longmapsto v_1 \cdots v_n v_{n+1} \cdots v_{n+m}$$

and unit the inclusion in the first summand $\mathbb{K} \hookrightarrow T(V)$.

The *free reduced tensor algebra* on V is the vector space

$$\overline{T}(V) := V \oplus V^{\otimes 2} \oplus \cdots \oplus V^{\otimes n} \oplus \cdots$$

endowed with the concatenation multiplication. It is not unital.

Proposition 1.5. *The free tensor algebra construction defines a functor $T(-)$: `Vect` $\to$ `uAs-alg` which is a left adjoint to the forgetful functor `uAs-alg` $\to$ `Vect`. In other words, for every vector space V and unital algebra A, there is a natural bijection*

$$\mathrm{Hom}_{\texttt{uAs-alg}}(T(V), A) = \mathrm{Hom}_{\texttt{Vect}}(V, A) \ .$$

Similarly, we have that $\mathrm{Hom}_{\texttt{As-alg}}(\overline{T}(V), A) = \mathrm{Hom}_{\texttt{Vect}}(V, A)$.

1.2. Coassociative coalgebras.

1.2.1. *Definitions.*

Definition 1.6. *(i) Let C be a vector space. A* coassociative coalgebra *structure on A corresponds to the datum of a map $\Delta : C \to C \otimes C$ such that $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$.*
(ii) A morphism of coalgebras is defined to be a linear map $f : C_1 \to C_2$ such that $(f \otimes f)\Delta_{C_1} = \Delta_{C_2} f$.

We will denote `As-cog` the category of coassociative coalgebras with morphisms of coalgebras. The field $\mathbb{K}$ is in particular a coassociative coalgebra.

Proposition 1.7. *(i) The dual of a coassociative coalgebra is an associative algebra.*
(ii) The dual of a finite-dimensional associative algebra is a coassociative coalgebra.

The image of an element $c \in C$ under the comultiplication Δ has the form

$$\Delta(c) := \sum_{i=1}^{n} c_i^{(1)} \otimes c_i^{(2)} \ .$$

This is often written for short as $\Delta(c) = c^{(1)} \otimes c^{(2)}$. Beware that it is a mere notation which does not mean that $\Delta(c)$ is a pure element of $C \otimes C$. It is called *Sweedler's notation.* Under this notation, a morphism of coalgebras is then simply a linear map $f : C_1 \to C_2$ such that $f(c)^{(1)} \otimes f(c)^{(2)} = f(c^{(1)}) \otimes f(c^{(2)})$.

If we write

$$(\Delta \otimes \mathrm{id})\Delta(c) = \Delta(c^{(1)}) \otimes c^{(2)} = c^{(1)(1)} \otimes c^{(2)(1)} \otimes c^{(2)}$$

and

$$(\mathrm{id} \otimes \Delta)\Delta(c) = c^{(1)} \otimes \Delta(c^{(2)}) = c^{(1)} \otimes c^{(1)(2)} \otimes c^{(2)(2)} \ ,$$

the coassociativity relation can be rephrased as

$$c^{(1)(1)} \otimes c^{(2)(1)} \otimes c^{(2)} = c^{(1)} \otimes c^{(1)(2)} \otimes c^{(2)(2)} \ .$$

This is often written as $(\mathrm{id} \otimes \Delta)\Delta(c) = (\Delta \otimes \mathrm{id})\Delta(c) = c^{(1)} \otimes c^{(2)} \otimes c^{(3)}$.

More generally, we denote $\Delta^n : C \to C^{\otimes n+1}$ the *iterated coproduct*

$$\Delta^n := (\Delta \otimes \mathrm{id}^{\otimes n-1}) \circ \cdots \circ \Delta \ ,$$

and write $\Delta^n(c) = c^{(1)} \otimes \cdots \otimes c^{(n+1)}$.

Definition 1.8. *(i) A coassociative coalgebra C endowed with a map $\varepsilon : C \to \mathbb{K}$ such that $(\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id}_C = (\mathrm{id} \otimes \varepsilon) \circ \Delta$ is said to be* counital. *The map ε is then called the* counit *of C.*

(ii) A morphism of coalgebras $f : C_1 \to C_2$ between counital coalgebras C_1 and C_2 is a morphism of counital coalgebras *if it preserves the counits, i.e. $\varepsilon_{C_1} = \varepsilon_{C_2} f$.*

(iii) A counital coalgebra C endowed with a morphism of counital coalgebras $u : \mathbb{K} \to C$ is said to be coaugmented. *The map u is then called its* coaugmentation.

(iv) A morphism of counital coalgebras $f : C_1 \to C_2$ between coaugmented coalgebras C_1 and C_2 is a morphism of coaugmented coalgebras *if it preserves the coaugmentations, i.e. $u_{C_1} = f u_{C_2}$.*

Writing the coproduct and counit respectively as  and , the axioms of a counital coalgebra can be represented as

Given a coaugmented coalgebra C, write $\overline{C} := \mathrm{Ker}(\varepsilon)$ and $1_C := u(1_{\mathbb{K}})$. Then $\Delta(1_C) = 1_C \otimes 1_C$ and the vector space C decomposes as $C := \overline{C} \oplus \mathbb{K}1_C$. The vector space $\overline{C}$ endowed with the coproduct $\overline{\Delta} : \overline{C} \to \overline{C} \otimes \overline{C}$,

$$\overline{\Delta}(x) := \Delta(x) - x \otimes 1_C - 1_C \otimes x \ ,$$

is then a coassociative coalgebra.

We denote $\overline{\Delta}^n : \overline{C} \to \overline{C}^{\otimes n+1}$ the iterated coproduct of $\overline{C}$.

Definition 1.9. *The coradical filtration of C is defined for $r \geqslant 0$ as*

$$F_r C := \mathbb{K}1_C \oplus \{x \in \overline{C},\ \overline{\Delta}^r(x) = 0\} \ ,$$

where $F_0 C := \mathbb{K}1_C$.

We point out that it is indeed a *filtration*, as $F_r C \subset F_{r+1} C$ for all $r \geqslant 0$.

Definition 1.10. *A coaugmented coalgebra C is said to be* conilpotent, *if $C = \cup_{r \geqslant 0} F_r C$.*

We denote `conil-As-cog` the category of conilpotent coalgebras with morphisms of coaugmented coalgebras between them. The coalgebra $\mathbb{K}$ is in particular a conilpotent coalgebra, with counit and coaugmentation the identity map.

1.2.2. *Cofree coassociative coalgebra.*

Definition 1.11. *The* cofree tensor (conilpotent) coalgebra *on a vector space V is defined to be the vector space*

$$T^c(V) := \mathbb{K} \oplus V \oplus V^{\otimes 2} \oplus \cdots \oplus V^{\otimes n} \oplus \cdots$$

endowed with the deconcatenation comultiplication

$$\Delta : v_1 \cdots v_n \longmapsto \sum_{i=0}^{n} v_1 \cdots v_i \otimes v_{i+1} \cdots v_n$$

where $\Delta(1) = 1 \otimes 1$, with counit the natural projection $T^c(V) \twoheadrightarrow \mathbb{K}$ and with coaugmentation the natural inclusion $\mathbb{K} \hookrightarrow T^c(V)$.

The cofree tensor coalgebra $T^c(V)$ is indeed conilpotent: its coradical filtration is in particular given by $F_r T^c(V) = \bigoplus_{n \leqslant r} V^{\otimes n}$.

Proposition 1.12. *The free tensor coalgebra construction defines a functor $T^c(-) : \mathtt{Vect} \to \mathtt{conil\text{-}As\text{-}cog}$ which is a right adjoint to the functor $\overline{(-)} : \mathtt{conil\text{-}As\text{-}cog} \to \mathtt{Vect}$ mapping C to $\overline{C}$. In other words, for every vector space V and conilpotent coalgebra C, there is a natural bijection*

$$\mathrm{Hom}_{\mathtt{Vect}}(\overline{C}, V) = \mathrm{Hom}_{\mathtt{conil\text{-}As\text{-}cog}}(C, T^c(V)) \ .$$

Proof. Let $f : C \to T^c(V)$ be a morphism of conilpotent coalgebras. We denote for every $n \geqslant 0$, $f_n := \pi_{V^{\otimes n}} f$. The fact that f preserves the counit and coaugmentation implies the following:

(1) $f_0(c) = \varepsilon_C(c)$ for every $c \in C$,
(2) $f(1_C) = 1_{\mathbb{K}}$,
(3) f maps $\overline{C}$ to $\overline{T}^c(V) = \bigoplus_{n \geqslant 1} V^{\otimes n}$.

We will denote $\overline{f} : \overline{C} \to \overline{T}^c(V)$ this induced morphism and $\overline{f}_n := \pi_{V^{\otimes n}} \overline{f}$ for $n \geqslant 1$. Every $c \in C$ moreover decomposes as $c = \varepsilon(c)1_C + \overline{c}$ where $\overline{c} := c - \varepsilon(c)1_c \in \overline{C}$. For every $c \in C$ we then have that

$$f(c) = \varepsilon(c) + \overline{f}(\overline{c}) \ .$$

The morphism $\overline{f}$ is in fact a morphism of coassociative coalgebras $(\overline{C}, \overline{\Delta}_C) \to (\overline{T}^c V, \overline{\Delta}_{T^c V})$. This implies that for all $n \geqslant 1$, the following diagram commutes

$$\begin{tikzcd}
\overline{C} \arrow[r, "\overline{f}"] \arrow[d, "\overline{\Delta}_C^{n-1}"'] & \overline{T}^c(V) \arrow[d, "\overline{\Delta}_{T^c V}^{n-1}"] \\
\overline{C}^{\otimes n} \arrow[r, "\overline{f}^{\otimes n}"] & \overline{T}^c(V)^{\otimes n}
\end{tikzcd} .$$

Projecting to the factor $V^{\otimes n}$ of $\overline{T}^c(V)^{\otimes n}$ yields for every $\overline{c} \in \overline{C}$ the equality

$$\overline{f}_n(\overline{c}) = \overline{f}_1(\overline{c}^{(1)}) \otimes \cdots \otimes \overline{f}_1(\overline{c}^{(n)}) \ ,$$

where the Sweedler's notation are w.r.t. $\overline{\Delta}_C$. Hence the morphism $\overline{f}_1 : \overline{C} \longrightarrow V$ completely determines $f : C \longrightarrow T^c(V)$.

Given conversely a morphism $f : \overline{C} \longrightarrow V$, one can define a morphism of conilpotent coalgebras $F : C \longrightarrow T^c(V)$ by the formula

$$F(c) := \varepsilon(c) + \sum_{n=1}^{+\infty} f(\overline{c}^{(1)}) \otimes \cdots \otimes f(\overline{c}^{(n)}) \ .$$

This morphism is well-defined: C is conilpotent hence the above sum is always finite. □

1.3. Differential graded (co)associative (co)algebras.

1.3.1. *Definitions.*

Definition 1.13. (i) *Let A be a graded vector space. A* graded associative algebra *structure on A corresponds to the datum of a map $\mu : A \otimes A \longrightarrow A$ such that (A, μ) is an associative algebra and the map μ has degree 0, i.e. for all $p, q \in \mathbb{Z}$, $\mu(A_p \otimes A_q) \subset A_{p+q}$.*

(ii) *Let (A, ∂) be a dg vector space. A* differential graded associative algebra *or* dg algebra *structure on A corresponds to the datum of a map $\mu : A \otimes A \longrightarrow A$ such that (A, μ) is a graded associative algebra and μ is a chain map, i.e. satisfies*

$$\partial\mu = \mu(\partial \otimes \mathrm{id}) + \mu(\mathrm{id} \otimes \partial) \ .$$

The equality of Item (ii) reads on two elements $a_1, a_2 \in A$ as

$$\partial\mu(a_1, a_2) = \mu(\partial a_1, a_2) + (-1)^{|a_1|}\mu(a_1, \partial a_2) \ .$$

The notions of *graded coassociative coalgebra* and of *dg coalgebra* can be defined in a similar fashion. We moreover point out that a standard (co)associative (co)algebra can then simply be seen as a graded associative coalgebra concentrated in degree 0.

Example 1.14. Let X be a topological space. The singular cochains $C^*(X)$ form a dg algebra for the *cup product* $\cup$ and the singular chains $C_*(X)$ form a dg coalgebra for the *Alexander-Whitney coproduct.*

Definition 1.15. (i) *Given A_1 and A_2 two graded associative algebras, a* morphism of graded associative algebras $A_1 \longrightarrow A_2$ *is defined to be a morphism of algebras of degree 0.*

(ii) *Given A_1 and A_2 two dg algebras, a* morphism of dg algebras $A_1 \longrightarrow A_2$ *is defined to be a morphism of algebras which is a chain map.*

The notion of *morphism of graded coassociative coalgebras* and of *morphism of dg coalgebras* is defined in a similar fashion. We will respectively denote `As-alg` and `As-cog` the category of dg algebras and the category of dg coalgebras. These notations are in conflict with the notation of the category of associative algebras and the category of coassociative coalgebras, but it will always be clear from the context which categories they refer to.

DEFINITION 1.16. *(i) A graded associative algebra* (A, μ) *is* unital *if there exists an element* $e \in A_0$ *which makes* (A, μ) *into an associative algebra.*
(ii) A dg algebra is unital if it is unital *as a graded associative algebra and its unit satisfies* $\partial(e) = 0$.

1.3.2. *Free graded associative algebra.* Let V be a graded vector space. The grading on V induces a grading on the free tensor algebra $T(V)$ defined as $|v_1 \cdots v_n| = \sum_{i=1}^n |v_i|$. This algebra is then a graded associative algebra with respect to this grading.

Proposition 1.17. *The free tensor algebra construction defines a functor* $T(-) : \mathtt{gr\ Vect} \to \mathtt{gr\ uAs\text{-}alg}$ *which is a left adjoint to the forgetful functor* $\mathtt{gr\ uAs\text{-}alg} \to \mathtt{gr\ Vect}$. *In other words, for every graded vector space* V *and unital graded associative algebra* A*, there is a natural bijection*

$$\mathrm{Hom}_{\mathtt{gr\ uAs\text{-}alg}}(T(V), A) = \mathrm{Hom}_{\mathtt{gr\ Vect}}(V, A)\ .$$

An analogous result holds for the free tensor coalgebra $T^c(V)$ seen as a conilpotent graded coassociative coalgebra with respect to the same grading.

We also point out that an alternative grading can be defined on $T(V)$, by setting the degree of an element of $V^{\otimes n}$ to be n. This grading is usually called the *weight* and the elements of weight greater that n will be denoted $T(V)^{\geqslant n}$. The algebra $T(V)$ is then a *weight-graded associative algebra*, meaning that the product preserves both the grading defined in the previous paragraph and the weight. The same holds for the coalgebra $T^c(V)$.

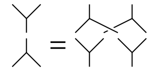
1.4. **Bialgebras and Hopf algebras.** Given two algebras A_1 and A_2, the tensor product $A_1 \otimes A_2$ can naturally be endowed with an algebra structure by setting $(x_1 \otimes x_2) \cdot (y_1 \otimes y_2) = x_1 \cdot y_1 \otimes x_2 \cdot y_2$. Using the switching map $\tau : A_1 \otimes A_2 \to A_2 \otimes A_1$ of Section 2.1, this can be rewritten as $\mu_{A_1 \otimes A_2} := (\mu_{A_1} \otimes \mu_{A_2})(\mathrm{id} \otimes \tau \otimes \mathrm{id})$. This last equality can be used to define an algebra structure on $A_1 \otimes A_2$ in the dg setting. We also point out that if A_1 and A_2 are unital, then $A_1 \otimes A_2$ is unital with unit $1_{A_1} \otimes 1_{A_2}$.

DEFINITION 1.18. *Given a vector space* H*, a* bialgebra *structure on* H *is defined to be the data of a unital algebra structure* (H, μ, u) *on* H *and of a counital coalgebra structure* (H, Δ, ε) *on* H *such that* Δ *and* ε *are morphisms of unital algebras, or equivalently such that* μ *and* u *are morphisms of counital coalgebras.*

The fact that Δ and ε are morphisms of unital algebras can be rephrased using the four following axioms

(1) $\Delta\mu = (\mu \otimes \mu)(\mathrm{id} \otimes \tau \otimes \mathrm{id})(\Delta \otimes \Delta)$, the *Hopf relation.*
(2) $\Delta u = u \otimes u$.
(3) $\varepsilon\mu = \varepsilon \otimes \varepsilon$.
(4) $\varepsilon u = \mathrm{id}_{\mathbb{K}}$.

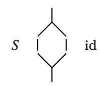
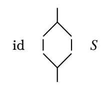
The Hopf relation can in particular be represented as


.

Definition 1.19. *A* Hopf algebra *is defined to be a bialgebra* $(H, \mu, u, \Delta, \varepsilon)$ *for which there exists a map* $S : H \to H$*, called the* antipode, *satisfying the relation*

$$\mu(S \otimes \mathrm{id})\Delta = u\varepsilon = \mu(\mathrm{id} \otimes S)\Delta \ .$$

The antipode relation can be represented as

 $= u\varepsilon =$  .

2. Commutative algebras

2.1. Symmetric groups actions on tensor products. Given a graded vector space V, the symmetric group $\mathfrak{S}_n$ acts on the left of $V^{\otimes n}$ as

$$\tau_\sigma(v_1 \otimes \cdots \otimes v_n) = (-1)^\varepsilon v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)} \ ,$$

where $\sigma \in \mathfrak{S}_n$ and $(-1)^\varepsilon$ is the sign obtained by rearranging $v_1 \otimes \cdots \otimes v_n$ into $v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}$ under the Koszul sign convention. The map $\tau := \tau_{(12)} : V \otimes V \to V \otimes V$ acts for instance on elements $v, w \in V$ as

$$\tau(v \otimes w) = (-1)^{|v|\cdot|w|} w \otimes v \ .$$

The symmetric group $\mathfrak{S}_n$ acts also on the right of $V^{\otimes n}$ as

$$(v_1 \otimes \cdots \otimes v_n) \cdot \sigma = (-1)^\varepsilon v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)} \ .$$

2.2. Commutative algebras.

2.2.1. *Definitions.*

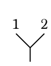
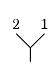
Definition 2.1. *(i) A (graded) associative algebra is said to be* (graded) commutative *if its multiplication* μ *satisfies* $\mu = \mu\tau$.

(ii) A dg algebra is said to be graded commutative *if it is graded commutative as a graded associative algebra. It is then called a* cdg algebra *for short.*

In the non-graded case with recover the usual equality $x \cdot y = y \cdot y$. In the graded case, the axioms of a graded commutative algebra read as

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z \qquad x \cdot y = (-1)^{|x||y|} y \cdot x \tag{2.2.1}$$

The second condition is usually represented as

 $=$  .

Remark 2.2. A graded commutative algebra is sometimes referred to as a *commutative superalgebra* or *supercommutative algebra*. This terminology, coming from the theory of supersymmetry in theoretical physics, will not be used in this course.

EXAMPLE 2.3. The singular cohomology $(H^*(X), \cup)$ of a topological space X is a graded commutative algebra. The singular cochains $(C^*(X), \cup)$ however do not form a cdg algebra, as the cup product is not graded commutative on the chain level. It is however graded commutative *up to homotopy*, i.e. there exists a degree -1 map $h : C^*(X) \otimes C^*(X) \to C^*(X)$ such that $\partial h + h\partial = \cup - \cup\tau$.

2.2.2. *Free graded commutative algebra.*

DEFINITION 2.4. *The free graded commutative algebra ΛV on a graded vector space V is defined to be the unital graded commutative algebra whose underlying vector space is*

$$T(V)/\langle x \otimes y - (-1)^{|x||y|} y \otimes x\rangle \ ,$$

whose grading is defined as in Section 1.3.2 and whose multiplication is given by the concatenation product.

The free graded commutative algebra ΛV admits the same weight grading as in Section 1.3.2, which determines a decomposition $\Lambda V := \bigoplus_{i=0}^{+\infty} \Lambda^i V$.

Proposition 2.5. *The functor* $\Lambda(-) : \mathtt{gr\ Vect} \to \mathtt{gr\ uCom\text{-}alg}$ *is left adjoint to the forgetful functor* $\mathtt{gr\ uCom\text{-}alg} \to \mathtt{gr\ Vect}$. *In other words, for every graded vector space V and graded commutative algebra C, there is a natural bijection*

$$\mathrm{Hom}_{\mathtt{gr\ uCom\text{-}alg}}(\Lambda V, C) = \mathrm{Hom}_{\mathtt{gr\ Vect}}(V, C) \ .$$

2.3. Sullivan models and rational homotopy theory. In this section, we work with cohomological conventions and set $\mathbb{K} = \mathbb{Q}$. We moreover assume that all cdg algebras are unital and that all morphisms are morphisms of unital cdg algebras.

2.3.1. *Sullivan models.*

DEFINITION 2.6. *A* Sullivan algebra *is a cdg algebra of the form* $(\Lambda V, \partial)$ *such that*

(1) $V = V^{\geqslant 1}$ *is a graded vector space concentrated in degree* $\geqslant 1$,
(2) $V = \cup_{k\geqslant 0} V(k)$ *where* $V(k)$ *is an increasing sequence of graded vector spaces* $V(0) \subset V(1) \subset \cdots$,
(3) $\partial(V(0)) = 0$ *and* $\partial(V(k)) \subset \Lambda V(k-1)$ *for* $k \geqslant 1$.

Item 3 is called the *nilpotence condition.*

DEFINITION 2.7. *(i) A* Sullivan model *for a cdg algebra C is a quasi-isomorphism* $(\Lambda V, \partial) \xrightarrow{\sim} C$.
(ii) A Sullivan model is said to be minimal *if* $\partial(V) \subset \Lambda^{\geqslant 2} V$.

In Definition 2.7, *quasi-isomorphism* means that the map is a morphism of cdg algebras which induces an isomorphism in cohomology.

Proposition 2.8. *Every cdg algebra A satisfying $H^0(A) = \mathbb{K}$ admits a minimal Sullivan model. It is moreover unique up to isomorphism.*

PROOF. The unicity up to isomorphism of a minimal Sullivan model stems from the fact that if two Sullivan models $(\Lambda V_1, \partial_1)$ and $(\Lambda V_2, \partial_2)$ are quasi-isomorphic, then (V_1, ∂_1) and

(V_2, ∂_2) are quasi-isomorphic. The minimality assumption then ensures that $\partial_i = 0$ on V_i, hence that V_1 and V_2 are isomorphic. □

2.3.2. *Sullivan models of topological spaces.* Recall from Example 2.3 that the singular cochains $C^*(X, \mathbb{Q})$ form a dg algebra which is not graded commutative.

THEOREM 2. *For any topological space X, there exists a dg algebra $D(X)$ and a cdg algebra $A_{PL}(X)$ that fit into the following diagram of quasi-isomorphisms of dg algebras*

$$C^*(X, \mathbb{Q}) \xrightarrow{\sim} D(X) \xleftarrow{\sim} A_{PL}(X) \ .$$

This diagram is moreover natural in X.

The cdg algebra $A_{PL}(X)$ is called the *algebra of polynomial differential forms* on X.

DEFINITION 2.9. *A* (minimal) Sullivan model *for a path connected topological space X is defined to be a (minimal) Sullivan model for the cdg algebra $A_{PL}(X)$.*

DEFINITION 2.10. *(i) Two cdg algebras A and B are said to be* weakly equivalent *if there exists a zig-zag of quasi-isomorphisms of cdg algebras*

$$A = C_0 \leftarrow C_1 \rightarrow \cdots \leftarrow C_{n-1} \rightarrow C_n = B \ .$$

(ii) A cdg algebra C satisfying $H^0(C) = \mathbb{K}$ is said to be formal *if it is weakly equivalent to the cdg algebra $H^*(C)$ with trivial differential.*

(iii) A path connected topological space X is said to be formal *if the cdg algebra $A_{PL}(X)$ is formal.*

Proposition 2.11. *Two cdg algebras A and B satisfying $H^0(A) = H^0(B) = \mathbb{K}$ are weakly equivalent if and only if their minimal Sullivan models are isomorphic.*

The minimal Sullivan model of a formal topological space X can thereby be directly computed from its cohomology $H^*(X, \mathbb{Q})$.

2.3.3. *Rational homotopy theory.*

DEFINITION 2.12. *(i) A topological space is of* finite rational type *if for every $n \geqslant 0$, the vector space $H^n(X, \mathbb{Q})$ is finite-dimensional.*

(ii) Two topological spaces are said to have the same rational homotopy type *if their exists a zig-zag of continuous maps*

$$X = X_0 \leftarrow X_1 \rightarrow \cdots \leftarrow X_{n-1} \rightarrow X_n = Y$$

inducing a weak equivalence zig-zag between $H^*(X, \mathbb{Q})$ and $H^*(Y, \mathbb{Q})$ in rational cohomology.

THEOREM 3. *Two simply connected spaces of finite rational type have same rational homotopy type if and only if their minimal Sullivan models are isomorphic as cdg algebras.*

In other words, the rational homotopy type of a simply connected topological spaces of finite rational type is completely determined by its minimal model.

DEFINITION 2.13. *Let X be a topological space. We define the* free loop space *LX to be the space of continuous maps $\mathbb{S}^1 \rightarrow X$.*

Theorem 4. *Let X be a simply connected topological space and $(\Lambda V, \partial)$ a minimal model of X. Then its free loop space LX admits a minimal model of the form $(\Lambda V \otimes \Lambda sV, \delta)$ where sV denotes the suspension of V.*

This last theorem is an important tool for the computation of singular cohomologies of free loop spaces. See also Section 3.4.2.

3. Lie algebras

3.1. Lie algebras.

3.1.1. *Lie algebras.*

Definition 3.1. *A* Lie algebra *is defined to be a vector space $\mathfrak{g}$ endowed with a map $[\cdot,\cdot] : \mathfrak{g}\otimes\mathfrak{g} \to \mathfrak{g}$, called the* Lie bracket*, such that*

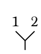
(1) $[x, y] = -[y, x]$ *: the bracket is* antisymmetric.
(2) $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ *: the bracket satifies the* Jacobi identity.

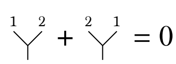
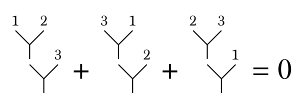
If we denote $c := [\cdot,\cdot]$ these two conditions can be rephrased as

$$c \circ (\mathrm{id} + \tau_{(12)}) = 0 \qquad c(c \otimes \mathrm{id}) \circ (\mathrm{id} + \tau_{(123)} + \tau_{(321)}) = 0 \ ,$$

where we point out that $\tau_{(12)}^2 = \mathrm{id}_{\mathfrak{g}^{\otimes 2}}$, $\tau_{(123)}^2 = \tau_{(321)}$ and $\tau_{(123)}^3 = \mathrm{id}$. We will denote these last two equalities as

$$c^{\mathrm{id}+(12)} = 0 \qquad\qquad c(c \otimes \mathrm{id})^{\mathrm{id}+(123)+(321)} = 0 \ , \tag{3.1.1}$$

Writing the bracket $[\cdot,\cdot]$ as  , they can also be represented as

(3.1.2)  

Remark 3.2. The Jacobi relation is also sometimes replaced by the *Leibniz relation*

$$[[x, y], z] = [[x, z], y] + [x, [y, z]] \ ,$$

which amounts to say that for all z the map $[\cdot, z] : \mathfrak{g} \to \mathfrak{g}$ is a derivation for the bracket $[\cdot,\cdot]$, i.e. satisfies the Leibniz relation w.r.t. this bracket.

Definition 3.3. *Given two Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$, a morphism of Lie algebras is defined to be a linear map $f : \mathfrak{g}_1 \to \mathfrak{g}_2$ such that $fc_1 = c_2(f \otimes f)$, i.e. such that $[f(x), f(y)]_{\mathfrak{g}_2} = f([x, y]_{\mathfrak{g}_1})$.*

We denote `Lie-alg` the category of Lie algebras with morphisms of Lie algebras.

Any associative algebra A can be endowed with a Lie algebra structure, by setting

$$[x, y] := xy - yx \ .$$

This defines a forgetful functor `As-alg` $\to$ `Lie-alg`.

Definition 3.4. *Let $\mathfrak{g}$ be a Lie algebra and denote $T(\mathfrak{g})$ the free tensor algebra over the vector space $\mathfrak{g}$. The* universal enveloping algebra *of $\mathfrak{g}$ is defined to be the quotient of $T(\mathfrak{g})$ by the two-sided ideal generated by the elements*

$$x \otimes y - y \otimes x - [x, y] \ ,$$

and is denoted $U(\mathfrak{g})$.

Proposition 3.5. *The universal enveloping algebra construction defines a functor $U(-) : \mathtt{Lie\text{-}alg} \to \mathtt{uAs\text{-}alg}$ which is left adjoint to the forgetful functor $\mathtt{uAs\text{-}alg} \to \mathtt{Lie\text{-}alg}$. In other words, for every unital associative algebra A and Lie algebra $\mathfrak{g}$, there is a natural bijection*

$$\mathrm{Hom}_{\mathtt{uAs\text{-}alg}}(U(\mathfrak{g}), A) = \mathrm{Hom}_{\mathtt{Lie\text{-}alg}}(\mathfrak{g}, A) \ .$$

3.1.2. *Lie algebra of a Lie group.* Let M be a smooth manifold. We denote $\Gamma(TM)$ the vector space of smooth vector fields on M, i.e. the vector space of sections of the tangent bundle $TM \to M$. Recall that $\Gamma(TM)$ is isomorphic to the vector space $\mathrm{Der}(\mathscr{C}^\infty(M, \mathbb{R}))$ of derivations of the algebra $\mathscr{C}^\infty(M, \mathbb{R})$. For $X \in \Gamma(TM)$ and f a smooth function, we will thereby denote Xf the image of f under the derivation X. The *Lie bracket* of two vector fields X and Y is then defined to be the vector field $[X, Y]$ acting on smooth functions as

$$[X, Y]f = X(Yf) - Y(Xf) \ .$$

Proposition 3.6. *The vector space of smooth vector fields $\Gamma(TM)$ endowed with the Lie bracket is a Lie algebra.*

Let now G be a Lie group, whose unit we denote $e \in G$. For $g \in G$ we denote $L_g : G \to G$ the left multiplication, i.e. $L_g(h) = g \cdot h$. A vector field X on G is then said to be *left-invariant* if for every $g \in G$ it is invariant under L_g, i.e. $(L_g)_*(X) = X$ or equivalently $d(L_g)_h(X_h) = X_{gh}$ for all $h \in G$. We denote $\mathrm{Lie}(G)$ the vector space of left-invariant vector fields on G.

Proposition 3.7.
(i) The vector space $\mathrm{Lie}(G)$ is a finite-dimensional vector space of dimension $\dim(G)$. More precisely, the linear map $\mathrm{Lie}(G) \to T_eG$ mapping X to X_e is an isomorphism.
(ii) The vector space $\mathrm{Lie}(G)$ is stable under the Lie bracket, i.e. is a finite-dimensional Lie subalgebra of $\Gamma(TG)$.

The functor Lie from the category of real finite-dimensional Lie algebras to the category of simply-connected Lie groups is in fact an equivalence of category:

Theorem 5.
(i) Let G and H be Lie groups. If G is simply-connected, then every morphism of Lie algebras $f : \mathrm{Lie}(G) \to \mathrm{Lie}(H)$ admits a unique lift to a morphism of Lie groups $F : G \to H$ such that $\mathrm{Lie}(F) = f$.
(ii) For every real finite-dimensional Lie algebra $\mathfrak{g}$ there exists a unique simply-connected Lie group G such that $\mathrm{Lie}(G) = \mathfrak{g}$.

Item (i) and Item (ii) are respectively called *Lie's second theorem* and the *Cartan-Lie theorem.*

3.1.3. *dg Lie algebras and Maurer-Cartan elements.*

Definition 3.8. *A* dg Lie algebra *is defined to be a dg vector space* $\mathfrak{g}$ *endowed with a linear map* $[\cdot,\cdot] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ *such that*

(1) The map $[\cdot,\cdot]$ *is a chain map.*
(2) $[x, y] = (-1)^{|x||y|+1}[y, x]$.
(3) $(-1)^{|x||z|}[[x, y], z] + (-1)^{|x||y|}[[y, z], x] + (-1)^{|y||z|}[[z, x], y] = 0$.

We point out that Items 2 and 3 can be reformulated using the Koszul sign rules as in Equation (3.1.2). The notion of a *graded Lie algebra* can be defined similarly. A *morphism of dg Lie algebras* is moreover defined as a chain map $f : \mathfrak{g}_1 \to \mathfrak{g}_2$ which preserves the Lie bracket.

Definition 3.9. *A* Maurer-Cartan element *of a dg Lie algebra* $\mathfrak{g}$ *over a field of characteristic* $\neq 2$ *is defined to be an element* $\alpha \in \mathfrak{g}_{-1}$ *of degree* -1 *such that*

$$\partial\alpha + \frac{1}{2}[\alpha, \alpha] = 0 \ .$$

The *set of Maurer-Cartan elements* of a dg Lie algebra $\mathfrak{g}$ will be denoted as $\mathrm{MC}(\mathfrak{g})$. We point out that a morphism of dg Lie algebras $f : \mathfrak{g}_1 \to \mathfrak{g}_2$ preserves the Maurer-Cartan equation hence induces a map $\mathrm{MC}(\mathfrak{g}_1) \to \mathrm{MC}(\mathfrak{g}_2)$.

Proposition 3.10. *Given a dg Lie algebra* $\mathfrak{g}$ *and* $\alpha \in \mathrm{MC}(\mathfrak{g})$*, the map*

$$\partial^\alpha := \partial + [\alpha, \cdot]$$

is a derivation w.r.t. the bracket $[\cdot,\cdot]$ *and satisfies* $\partial^\alpha \circ \partial^\alpha = 0$.

The differential ∂^α is called the *twisted differential.* The dg Lie algebra $(\mathfrak{g}, \partial^\alpha, [\cdot,\cdot])$ will then be denoted $\mathfrak{g}^\alpha$ and called the *twisted dg Lie algebra.*

Proposition 3.11. *For a dg Lie algebra* $\mathfrak{g}$ *and* $\alpha \in \mathrm{MC}(\mathfrak{g})$*, we have that* $\beta \in \mathrm{MC}(\mathfrak{g}^\alpha)$ *if and only if* $\beta + \alpha \in \mathrm{MC}(\mathfrak{g})$.

We will see in Section 3 that the problem of the *deformation* of an algebraic structure is usually encoded by a dg Lie algebra whose Maurer-Cartan elements correspond to the deformations of the considered structure.

3.2. pre-Lie algebras.

Definition 3.12. *A* pre-Lie algebra *is defined to be a vector space* A *endowed with a map* $\{\cdot,\cdot\} : A \otimes A \to A$ *satisfying the relation*

$$\{\{x, y\}, z\} - \{x, \{y, z\}\} = \{\{x, z\}, y\} - \{x, \{z, y\}\} \ .$$

An associative algebra is in particular a pre-Lie algebra with bracket its multiplication.

Proposition 3.13. *Let* $(A, \{\cdot,\cdot\})$ *be a pre-Lie algebra. The map* $[\cdot,\cdot] : A \otimes A \to A$ *defined as* $[x, y] := \{x, y\} - \{y, x\}$ *endows* A *with a Lie algebra structure.*

A dg pre-Lie algebra $(A, \partial, \{\cdot,\cdot\})$ is defined similarly. The bracket of the induced dg Lie algebra then reads as $[x, y] := \{x, y\} - (-1)^{|x||y|}\{y, x\}$. The Maurer-Cartan equation for a dg pre-Lie algebra hence reads as $\partial\alpha + \{\alpha, \alpha\} = 0$.

3.3. Poisson algebras.

Definition 3.14. *A* Poisson algebra *structure on a vector space P corresponds to the data of a multiplication $\mu : P \otimes P \to P$ and of a bracket $[\cdot,\cdot] : P \otimes P \to P$ such that*

(1) (P, μ) is an associative algebra,
(2) $(P, [\cdot,\cdot])$ is a Lie algebra,
(3) for all $x \in P$ the map $[x, \cdot] : P \to P$ is a derivation w.r.t. the multiplication, i.e.

$$[x, y \cdot z] = [x, y] \cdot z + y \cdot [x, z] \ .$$

The relation of Item 3 is called the *Poisson identity.*

A *morphism of Poisson algebras* is defined to be a linear map $f : P_1 \to P_2$ that commutes with both the brackets and the multiplications of P_1 and P_2. We denote `Pois` the category of Poisson algebras.

Example 3.15. See Exercise sheet 2.

3.4. Gerstenhaber and Batalin-Vilkovisky algebras.

3.4.1. *Definitions.*

Definition 3.16. *A* Gerstenhaber algebra *structure on a graded vector space G is defined to be the data of a multiplication $\mu : G \otimes G \to G$ and of a bracket $[\cdot,\cdot] : sG \otimes sG \to sG$ such that*

(1) (G, μ) is a graded commutative algebra,
(2) $(sG, [\cdot,\cdot])$ is a graded Lie algebra,
(3) for all $x \in G$ the map $[x, \cdot] : G \to G$ is a derivation w.r.t. the multiplication, i.e. satisfies

$$[x, y \cdot z] = [x, y] \cdot z + (-1)^{(|y|+1)(|x|+1)} y \cdot [x, z] \ .$$

Let us make explicit the relations that a Gerstenhaber algebra has to satisfy:

(1) the relations of Equation (2.2.1) for the graded commutative algebra (G, μ),
(2) the degree 0 map $[\cdot,\cdot] : sG \otimes sG \to sG$ induces a degree 1 map $[\cdot,\cdot] : G \otimes G \to G$ which satisfies the relations for the graded Lie algebra $(sG, [\cdot,\cdot])$,

$$[x, y] = (-1)^{(|x|+1)(|y|+1)+1}[y, x]$$

$$(-1)^{(|x|+1)(|z|+1)}[[x, y], z] + (-1)^{(|x|+1)(|y|+1)}[[y, z], x] + (-1)^{(|y|+1)(|z|+1)}[[z, x], y] = 0 \ ,$$

where we use the fact that $|x|_{sG} = |x|_G + 1$,
(3) the *Gerstenhaber relation* of Item 3.

Beware that a Gerstenhaber algebra is not the same thing as a graded Poisson algebra !

Example 3.17. See Exercise sheet 2.

DEFINITION 3.18. *A* Batalin-Vilkovisky algebra *structure or* BV algebra *structure on a graded vector space A is the data of a multiplication $\mu : A \otimes A \to A$ and of a linear map $\Delta : A \to A$, called the* BV operator, *such that*

(1) (A, μ) *is a graded commutative algebra,*
(2) Δ *has degree 1 and* $\Delta^2 = 0$,
(3) $\Delta(- \cdot - \cdot -) = (\Delta(- \cdot -) \cdot -)^{\mathrm{id}+(123)+(321)} - (\Delta(-) \cdot - \cdot -)^{\mathrm{id}+(123)+(321)}$

The notations of Item 3 are as in Equation (3.1.1). The BV relation moreover reads on elements as

$$\begin{aligned}\Delta(abc) =&\Delta(ab)c + (-1)^{|a|}a\Delta(bc) + (-1)^{(|a|+1)|b|}b\Delta(ac)\\ &- \Delta(a)bc - (-1)^{|a|}a\Delta(b)c - (-1)^{|a|+|b|}ab\Delta(c)\end{aligned}$$

Proposition 3.19. *A BV algebra* (A, μ, Δ) *is in particular a Gerstenhaber algebra* $(A, \mu, [\cdot, \cdot])$ *whose bracket is defined as*

$$[a, b] = (-1)^{|a|} \Big(\Delta(a \cdot b) - (-1)^{|a|}a \cdot \Delta(b) - \Delta(a) \cdot b\Big) \ .$$

The bracket can be interpreted as the obstruction to Δ being a derivation. It can also be checked that Δ is then a derivation for $[\cdot, \cdot]$.

3.4.2. *Loop homology.* Let M be a smooth orientable closed manifold of dimension m. The free loop space LM defined in Definition 2.13 comes with an evaluation map $\mathrm{ev} : \gamma \in LM \mapsto \gamma(0) \in M$, where we define $\mathbb{S}^1$ as the quotient of $[0, 1]$ by the relation $0 = 1$.

DEFINITION 3.20. *We denote* $\mathbb{H}_*(LM) := H_{*+m}(LM)$ *and call it the* loop homology *of* M.

Consider two singular chains $\sigma_i \in C_i(LM)$ and $\sigma_j \in C_j(LM)$. We introduce the map

$$\phi := (\mathrm{ev} \circ \sigma_i, \mathrm{ev} \circ \sigma_j) : \Delta^i \times \Delta^j \to M \times M \ ,$$

and denote $D := \{(x, x), x \in M\} \subset M \times M$. The *Chas-Sullivan product* $\sigma_i \bullet \sigma_j$ is then the chain in $C_{i+j-m}(LM)$ defined as

$$\begin{aligned}&\phi^{-1}(D) \times \mathbb{S}^1 \to M\\ &(\delta_1, \delta_2, t) \mapsto \begin{cases}\sigma_1(\delta_1, 2t) \text{ if } t \in [0, 1/2]\\ \sigma_2(\delta_2, 2t - 1) \text{ if } t \in [1/2, 1]\end{cases}\end{aligned} \ .$$

REMARK 3.21. For a smooth map $\phi : M \to N$ transverse to a smooth submanifold $S \subset N$, the space $\phi^{-1}(S) \subset M$ is a submanifold of M of dimension

$$\dim(\phi^{-1}(S)) = \dim(M) + \dim(S) - \dim(N) \ .$$

Applying this idea to the map $\phi : \Delta^i \times \Delta^j \to M \times M$ and the submanifold $D \subset M \times M$, we find

$$\dim(\phi^{-1}(D)) = i + j + m - 2m = i + j - m \ .$$

For a singular chain $\sigma \in C_i(LM)$ we moreover define the chain $\Delta(\sigma) \in C_{i+1}(M)$ as

$$\begin{aligned}(\mathbb{S}^1 \times \Delta^i) \times \mathbb{S}^1 &\to M \\ (s, \delta, t) &\mapsto \sigma(\delta, s+t) \ .\end{aligned}$$

The Chas-Sullivan product and the operator Δ induce operations on loop homology

$$\bullet : \mathbb{H}_*(LM) \otimes \mathbb{H}_*(LM) \to \mathbb{H}_*(LM) \qquad \Delta : \mathbb{H}_*(LM) \to \mathbb{H}_{*+1}(LM) \ .$$

We point out that the degree shift $H_{*+m}(LM)$ was necessary in order for $\bullet$ to have degree 0.

We refer to **[CS99]**, **[LO15]** and **[CHV06]** for more details on these two operations.

THEOREM 6. *The loop homology $\mathbb{H}_*(LM)$ endowed with the Chas-Sullivan product $\bullet$ and the operator Δ is a BV algebra.*

We have proven in Exercise sheet 2 that for an associative algebra A, the Hochschild cohomology $HH^*(A, A)$ is a Gerstenhaber algebra. The same result holds for Hochschild homology of a dg algebra (see Exercise sheet 4). We also recall from Example 1.14 that the singular cochains $C^*(M)$ form a dg algebra.

THEOREM 7. *We assume that M is simply connected and that* $\mathrm{char}(\mathbb{K}) = 0$. *Then there exists a BV algebra structure on $HH^*(C^*(M), C^*(M))$ such that*

(1) the BV algebras $HH^*(C^*(M), C^*(M))$ and $\mathbb{H}_*(LM)$ are isomorphic as BV algebras,
(2) the induced Gerstenhaber algebra structure on $HH^*(C^*(M), C^*(M))$ is its standard Gerstenhaber algebra structure.

CHAPTER 3

Operads

1. Operads and $\mathscr{P}$-algebras

We fix a closed symmetric monoidal category $(\mathtt{C}, \boxtimes, I)$ in this chapter. Recall from Section 1.3.1 that given n objects $C_1, \ldots, C_n$ in the monoidal category $\mathtt{C}$, two bracketings of $C_1 \boxtimes \cdots \boxtimes C_n$ are always equivalent through a unique natural equivalence made of associators. We will hence write $C_1 \boxtimes \cdots \boxtimes C_n$ for any representative of this equivalence class of bracketings.

1.1. Group actions. Let $\mathtt{C}$ be a category and $X \in \mathtt{C}$. Then the composition map $(g, f) \mapsto g \circ f$ of $\mathtt{C}$ and the identity id_X naturally endow the set $\mathtt{C}(X, X)$ with a structure of monoid.

Definition 1.1. *Let G be a group.*

(i) A left group action *of G on X is defined to be a morphism of monoids $G \to \mathtt{C}(X, X)$.*
(ii) A right group action *of G on X is defined to be a morphism of monoids $G^{op} \to \mathtt{C}(X, X)$, where G^{op} denotes the set G endowed with the multiplication $(g_1, g_2) \mapsto g_2 g_1$.*

This definition recovers the usual notions of left and right group actions when X is a set. We will write the image of $g \in G$ in $\mathtt{C}(X, X)$ as τ_g.

Definition 1.2. *(i) A morphism $f : X_1 \to X_2$ between two objects $X_1, X_2 \in \mathtt{C}$ with a left/right G-action is said to be G-*equivariant *if $f \tau_g^{X_1} = \tau_g^{X_2} f$ for every $g \in G$.*
*(ii) A morphism $f : X \to Y$ between two objects $X, Y \in \mathtt{C}$ where G acts on the left/right of X is said to be G-*invariant *if $f \tau_g = f$ for every $g \in G$.*

Definition 1.3. *Let V be a vector space together with a left action of a group G on V.*

(i) The vector space of coinvariants *V_G is defined as $V_G := V / \langle v - gv, g \in G \rangle$.*
(ii) The vector space of invariants *V^G is defined as $V^G := \{v \in V,\ gv = v\ \forall g \in G\}$.*

Remark 1.4. A left group action of G on V is in fact equivalent to a left $\mathbb{K}[G]$-module structure on V. Endowing $\mathbb{K}$ with its trivial right $\mathbb{K}[G]$-algebra structure, we then have that

$$V_G = \mathbb{K} \otimes_{\mathbb{K}[G]} V \qquad\qquad V^G = \mathrm{Hom}_{\mathbb{K}[G]\text{-}\mathtt{mod}}(\mathbb{K}, V)\ .$$

1.2. May's original definition. We set $\mathfrak{S}_0 := \{*\}$ to be the trivial group with one element.

Definition 1.5. *A $\mathfrak{S}$-* module *is defined to be a sequence of objects $\{\mathscr{M}(n)\}_{n \geqslant 0}$ in $\mathtt{C}$ together with a right action of $\mathfrak{S}_n$ on $\mathscr{M}(n)$ for $n \geqslant 0$.*

Let $\{\mathcal{M}(n)\}_{n\geqslant 0}$ be a $\mathfrak{S}$-module, $\boldsymbol{l} = l_1, \ldots, l_k$ be a sequence of integers $\geqslant 0$ and $\sigma \in \mathfrak{S}_k$. We will denote $\mathcal{M}(\boldsymbol{l}) := \mathcal{M}(l_1) \boxtimes \cdots \boxtimes \mathcal{M}(l_s)$, $|\boldsymbol{l}| := \sum_{i=1}^{k} l_i$ and $\boldsymbol{l}\sigma := l_{\sigma(1)}, \ldots, l_{\sigma(k)}$.

Definition 1.6. *For $k \geqslant 1$, $\boldsymbol{i} = i_1, \ldots, i_k$ a sequence of integers $\geqslant 0$ and $\sigma \in \mathfrak{S}_k$, we define the* block permutation $\sigma_{\boldsymbol{i}} \in \mathfrak{S}_{i_1+\cdots+i_k}$ *as the permutation which permutes the k intervals*

$$[[i_1 + \cdots + i_{s-1} + 1, i_1 + \cdots + i_s]] \ , \ 1 \leqslant s \leqslant k$$

as prescribed by the permutation σ.

Definition 1.7. *An* operad *structure on a $\mathfrak{S}$-module $\{\mathcal{P}(n)\}_{n\geqslant 0}$ is defined to be the following data:*

(1) A morphism

$$\gamma_{i_1,\cdots,i_k} : \mathcal{P}(k) \boxtimes \mathcal{P}(i_1) \boxtimes \cdots \boxtimes \mathcal{P}(i_k) \to \mathcal{P}(i_1 + \cdots + i_k)$$

for all $k \geqslant 1$ and $i_1, \ldots, i_k \geqslant 0$, called a composition morphism.

(2) A morphism $\eta : I \to \mathcal{P}(1)$ called the unit.

These data have to satisfy the following properties:

(1) The maps $\gamma_{i_1,\cdots,i_k}$ are equivariant under the right action of $\mathfrak{S}_{i_1} \times \cdots \times \mathfrak{S}_{i_k}$, where $\mathfrak{S}_{i_1} \times \cdots \times \mathfrak{S}_{i_k}$ is seen as a subgroup of $\mathfrak{S}_{i_1+\cdots+i_k}$.

(2) Given an integer $k \geqslant 1$, a sequence $\boldsymbol{i} = i_1, \ldots, i_k$ of integers $\geqslant 1$ and k sequences $\boldsymbol{j}_h$ of integers $\geqslant 0$ of length i_h for $1 \leqslant h \leqslant k$, the following diagram commutes

$$\begin{tikzcd}
\mathcal{P}(k) \boxtimes \mathcal{P}(\boldsymbol{i}) \boxtimes \mathcal{P}(\boldsymbol{j}) \arrow[r] \arrow[dd, "\gamma_{i_1,\ldots,i_k} \boxtimes \mathrm{id}_{\mathcal{P}(\boldsymbol{j})}"] & \mathcal{P}(k) \boxtimes \mathcal{P}(i_1) \boxtimes \mathcal{P}(\boldsymbol{j}_1) \boxtimes \cdots \boxtimes \mathcal{P}(i_k) \boxtimes \mathcal{P}(\boldsymbol{j}_k) \arrow[d, "\mathrm{id}_{\mathcal{P}(k)} \boxtimes \gamma_{\boldsymbol{j}_1} \boxtimes \cdots \boxtimes \gamma_{\boldsymbol{j}_k}"] \\
 & \mathcal{P}(k) \boxtimes \mathcal{P}(|\boldsymbol{j}_1|) \boxtimes \cdots \boxtimes \mathcal{P}(|\boldsymbol{j}_k|) \arrow[d, "\gamma_{|\boldsymbol{j}_1|,\ldots,|\boldsymbol{j}_k|}"] \\
\mathcal{P}(i_1 + \cdots + i_k) \boxtimes \mathcal{P}(\boldsymbol{j}) \arrow[r, "\gamma_{\boldsymbol{j}}"] & \mathcal{P}(|\boldsymbol{j}_1| + \cdots + |\boldsymbol{j}_k|)
\end{tikzcd}$$

where $\boldsymbol{j}$ denotes the concatenated sequence $\boldsymbol{j} = \boldsymbol{j}_1 \ldots \boldsymbol{j}_k$ and the top arrow is the composition of braidings rearranging $\mathcal{P}(\boldsymbol{i}) \boxtimes \mathcal{P}(\boldsymbol{j})$ into $\mathcal{P}(i_1) \boxtimes \mathcal{P}(\boldsymbol{j}_1) \boxtimes \cdots \boxtimes \mathcal{P}(i_k) \boxtimes \mathcal{P}(\boldsymbol{j}_k)$.

(3) For $k \geqslant 1$, $i_1, \ldots, i_k \geqslant 0$ and $\sigma \in \mathfrak{S}_k$, the following diagram commutes

$$\begin{tikzcd}
\mathcal{P}(k) \boxtimes \mathcal{P}(i_1) \boxtimes \cdots \boxtimes \mathcal{P}(i_k) \arrow[r] \arrow[dd, "\sigma \boxtimes \mathrm{id}_{\mathcal{P}(i_1)\boxtimes\cdots\boxtimes\mathcal{P}(i_k)}"] & \mathcal{P}(k) \boxtimes \mathcal{P}(i_{\sigma^{-1}(1)}) \boxtimes \cdots \boxtimes \mathcal{P}(i_{\sigma^{-1}(k)}) \arrow[d, "\gamma_{\boldsymbol{i}\sigma^{-1}}"] \\
 & \mathcal{P}(i_1 + \cdots + i_k) \arrow[d, "\sigma_{\boldsymbol{i}}"] \\
\mathcal{P}(k) \boxtimes \mathcal{P}(i_1) \boxtimes \cdots \boxtimes \mathcal{P}(i_k) \arrow[r, "\gamma_{i_1,\ldots,i_k}"] & \mathcal{P}(i_1 + \cdots + i_k) \ ,
\end{tikzcd}$$

where the top arrow is the composition of braidings rearranging $\mathcal{P}(i_1) \boxtimes \cdots \boxtimes \mathcal{P}(i_k)$ into $\mathcal{P}(i_{\sigma^{-1}(1)}) \boxtimes \cdots \boxtimes \mathcal{P}(i_{\sigma^{-1}(k)})$.

(4) For $n \geqslant 0$, the composite morphisms

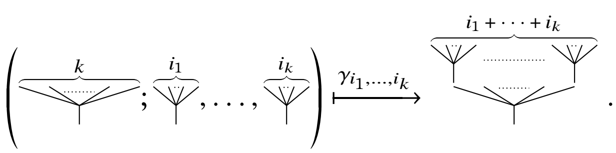
$$\mathcal{P}(n) \boxtimes I^{\boxtimes n} \xrightarrow{\mathrm{id}_{\mathcal{P}(n)} \boxtimes \eta^{\boxtimes n}} \mathcal{P}(n) \boxtimes \mathcal{P}(1) \boxtimes \cdots \boxtimes \mathcal{P}(1) \xrightarrow{\gamma_{1,\ldots,1}} \mathcal{P}(n),$$

$$I \boxtimes \mathcal{P}(n) \xrightarrow{\eta \boxtimes \mathrm{id}_{\mathcal{P}(n)}} \mathcal{P}(1) \boxtimes \mathcal{P}(n) \xrightarrow{\gamma_n} \mathcal{P}(n)$$

are respectively equal to the iterations of the right unit of $\mathtt{C}$ *and to the left unit of* $\mathtt{C}$.

The object $\mathcal{P}(n)$ *is then called the* space of operations of arity n *of the operad* $\mathcal{P}$.

Assuming that the objects of $\mathtt{C}$ are sets and representing the elements of $\mathcal{P}(n)$ as corollae of arity n, the composition morphisms $\gamma_{i_1,\cdots,i_k} : \mathcal{P}(k) \boxtimes \mathcal{P}(i_1) \boxtimes \cdots \boxtimes \mathcal{P}(i_k) \to \mathcal{P}(i_1 + \cdots + i_k)$ can naturally be represented as



The elements of $\mathcal{P}(0)$ will moreover be represented as .

Example 1.8. (i) The operad $uAss$ is the operad in $\mathtt{Set}$ whose arity n set of operations is $\mathfrak{S}_n$ for $n \geqslant 0$. The group $\mathfrak{S}_n$ acts on the right of $uAss(n)$ by multiplication on the right. The composition maps are given by mapping a permutation $\sigma \in uAss(k)$ and permutations $\sigma_j \in uAss(i_j)$ for $1 \leqslant j \leqslant k$ to the composite permutation $\sigma_{\boldsymbol{i}}(\sigma_1 \times \cdots \times \sigma_k)$ of $uAss(i_1 + \cdots + i_k)$, where $\mathfrak{S}_{i_1} \times \cdots \times \mathfrak{S}_{i_k}$ is seen as a subgroup of $\mathfrak{S}_{i_1+\cdots+i_k}$.

(ii) The operad Com is the operad in $\mathtt{Set}$ whose arity n set of operations is a singleton $\{*\}$ for $n \geqslant 1$ and the empty set for $n = 0$. The action of the symmetric groups as well as the composition maps and unit are then all trivial.

(iii) A monoid structure on an object $C \in \mathtt{C}$ is equivalent to a structure of operad on the $\mathfrak{S}$-module $(0, M, 0, \ldots, 0, \ldots)$, where we assume that the closed symmetric monoidal category $\mathtt{C}$ has an initial object $0 \in \mathtt{C}$.

We will denote $I := (0, I, 0, \ldots)$ the $\mathfrak{S}$-module concentrated in arity 1. The object I being a monoid in $\mathtt{C}$, the previous example implies that the $\mathfrak{S}$-module I is an operad.

Definition 1.9. *A* morphism of operads $\mathcal{P} \to \mathbb{Q}$ *is defined to be a sequence of* $\mathfrak{S}_n$*-equivariant maps* $\mathcal{P}(n) \to \mathbb{Q}(n)$ *for* $n \geqslant 0$ *which commute with the composition maps and preserve the units. It is an* isomorphism of operads *if each map* $\mathcal{P}(n) \to \mathbb{Q}(n)$ *is an isomorphism.*

Definition 1.10. *An* augmented *operad is defined to be the data of an operad* $\mathcal{P}$ *together with a morphism of operads* $\varepsilon : \mathcal{P} \to I$ *called the* augmentation.

Definition 1.11. *Let* $\mathcal{P}$ *and* $\mathbb{Q}$ *be two operads. We define their* Hadamard product $\mathcal{P} \boxtimes \mathbb{Q}$ *as the operad whose underlying* $\mathfrak{S}$*-module is* $(\mathcal{P} \boxtimes \mathbb{Q})(n) := \mathcal{P}(n) \boxtimes \mathbb{Q}(n)$*, whose unit is*

$$\eta_{\mathcal{P}\boxtimes\mathbb{Q}} : I \xrightarrow{\lambda_I^{-1} = \rho_I^{-1}} I \boxtimes I \xrightarrow{\eta_{\mathcal{P}} \boxtimes \eta_{\mathbb{Q}}} \mathcal{P}(1) \boxtimes \mathbb{Q}(1) = (\mathcal{P} \boxtimes \mathbb{Q})(1)$$

and whose composition maps $\gamma^{\mathcal{P}\boxtimes\mathbb{Q}}_{i_1,\ldots,i_k}$ *are defined as*

$$\begin{array}{c}\mathcal{P}(k)\boxtimes\mathbb{Q}(k)\boxtimes\mathcal{P}(i_1)\boxtimes\mathbb{Q}(i_1)\boxtimes\cdots\boxtimes\mathcal{P}(i_k)\boxtimes\mathbb{Q}(i_k)\\ \big\downarrow\\ \mathcal{P}(k)\boxtimes\mathcal{P}(i_1)\boxtimes\cdots\boxtimes\mathcal{P}(i_k)\boxtimes\mathbb{Q}(k)\boxtimes\mathbb{Q}(i_1)\boxtimes\cdots\boxtimes\mathbb{Q}(i_k)\\ \big\downarrow{\scriptstyle \gamma^{\mathcal{P}}_{i_1,\ldots,i_k}\boxtimes\gamma^{\mathbb{Q}}_{i_1,\ldots,i_k}}\\ (\mathcal{P}\boxtimes\mathbb{Q})(i_1+\cdots+i_k)\end{array}$$

where the top arrow is the composition of braidings rearranging the factors of the top expression into those of the bottom expression.

Proposition 1.12. *Let* $\mathtt{C}$ *and* $\mathtt{D}$ *be two closed symmetric monoidal categories and* $\mathcal{F}:\mathtt{C}\longrightarrow\mathtt{D}$ *be a strong symmetric monoidal functor. Then* $\mathcal{F}$ *maps operads in the category* $\mathtt{C}$ *to operads in the category* $\mathtt{D}$.

1.3. Algebras over an operad.

Definition 1.13. *Let* C *be an object of* $\mathtt{C}$.

(i) The endormorphism operad *of* C *is defined to be the* $\mathfrak{S}$*-module* $\mathrm{End}_C(n):=\underline{\mathrm{Hom}}(C^{\boxtimes n},C)$ *where the right action of* $\mathfrak{S}_n$ *is defined by permuting the inputs using composites of braidings, whose unit is* id_C *and whose compositions are defined for* $g:C^{\boxtimes k}\longrightarrow C$ *and* $f_j:C^{\boxtimes i_j}\longrightarrow C$ *as*

$$\gamma_{i_1,\ldots,i_k}(g;f_1,\ldots,f_k)=g\circ(f_1\boxtimes\cdots\boxtimes f_k)\ .$$

(ii) The coendormorphism operad *of* C *is defined to be the* $\mathfrak{S}$*-module* $\mathrm{coEnd}_C(n):=\underline{\mathrm{Hom}}(C,C^{\boxtimes n})$ *where* $\mathfrak{S}_n$ *acts on the right by permuting the outputs, whose unit is* id_C *and whose compositions* $\gamma_{i_1,\ldots,i_k}$ *are defined for* $g:C\longrightarrow C^{\boxtimes k}$ *and* $f_j:C\longrightarrow C^{\boxtimes i_j}$, $1\leqslant j\leqslant k$ *as*

$$\gamma_{i_1,\ldots,i_k}(g;f_1,\ldots,f_k)=(f_1\boxtimes\cdots\boxtimes f_k)\circ g\ .$$

We point out that for the sake of clarity, we have written the definition as if the internal homs are sets. It is however possible to spell out Definition 1.13 without this assumption, using diagrams in $\mathtt{C}$ involving internal homs.

Definition 1.14. *Let* $\mathcal{P}$ *be an operad.*

(i) A structure of $\mathcal{P}$-algebra *on* $A\in\mathtt{C}$ *is defined to be a morphisms of operads* $\mathcal{P}\longrightarrow\mathrm{End}_A$.
(ii) A structure of $\mathcal{P}$-coalgebra *on* $C\in\mathtt{C}$ *is defined to be a morphism of operads* $\mathcal{P}\longrightarrow\mathrm{coEnd}_C$.

In other words, a $\mathcal{P}$-algebra structure on A is a way to translate each operation of arity n in $\mathcal{P}(n)$ as an arity n morphism $A^{\boxtimes n}\longrightarrow A$, such that a composition of operations in $\mathcal{P}$ translates into a composition of morphisms in $\mathtt{C}$.

Using the adjunction $-\boxtimes A^{\boxtimes n}\dashv\underline{\mathrm{Hom}}(A^{\boxtimes n},\cdot)$, a $\mathcal{P}$-algebra structure on A induces in particular a morphism $\mathcal{P}(n)\boxtimes A^{\boxtimes n}\longrightarrow A$ that is $\mathfrak{S}_n$-invariant for every $n\geqslant 0$. We refer to Section 1.4.2 for more details on that viewpoint.

Definition 1.15. *Given A_1 and A_2 two $\mathcal{P}$-algebras, a morphism $f \in \mathtt{C}(A_1, A_2)$ is said to be a* morphism of $\mathcal{P}$-algebras *if for every $n \geqslant 0$ the following diagram commutes*

$$\begin{tikzcd} \mathcal{P}(n) \boxtimes A_1^{\boxtimes n} \arrow[r] \arrow[d, "\mathrm{id}_{\mathcal{P}(n)} \boxtimes f^{\boxtimes n}"] & A_1 \arrow[d, "f"] \\ \mathcal{P}(n) \boxtimes A_2^{\boxtimes n} \arrow[r] & A_2 \end{tikzcd} .$$

We will denote $\mathcal{P}$-$\mathtt{alg}$ the *category of $\mathcal{P}$-algebras* with morphisms of $\mathcal{P}$-algebras between them. The category $\mathcal{P}$-$\mathtt{cog}$ of $\mathcal{P}$-coalgebras can be defined in a similar fashion.

Example 1.16. (i) A $uAss$-algebra structure on a set X is exactly a monoid structure on X.

(ii) Denote $u\mathcal{A}ss := \mathbb{K}[uAss]$ and $\mathcal{C}om := \mathbb{K}[Com]$. Then a vector space A with a $u\mathcal{A}ss$-algebra/$\mathcal{C}om$-algebra structure is exactly a unital associative algebra/a commutative algebra.

(iii) Seeing the operads $u\mathcal{A}ss$ and $\mathcal{C}om$ as operads in $\mathtt{dg\ Vect}$ concentrated in degree 0 with null differential, a $u\mathcal{A}ss$-algebra/$\mathcal{C}om$-algebra structure is exactly a unital dg algebra/a cdg algebra.

Proposition 1.17. *A morphism of operads $\mathbb{Q} \to \mathcal{P}$ induces a functor $\mathcal{P}$-$\mathtt{alg} \to \mathbb{Q}$-$\mathtt{alg}$.*

1.4. Operads as monoids. In this section, we let $\mathtt{C}$ be one the following three categories: $\mathtt{Vect}$, $\mathtt{gr\ Vect}$ and $\mathtt{dg\ Vect}$. An operad in one of these categories is usually called an *algebraic operad.*

1.4.1. *Monoidal category structure on $\mathfrak{S}$-$\mathtt{mod}$.* Recall that for a group G, a vector space with a left/right G-action is equivalent to a left/right $\mathbb{K}[G]$-module.

Definition 1.18. *Let G be a group.*

(1) Given two vector spaces V and W respectively with a right and left G-action, we define

$$V \otimes_G K := V \otimes_{\mathbb{K}[G]} W .$$

(2) Given a subgroup $H \subset G$ and a vector space V with a right H-action, we define

$$\mathrm{Ind}_H^G V := V \otimes_H \mathbb{K}[G] .$$

(3) Given a subgroup $H \subset G$ and a vector space V with a left H-action, we define

$$\mathrm{Coind}_H^G V := \mathrm{Hom}_{\mathbb{K}[H]}(\mathbb{K}[G], V) .$$

Definition 1.19. *Given two $\mathfrak{S}$-modules $\mathcal{M}$ and $\mathcal{N}$, we define their* composite *as the $\mathfrak{S}$-module $\mathcal{M} \circ \mathcal{N}$ given in arity $n \geqslant 0$ by*

$$\mathcal{M} \circ \mathcal{N}(n) = \bigoplus_{k \geqslant 0} \mathcal{M}(k) \otimes_{\mathfrak{S}_k} \bigoplus_{i_1 + \cdots + i_k = n} \mathrm{Ind}_{\mathfrak{S}_{i_1} \times \cdots \times \mathfrak{S}_{i_k}}^{\mathfrak{S}_n} (\mathcal{N}(i_1) \otimes \cdots \otimes \mathcal{N}(i_k)) ,$$

where the summand is equal to $\mathcal{M}(0)$ when $k = 0$.

The composite defines in fact a bifunctor $- \circ - : \mathfrak{S}$-$\mathtt{mod} \times \mathfrak{S}$-$\mathtt{mod} \to \mathfrak{S}$-$\mathtt{mod}$. We moreover denote $\mathbb{K} := (0, \mathbb{K}, 0, \dots, 0, \dots)$.

Proposition 1.20. *The category $(\mathfrak{S}\text{-}\mathtt{mod}, \circ, \mathbb{K})$ is a monoidal category.*

Proposition 1.21. *Given a $\mathfrak{S}$-module $\mathcal{P}$, a structure of operad on $\mathcal{P}$ corresponds exactly to a structure of monoid on $\mathcal{P}$ in $(\mathfrak{S}\text{-}\mathtt{mod}, \circ, \mathbb{K})$.*

1.4.2. *Schur functor.*

Definition 1.22. *Let $\mathcal{M}$ be a $\mathfrak{S}$-module. The* Schur functor *associated to $\mathcal{M}$ is the endofunctor $S_{\mathcal{M}} : \mathtt{C} \to \mathtt{C}$ defined as*

$$S_{\mathcal{M}}(V) := \bigoplus_{n\geqslant 0} (\mathcal{M}(n) \otimes V^{\otimes n})_{\mathfrak{S}_n} \ ,$$

where $\mathcal{M}(n) \otimes V^{\otimes n}$ is endowed with the diagonal right $\mathfrak{S}_n$-action.

This construction defines in fact a functor $S_{-} : \mathfrak{S}\text{-}\mathtt{mod} \to \mathrm{EndoFun}(\mathtt{C})$. Recall moreover from Exercice sheet 1 that the category $(\mathrm{EndoFun}(\mathtt{C}), \circ, \mathrm{id}_{\mathtt{C}})$ is a strict monoidal category.

Proposition 1.23. *The functor S_{-} is strong monoidal. In particular, an operad structure on $\mathcal{P}$ induces a monoid structure on its Schur functor $S_{\mathcal{P}}$.*

In other words, an operad $\mathcal{P}$ defines a *monad* $S_{\mathcal{P}}$.

1.4.3. *Free algebra over an operad.*

Proposition 1.24. *(i) A $\mathcal{P}$-algebra structure on an object A of $\mathtt{C}$ is equivalent to a $S_{\mathcal{P}}$-algebra structure on A, i.e. to a morphism $\mu : S_{\mathcal{P}}(A) \to A$ such that the following diagrams commute*

$$\begin{tikzcd} (S_{\mathcal{P}} \circ S_{\mathcal{P}})(A) = S_{\mathcal{P}}(S_{\mathcal{P}}(A)) \arrow[r, "S_{\mathcal{P}}(\mu)"] \arrow[d, "\gamma_A"] & S_{\mathcal{P}}(A) \arrow[d, "\mu"] \\ S_{\mathcal{P}}(A) \arrow[r, "\mu"] & S_{\mathcal{P}}(A) \end{tikzcd} \qquad \begin{tikzcd} A = \mathrm{id}_{\mathtt{C}}(A) \arrow[r, "\eta_A"] \arrow[dr, "\mathrm{id}_A"'] & S_{\mathcal{P}}(A) \arrow[d, "\mu"] \\ & A \end{tikzcd} .$$

(ii) A morphism $f : A_1 \to A_2$ is then a morphism of $\mathcal{P}$-algebras if and only if the following diagram commutes

$$\begin{tikzcd} S_{\mathcal{P}}(A_1) \arrow[r, "\gamma_{A_1}"] \arrow[d, "S_{\mathcal{P}}(f)"] & A_1 \arrow[d, "f"] \\ S_{\mathcal{P}}(A_2) \arrow[r, "\gamma_{A_2}"] & A_2 \end{tikzcd} .$$

Definition 1.25. *For an operad $\mathcal{P}$ and a vector space V of $\mathtt{C}$, the* free $\mathcal{P}$-algebra *on V is defined to be the object $S_{\mathcal{P}}(V)$ whose $\mathcal{P}$-algebra structure is given by the morphism*

$$S_{\mathcal{P}}(S_{\mathcal{P}}(V)) = (S_{\mathcal{P}} \circ S_{\mathcal{P}})(V) \xrightarrow{\gamma_V} S_{\mathcal{P}}(V) \ .$$

Proposition 1.26. *The free $\mathcal{P}$-algebra construction defines a functor $S_{\mathcal{P}} : \mathtt{C} \to \mathcal{P}\text{-}\mathtt{alg}$ which is left adjoint to the forgetful functor $\mathcal{P}\text{-}\mathtt{alg} \to \mathtt{C}$. In other words, for every $V \in \mathtt{C}$ and $\mathcal{P}$-algebra A, there is a natural bijection*

$$\mathrm{Hom}_{\mathcal{P}\text{-}\mathtt{alg}}(S_{\mathcal{P}}(V), A) = \mathrm{Hom}_{\mathtt{C}}(V, A) \ .$$

Example 1.27. If $\mathtt{C} = \mathtt{gr\ Vect}$, the free $u\mathcal{A}ss$-algebra on a graded vector space V is the free graded tensor algebra $T(V)$ and the free $\mathcal{C}om$-algebra on V is the reduced free graded commutative algebra $\overline{\Lambda}V = \bigoplus_{n\geqslant 1} \Lambda^n V$.

1.5. Nonsymmetric operads. Let $\mathtt{C}$ be a closed symmetric monoidal category.

Definition 1.28. *A* $\mathbb{N}$- module *is defined to be a sequence of objects* $\{\mathcal{M}(n)\}_{n\geqslant 0}$ *in* $\mathtt{C}$.

Definition 1.29. *A* nonsymmetric operad *or* ns operad *structure on a* $\mathbb{N}$*-module* $\{\mathcal{P}(n)\}_{n\geqslant 0}$ *is defined to be the data of composition morphisms*

$$\gamma_{i_1,\cdots,i_k} : \mathcal{P}(k) \boxtimes \mathcal{P}(i_1) \boxtimes \cdots \boxtimes \mathcal{P}(i_k) \to \mathcal{P}(i_1 + \cdots + i_k)$$

and of a unit morphism $\eta : I \to \mathcal{P}(1)$ *which satisfy Item 2 and Item 4 of Definition 1.7.*

Every symmetric operad yields a ns operad by forgetting about the symmetric groups actions: this defines a forgetful functor $\mathtt{Op} \to \mathtt{nsOp}$. The endomorphism and coendormophism operads End_C and coEnd_C of an object $C \in \mathtt{C}$ are thereby in particular ns operads. Given a ns operad $\mathcal{P}$, a $\mathcal{P}$*-algebra* structure on A is then defined as a morphism of ns operads $\mathcal{P} \to \mathrm{End}_A$, and a $\mathcal{P}$*-coalgebra* structure on C is defined as a morphism of ns operads $\mathcal{P} \to \mathrm{coEnd}_C$.

Example 1.30.
(i) The ns operad uAs is the ns operad in $\mathtt{Set}$ whose arity n set of operations is a singleton $\{*\}$ for every $n \geqslant 0$. A uAs-algebra X is then a monoid X in $\mathtt{Set}$.
(ii) The ns operad $u\mathcal{A}s := \mathbb{K}[uAs]$ is the ns operad in vector spaces which encodes unital associative/unital graded associative/unital dg algebras. Beware that $u\mathcal{A}s$ is not the image of the operad $u\mathcal{A}ss$ under the forgetful functor $\mathtt{Op} \to \mathtt{nsOp}$, but the image of the operad $u\mathcal{C}om$!

Assume now that the category $\mathtt{C}$ is either $\mathtt{Vect}$, $\mathtt{gr\ Vect}$ or $\mathtt{dg\ Vect}$. The *Schur functor* associated to a $\mathbb{N}$-module $\mathcal{M}$ is the endofunctor $S_{\mathcal{M}} : \mathtt{C} \to \mathtt{C}$ defined as

$$S_{\mathcal{M}}(V) := \bigoplus_{n\geqslant 0} \mathcal{M}(n) \otimes V^{\otimes n} \ .$$

All constructions and propositions of Section 1.4 then still hold in the ns case. For instance, the free $u\mathcal{A}s$-algebra on a vector space V is again the free tensor algebra $S_{u\mathcal{A}s}(V) = T(V)$.

2. Free operad

In this section, $\mathtt{C}$ either denotes $\mathtt{Vect}$, $\mathtt{gr\ Vect}$ or $\mathtt{dg\ Vect}$.

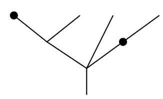
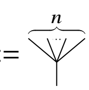
2.1. Trees.

2.1.1. *Planar and nonplanar trees.*

Definition 2.1. *A* planar tree t *is defined to be a tree which satisfies the following conditions:*

(1) it has a distinguished outgoing edge called the root*,*
(2) each vertex v *of* t *is endowed with a linear order on the set of its incoming edges* $\mathrm{inc}(v)$ *(a way to embed the tree in the plane),*
(3) its input edges can be capped by stumps *i.e. vertices with no incoming edge,*
(4) vertices with only one incoming edge are allowed.

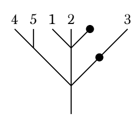
These conditions imply in particular that the set of non-capped input edges of t is linearly ordered. We will moreover denote $\mathrm{Vert}(t)$ the set of vertices of t, and PT_n the *set of planar trees* with n non-capped input edges. The *trivial tree* | is then an element of PT_1. The corolla of arity $n \geqslant 1$ will moreover be denoted $c_n \in \mathrm{PT}_n$.

 $\in \mathrm{PT}_3$ $\in \mathrm{PT}_0$ $c_n :=$  $\in \mathrm{PT}_n$

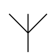
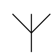
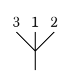
Unary vertices and capping vertices are represented with bullets in the above planar trees.

Definition 2.2. *A* nonplanar tree *is defined to be a planar tree t together with a permutation of its non-capped input edges.*

We will denote $\mathrm{T}_n \simeq \mathrm{PT}_n \times \mathfrak{S}_n$ the *set of nonplanar trees* with n non-capped input edges.

 $\in \mathrm{T}_5$

We will use three different representation for an element of T_n depending on the context, as illustrated below.

 $\cdot\, (123) =$  $^{(123)} =$  $\in \mathrm{T}_3$

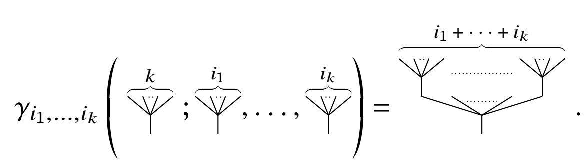
An element of T_n will moreover either be denoted as $t\cdot\sigma$ where t denotes its underlying planar tree and σ the associated permutation, or simply as t depending on the context.

2.1.2. *The ns operad $\mathcal{PT}$.*

Definition 2.3. *The ns operad $\mathcal{PT}$ is the $\mathbb{N}$-module $\mathcal{PT} := \{\mathrm{PT}_n\}_{n\geqslant 0}$ with composition maps*

$$\gamma_{i_1,\dots,i_k} : \mathrm{PT}_k \times \mathrm{PT}_{i_1} \times \cdots \times \mathrm{PT}_{i_k} \longrightarrow \mathrm{PT}_{i_1+\cdots+i_k}$$

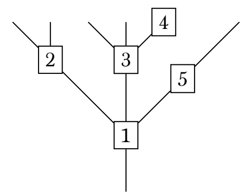
given by grafting the root of a tree in PT_{i_j} to the j-th non-capped input edge of a tree in PT_k.



This ns operad structure permits us to define an order on the vertices of a planar tree as follows. Every planar tree $\neq$ ⫯, | can be written as a composition $t := \gamma(c_k; t_1, \dots, t_k)$, where c_k is the corolla of arity $k \geqslant 1$. If $t_1, \dots, t_k$ are all equal to the trivial tree |, $t = c_k$ and there is only one vertex. Otherwise, we put the vertex v_{c_k} of c_k in first position and then proceed by induction to concatenate the orders on the vertices of the t_i, i.e.

$$\mathrm{Vert}(t) = v_{c_k} < \mathrm{Vert}(t_1) < \cdots < \mathrm{Vert}(t_k)\ .$$

We represent an example of an ordering of the vertices of a planar tree below.



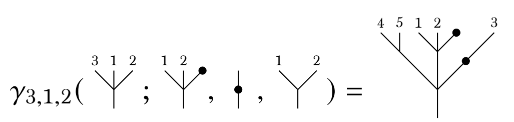
2.1.3. *The operad $\mathcal{T}$.*

Definition 2.4. *The sets T_n for $n \geqslant 0$ define a $\mathfrak{S}$-module $\mathcal{T}$ in* `Set`*, where $\mathfrak{S}_n$ acts on the right of T_n by permuting the n non-capped input edges of the nonplanar tree. This $\mathfrak{S}$-module is an operad with composition maps*

$$\gamma_{i_1,\ldots,i_k} : \mathrm{T}_k \times \mathrm{T}_{i_1} \times \cdots \times \mathrm{T}_{i_k} \to \mathrm{T}_{i_1+\cdots+i_k}$$

given by grafting the root of a tree $t_j \cdot \sigma_j \in \mathrm{T}_{i_j}$ to the j-th non-capped input edge of a tree $t \cdot \sigma \in \mathrm{T}_k$. The choice of permutation on the planar tree obtained under this composition is then defined to be $\sigma_{\boldsymbol{i}}(\sigma_1 \times \cdots \times \sigma_k)$.

We have for instance that



We moreover linearly order the vertices of a nonplanar tree by ordering the vertices of its underlying planar tree as in the previous section.

2.2. Free operad.

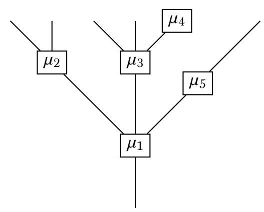
2.2.1. *Free ns operad.*

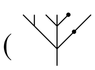
Definition 2.5. *Let $\mathcal{M}$ be a $\mathbb{N}$-module and t a planar tree. We define*

$$\mathcal{M}(t) := \bigotimes_{v \in \mathrm{Vert}(t)} \mathcal{M}(|\mathrm{inc}(v)|)\ ,$$

where the set of vertices is ordered as explained in Section 2.1.2 and we set $\mathcal{M}(\,|\,) := \mathbb{K}$.

An element of $\mathcal{M}(t)$ can be represented as a linear combination of labelings of the vertices of t by operations in $\mathcal{M}$. For instance



$\in \mathcal{M}($  $)$

where $\mu_1 \in \mathcal{M}(3)$, $\mu_2 \in \mathcal{M}(2)$, $\mu_3 \in \mathcal{M}(3)$, $\mu_4 \in \mathcal{M}(0)$ and $\mu_5 \in \mathcal{M}(1)$.

Definition 2.6. *The* free ns operad $\mathcal{T}_{ns}(\mathcal{M})$ *on a* $\mathbb{N}$*-module* $\mathcal{M}$ *is defined to be the* $\mathbb{N}$*-module*

$$\mathcal{T}_{ns}(\mathcal{M})(n) := \bigoplus_{t\in \mathrm{PT}_n} \mathcal{M}(t) \ ,$$

endowed with the composition maps

$$\mathcal{T}_{ns}(\mathcal{M})(k) \otimes \mathcal{T}_{ns}(\mathcal{M})(i_1) \otimes \cdots \otimes \mathcal{T}_{ns}(\mathcal{M})(i_k) \to \mathcal{T}_{ns}(\mathcal{M})(i_1 + \cdots + i_k)$$

defined on each summand as

$$\mathcal{M}(t') \otimes \mathcal{M}(t_1) \otimes \cdots \otimes \mathcal{M}(t_k) \to \mathcal{M}(\gamma(t; t_1, \ldots, t_k)) \hookrightarrow \mathcal{T}_{ns}(\mathcal{M})(i_1 + \cdots + i_k)$$

where $t' \in \mathrm{PT}_k$, $t_j \in \mathrm{PT}_{i_j}$ *for* $1 \leqslant j \leqslant k$ *and the left arrow corresponds to reordering the factors of* $\mathcal{M}(t') \otimes \mathcal{M}(t_1) \otimes \cdots \otimes \mathcal{M}(t_k)$ *into the factors of* $\mathcal{M}(\gamma(t; t_1, \ldots, t_k))$.

In other words, the composition of the operad $\mathcal{T}_{ns}(\mathcal{M})$ is given by the grafting of trees whose vertices are labeled by operations of $\mathcal{M}$.

Example 2.7. (i) The free ns operad on the $\mathbb{N}$-module $V = (0, V, 0, \ldots, 0, \ldots)$ is

$$\mathcal{T}_{ns}(V) = (0, T(V), 0, \ldots, 0)$$

where $T(V)$ is the free tensor algebra of Definition 1.4.

(ii) The free ns operad on the $\mathbb{N}$-module $(\mathbb{K}, \mathbb{K}, \mathbb{K}, \ldots)$ is the operad $\mathbb{K}[\mathcal{PT}]$.

A free ns operad $\mathcal{T}_{ns}(\mathcal{M})$ is in particular weight graded, where for $m \geqslant 0$ the $\mathbb{N}$-module $\mathcal{T}_{ns}(\mathcal{M})^{(m)}$ is defined as

$$\mathcal{T}_{ns}(\mathcal{M})^{(m)}(n) = \bigoplus_{\substack{t\in \mathrm{PT}_n \\ |\mathrm{Vert}(t)|=m}} \mathcal{M}(t) \ .$$

Proposition 2.8. *The functor* $\mathcal{T}_{ns}(-) : \mathbb{N}\text{-}\mathtt{mod} \to \mathtt{nsOp}$ *is left adjoint to the forgetful functor* $\mathtt{nsOp} \to \mathbb{N}\text{-}\mathtt{mod}$. *In other words, for every* $\mathbb{N}$*-module* $\mathcal{M}$ *and ns operad* $\mathcal{P}$ *there is a natural bijection*

$$\mathrm{Hom}_{\mathtt{nsOp}}(\mathcal{T}_{ns}(\mathcal{M}), \mathcal{P}) \simeq \mathrm{Hom}_{\mathbb{N}\text{-}\mathtt{mod}}(\mathcal{M}, \mathcal{P}) \ .$$

In particular, a structure of $\mathcal{T}_{ns}(\mathcal{M})$-algebra on A simply corresponds to a morphism of $\mathbb{N}$-modules $\mathcal{M}(n) \to \underline{\mathrm{Hom}}(A^{\otimes n}, A)$.

2.2.2. *Free operad.* For a nonplanar tree $t\cdot\sigma$ and a $\mathfrak{S}$-module $\mathcal{M}$, we define $\mathcal{M}(t\cdot\sigma) := \mathcal{M}(t)$ as in Definition 2.5, where the set of vertices is ordered as in Section 2.1.3 and we set $\mathcal{M}(\,|\,) := \mathbb{K}$. We will now define the underlying $\mathfrak{S}$-module $\mathcal{T}(\mathcal{M})$ of the free operad on $\mathcal{M}$. For every $n \geqslant 0$, the vector space $\bigoplus_{t\in \mathrm{T}_n} \mathcal{M}(t)$ admits a right $\mathfrak{S}_n$-action defined on a summand as

$$\tau_\sigma : \mathcal{M}(t') \xrightarrow{\mathrm{id}} \mathcal{M}(t'\cdot\sigma) \hookrightarrow \bigoplus_{t\in\mathrm{T}_n} \mathcal{M}(t)\ .$$

Definition 2.9. *We define the $\mathfrak{S}$-module $\mathcal{T}(\mathcal{M})$ as*

$$\mathcal{T}(\mathcal{M})(n) := (\bigoplus_{t\in\mathrm{T}_n} \mathcal{M}(t))/\sim\ .$$

where the quotient is defined by induction as explained below.

Any nonplanar tree $\neq$ ⫯, $|$ can be written as a composition $t := \gamma(c_k\cdot\sigma; t_1,\ldots,t_k)\cdot\sigma'$ where:

(1) c_k is the corolla of arity $k \geqslant 1$ and $\sigma \in \mathfrak{S}_k$ is a permutation of its incoming edges,
(2) $t_j \in \mathrm{PT}_{i_j}$ for $1 \leqslant j \leqslant k$ and $i_1+\cdots+i_k = n$,
(3) σ' is a permutation of the non-capped input edges of $\gamma(c_k\cdot\sigma; t_1,\ldots,t_k)$.

Write $\mu\otimes\boldsymbol{\nu}$ an element of $\mathcal{M}(t) = \mathcal{M}(k)\otimes\cdots$ where μ is the operation labeling the unique vertex of the corolla c_k. The quotient is then defined by induction on the number of vertices of the tree t by the identification

$$\mathcal{M}(t) \ni \mu\otimes\boldsymbol{\nu} = \mu\cdot\sigma\otimes\boldsymbol{\nu}' \in \mathcal{M}(t')$$

where

(1) $t' = \gamma(c_k; t_{\sigma^{-1}(1)},\ldots,t_{\sigma^{-1}(k)})\sigma_{\boldsymbol{i}}\sigma'$ where $\sigma_{\boldsymbol{i}}$ is defined in Definition 1.6,
(2) $\boldsymbol{\nu}'$ is obtained by reordering the factors of $\boldsymbol{\nu}$ following the ordering on the vertices of t'.

We moreover check that the action of $\mathfrak{S}_n$ is still well-defined after quotienting.

The operations of $\mathcal{T}(\mathcal{M})$ are thereby to be understood as linear combinations of labelings of the vertices of nonplanar trees by operations of $\mathcal{M}$, such that permuting the incoming edges at a vertex is equal to applying this permutation to the label of this vertex.

Definition 2.10. *The* free operad *on the $\mathfrak{S}$-module $\mathcal{M}$ is defined to be the $\mathfrak{S}$-module $\mathcal{T}(\mathcal{M})$ endowed with the composition given by the grafting of nonplanar trees as in Definition 2.6.*

The operad $\mathcal{T}(\mathcal{M})$ is again weight graded by the number of vertices of a nonplanar tree $\mathcal{T}_{ns}(\mathcal{M}) = \bigoplus_{m\geqslant 0}\mathcal{T}_{ns}(\mathcal{M})^{(m)}$.

Example 2.11. (i) The free operad on the $\mathfrak{S}$-module $V = (0, V, 0,\ldots,0,\ldots)$ is

$$\mathcal{T}(V) = (0, T(V), 0,\ldots,0)\ .$$

(ii) The free operad on the $\mathfrak{S}$-module $\{K[\mathfrak{S}_n]\}_{n\geqslant 0}$ is the operad $\mathbb{K}[\mathcal{T}]$.

Proposition 2.12. *The functor $\mathcal{T}(-) : \mathfrak{S}\text{-}\mathtt{mod} \to \mathtt{Op}$ is left adjoint to the forgetful functor $\mathtt{Op} \to \mathfrak{S}\text{-}\mathtt{mod}$. In other words, for every $\mathfrak{S}$-module $\mathcal{M}$ and operad $\mathcal{P}$ there is a natural bijection*

$$\mathrm{Hom}_{\mathtt{Op}}(\mathcal{T}(\mathcal{M}),\mathcal{P}) \simeq \mathrm{Hom}_{\mathfrak{S}\text{-}\mathtt{mod}}(\mathcal{M},\mathcal{P})\ .$$

2.3. Presentation of an operad.

DEFINITION 2.13. *Let $\mathcal{P}$ be an operad.*

(i) An ideal *of $\mathcal{P}$ is defined to a $\mathfrak{S}$-module $\mathcal{M}$ such that $\mathcal{M}(n) \subset \mathcal{P}(n)$ for every $n \geqslant 0$ and such that for any family of operations $\nu; \mu_1, \ldots, \mu_k$ of $\mathcal{P}$, if one of them is in $\mathcal{M}$ then theire composition $\gamma(\nu; \mu_1, \ldots, \mu_k)$ is in $\mathcal{M}$.*

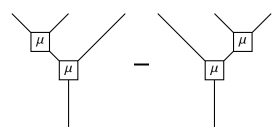
(ii) Given a collection of operations $r_j \in \mathcal{P}$, the ideal generated *by the r_j is defined to be the smallest ideal of $\mathcal{P}$ which contains the r_j.*

We will say that an operad $\mathcal{P}$ admits a *presentation*, if there exists a $\mathfrak{S}$-module $\mathcal{M}$ and an ideal $\mathcal{I} \subset \mathcal{T}(\mathcal{M})$ such that

$$\mathcal{P} = \mathcal{T}(\mathcal{M})/\mathcal{I} \ .$$

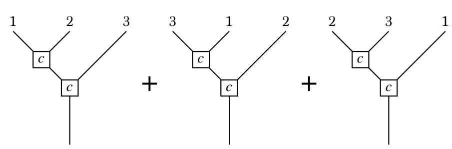
In most cases, we will assume that the ideal $\mathcal{I}$ is generated by some r_j in $\mathcal{T}(\mathcal{M})$. Choosing for every $n \geqslant 0$ a basis μ_i^n for the vector space $\mathcal{M}(n)$, we will call the elements $\mu_i^n \in \mathcal{M}(n)$ the *generating operations* of the operad $\mathcal{P}$ and the elements $r_j \in \mathcal{T}(\mathcal{M})$ its *relators.*

EXAMPLE 2.14. (i) The ns operad $\mathcal{A}s$ encoding associative algebras admits a presentation with generating $\mathbb{N}$-module $(0, 0, \mathbb{K}\mu, 0, \ldots)$ and with ideal generated by the relator



.

(ii) The operad $\mathcal{L}ie$ encoding Lie algebra admits a presentation with generating $\mathfrak{S}$-module $(0, 0, \mathbb{K}c, 0, \ldots)$ where $c \cdot (12) = -c$ and with ideal generated by the relator



.

3. Cooperads

In this section, we let `C` be one the following three closed symmetric monoidal categories: `Vect`, `gr Vect` and `dg Vect`.

3.1. Definitions.

3.1.1. *May cooperad.*

DEFINITION 3.1. *A* May cooperad *structure on a $\mathfrak{S}$-module $\{\mathcal{C}(n)\}_{n \geqslant 0}$ is defined to be data of:*

(1) A morphism

$$\delta^{i_1, \ldots, i_k} : \mathcal{C}(n) \to \mathcal{C}(k) \otimes \mathcal{C}(i_1) \otimes \cdots \otimes \mathcal{C}(i_k)$$

for all $k \geqslant 1$ and $i_1, \ldots, i_k \geqslant 0$ such that $i_1 + \cdots + i_k = n$, called a decomposition morphism.

(2) A morphism $\varepsilon : \mathcal{C}(1) \to \mathbb{K}$ called the counit.

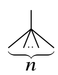
These data have to satisfy the following properties:

(1) The maps $\delta^{i_1,\dots,i_k}$ *are equivariant under the right action of* $\mathfrak{S}_{i_1} \times \cdots \times \mathfrak{S}_{i_k}$.
(2) The maps

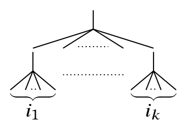
$$\prod_{i_1+\cdots+i_k=n} \delta^{i_1,\dots,i_k} : \mathscr{C}(n) \to \prod_{i_1+\cdots+i_k=n} \mathscr{C}(k) \otimes \mathscr{C}(i_1) \otimes \cdots \otimes \mathscr{C}(i_k)$$

factor through the invariants of $\mathfrak{S}_k$ *on the target.*
(3) The decomposition morphisms and the counit satisfy coassociativity and counit axioms dual to the ones of an operad.

The vector space $\mathscr{C}(n)$ is then called the *space of cooperations of arity* n of the May cooperad $\mathscr{C}$. The decomposition morphisms can be represented as



$$\in \mathscr{C}(n) \xmapsto{\delta_{i_1,\dots,i_k}}$$



$$\in \mathscr{C}(k) \otimes \mathscr{C}(i_1) \otimes \cdots \otimes \mathscr{C}(i_k)\ .$$

Example 3.2. A May cooperad structure on the $\mathfrak{S}$-module $(0, C, 0, \dots, 0, \dots)$ is exactly a counital coassociative coalgebra structure on C

We will moreover denote Δ^n the arity n *total decomposition map*

$$\Delta^n = \prod_{k\geqslant 1} \prod_{i_1+\cdots+i_k=n} \delta_{i_1,\dots,i_k} : \mathscr{C}(n) \to \prod_{k\geqslant 1} \prod_{i_1+\cdots+i_k=n} \mathscr{C}(k) \otimes \mathscr{C}(i_1) \otimes \cdots \mathscr{C}(i_k)\ .$$

3.1.2. *Cooperads and $\mathscr{C}$-coalgebras.*

Definition 3.3. *(i) The* cocomposite product *on* $\mathfrak{S}$-mod *is defined as*

$$\mathscr{M} \bar{\circ} \mathscr{N}(n) = \bigoplus_{k\geqslant 0} \left(\mathscr{M}(k) \otimes \bigoplus_{i_1+\cdots+i_k=n} \mathrm{Ind}^{\mathfrak{S}_n}_{\mathfrak{S}_{i_1}\times\cdots\times\mathfrak{S}_{i_k}} \left(\mathscr{N}(i_1) \otimes \cdots \otimes \mathscr{N}(i_k)\right) \right)^{\mathfrak{S}_k}\ .$$

(ii) The coSchur functor *associated to* $\mathscr{M}$ *is the endofunctor* $S^{\mathscr{M}} : \mathtt{C} \to \mathtt{C}$ *defined as*

$$S^{\mathscr{M}}(V) := \bigoplus_{n\geqslant 0} (\mathscr{M}(n) \otimes V^{\otimes n})^{\mathfrak{S}_n}\ .$$

Proposition 3.4. *(i) The cocomposite product* $\bar{\circ}$ *defines a monoidal structure on the category* $\mathfrak{S}$-mod.
(ii) The functor $S^- : (\mathfrak{S}\text{-mod}, \bar{\circ}) \to (\mathrm{EndoFun}(\mathtt{C}), \circ)$ *is strong monoidal i.e.* $S^{\mathscr{M}\bar{\circ}\mathscr{N}} = S^{\mathscr{M}} \circ S^{\mathscr{N}}$

Definition 3.5. *A* cooperad $\mathscr{C}$ *is defined to be a May cooperad such that the total decomposition maps* Δ^n *factor through*

$$\Delta^n : \mathscr{C}(n) \to \bigoplus_{k\geqslant 1} \bigoplus_{i_1+\cdots+i_k=n} \mathscr{C}(k) \otimes \mathscr{C}(i_1) \otimes \cdots \mathscr{C}(i_k)\ .$$

In other words, a cooperad is a May cooperad such that for each $n \geqslant 0$ and $\mu \in \mathscr{C}(n)$ there exists only a finite numbers of $i_1, \dots, i_k \geqslant 0$ with $i_1 + \cdots + i_k = n$ such that $\delta^{i_1,\dots,i_k}(\mu) \neq 0$.

Generalizing Sweedler's notation to cooperads, we will denote the image of a cooperation $\mu \in \mathscr{C}(n)$ under Δ^n as the finite sum

$$\Delta^n(\mu) = \sum(\nu; \nu_1, \ldots, \nu_k) \in \bigoplus_{k\geqslant 1} \bigoplus_{\substack{i_1+\cdots+i_k=n \\ i_1,\ldots,i_k\geqslant 0}} \mathscr{C}(k) \otimes \mathscr{C}(i_1) \otimes \cdots \otimes \mathscr{C}(i_k) \ .$$

Proposition 3.6. *A cooperad structure on a $\mathfrak{S}$-module $\mathscr{C}$ is equivalent to a comonoid structure on $\mathscr{C}$ in $(\mathfrak{S}\text{-}\mathtt{mod}, \bar{\circ})$.*

A cooperad structure on a $\mathfrak{S}$-module $\mathscr{C}$ then induces in particular a comonad structure on its coSchur functor $S^{\mathscr{C}}$.

Definition 3.7. *Let $\mathscr{C}$ be a cooperad. A $\mathscr{C}$-*coalgebra *structure on a vector space C is defined to be a $S^{\mathscr{C}}$-coalgebra structure on C, i.e. the datum of a linear map $\Delta_C : C \to \bigoplus_{n\geqslant 0}(\mathscr{C}(n) \otimes C^{\otimes n})^{\mathfrak{S}_n} = S^{\mathscr{C}}(C)$ such that the following diagrams commute*

$$\begin{tikzcd} C \arrow[r, "\Delta_C"] \arrow[d, "\Delta_C"] & S^{\mathscr{C}}(C) \arrow[d, "S^{\mathscr{C}}(\Delta_C)"] \\ S^{\mathscr{C}} \arrow[r, "\Delta_C^{\mathscr{C}}"] & S^{\mathscr{C}}(S^{\mathscr{C}}(C)) \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "\Delta_C"] \arrow[dr, "\mathrm{id}_C"'] & S^{\mathscr{C}}(C) \arrow[d, "\varepsilon_C"] \\ & C \end{tikzcd}$$

In other words, a $\mathscr{C}$-coalgebra C corresponds to the data of $\mathfrak{S}_n$-invariant linear maps

$$\Delta^n : C \to \mathscr{C}(n) \otimes C^{\otimes n}$$

for every $n \geqslant 0$ that are compatible with the decomposition of the cooperad $\mathscr{C}$ and such that for every $c \in C$, only a finite number of $\Delta^n(c)$ are $\neq 0$.

Remark 3.8. Hence, a $\mathscr{C}$-coalgebra is in some sense always conilpotent. See Exercice 8 in Exercice sheet 3 for more details.

3.2. Conilpotent cooperads. Recall that the $\mathfrak{S}$-module $\mathbb{K}$ is defined as the $\mathfrak{S}$-module concentrated in arity 1 $(0, \mathbb{K}, 0, \ldots)$. It carries an obvious structure of cooperad.

Definition 3.9. *A* coaugmented cooperad *is defined to be the data of a cooperad $\mathscr{C}$ together with a morphism of cooperads $\eta : \mathbb{K} \to \mathscr{C}$ called the* coaugmentation.

Let $\mathscr{C}$ be a coaugmented cooperad. We define $\mathrm{id} := \eta(1_{\mathbb{K}}) \in \mathscr{C}(1)$ and call it the *identity*. We also define $\bar{\mathscr{C}}(n) := \mathscr{C}(n)$ for $n \neq 1$ and $\bar{\mathscr{C}}(1) := \mathrm{Ker}(\varepsilon)$, where ε is the counit of $\mathscr{C}$.

We moreover set $\bar{T}_n := T_n$ for $n \neq 1$ and $\bar{T}_1 := T_1 - \{\,|\,\}$. Given $\mathscr{M}$ a $\mathfrak{S}$-module we then denote $\bar{\mathscr{T}}^{\wedge}(\mathscr{M})$ the $\mathfrak{S}$-module given in arity n by

$$\bar{\mathscr{T}}^{\wedge}(\mathscr{M}) := \prod_{t\in\bar{T}_n} \mathscr{M}(t)/\sim \ ,$$

where $\sim$ is defined as in Definition 2.9. For $m \geqslant 0$, we also denote

$$\bar{\mathscr{T}}^{\wedge}(\mathscr{M})^{(m)}(n) := \prod_{\substack{t\in\bar{T}_n \\ |\mathrm{Vert}(t)|=m}} \mathscr{M}(t)/\sim \ .$$

Proposition 3.10. *A coaugmented cooperad $\mathscr{C}$ determines a morphism of $\mathfrak{S}$-modules*

$$\bar{\Delta} : \bar{\mathscr{C}} \to \bar{\mathscr{T}}^\wedge(\bar{\mathscr{C}}) \ .$$

Proof. The projection $\bar{\mathscr{C}} \to \bar{\mathscr{T}}^\wedge(\bar{\mathscr{C}})^{(1)} = \mathscr{C}$ is equal to the identity. The projection $\bar{\mathscr{C}} \to \bar{\mathscr{T}}^\wedge(\bar{\mathscr{C}})^{(2)}$ is defined for $\mu \in \mathscr{C}(n)$ as the sum $\sum \nu_1 \otimes \nu_2$ for all decompositions of the form $(\nu_1; \mathrm{id}, \ldots, \mathrm{id}, \nu_2, \mathrm{id}, \ldots, \mathrm{id})$ appearing in $\Delta^n(\mu)$. The sum $\Delta^n(\mu) - (\mu; \mathrm{id}, \ldots, \mathrm{id}) - (\mathrm{id}; \mu)$ determines in fact the projection of $\bar{\Delta}$ to the factors associated to 2-leveled trees in $\bar{\mathscr{T}}^\wedge(\bar{\mathscr{C}})$. More generally, the factors of $\bar{\Delta}(\mu)$ are obtained by iterating the total decomposition maps Δ^n and discarding the summands featuring an intermediate cooperation equal to id or a final level of cooperations equal to id. □

In other words, the map $\bar{\Delta}$ is defined by iterating the total decomposition maps Δ^n and ensuring that the equivalent decompositions of a cooperation contribute exactly once to the associated factor in $\bar{\mathscr{T}}^\wedge(\bar{\mathscr{C}})$.

Definition 3.11. *Let $\mathscr{C}$ be a coaugmented cooperad. The* coradical filtration *on $\bar{\mathscr{C}}$ is defined as*

$$F_r\bar{\mathscr{C}} := \mathrm{Ker}\left(\bar{\Delta}^{\geqslant r} : \bar{\mathscr{C}} \to \bar{\mathscr{T}}^\wedge(\bar{\mathscr{C}})^{(\geqslant r)}\right) \ ,$$

where $\bar{\Delta}^{\geqslant r}$ denotes the projection of $\bar{\Delta}$ to $\bar{\mathscr{T}}^\wedge(\bar{\mathscr{C}})^{(\geqslant r)}$.

It is indeed a filtration as

$$0 = F_0\bar{\mathscr{C}} \subset F_1\bar{\mathscr{C}} \subset \cdots \subset F_r\bar{\mathscr{C}} \subset \cdots \ .$$

Definition 3.12. *A coaugmented cooperad is said to be* conilpotent *if its coradical filtration is complete, i.e. if $\bar{\mathscr{C}} = \bigcup_{r\geqslant 0} F_r\bar{\mathscr{C}}$.*

In other words, a coaugmented cooperad $\mathscr{C}$ is *conilpotent* if a sequence of nontrivial decompositions of any cooperation in $\bar{\mathscr{C}}$ always terminates.

Remark 3.13. There are two main obstructions for a cooperad to be conilpotent.

(1) The *vertical obstruction* coming from the coassociative coalgebra structure on $\mathscr{C}(1)$ with coproduct δ^1. This is in particular one of the reasons why we define conilpotency using $\bar{\mathscr{C}}$ and not $\mathscr{C}$, as the identity can be decomposed ad libitum.
(2) The *horizontal obstruction* which comes from the existence of arity 0 elements.

Proposition 3.14. *(i) A May cooperad $\mathscr{C}$ with $\mathscr{C}(0) = 0$ is a cooperad.*
(ii) A coaugmented cooperad $\mathscr{C}$ with $\mathscr{C}(0) = 0$ and $\mathscr{C}(1) = \mathbb{K}\mathrm{id}$ is conilpotent.

3.3. Cofree cooperad. Let t be a nonplanar tree. A *degrafting* of t is defined to be a collection of trees $t', t_1, \ldots, t_k$ such that $t = \gamma(t'; t_1, \ldots, t_k)$. Degrafting of trees defines a cooperad structure on the $\mathfrak{S}$-module $\{\mathrm{T}_n\}_{n\geqslant 0}$ in `Set`. We will denote it as $\mathscr{T}^c$.

Definition 3.15. *The* cofree cooperad *on a $\mathfrak{S}$-module $\mathscr{M}$ is defined to be the $\mathfrak{S}$-module $\mathscr{T}(\mathscr{M})$ endowed with the decomposition given by the degrafting of nonplanar trees. Its is denoted $\mathscr{T}^c(\mathscr{M})$.*

The cofree cooperad $\mathcal{T}^c(\mathcal{M})$ on $\mathcal{M}$ is moreover conilpotent: a sequence of nontrivial degraftings of a nonplanar tree whose vertices are labeled by operations of $\mathcal{M}$ always terminates when all the pieces of the degrafted tree are equal to corollae labeled by an operation of $\mathcal{M}$.

Proposition 3.16. *The functor $\mathcal{T}^c(-)$: $\mathfrak{S}$-$\mathtt{mod} \to \mathtt{conil\ Coop}$ is right adjoint to the functor $\mathcal{C} \in \mathtt{conil\ Coop} \to \bar{\mathcal{C}} \in \mathfrak{S}$-$\mathtt{mod}$. In other words, for every $\mathfrak{S}$-module $\mathcal{M}$ and conilpotent cooperad $\mathcal{C}$ there is a natural bijection*

$$\mathrm{Hom}_{\mathfrak{S}\text{-}\mathtt{mod}}(\bar{\mathcal{C}}, \mathcal{M}) \simeq \mathrm{Hom}_{\mathtt{conil\ Coop}}(\mathcal{C}, \mathcal{T}^c(\mathcal{M})) \ .$$

Example 3.17. (i) The cofree cooperad on the $\mathfrak{S}$-module $V = (0, V, 0, \dots, 0, \dots)$ is the cofree coalgebra

$$\mathcal{T}^c(V) = (0, T^c(V), 0, \dots, 0) \ .$$

(ii) The cofree cooperad on the $\mathfrak{S}$-module $\{K[\mathfrak{S}_n]\}_{n\geqslant 0}$ is the cooperad $\mathbb{K}[\mathcal{T}^c]$.

4. Applications in algebraic topology

4.1. Recognition principle for k-fold loop spaces.

4.1.1. *k-fold loop spaces.*

Definition 4.1. *Let X be a topological space and $x \in X$. The* (based) loop space $\Omega_x X$ *is defined to be the topological space of pointed continuous maps* $(\mathbb{S}^1, 0) \to (X, x)$.

The topology taken on $\Omega_x X$ is the compact-open topology and it is naturally pointed by the constant loop at x. We will write $\Omega X := \Omega_x X$ in the rest of this section for the sake of readability. The space ΩX being pointed, we can define $\Omega^2 X := \Omega(\Omega X)$ and $\Omega^k X$ for any $k \geqslant 0$ inductively. We then call a space of the form $\Omega^k X$ a *k-fold loop space.*

4.1.2. *Little k-cubes operad.*

Definition 4.2. *A* linear embedding with parallel axes $I^n \to I^n$ *is a map of the form*

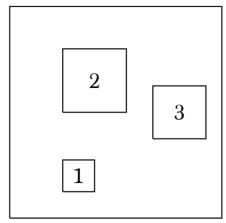
$$(t_1, \dots, t_n) \mapsto (c_1(t_1), \dots, c_n(t_n))$$

where $c_i(t_i) = (1 - t_i)x_i + t_i y_i$ *for fixed* $0 < x_i < y_i \leqslant 1$.

Definition 4.3. *The* little k-cubes operad $\mathcal{C}_k := \{\mathcal{C}_k(n)\}_{n\geqslant 1}$ *is the operad in topological spaces defined as follows:*

(1) The arity n space $\mathcal{C}_k(n)$ is the space of ordered collections of n linear embeddings with parallel axes $I^k \to I^k$ with disjoint interiors. A collection of n linear embeddings will be written $\bigsqcup_{i=1}^n I^k \to I^k$.
(2) The action of $\mathfrak{S}_n$ on $\mathcal{C}_k(n)$ is given by permuting the linear embeddings.
(3) Given $l \geqslant 1$, collections of linear embeddings $\bigsqcup_{i=1}^{n_j} I^k \to I^k$ in $\mathcal{C}_k(n_j)$ for $1 \leqslant l \leqslant k$ and a collection of linear embeddings $\bigsqcup_{i=1}^{l} I^k \to I^k$ in $\mathcal{C}_k(l)$, their composition in $\mathcal{C}_k(n_1 + \cdots + n_l)$ is given by inserting each collection of linear embeddings of $\mathcal{C}_k(n_j)$ in the j-th slot of $\bigsqcup_{i=1}^{l} I^k \to I^k$.

An element of $\mathscr{C}_2(3)$ can for instance be represented as

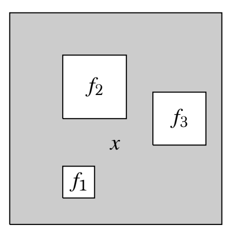


.

4.1.3. *Recognition principle for k-fold loop spaces.*

Proposition 4.4. *Every k-fold loop space $\Omega^k X$ is an algebra over the little k-cubes operad.*

Proof. An element of the k-fold loop space $\Omega^k X$ can be equivalently defined as a map $I^n \to X$ which maps the boundary of I^n to the base point x. The structure maps $\mathscr{C}_k(n) \times (\Omega^k X)^{\times n} \to \Omega^k X$ endowing $\Omega^k X$ with a $\mathscr{C}_k$-algebra structure are then defined as follows. Consider an element $\bigsqcup_{i=1}^n I^k \to I^k$ in $\mathscr{C}_k(n)$ and n elements $f_i : I^k \to X$ of $\Omega^k X$. Their image under the $\mathscr{C}_k$-algebra structure map is defined to be the map $I^k \to X$ whose restriction to the i-th embedded cube is f_i and whose restriction to the complement of the interiors of embedded cubes is the constant map x. An element of $\mathscr{C}_2(3)$ acts for instance on f_1, f_2 and f_3 as



.

□

Theorem 8. *If a connected space X is an algebra over the little k-cubes operad then there exists a pointed topological space X_k such that X is homotopy equivalent to $\Omega^k X_k$.*

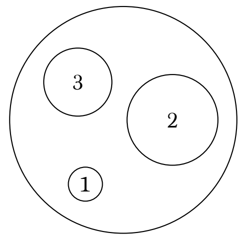
4.2. Framed little disks operad.

4.2.1. *Little k-disks operad.*

Definition 4.5. *The* little k-disks operad $\mathscr{D}_k := \{\mathscr{D}_k(n)\}_{n \geqslant 1}$ *is the operad in topological spaces defined as follows:*

(1) The arity n space $\mathscr{D}_k(n)$ is the space of ordered collections of n embeddings $\mathbb{D}^k \to \mathbb{D}^k$ by translation and dilation with disjoint interiors.
(2) The action of $\mathfrak{S}_n$ on $\mathscr{D}_k(n)$ and the composition maps are defined as in Definition 4.3.

An element of $\mathscr{D}_2(3)$ can for instance be represented as



The functor $H_*(\cdot) : \mathtt{Top} \to \mathtt{gr\ Vect}$ mapping a topological space to its singular homology with coefficients in $\mathbb{K}$ is strong monoidal, i.e. $H_*(X \times Y) \simeq H_*(X) \otimes H_*(Y)$. Following Proposition 1.12, the functor $H_*(\cdot)$ thereby maps operads in topological spaces to operads in graded vector spaces.

Proposition 4.6. *For every $k \geqslant 1$ the little k-disks operad $\mathscr{D}_k$ and the little k-cubes operad $\mathscr{C}_k$ are weakly equivalent, i.e. there exists a zig-zag of morphisms of operads in topological spaces*

$$\mathscr{D}_k =: \mathscr{P}_0 \leftarrow \mathscr{P}_1 \rightarrow \cdots \leftarrow \mathscr{P}_{n-1} \rightarrow \mathscr{P}_n := \mathscr{C}_k$$

that induces a zig-zag of isomorphisms of operads in graded vector spaces

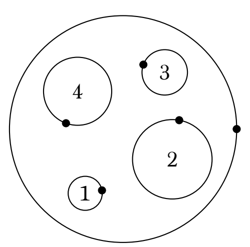
$$H_*(\mathscr{D}_k) = H_*(\mathscr{P}_0) \xleftarrow{\sim} H_*(\mathscr{P}_1) \xrightarrow{\sim} \cdots \xleftarrow{\sim} H_*(\mathscr{P}_{n-1}) \xrightarrow{\sim} H_*(\mathscr{P}_n) = H_*(\mathscr{C}_k) \ .$$

4.2.2. *Framed little disks operad.*

Definition 4.7. *The framed little disks operad $f\mathscr{D}_2 := \{f\mathscr{D}_2(n)\}_{n \geqslant 1}$ is the operad in topological spaces defined as follows:*

(1) The arity n space $f\mathscr{D}_2(n)$ is the space of ordered collections of n embeddings $\mathbb{D}^2 \to \mathbb{D}^2$ by translation, dilation AND rotation with disjoint interiors.
(2) The action of $\mathfrak{S}_n$ on $f\mathscr{D}_2(n)$ and the composition maps are defined as in Definition 4.3 .

An element of $f\mathscr{D}_2(4)$ can for instance be represented as

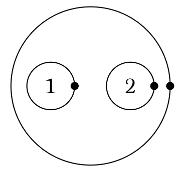


where the marked points on the boundaries of the disks represents the rotations used in the embeddings.

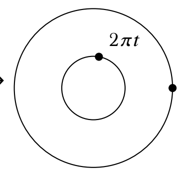
Theorem 9. *The singular homology of the framed little disks operad is isomorphic to the BV operad as an operad in graded vector spaces*

$$H_*(f\mathscr{D}_2) \simeq BV \ .$$

Proof. The following element of $f\mathcal{D}_2(2)$ seen as a 0-chain



is mapped in homology to the multiplication operation of the operad BV, while the following 1-cycle in $f\mathcal{D}_2(1)$

$t \in [0,1] \mapsto$ 

is mapped in homology to the Δ operation of the operad BV. $\square$

In particular if X is a topological space with a $f\mathcal{D}_2$-algebra structure, its singular homology $H_*(X)$ inherits a BV algebra structure.

CHAPTER 4

Twisting morphisms

1. Twisting morphisms

1.1. Convolution algebra and twisting morphisms. Let A be a dg algebra and C be a dg coalgebra.

Definition 1.1. *We define the* convolution algebra *of A and C to be the dg vector space*

$$\underline{\mathrm{Hom}}(C, A) := \underline{\mathrm{Hom}}_{\mathtt{dg\ Vect}}(C, A) = \bigoplus_{r\in\mathbb{Z}} \prod_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathtt{Vect}}(C_n, A_{n+r})$$

endowed with the differential $\partial := [\partial, \cdot]$ and the convolution product

$$f \star g := \mu_A(f \otimes g)\Delta_C \ .$$

The convolution algebra $(\underline{\mathrm{Hom}}(C, A), \star, \partial)$ is a dg algebra. If A is unital with unit u_A and C is counital with counit ε_C, it is moreover a unital dg algebra with unit $u_A\varepsilon_C$.

Definition 1.2. *A* twisting morphism *$\alpha : C \longrightarrow A$ is defined to be a linear map of degree -1 such that*

$$\partial(\alpha) + \alpha \star \alpha = 0 \ .$$

If C is coaugmented with coaugmentation u_C and A is augmented with augmentation ε_A, the linear map α also has to satisfy

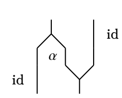
$$\varepsilon_A\alpha = 0 \qquad\qquad \alpha u_B = 0 \ .$$

We denote $\mathrm{Tw}(C, A)$ the *set of twisting morphisms $C \longrightarrow A$.* We point out that a twisting morphism is exactly a Maurer-Cartan element in the dg algebra $\underline{\mathrm{Hom}}(C, A)$.

We now define the map $\partial^r_\alpha : C \otimes A \longrightarrow C \otimes A$ as

$$\partial^r_\alpha := (\mathrm{id}_C \otimes \mu_A)(\mathrm{id}_C \otimes \alpha \otimes \mathrm{id}_A)(\Delta_C \otimes \mathrm{id}_A) \ ,$$

which can be represented as

$\partial^r_\alpha =$ 

.

In other words $\partial^r_\alpha(c \otimes a) = c^{(1)} \otimes \mu_A(c^{(2)}, a)$.

Proposition 1.3. *Let α be a twisting morphism. Then the degree -1 map*

$$\partial_\alpha := \partial^r_\alpha + \partial_{C\otimes A}$$

defines a differential on $C \otimes A$.

Proof. $\partial_\alpha^2 = (\partial_\alpha^r)^2 + \partial_{C\otimes A}\partial_\alpha^r + \partial_\alpha^r\partial_{C\otimes A} + \partial_{C\otimes A}^2 = \partial_{\alpha\star\alpha}^r + \partial_{\partial(\alpha)}^r + 0 = \partial_{\partial\alpha+\alpha\star\alpha}^r = 0.$ □

We will denote $C \otimes_\alpha A := (C \otimes A, \partial_\alpha)$ and call it the *(right) twisted tensor product*. The *(left) twisted tensor product* $A_\alpha \otimes C$ can be defined in a similar fashion.

1.2. Homology of fiber spaces. Twisted differentials are used in the computation of the singular homology of fiber spaces. We sketch the results of [**Bro59**] in this section.

1.2.1. *Space of Moore loops.*

Definition 1.4. *Let (B, b_0) be a pointed topological space. We define the* space of Moore loops $\Omega^M_{b_0}X$ *as*

$$\Omega^M_{b_0}B := \{\gamma : [0,r] \to B, \gamma(0) = \gamma(r) = b_0, r \geqslant 0\} .$$

We then have a natural inclusion $\Omega B \subset \Omega^M B$ where ΩB is the standard based loop space (we drop the subscript $\cdot_{b_0}$ for the sake of readability). This inclusion is in fact a strong deformation retract, which implies in particular that these two spaces are homotopy equivalent.

The space of Moore loops carries a topological monoid structure, given by the concatenation of loops: for $\gamma_1 : [0,r] \to B$ and $\gamma_2 : [0,s] \to B$, we define

$$\gamma_1 * \gamma_2 : t \in [0, r+s] \mapsto \begin{cases} \gamma_1(t) \text{ if } t \in [0,r] \\ \gamma_2(t-r) \text{ if } t \in [r, r+s] \end{cases} .$$

Its unit is moreover given by e_{b_0} the *constant path* at b_0 of length 0. This implies that the singular chains $C_*(\Omega^M B)$ form a dg algebra.

Remark 1.5. The based loop space ΩB endowed with the concatenation of based loops is not a topological monoid, as concatenation of loops is not associative. We will however see in Section 2 that it is an A$_\infty$-space, i.e. a monoid whose multiplication is associative up to homotopy and higher coherent homotopies.

Definition 1.6. *(i) The* space of Moore paths *of a topological space B is defined as*

$$P(B) := \{\gamma : [0,r] \to B, r \geqslant 0\} .$$

(ii) The space of based Moore paths *of a pointed topological space (B, b_0) is the subspace $E(B) \subset P(B)$ of Moore paths that end at b_0.*

We point out that two paths $\alpha, \beta \in P(B)$ such that $\alpha(r) = \beta(0)$ can still be concatenated to a path $\alpha * \beta$.

1.2.2. *Weakly transitive Hurewicz fibrations.*

Definition 1.7. *A continuous map $p : E \to B$ is said to be a* Hurewicz fibration *if it admits the* homotopy-lifting property, *i.e. if for every topological space X the following diagram can be filled*

$$\begin{tikzcd} X \times \{0\} \arrow[r] \arrow[d, hook] & E \arrow[d, "p"] \\ X \times [0,1] \arrow[r] \arrow[ur, dashed] & B \end{tikzcd} .$$

We define the *fiber* over a point $b \in B$ to be the space $p^{-1}(b)$. For a Hurewicz fibration, all fibers over a path component of B can be proven to be homotopy equivalent: one can thereby speak of the fiber $F := p^{-1}(b)$ of a Hurewicz fibration when B is path connected. This is usually denoted as $F \hookrightarrow E \to B$.

Example 1.8. (i) The *path space fibration* is defined as $\mathrm{ev}_0 : \gamma \in E(B) \mapsto \gamma(0) \in B$. It is an Hurewicz fibration with fiber $ev_0^{-1}(b_0) = \Omega^M B$, i.e.

$$\Omega^M B \hookrightarrow E(B) \to B \ .$$

(ii) Covering spaces and vector bundles are Hurewicz fibrations

Let $p : E \to B$ be a Hurewicz fibration and assume that we have chosen a point $b_0 \in B$. We consider the fiber product

$$U_p := P(B)_{\mathrm{ev}} \times_p E = \{(\gamma, e) \in P(B) \times E, \gamma(r) = p(e)\} \ .$$

We define a *lifting function* of a Hurewicz fibration $p : E \to B$ to be a map $\lambda : U_p \to E$ such that $p\lambda = \mathrm{ev}_0$. The homotopy lifting property ensures that a Hurewicz fibration always admits a lifting function.

Definition 1.9. *A lifting function for p is said to be* weakly transitive *if*

(1) $\lambda(e_{b_0}, x) = x$ for every $x \in p^{-1}(b_0)$,
(2) $\lambda(\alpha * \beta, x) = \lambda(\alpha, \lambda(\beta, x))$ for every $\alpha, \beta \in P(B)$ such that $\alpha(r) = \beta(0) = b_0$.

A Hurewicz fibration is said to be *weakly transitive* if it admits a weakly transitive lifting function. A weakly transitive lifting function defines in particular a left action of $\Omega^M B$ on the fiber $F = p^{-1}(b_0)$ by setting

$$\alpha \cdot x = \lambda(\alpha, x) \ .$$

This action induces a left $C_*(\Omega^M B)$-module structure on $C_*(F)$.

Example 1.10. The path space fibration is weakly transitive. The action of $\Omega^M B$ on the fiber $\Omega^M B \hookrightarrow E(B)$ is then simply given by the concatenation of Moore loops.

Remark 1.11. The weakly transitive property for a Hurewicz fibration should be compared to the the fact the the fundamental group of a connected topological space B acts on the fiber of its universal cover $\tilde{B} \to B$.

1.2.3. *Brown's twisting morphism.* Given A a dg algebra, C a dg coalgebra, (M, μ_M) a dg A-module and $\alpha : C \to A$ a twisting morphism. We can again define a twisted differential on $C \otimes M$ as $\partial_\alpha := \partial_\alpha^r + \partial_{C \otimes M}$ where $\partial_\alpha^r = (\mathrm{id}_C \otimes \mu_M)(\mathrm{id}_C \otimes \alpha \otimes \mathrm{id}_M)(\Delta_C \otimes \mathrm{id}_M)$.

Theorem 10. *Let B be a path connected topological space and $b_0 \in B$.*

(i) There exists a twisting morphism $\Phi_B : C_*(B) \to C_*(\Omega^M B)$, where $C_*(B)$ is endowed with the Alexander-Whitney coproduct.
(ii) For every weakly transitive Hurewicz fibration $F \to E \to B$, there is a quasi-isomorphism

$$C_*(B) \otimes_{\Phi_B} C_*(F) \to C_*(E) \ .$$

This theorem should be understood as a generalization of Künneth's theorem: if the fibration is trivial i.e. if $E = B \times F$, then the twisted tensor product $C_*(B) \otimes_{\Phi_B} C_*(F)$ is exactly $C_*(B) \otimes C_*(F)$ and we recover that $H_*(B \times F) \simeq H_*(B) \otimes H_*(F)$.

1.3. The cobar-bar adjunction.

1.3.1. *Bar and cobar constructions.* Let (A, μ, ∂) be a dg algebra. We define two degree -1 maps $d_1, d_2 : T^c(sA) \to T^c(sA)$ as

$$d_1(sa_1, \ldots, sa_n) = \sum_{i=1}^{n} (-1)^{i+|a_1|+\cdots+|a_{i-1}|} sa_1 \otimes \cdots \otimes s\partial a_i \otimes \cdots \otimes sa_n$$

$$d_2(sa_1, \ldots, sa_n) = \sum_{i=1}^{n-1} (-1)^{i-1+|a_1|+\cdots+|a_i|} sa_1 \otimes \cdots \otimes s\mu(a_i, a_{i+1}) \otimes \cdots \otimes sa_n \ .$$

We check that $d_1^2 = 0$, $d_2^2 = 0$ and $d_1 d_2 + d_2 d_1 = 0$. The map $d_1 + d_2$ hence defines a differential on the graded vector space $T^c(sA)$. This differential is moreover compatible with the deconcatenation coproduct.

Definition 1.12. *The* bar construction *of an augmented dg algebra A is defined to be the dg coalgebra*

$$BA := (T^c(s\bar{A}), d_1 + d_2) \ .$$

In a similar fashion, let (C, Δ, ∂) be a dg coalgebra. We define this time $d_2 : T(s^{-1}C) \to T(s^{-1}C)$ as

$$d_2(s^{-1}c_1, \ldots, s^{-1}c_n) = \sum_{i=1}^{n} (-1)^{i+|c_1|+\cdots+|c_i^{(1)}|} s^{-1}c_1 \otimes \cdots \otimes s^{-1}c_i^{(1)} \otimes s^{-1}c_i^{(2)} \otimes \cdots \otimes s^{-1}c_n \ .$$

Defining d_1 as previously, we prove that $d_1 + d_2$ defines a differential which is compatible with the concatenation product of $T(s^{-1}C)$.

Definition 1.13. *The* cobar construction *of a coaugmented dg coalgebra C is defined to be the dg algebra*

$$\Omega C := (T(s^{-1}\bar{C}), d_1 + d_2) \ .$$

1.3.2. *Cobar-bar adjunction and universal twisting morphisms.* The cobar and bar constructions define functors

$$\Omega : \texttt{conil dg cog} \leftrightarrows \texttt{aug dg alg} : B \ .$$

Proposition 1.14. *The cobar and bar constructions define an adjunction $\Omega \dashv B$ such that*

$$\mathrm{Hom}_{\texttt{aug dg alg}}(\Omega C, A) \simeq \mathrm{Tw}(C, A) \simeq \mathrm{Hom}_{\texttt{conil dg cog}}(C, BA) \ .$$

Proof. A twisting morphism $\alpha : C \to A$ satifies $\varepsilon_A \alpha = \alpha u_C = 0$, hence is determined by its restriction $\bar{\alpha} : \bar{C} \to \bar{A}$ which is a linear map of degree -1. Consider a morphism of augmented dg algebras $F : \Omega C \to A$. Following Proposition 1.5, it is completely determined as a morphism of unital algebras by its restriction $s^{-1}\bar{C} \to A$. This degree 0 map is equivalent to a degree -1 map $f : C \to A$ such that $\varepsilon_A \alpha = \alpha u_C = 0$. The fact that F preserves the differentials then implies that

$$\partial(f) + f \star f = 0 \ .$$

A similar argument using Proposition 1.12 proves the second bijection. □

1.3.3. *Universal twisting morphisms.* The unit $C \to B\Omega C$ and counit $\Omega BA \to A$ of the adjunction of Proposition 1.14 respectively induce twisting morphisms $\iota : C \to \Omega C$ and $\pi : BA \to A$. They can be explicitly computed: the twisting morphism π is equal to the composition

(1) $$BA = T^c(s\bar{A}) \twoheadrightarrow s\bar{A} \xrightarrow{s^{-1}} \bar{A} \hookrightarrow A$$

while the twisting morphism ι is equal to the composition

(2) $$C \twoheadrightarrow \bar{C} \xrightarrow{s^{-1}} s^{-1}\bar{C} \hookrightarrow T(s\bar{C}) = \Omega C \ .$$

They are called the *universal twisting morphisms* as every twisting morphism $\alpha : C \to A$ factors through them:

(3) $$\begin{tikzcd} C \arrow[r, "\iota"] \arrow[d, dashed, "f_\alpha"'] \arrow[dr, "\alpha"] & \Omega C \arrow[d, dashed, "g_\alpha"] \\ BA \arrow[r, "\pi"'] & A \end{tikzcd} ,$$

where f_α is the morphism of augmented dg algebras and g_α is the morphism of conilpotent dg coalgebras of Proposition 1.14.

Proposition 1.15. *The right twisted tensor products $BA \otimes_\pi A$ and $C \otimes_\iota \Omega C$ and the left twisted tensor products $A_\pi \otimes BA$ and $\Omega C_\iota \otimes C$ are acyclic.*

PROOF. We refer to Exercise sheet 4 for the proof of this result, and to Section 1.4.1 for a comment on the notion of acyclicity in this context. □

REMARK 1.16. The twisted tensor products $BA \otimes_\pi A$ and $C \otimes_\iota \Omega C$ are usually respectively called the *augmented bar construction* of A and the *coaugmented cobar construction* of C.

1.4. Koszul morphisms. We assume that every dg vector space is concentrated in non-negative degree in this section.

1.4.1. *Koszul criterion.*

DEFINITION 1.17. *Let C be a coaugmented dg coalgebra and A be an augmented dg algebra. A twisting morphism $\alpha : C \to A$ is said to be* Koszul *if the twisted complexes $C \otimes_\alpha A$ and $A_\alpha \otimes C$ are acyclic.*

In Proposition 1.15 and Definition 1.17, the left twisted Koszul complex $C \otimes_\alpha A$ is said to be *acyclic* if $H_n(C \otimes_\alpha A) = 0$ and $H_0(C \otimes_\alpha A) = \mathbb{K}1_C \otimes 1_A$.

REMARK 1.18. Unless C is cocommutative and A is commutative, the twisted tensor products $C \otimes_\alpha A$ and $A_\alpha \otimes C$ are in general not quasi-isomorphic.

EXAMPLE 1.19. The universal twisting morphisms $\iota : C \to \Omega C$ and $\pi : BA \to A$ are Koszul.

Recall that we have defined weight graded differential (co)algebras in Section 1.3.2. We will use the abbreviation *wgd* in the rest of these notes. A unital wgd (co)algebra A is then said to be *connected* if A is concentrated in nonnegative degree and nonnegative weight $A = \bigoplus_{i \geqslant 0} A^{(i)}$ with $A^{(0)} = \mathbb{K}1_A$.

THEOREM 11. *Consider a twisting morphism $\alpha : C \to A$, where C is a connected coaugmented wgd coalgebra and A is a connected augmented wgd algebra. Then the following conditions are equivalent:*

(1) the twisted complex $C \otimes_\alpha A$ is acyclic,
(2) the twisted complex $A \otimes_\alpha C$ is acyclic,
(3) the induced morphism $\Omega C \to A$ is a quasi-isomorphism,
(4) the induced morphism $C \to BA$ is a quasi-isomorphism.

PROOF. Consider A_1, A_2 two connected augmented wgd algebras, C_1, C_2 two connected coaugmented wdg coalgebras, a morphism of coaugmented wgd coalgebras $f : C_1 \to C_2$ and a morphism of augmented wgd algebras $g : A_1 \to A_2$. Let $\alpha_1 : C_1 \to A_1$ and $\alpha_2 : C_2 \to A_2$ be twisting morphisms such that $g\alpha_1 = \alpha_2 f$. The map $g \otimes f$ is then a chain map

$$C_1 \otimes_{\alpha_1} A_1 \xrightarrow{g \otimes f} C_2 \otimes_{\alpha_2} A_2 \ .$$

The *comparison lemma for twisted tensor products* states that if two of the three chain maps f, g and $f \otimes g$ are quasi-isomorphism, then so is the third one.

Consider now a twisting morphism $\alpha : C \to A$. We have seen that it factors as $\alpha = g_\alpha \iota$ where $g_\alpha : \Omega C \to A$ is the morphism of coaugmented wgd coalgebras given by the cobar-bar adjunction. This implies that the map $\mathrm{id}_C \otimes g_\alpha : C \otimes_\iota \Omega C \to C \otimes_\alpha A$ is a chain map. The twisted tensor product $C \otimes_\alpha A$ is acyclic if and only if the chain map $\mathrm{id}_C \otimes g_\alpha$ is a quasi-isomorphism, hence if and only if the morphism g_α is a quasi-isomorphism by the comparison lemma. Thereby Item 1 $\Leftrightarrow$ Item 3.

We prove in a similar fashion that Item 2 $\Leftrightarrow$ Item 4 and that Item 3 $\Leftrightarrow$ Item 4. $\square$

This theorem is called the *Koszul criterion.* It is usually represented as

$$\begin{array}{ccccc}
\mathrm{Hom}_{\texttt{aug dg alg}}(\Omega C, A) & \simeq & \mathrm{Tw}(C, A) & \simeq & \mathrm{Hom}_{\texttt{conil dg cog}}(C, BA) \\
\cup & & \cup & & \cup \\
\mathrm{qIso}(\Omega C, A) & \simeq & \mathrm{Kosz}(C, A) & \simeq & \mathrm{qIso}(C, BA)
\end{array} .$$

Proposition 1.20. *Given an augmented dg algebra A and a coaugmented dg coalgebra C, the unit $\Omega BA \to A$ and counit $C \to B\Omega C$ are quasi-isomorphisms.*

PROOF. If C and A satisfy the assumptions of Theorem 11, this proposition is a simple corollary obtained by applying the theorem to the universal twisting morphisms. The result still holds when C and A are not necessarily weight graded and connected, but relies on a different proof. $\square$

1.4.2. *Cobar construction and the based loop space.* The following theorem was proven recently in [**Riv22**] drawing from [**Ada56**].

THEOREM 12. *For every path connected topological space B with base point b_0, there exists a quasi-isomorphism of dg algebras*

$$\Omega C_*(B) \simeq C_*(\Omega^M B) \ .$$

PROOF. The twisted complex $C_*(B) \otimes_{\Phi_B} C_*(\Omega^M B) \simeq C_*(E(B))$ is acyclic, as $E(B)$ is homotopy equivalent to $\{b_0\}$. [**Ada56**] proves a theorem similar to Theorem 11 if B is simply connected: the acyclicity of the twisted complex implies that the induced morphism $\Omega C_*(B) \simeq C_*(\Omega^M B)$ is a quasi-isomorphism. The general proof of [**Riv22**] uses more advanced tools. □

2. Graded operads and dg operads

2.1. Graded $\mathfrak{S}$-modules and dg $\mathfrak{S}$-modules. We respectively define a *graded $\mathfrak{S}$-module* and a *dg $\mathfrak{S}$-module* to be a $\mathfrak{S}$-module in `gr Vect` and a $\mathfrak{S}$-module in `dg Vect`. Let us spell out the definitions of Chapter 3 for dg $\mathfrak{S}$-modules.

A dg $\mathfrak{S}$-module $\mathcal{M}$ corresponds to the data of dg vector spaces $\mathcal{M}_*(n)$ with a right $\mathfrak{S}_n$-action for $n \geqslant 0$. The composite product of two dg $\mathfrak{S}$-modules then reads as

$$(\mathcal{M} \circ \mathcal{N})_p(n) = \bigoplus_{q+r_1+\cdots+r_k=p} \ \bigoplus_{k\geqslant 0} \mathcal{M}_q(k) \otimes_{\mathfrak{S}_k} \bigoplus_{i_1+\cdots+i_k=n} \mathrm{Ind}^{\mathfrak{S}_n}_{\mathfrak{S}_{i_1}\times\cdots\times\mathfrak{S}_{i_k}} \left(\mathcal{N}_{r_1}(i_1) \otimes \cdots \otimes \mathcal{N}_{r_k}(i_k)\right) \ .$$

with differential given by

$$\partial(\nu;\mu_1,\ldots,\mu_k) = (\partial\nu;\mu_1,\ldots,\mu_k) + \sum_{i=1}^{k} (-1)^{|\nu|+|\mu_1|+\cdots+|\mu_{i-1}|}(\nu;\mu_1,\ldots,\partial\mu_i,\ldots,\mu_k) \ .$$

Proposition 2.1. *If* $\mathrm{char}(\mathbb{K}) = 0$ *then* $H_*(\mathcal{M} \circ \mathcal{N}) \simeq H_*(\mathcal{M}) \circ H_*(\mathcal{N})$.

PROOF. The assumption that $\mathrm{char}(\mathbb{K}) = 0$ implie that every $\mathbb{K}[\mathfrak{S}_n]$-module is projective, hence that one can apply Künneth's theorem to the formula for $(\mathcal{M} \circ \mathcal{N})_*(n)$. □

This proposition should be understood as a generalization of Künneth's theorem for dg $\mathfrak{S}$-modules.

2.2. Graded operads and dg operads. We respectively define a *graded operad* and a *dg operad* to be an operad in `gr Vect` and an operad in `dg Vect`. The categories whose objects are graded operads and dg operads are then respectively denoted as `gr Op` and `dg Op`.

For a dg operad, the composition maps have in particular to satisfy

$$|\gamma(\nu;\mu_1,\ldots,\mu_k)| = |\nu| + |\mu_1| + \cdots + |\mu_k| \ ,$$

$$\partial(\gamma(\nu;\mu_1,\ldots,\mu_k)) = \gamma(\partial\nu;\mu_1,\ldots,\mu_k) + \sum_{i=1}^{k} (-1)^{|\nu|+|\mu_1|+\cdots+|\mu_{i-1}|}\gamma(\nu;\mu_1,\ldots,\partial\mu_i,\ldots,\mu_k) \ .$$

Proposition 2.2. *(i) A dg operad $\mathcal{P}$ induces a graded operad $H_*(\mathcal{P})$ in homology.*
(ii) A $\mathcal{P}$-algebra A induces a $H_*(P)$-algebra $H_*(A)$ in homology.

REMARK 2.3. We point out that a $\mathcal{P}$-algebra A also always induces a $\mathcal{P}$-algebra structure on $H_*(A)$ in homology.

2.3. Minimal models of operads.

2.3.1. *Quasi-free operads.* Let $\mathcal{M}$ be a graded $\mathfrak{S}$-module. The grading on $\mathcal{M}$ induces a grading on the free operad $\mathcal{T}(\mathcal{M})$: a nonplanar tree t whose vertices are labeled by operations μ_v of degree $|\mu_v|$ for $v \in \mathrm{Vert}(t)$ is defined to have degree

$$| \otimes_{v\in \mathrm{Vert}(t)} \mu_v| = \sum_{v\in \mathrm{Vert}(t)} |\mu_v| \ .$$

This induced grading together with the weight grading $\mathcal{T}(\mathcal{M})^{(m)}$ define a weight-graded operad structure on $\mathcal{T}(\mathcal{M})$.

Proposition 2.4. *The functor $\mathcal{T}(-)$: $\mathtt{gr}\ \mathfrak{S}\mathtt{-mod} \to \mathtt{gr}\ \mathtt{Op}$ is left adjoint to the forgetful functor $\mathtt{gr}\ \mathtt{Op} \to \mathtt{gr}\ \mathfrak{S}\mathtt{-mod}$. In other words, for every graded $\mathfrak{S}$-module $\mathcal{M}$ and graded operad $\mathcal{P}$ there is a natural bijection*

$$\mathrm{Hom}_{\mathtt{gr}\ \mathtt{Op}}(\mathcal{T}(\mathcal{M}), \mathcal{P}) \simeq \mathrm{Hom}_{\mathtt{gr}\ \mathfrak{S}\mathtt{-mod}}(\mathcal{M}, \mathcal{P}) \ .$$

DEFINITION 2.5. *A* quasi-free operad *is a dg operad whose underlying graded operad is a free operad.*

A *derivation* on a graded operad $\mathcal{P}$ is a morphism of graded $\mathfrak{S}$-modules $d : \mathcal{P} \to \mathcal{P}$ such that

$$d(\gamma(\nu;\mu_1,\ldots,\mu_k)) = \gamma(d\nu;\mu_1,\ldots,\mu_k) + \sum_{i=1}^{k}(-1)^{|\nu|+|\mu_1|+\cdots+|\mu_{i-1}|}\gamma(\nu;\mu_1,\ldots,d\mu_i,\ldots,\mu_k) \ .$$

A dg operad can then be equivalently defined as a graded operad endowed with a degree -1 derivation ∂ such that $\partial^2 = 0$.

Proposition 2.6. *For a $\mathfrak{S}$-module $\mathcal{M}$, there is a correspondence between derivations of the operad $\mathcal{T}(\mathcal{M})$ and morphisms of $\mathfrak{S}$-modules $\mathcal{M} \to \mathcal{T}(\mathcal{M})$.*

PROOF. Let $d : \mathcal{T}(\mathcal{M}) \to \mathcal{T}(\mathcal{M})$ be a derivation and consider $\delta : \mathcal{M} \hookrightarrow \mathcal{T}(\mathcal{M}) \to \mathcal{T}(\mathcal{M})$ its restriction to $\mathcal{M}$. Then the operadic Leibniz relation implies that applying d to a nonplanar tree t whose vertices are labeled by operations of $\mathcal{M}$ corresponds exactly to applying δ to each vertices of t separately. In other words, the derivation d is completely determined by the morphism of $\mathfrak{S}$-modules δ. □

A *coderivation* on a cooperad $\mathcal{C}$ can be defined in a similar fashion.

Proposition 2.7. *For a $\mathfrak{S}$-module $\mathcal{M}$, there is a correspondence between coderivations of the cofree cooperad $\mathcal{T}^c(\mathcal{M})$ and morphisms of $\mathfrak{S}$-modules $\mathcal{T}^c(\mathcal{M}) \to \mathcal{M}$.*

PROOF. See Exercise sheet 4. □

2.3.2. *Minimal models.*

DEFINITION 2.8. *A* minimal operad *is a quasi-free operad* $(\mathcal{T}(\mathcal{M}), \partial)$ *such that*

(1) $\partial(\mathcal{M}) \subset \mathcal{T}(\mathcal{M})^{(\geqslant 2)}$ *i.e.* ∂ *is decomposable,*
(2) $\mathcal{M} = \cup_{k\geqslant 0}\mathcal{M}^{(k)}$ *where* $\mathcal{M}^{(k)}$ *is an increasing sequence of graded* $\mathfrak{S}$*-modules* $\mathcal{M}^{(0)} \subset \mathcal{M}^{(1)} \subset \cdots$,
(3) $\partial(\mathcal{M}^{(0)}) = 0$ *and* $\partial(\mathcal{M}^{(k)}) \subset \mathcal{T}(\mathcal{M}^{(k-1)})$ *for* $k \geqslant 1$.

The additional grading $\mathcal{M}^{(k)}$ is called the *syzygy grading.*

DEFINITION 2.9. *A* minimal model *for a dg operad* $\mathcal{P}$ *is the data of a minimal operad* $(\mathcal{T}(\mathcal{M}), \partial)$ *together with a quasi-isomorphism of dg operads* $(\mathcal{T}(\mathcal{M}), \partial) \xrightarrow{\sim} \mathcal{P}$.

THEOREM 13. *A minimal model for an operad is unique up to (non-unique) isomorphism of dg operads.*

PROOF. As for Proposition 2.8, the proof comes down to proving that two minimal models of an operad $\mathcal{P}$ are always quasi-isomorphic, which implies that their generating graded $\mathfrak{S}$-modules are quasi-isomorphic hence isomorphic. □

The goal of Chapter 5 will be to construct explicit and computable minimal models $\mathcal{P}_\infty \xrightarrow{\sim} \mathcal{P}$ for a particular class of operads $\mathcal{P}$ (Koszul operads), using the framework introduced in this chapter. We will then see that $\mathcal{P}_\infty$-algebras provide a satisfactory notion of homotopy $\mathcal{P}$-algebras. This stems partly from the fact that quasi-free operads are the fibrant-cofibrant objects in the model category `dg Op`.

3. Operadic twisting morphisms

3.1. Operadic convolution pre-Lie algebra and twisting morphisms.

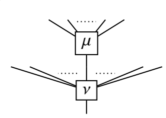
3.1.1. *Infinitesimal composite.*

DEFINITION 3.1. *Let* $\mathcal{M}$ *and* $\mathcal{N}$ *be two* $\mathfrak{S}$*-modules. We define the* infinitesimal composite $\mathcal{M} \circ_{(1)} \mathcal{N}$ *to be the* $\mathfrak{S}$*-module given in arity* n *by*

$$\mathcal{M} \circ_{(1)} \mathcal{N}(n) := \left(\bigoplus_{i_1+i_2+i_3=n} \mathrm{Ind}_{\mathfrak{S}_{i_2}}^{\mathfrak{S}_n} \left(\mathcal{M}(i_1+1+i_3) \otimes \mathcal{N}(i_2)\right)\right) / \sim$$

where $\mathfrak{S}_{i_2}$ *is seen as the subgroup of* $\mathfrak{S}_n$ *permuting the elements of* $[[i_1+1, i_1+i_2]] \subset [[1,n]]$ *and the quotient is defined as in Definition 2.9.*

The infinitesimal composite defines in fact a functor $- \circ_{(1)} - : \mathfrak{S}\text{-}\mathtt{mod} \times \mathfrak{S}\text{-}\mathtt{mod} \to \mathfrak{S}\text{-}\mathtt{mod}$. An operation $\nu \otimes \mu$ of $\mathcal{M} \circ_{(1)} \mathcal{N}$ can simply be seen as an operation of the form



.

The infinitesimal composite of a $\mathfrak{S}$-module $\mathcal{M}$ with itself $\mathcal{M} \circ_{(1)} \mathcal{M}$ can moreover be seen to be equal to the weight 2 part of the free operad $\mathcal{T}(\mathcal{M})^{(2)}$.

Definition 3.2. *(i) Let $\mathcal{P}$ be an operad. We define the* infinitesimal composition *of $\mathcal{P}$ as*

$$\gamma_{(1)} : \mathcal{P} \circ_{(1)} \mathcal{P} \to \mathcal{P} \circ \mathcal{P} \xrightarrow{\gamma^{\mathcal{P}}} \mathcal{P} \ .$$

(ii) Let $\mathcal{C}$ be a cooperad. We define the infinitesimal decomposition *of $\mathcal{C}$ as*

$$\Delta_{(1)} : \mathcal{C} \xrightarrow{\Delta^{\mathcal{C}}} \mathcal{C} \bar{\circ} \mathcal{C} \to \mathcal{C} \circ_{(1)} \mathcal{C} \ .$$

The "projection" $\mathcal{C} \bar{\circ} \mathcal{C} \to \mathcal{C} \circ_{(1)} \mathcal{C}$ is defined as

$$(\nu; \mu_1, \ldots, \mu_k) \mapsto \sum_{i=1}^{k} \nu \otimes \varepsilon(\mu_1) \cdots \varepsilon(\mu_{i-1}) \mu_i \varepsilon(\mu_{i+1}) \cdots \varepsilon(\mu_k)$$

where ε is the counit of $\mathcal{C}$. If $\mathcal{C}$ is coaugmented, for $\mu \in \mathcal{C}$ the element $\Delta_{(1)}(\mu)$ in $\mathcal{C} \circ_{(1)} \mathcal{C}$ can be identified to the sum $\sum (\nu; \mathrm{id}, \ldots, \mathrm{id}, \mu_i, \mathrm{id}, \ldots, \mathrm{id})$ obtained by discarding all terms in $\Delta(\mu) = \sum(\nu; \mu_1, \ldots, \mu_k)$ which are not of the form $(\nu; \mathrm{id}, \ldots, \mathrm{id}, \mu_i, \mathrm{id}, \ldots, \mathrm{id})$. The "inclusion" $\mathcal{P} \circ_{(1)} \mathcal{P} \to \mathcal{P} \circ \mathcal{P}$ is defined in a similar fashion using the unit of $\mathcal{P}$.

3.1.2. *Operadic convolution pre-Lie algebra.*

Definition 3.3. *Let $\mathcal{P}$ be a dg operad and $\mathcal{C}$ be a dg cooperad. We define the* convolution operad *of $\mathcal{P}$ and $\mathcal{C}$ to be the following dg operad:*

(1) the arity n space of operations is

$$\underline{\mathrm{Hom}}(\mathcal{C}, \mathcal{P})(n) := \underline{\mathrm{Hom}}_{\mathtt{dg\ Vect}}(\mathcal{C}(n), \mathcal{P}(n))$$

endowed with the right $\mathfrak{S}_n$-action defined by the formula $(f \cdot \sigma)(x) = f(x \cdot \sigma^{-1}) \cdot \sigma$;

(2) the composition $\gamma(f; g_1, \ldots, g_k)$ of $g : \mathcal{C}(k) \to \mathcal{P}(k)$ and $f_j : \mathcal{C}(i_j) \to \mathcal{P}(i_j)$ where $1 \leqslant j \leqslant k$ is defined as the composite linear map

$$\mathcal{C}(i_1 + \cdots + i_k) \xrightarrow{\delta^{i_1, \ldots, i_k}} \mathcal{C}(k) \otimes \mathcal{C}(i_1) \otimes \cdots \otimes \mathcal{C}(i_k) \xrightarrow{g \otimes f_1 \otimes \cdots \otimes f_k} \mathcal{P}(k) \otimes \mathcal{P}(i_1) \otimes \cdots \otimes \mathcal{P}(i_k) \xrightarrow{\gamma^{i_1, \ldots, i_k}} \mathcal{P}(i_1 + \cdots + i_k) \ ;$$

(3) the unit is equal to $u_{\mathcal{P}} \varepsilon_{\mathcal{C}}$ where $\varepsilon_{\mathcal{C}}$ is the counit of $\mathcal{C}$ and $u_{\mathcal{P}}$ is the unit of $\mathcal{P}$.

For two morphisms of $\mathfrak{S}$-modules $f, g : \mathcal{C} \to \mathcal{P}$, we define the morphism of $\mathfrak{S}$-modules $f \star g : \mathcal{C} \to \mathcal{P}$ as the composite

$$f \star g : \mathcal{C} \xrightarrow{\Delta_{(1)}} \mathcal{C} \circ_{(1)} \mathcal{C} \xrightarrow{f \circ_{(1)} g} \mathcal{P} \circ_{(1)} \mathcal{P} \xrightarrow{\gamma_{(1)}} \mathcal{P} \ .$$

We moreover define the dg vector space $\underline{\mathrm{Hom}}_{\mathtt{gr}\ \mathfrak{S}\text{-}\mathtt{mod}}(\mathcal{C}, \mathcal{P})_*$ as the graded vector space

$$\underline{\mathrm{Hom}}_{\mathtt{gr}\ \mathfrak{S}\text{-}\mathtt{mod}}(\mathcal{C}, \mathcal{P})_r := \prod_{n \geqslant 0} \underline{\mathrm{Hom}}^{\mathfrak{S}_n}_{\mathtt{dg\ Vect}}(\mathcal{C}(n), \mathcal{P}(n))_r$$

with differential the standard component-wise differential $[\partial, -]$ and the upper script $\mathfrak{S}_n$ denotes the fact that the graded chain maps are $\mathfrak{S}_n$-equivariant.

Proposition 3.4. *The operation $\star$ endows the dg vector space $\underline{\mathrm{Hom}}_{\mathtt{gr}\ \mathfrak{S}\text{-}\mathtt{mod}}(\mathcal{C}, \mathcal{P})_*$ with a dg pre-Lie algebra structure. It is called the* operadic convolution pre-Lie algebra *of $\mathcal{C}$ and $\mathcal{P}$.*

Remark 3.5. We will see in Exercise sheet 4 that there exists a functor $\mathtt{dg\ Op} \to \mathtt{dg\ preLie\text{-}alg}$ which maps the convolution operad to the operadic convolution pre-Lie algebra.

3.1.3. *Operadic twisting morphisms and twisted composite products.*

Definition 3.6. *Let $\mathcal{P}$ be a dg operad and $\mathcal{C}$ be a dg cooperad. A* twisting morphism $\alpha : \mathcal{C} \to \mathcal{P}$ *is defined to be a morphism of $\mathfrak{S}$-modules of degree -1 which is a Maurer-Cartan element of the operadic convolution pre-Lie algebra $\underline{\mathrm{Hom}}_{\mathtt{gr}\ \mathfrak{S}\text{-}\mathtt{mod}}(\mathcal{C}, \mathcal{P})_*$, i.e. such that*

$$\partial(\alpha) + \alpha \star \alpha = 0 \ .$$

If $\mathcal{C}$ is coaugmented with coaugmentation $u_{\mathcal{C}}$ and $\mathcal{P}$ is augmented with augmentation $\varepsilon_{\mathcal{P}}$, the morphism α also has to satisfy

$$\varepsilon_{\mathcal{P}}\alpha = 0 \qquad\qquad \alpha u_{\mathcal{C}} = 0 \ .$$

Remark 3.7. If $\mathcal{C}$ and $\mathcal{P}$ are concentrated in arity 1 i.e. are respectively a dg coalgebra and a dg algebra, we recover exactly Definition 1.2.

Let $\alpha : \mathcal{C} \to \mathcal{P}$ be a twisting morphism. We define a degree -1 morphism of graded $\mathfrak{S}$-modules as the composite

$$\partial^r_\alpha : \mathcal{C} \circ \mathcal{P} \xrightarrow{\Delta_{(1)} \circ \mathrm{id}_{\mathcal{P}}} (\mathcal{C} \circ_{(1)} \mathcal{C}) \circ \mathcal{P} \xrightarrow{(\mathrm{id}_{\mathcal{C}} \circ_{(1)} \alpha) \circ \mathrm{id}_{\mathcal{P}}} (\mathcal{C} \circ_{(1)} \mathcal{P}) \circ \mathcal{P} \to \mathcal{C} \circ \mathcal{P} \circ \mathcal{P} \xrightarrow{\mathrm{id}_{\mathcal{C}} \circ \gamma} \mathcal{C} \circ \mathcal{P} \ .$$

Proposition 3.8. *Let $\alpha : \mathcal{C} \to \mathcal{P}$ be a twisting morphism. Then the degree -1 morphism of graded $\mathfrak{S}$-modules $\partial_\alpha := \partial^r_\alpha + \partial_{\mathcal{C} \circ \mathcal{P}}$ defines a differential on the graded $\mathfrak{S}$-module $\mathcal{C} \circ \mathcal{P}$.*

The dg $\mathfrak{S}$-module $(\mathcal{C} \circ \mathcal{P}, \partial_\alpha)$ is then called the *(right) twisted composite product* of $\mathcal{C}$ and $\mathcal{P}$ and denoted $\mathcal{C} \circ_\alpha \mathcal{P}$. The morphism of $\mathfrak{S}$-modules ∂^l_α and the *(left) twisted composite product* $\mathcal{P}_\alpha \circ \mathcal{C}$ can defined in a similar fashion.

3.2. Operadic cobar-bar adjunction. Let s be an element of degree $|s| = 1$. We introduce the degree -1 map $\mu_s : \mathbb{K}s \otimes \mathbb{K}s \to \mathbb{K}s$ defined as $\mu_s(s \otimes s) = s$. We also point out that the suspension sV of a dg vector space V then corresponds to the dg tensor product $\mathbb{K}s \otimes V$. We also introduce the degree -1 linear map $\Delta_s : \mathbb{K}s^{-1} \to \mathbb{K}s^{-1} \otimes \mathbb{K}s^{-1}$ where $|s^{-1}| = -1$. For a graded $\mathfrak{S}$-module $\mathcal{M}$ we moreover denote $s\mathcal{M}$ the graded $\mathfrak{S}$-module defined as the arity-wise suspension of $\mathcal{M}$.

Let $\mathcal{P}$ be an operad. For $\nu \in \mathcal{P}(n)$, $\mu \in \mathcal{P}(m)$ and $1 \leqslant i \leqslant m$ we define

$$\nu \circ_i \mu := \gamma(\nu; \mathrm{id}, \ldots, \mathrm{id}, \mu, \mathrm{id}, \ldots, \mathrm{id}) \in \mathcal{P}(n + m - 1)$$

where μ is in i-th position. It is called the *partial composition* of ν with μ at i-th position.

3.2.1. *Operadic bar and cobar constructions.* Let $\mathcal{P}$ be a dg operad. Using Proposition 2.7, we define d_1 and d_2 to be the unique degree -1 coderivations of $\mathcal{T}^c(\mathcal{P})$ extending the degree -1 morphisms of graded $\mathfrak{S}$-modules

$$d_1 : \mathcal{T}^c(s\mathcal{P}) \twoheadrightarrow s\mathcal{P} \xrightarrow{\partial_{s\mathcal{P}}} s\mathcal{P} \ .$$

$$d_2 : \mathcal{T}^c(s\mathcal{P}) \twoheadrightarrow \mathcal{T}^c(s\mathcal{P})^{(2)} = s\mathcal{P} \circ_{(1)} s\mathcal{P} = (\mathbb{K}s \otimes \mathcal{P}) \circ_{(1)} (\mathbb{K}s \otimes \mathcal{P})$$
$$\xrightarrow{\sim} (\mathbb{K}s \otimes \mathbb{K}s) \otimes (\mathcal{P} \circ_{(1)} \mathcal{P}) \xrightarrow{\gamma_s \otimes \gamma_{(1)}} \mathbb{K}s \otimes \mathcal{P} = s\mathcal{P} \ .$$

We then check that $d_2^2 = d_1 d_2 + d_2 d_1 = d_1^2 = 0$.

Definition 3.9. *The* bar construction *of an augmented dg operad $\mathcal{P}$ is defined to be the conilpotent dg cooperad*

$$B\mathcal{P} := (\mathcal{T}^c(s\bar{\mathcal{P}}), d_1 + d_2)\ .$$

Let $\mathcal{C}$ be a dg cooperad. Using Proposition 2.6, we define d_1 and d_2 to be the unique degree -1 derivations of $\mathcal{T}^c(s^{-1}\mathcal{C})$ extending the degree -1 morphisms of graded $\mathfrak{S}$-modules

$$d_1 : s^{-1}\mathcal{C} \xrightarrow{\partial_{s^{-1}\mathcal{C}}} s^{-1}\mathcal{C} \hookrightarrow \mathcal{T}(s^{-1}\mathcal{C})$$

$$d_2 : s^{-1}\mathcal{C} = \mathbb{K}s^{-1} \otimes \mathcal{C} \xrightarrow{\Delta_s \otimes \Delta_{(1)}} (\mathbb{K}s^{-1} \otimes \mathbb{K}s^{-1}) \otimes (\mathcal{C} \circ_{(1)} \mathcal{C}) \tilde{\to} s^{-1}\mathcal{C} \circ_{(1)} s^{-1}\mathcal{C} \hookrightarrow \mathcal{T}(s^{-1}\mathcal{C})\ .$$

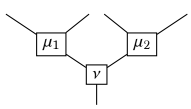
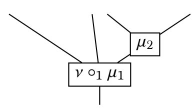
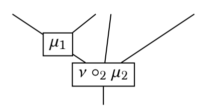
We check again that $d_2^2 = d_1 d_2 + d_2 d_1 = d_1^2 = 0$.

Definition 3.10. *The* cobar construction *of a coaugmented dg cooperad $\mathcal{C}$ is defined to be the augmented dg operad*

$$\Omega\mathcal{C} := (\mathcal{T}(s^{-1}\bar{\mathcal{C}}), d_1 + d_2)\ .$$

Remark 3.11. If $\mathcal{C}$ and $\mathcal{P}$ are concentrated in arity 1, we recover exactly the cobar and bar constructions for standard coaugmented dg coalgebras and augmented dg algebras, as proven in Exercise sheet 4.

Recall that an operation of $B\mathcal{P}$ can be represented as a nonplanar tree whose vertices are labeled by operations of $s\bar{\mathcal{P}}$. The differential d_2 of such an operation then corresponds to the sum of all possible ways to contract exactly one edge of the nonplanar tree using a partial composition. For instance,

$d_2($  $) = (-1)^{|\nu|}$  $+ (-1)^{1+|\nu|+|\mu_1|}$  .

Similarly, the differential d_2 on $\Omega\mathcal{C}$ corresponds to the sum of all possible ways to expand exactly one vertex of the nonplanar tree using the infinitesimal decomposition map $\Delta_{(1)}$.

3.2.2. *Operadic cobar-bar adjunction.*

Proposition 3.12. *The operadic cobar and bar constructions define an adjunction $\Omega \dashv B$ such that*

$$\mathrm{Hom}_{\texttt{aug dg Op}}(\Omega\mathcal{C}, \mathcal{P}) \simeq \mathrm{Tw}(\mathcal{C}, \mathcal{P}) \simeq \mathrm{Hom}_{\texttt{conil dg Coop}}(\mathcal{C}, B\mathcal{P})\ .$$

Proof. The proof is a simple adaptation of the proof of Proposition 1.14. □

The unit $\mathcal{C} \to B\Omega\mathcal{C}$ and counit $\Omega B\mathcal{P} \to \mathcal{P}$ and the induced twisting morphisms $\iota : \mathcal{C} \to \Omega\mathcal{C}$ and $\pi : B\mathcal{P} \to \mathcal{P}$ then satisfy the same properties as in Section 1.3.3:

(1) they can be computed by replacing A by $\mathcal{P}$ and C by $\mathcal{C}$ in Equations (1) and (2) ;
(2) the twisting morphisms ι and π are universal, meaning that every twisting morphism $\alpha : \mathcal{C} \to \mathcal{P}$ factors through them as in Equation (3) ;
(3) the twisted composite products $B\mathcal{P} \circ_\pi \mathcal{P}$, $\mathcal{C} \circ_\iota \Omega\mathcal{C}$, $\mathcal{P}_\pi \circ B\mathcal{P}$ and $\Omega\mathcal{C}_\iota \circ \mathcal{C}$ are all acyclic.

3.3. Operadic Koszul morphisms. We assume again in this section that the dg cooperad $\mathscr{C}$ and the dg operad $\mathscr{P}$ are concentrated in nonnegative degree. A twisting morphism $\alpha : \mathscr{C} \to \mathscr{P}$ is said to be *Koszul* if the twisted composite products $\mathscr{C} \circ_\alpha \mathscr{P}$ and $\mathscr{P}_\alpha \circ \mathscr{C}$ are acyclic, in the sense of Section 1.4.1. The universal twisting morphisms ι and π are in particular Koszul.

THEOREM 14. *Let $\alpha : \mathscr{C} \to \mathscr{P}$ be a twisting morphism between a connected coaugmented wgd cooperad and a connected augmented wgd operad $\mathscr{P}$. Then the following conditions are equivalent:*

(1) the twisted complex $\mathscr{C} \circ_\alpha \mathscr{P}$ is acyclic,
(2) the twisted complex $\mathscr{P} \circ_\alpha \mathscr{C}$ is acyclic,
(3) the induced morphism $\Omega\mathscr{C} \to \mathscr{P}$ is a quasi-isomorphism,
(4) the induced morphism $\mathscr{C} \to B\mathscr{P}$ is a quasi-isomorphism.

This theorem can be represented as

$$\begin{array}{ccccc} \mathrm{Hom}_{\texttt{aug dg Op}}(\Omega\mathscr{C}, \mathscr{P}) & \simeq & \mathrm{Tw}(\mathscr{C}, \mathscr{P}) & \simeq & \mathrm{Hom}_{\texttt{conil dg Coop}}(\mathscr{C}, B\mathscr{P}) \\ \cup & & \cup & & \cup \\ \mathrm{qIso}(\Omega\mathscr{C}, \mathscr{P}) & \simeq & \mathrm{Kosz}(\mathscr{C}, \mathscr{P}) & \simeq & \mathrm{qIso}(\mathscr{C}, B\mathscr{P}) \end{array}.$$

Proposition 3.13. *Given an augmented dg operad $\mathscr{P}$, the counit $\Omega B\mathscr{P} \to \mathscr{P}$ is a quasi-isomorphism.*

Hence the bar-cobar construction ΩB gives a quasi-free resolution for any augmented dg operad. We will see in Exercise sheet 4 that it is however not minimal in general.

CHAPTER 5

Koszul duality

1. Koszul duality for associative algebras

1.1. Quadratic data.

1.1.1. *Quadratic algebra and coalgebra.* A *graded subspace* W of a graded vector space $V = \bigoplus_{r\in\mathbb{Z}} V_r$ is defined to be a subspace $W \subset V$ such that $W = \bigoplus_{r\in\mathbb{Z}} W \cap V_r$.

Definition 1.1. *A* (graded) quadratic data *corresponds to the data of a (graded) vector space* V *together with a (graded) subspace* $R \subset V \otimes V$.

A *morphism of quadratic data* $f : (V, R) \to (W, S)$ is then a degree 0 linear map $f : V \to W$ such that $(f \otimes f)(R) \subset S$.

Definition 1.2. *The* quadratic algebra *associated to the quadratic data* (V, R) *is the quotient of the free tensor algebra* $T(V)$ *by the two-sided ideal generated by* $R \subset V \otimes V$,

$$A(V, R) := T(V)/\langle R\rangle \ .$$

More precisely, the quadratic algebra $A(V, R)$ is the augmented graded associative algebra

$$A(V, R) := \mathbb{K} \oplus V \oplus (V^{\otimes 2}/R) \oplus \cdots \oplus (V^{\otimes n}/\sum_{i=0}^{n-2} V^{\otimes i} \otimes R \otimes V^{\otimes n-i-2}) \oplus \cdots$$

whose product is the concatenation product.

Proposition 1.3. *The quadratic algebra* $A(V, R)$ *has the following* universal property*: for every augmented graded associative algebra* A *together with a morphism* $p : T(V) \to A$ *such that the composition*

$$R \hookrightarrow T(V) \to A$$

is equal to zero, there exists a unique morphism $A(V, R) \to A$ *such that the composition* $T(V) \twoheadrightarrow A(V, R) \to A$ *is equal to the morphism* p.

Definition 1.4. *The* quadratic coalgebra $C(V, R)$ *associated to the quadratic data* (V, R) *is the coaugmented graded coassociative coalgebra satisfying the following universal property: for every coaugmented graded coassociative coalgebra* C *together with a morphism* $\iota : C \to T^c(V)$ *such that the composition*

$$C \to T^c(V) \twoheadrightarrow V^{\otimes 2}/R$$

is equal to zero, there exists a unique morphism $C \to C(V, R)$ *such that the composition* $C \to C(V, R) \hookrightarrow T^c(V)$ *is equal to the morphism* ι.

The quadratic coalgebra $C(V,R)$ is in fact equal to the conilpotent coassociative coalgebra

$$C(V,R) := \mathbb{K} \oplus V \oplus R \oplus \cdots \oplus \bigcap_{i=0}^{n-2} V^{\otimes i} \otimes R \otimes V^{\otimes n-i-2} \oplus \cdots$$

whose coproduct is the deconcatenation coproduct.

We also point out that a morphism of quadratic data $f : (V,R) \to (W,S)$ naturally induces morphisms $A(V,R) \to A(W,S)$ and $C(V,R) \to C(W,S)$.

Example 1.5. Let V be a finite-dimensional vector space concentrated in degree 0. Then the free commutative algebra on V is a quadratic algebra presented by the quadratic data $(V, \langle x \otimes y - y \otimes x,\ x, y \in V\rangle)$. It is sometimes denoted $S(V)$ and called the *symmetric algebra* on the vector space V.

1.1.2. *Koszul dual of a quadratic (co)algebra.*

Definition 1.6. (i) *The* Koszul dual coalgebra *of the quadratic algebra $A := A(V,R)$ is the quadratic coalgebra*

$$A^{¡} := C(sV, s^2R)$$

where s^2R denotes the image of R in $(sV)^{\otimes 2}$ under the map $v \otimes w \mapsto (-1)^{|v|} sv \otimes sw$.

(ii) The Koszul dual algebra *of the quadratic coalgebra $C := C(V,R)$ is the quadratic algebra*

$$C^{¡} := A(s^{-1}V, s^{-2}R)\ .$$

Definition 1.7. *(i) For a graded vector space $V = \bigoplus_{r\in\mathbb{Z}} V_r$ we define its* graded dual *to be the graded vector space $V^\circ := \bigoplus_{r\in\mathbb{Z}} V_{-r}^\vee$ with $V_r^\circ = V_{-r}^\vee$.*

(ii) For a weight-graded vector space V we define the weight-graded dual *V° to be the weight-graded vector space given by $(V^\circ)_r^{(n)} = (V_{-r}^{(n)})^\vee$.*

We have in particular that $V^\circ \subset V^\vee$ as vector spaces, as

$$(\bigoplus_{r\in\mathbb{Z}} V_{-r})^\vee = \prod_{r\in\mathbb{Z}} V_{-r}^\vee\ .$$

The choice of grading on V° moreover ensures that the natural evaluation map $V^\circ \otimes V \to \mathbb{K}$ is a linear map of degree 0.

Definition 1.8. *The* Koszul dual algebra *of the quadratic algebra $A := A(V,R)$ is defined to be the weight-graded dual $A^! := A^{¡\circ}$ endowed with the weight grading*

$$(A^!)^{(n)} := s^n (A^{¡\circ})^{(n)}\ .$$

For $R \subset V \otimes V$ we denote $R^\perp \subset (V^{\otimes 2})^\vee$ the subspace of linear forms that vanish on R. A quadratic data (V,R) with V a finite-dimensional graded vector space is said to be *finite-dimensional*: the quadratic algebra $A(V,R)$ is then said to be *finitely generated.*

Proposition 1.9. *Let (V,R) be a finite-dimensional quadratic data. Then the Koszul dual algebra $A^!$ is quadratic,*

$$A^! = A(V^\circ, R^\perp)\ .$$

Proof. See Exercise sheet 5. □

It is then easy to check that $(A^{¡})^{¡} = A$, $(C^{¡})^{¡} = C$ and $(A^!)^! = A$ when V is finite-dimensional.

Example 1.10. (i) Let V be a finite-dimensional vector space and $R = 0$. Then $A = T(V)$ and its Koszul dual algebra is the algebra $A^! = \mathbb{K} \oplus V^\vee =: D(V^\vee)$ with trivial multiplication. It is sometimes called the *algebra of dual numbers.*

(ii) When $\mathrm{char}(\mathbb{K}) = 0$, the Koszul dual coalgebra of the symmetric algebra $S(V)$ is the cofree graded cocommutative coalgebra $\Lambda^c(sV) \subset T^c(sV)$. It is equal in weight n to the subspace

$$\Lambda^c(sV)^{(n)} = \langle \sum_{\sigma \in \mathfrak{S}_n} \sigma \cdot (sv_1 \otimes \cdots \otimes sv_n),\ v_1, \ldots, v_n \in V\rangle \subset (sV)^{\otimes n}$$

and presented by the quadratic data $(sV, \langle sx \otimes sy - sy \otimes sx,\ x, y \in V\rangle)$.

1.2. Koszul algebras.

1.2.1. *Koszul complex of a quadratic data.* Let (V, R) be a quadratic data and $A := A(V, R)$. We define the degree -1 morphism $\kappa : A^{¡} \to A$ as the composition

$$\kappa : A^{¡} = C(sV, s^2R) \twoheadrightarrow sV \xrightarrow{s^{-1}} V \hookrightarrow A\ .$$

Proposition 1.11. *The morphism $\kappa : A^{¡} \to A$ is a twisting morphism.*

Proof. The graded vector spaces $A^{¡}$ and A are here seen as dg vector spaces with null differential. We thereby have to check that

$$\kappa \star \kappa = 0\ .$$

By definition of the morphism κ, this equality holds in weight $\neq 2$. We only need to check it in weight 2: the morphism $\kappa \star \kappa$ is then equal to the composition

$$C(sV, s^2R)^{(2)} = s^2R \hookrightarrow sV \otimes sV \xrightarrow{s^{-1} \otimes s^{-1}} V \otimes V \twoheadrightarrow V^{\otimes 2}/R = A(V, R)^{(2)}$$

which is equal to zero. This concludes the proof of the proposition. □

The chain complexes $A^{¡} \otimes_\kappa A$ and $A_\kappa \otimes A^{¡}$ are then respectively called the *left* and *right Koszul complexes* of the quadratic data (V, R).

Proposition 1.12. *Under the bijection of Proposition 1.14 we have that*

(1) the morphism of algebras $f_\kappa : \Omega A^{¡} \to A$ is given by the natural projection $\Omega A^{¡} \twoheadrightarrow A$,
(2) the morphism of coalgebras $g_\kappa : A^{¡} \to BA$ is given by the natural inclusion $A^{¡} \hookrightarrow BA$.

1.2.2. *Koszul algebras.* We assume for all the results of this section that the quadratic data (V, R) is concentrated in nonnegative degree. Then the augmented dg algebra $A = A(V, R)$ and the conilpotent dg coalgebra $A^{¡} = C(sV, s^2R)$ are both connected wgd. As a direct application of Theorem 11 we get that:

THEOREM 15 (Koszul criterion). *Let (V, R) be a quadratic data. Then the following conditions are equivalent:*

(1) the left Koszul complex $A^{¡} \otimes_{\kappa} A$ is acyclic,
(2) the right Koszul complex $A_{\kappa} \otimes A^{¡}$ is acyclic,
(3) the projection $\Omega A^{¡} \twoheadrightarrow A$ is a quasi-isomorphism,
(4) the inclusion $A^{¡} \hookrightarrow BA$ is a quasi-isomorphism.

A quadratic algebra satisfying one of the four equivalent conditions of Theorem 15 is then said to be *Koszul*.

EXAMPLE 1.13. We will prove in Exercise sheet 5 that the free graded tensor algebra $T(V)$ of a graded vector space V and the symmetric algebra $S(V)$ of a vector space V are Koszul by explicitly computing their left Koszul complexes.

Proposition 1.14. *If $A = A(V, R)$ is a Koszul quadratic algebra then $\Omega A^{¡} \twoheadrightarrow A$ is a minimal model for A, where the differential on the graded algebra A is defined to be null.*

PROOF. A *minimal model* of a dg algebra A is defined to be the data of a quasi-free algebra $(T(V), \partial)$ and of a quasi-isomorphism $(T(V), \partial) \xrightarrow{\sim} A$ satisfying the same conditions as in Definition 2.8. This proposition then stems from the general fact that for a coaugmented connected wgd coalgebra C with null differential and a Koszul morphism $C \to A$, the induced quasi-isomorphism $\Omega C \to A$ defines a minimal model for A. □

Since $A^{¡} \hookrightarrow BA$ the resolution $\Omega A^{¡} \twoheadrightarrow A$ provides a resolution smaller and simpler than the resolution $\Omega BA \to A$ of Proposition 1.20, which has moreover the property to be minimal.

Proposition 1.15. *Let (V, R) be a finite-dimensional quadratic data. Then the quadratic algebra $A(V, R)$ is Koszul if and only if its Koszul dual algebra $A(V, R)^! = A(V^\circ, R^\perp)$ is Koszul.*

PROOF. See Exercise sheet 5. □

Proposition 1.16. *Let (V, R) be a quadratic data where V is concentrated in degree 0. Then the Hochschild homology of A with coefficients in $\mathbb{K}$ is equal to $A^{¡}$ as a graded vector space.*

PROOF. The proof is a direct application of Exercise 2 in Exercise sheet 4. □

1.2.3. *Inhomogeneous Koszul duality.* The exists a more general version of Koszul duality, called *inhomogeneous Koszul duality*, where we consider a data (V, R) which is quadratic linear i.e. such that $R \subset V \oplus V^{\otimes 2}$. The enveloping algebra of a Lie algebra $\mathfrak{g}$

$$U(\mathfrak{g}) = T(\mathfrak{g})/\langle x \otimes y - y \otimes x - [x, y],\ x, y \in \mathfrak{g}\rangle$$

is then a quadratic linear algebra which has the property to be Koszul.

1.3. The rewriting method. It is generally quite difficult to compute the homology of the left/right Koszul complex of a quadratic data. The rewriting method that we now expose offers an alternative algorithmic method to prove Koszulity of a finitely generated quadratic algebra. We illustrate it on the example of the quadratic algebra

$$A(V,R)=A(v_1,v_2,v_3;v_1v_1-v_1v_2,v_2v_3+v_2v_2,v_1v_3+2v_1v_2-v_1v_1)\ ,$$

where $|v_1|=|v_2|=|v_3|=0$.

Step 1. We choose an ordered basis $v_1<\cdots<v_n$ for V, here $v_1<v_2<v_3$.

Step 2. The basis $\{v_iv_j\}$ of $V^{\otimes 2}$ is then endowed with a suitable order, here the lexicographic order

$$v_1v_1<v_1v_2<v_1v_3<v_2v_1<v_2v_2<v_2v_3<v_3v_1<v_3v_2<v_3v_3\ .$$

We also choose a basis of R and write any element r of the basis as

$$r=\lambda v_iv_j-\sum_{(k,l)<(i,j)}\lambda_{k,l}v_kv_l$$

where $\lambda\neq 0$. The monomial v_iv_j is called the leading term of r. The basis of R can moreover be normalized i.e. replaced by a basis such that the coefficient of the leading term λ is equal to 1, two elements of the basis of R have distinct leading terms and the sum in the right-hand side of a basis vector of R does not contain any leading term.

In our example, the leading terms are taken to be v_1v_2, v_1v_3 and v_2v_3 and a normalized basis for R is

$$v_1v_2-v_1v_1,\ v_2v_3-(-v_2v_2),\ v_1v_3-(-v_1v_1)\ .$$

Step 3. These normalized basis relators of R provide rewriting rules for monomials

$$v_iv_j\mapsto\sum_{(k,l)<(i,j)}\lambda_{k,l}v_kv_l\ .$$

A monomial $v_iv_jv_k$ is then said to be *critical* if both v_iv_j and v_jv_k are leading terms. The successive applications of rewriting rules to a critical monomial then define a graph, which is said to be *confluent* if it has only one terminal vertex.

In our example, the only critical monomial is $v_1v_2v_3$ and the rewriting graph has the following form

$$\begin{tikzcd}
 & \boxed{v_1v_2v_3} \arrow[dl] \arrow[dr] & \\
\boxed{v_1v_1v_3} \arrow[ddr] & & \boxed{-v_1v_2v_2} \arrow[d] \\
 & & \boxed{-v_1v_1v_2} \arrow[dl] \\
 & \boxed{-v_1v_1v_1} &
\end{tikzcd}$$

hence is confluent.

THEOREM 16. *If every critical monomial of the quadratic data (V, R) is confluent then the quadratic algebra $A(V, R)$ is Koszul.*

Many technical details and assumptions were omitted in this exposure of the rewriting method.

2. Koszul duality for operads

2.1. Operadic quadratic data.

2.1.1. *Quadratic operad and cooperad.*

DEFINITION 2.1. *An* operadic quadratic data *corresponds to the data of a graded $\mathfrak{S}$-module $\mathcal{M}$ together with a graded $\mathfrak{S}$-submodule $\mathcal{R} \subset \mathcal{T}(\mathcal{M})^{(2)}$.*

DEFINITION 2.2. *The* quadratic operad *associated to the quadratic data $(\mathcal{M}, \mathcal{R})$ is the quotient of the free operad $\mathcal{T}(\mathcal{M})$ by the operadic ideal $\mathcal{R}$ generated by $\mathcal{R}$,*

$$\mathcal{P}(\mathcal{M}, \mathcal{R}) = \mathcal{T}(\mathcal{M})/\langle\mathcal{R}\rangle \ .$$

The assumption $\mathcal{R} \subset \mathcal{T}(\mathcal{M})^{(2)} = \mathcal{M} \circ_{(1)} \mathcal{M}$ means that the relators are made up of elements involving only one partial composition. The quadratic operad $\mathcal{P}(\mathcal{M}, \mathcal{R})$ can moreover be equivalently defined by the universal property of the quotient operad $\mathcal{T}(\mathcal{M})/\langle\mathcal{R}\rangle$.

DEFINITION 2.3. *The* quadratic cooperad $\mathcal{C}(\mathcal{M}, \mathcal{R})$ *associated to the quadratic data $(\mathcal{M}, \mathcal{R})$ is the graded cooperad which is universal among the conilpotent cooperads $\mathcal{C}$ endowed with a morphism of conilpotent cooperads $\mathcal{C} \to \mathcal{T}^c(\mathcal{M})$ such that the following composition is equal to 0*

$$\mathcal{C} \to \mathcal{T}^c(\mathcal{M}) \twoheadrightarrow \mathcal{T}^c(\mathcal{M})^{(2)}/\mathcal{R} \ .$$

In other words, the cooperad $\mathcal{C}(\mathcal{M}, \mathcal{R})$ is the subcooperad of the cofree cooperad $\mathcal{T}^c(\mathcal{M})$ that can be described as follows: an operation of $\mathcal{T}^c(\mathcal{M})$ seen as a nonplanar tree whose vertices are labeled by operations of $\mathcal{M}$ belongs to $\mathcal{C}(\mathcal{M}, \mathcal{R})$ if each each subtree with two vertices of this nonplanar tree belongs to $\mathcal{R} \subset \mathcal{T}(\mathcal{M})^{(2)}$.

EXAMPLE 2.4. The operads $\mathcal{A}ss$, $\mathcal{L}ie$ and $\mathcal{C}om$ are quadratic operads. See Example 2.14.

An operadic quadratic data is said to be *finite-dimensional* if for each $n \geqslant 0$ the vector space $\mathcal{M}(n)$ is finite-dimensional. The quadratic operad $\mathcal{P}(\mathcal{M}, \mathcal{R})$ is then said to *finitely generated.*

2.1.2. *Koszul dual of a quadratic operad.*

DEFINITION 2.5. *The* Koszul dual cooperad *of a quadratic operad $\mathcal{P} := \mathcal{P}(\mathcal{M}, \mathcal{R})$ is defined to be the quadratic cooperad*

$$\mathcal{P}^{i} := \mathcal{C}(s\mathcal{M}, s^2\mathcal{R}) \ .$$

The arity-wise suspension $s\mathcal{P}$ of a graded operad $\mathcal{P}$ is in general not a graded operad, as the induced composition maps of $s\mathcal{P}$ do not have the correct degree. We thereby introduce $\mathcal{S} := \mathrm{End}_{\mathbb{K}s}$ the endomorphism operad of the 1-dimensional graded vector space $\mathbb{K}s$ concentrated in degree 1. In each arity $n \geqslant 1$ the vector space $\mathcal{S}(n)$ is generated by one operation of degree

$1-n$ sending s^n to s. We also denote $\mathcal{S}^{-1} := \mathrm{End}_{\mathbb{K}s^{-1}}$ where $|s^{-1}| = -1$. In each arity $n \geqslant 1$ the vector space $\mathcal{S}^{-1}(n)$ is generated by one operation of degree $n-1$ sending s^{-n} to s^{-1}.

Definition 2.6. *The* operadic suspension *and the* operadic desuspension *of a graded operad $\mathcal{P}$ are respectively defined to be the Hadamard product $\mathcal{S} \otimes \mathcal{P}$ and $\mathcal{S}^{-1} \otimes \mathcal{P}$.*

We check in particular that $(\mathcal{S} \otimes \mathcal{P})(n) = s^{1-n}\mathcal{P}(n)$, $(\mathcal{S}^{-1} \otimes \mathcal{P})(n) = s^{n-1}\mathcal{P}(n)$ and that $\mathcal{S}^{-1} \otimes \mathcal{S} \otimes \mathcal{P} \simeq \mathcal{P}$.

Definition 2.7. *The* Koszul dual operad *of the quadratic operad $\mathcal{P} := \mathcal{P}(\mathcal{M}, \mathcal{R})$ is defined to be the operad*

$$\mathcal{P}^! = \mathcal{S}^{-1} \otimes (\mathcal{P}^¡)^\circ$$

where $-^\circ$ denotes the arity-wise weight-graded dual.

Beware that when the quadratic data is concentrated in arity 1, this definition does not coincide with Definition 1.8.

Example 2.8. We will prove the following equalities in Section 2.2

$$\mathcal{A}ss^! = \mathcal{A}ss \qquad \mathcal{L}ie^! = \mathcal{C}om \qquad \mathcal{C}om^! = \mathcal{L}ie \,.$$

Remark 2.9. There also exists a notion of nonsymmetric operadic quadratic data. The operad $\mathcal{A}s$ is then in particular quadratic as proven in Example 2.14. We will prove in Section 2.4 that $\mathcal{A}s^! = \mathcal{A}s$.

2.2. Binary quadratic operads.

2.2.1. *Binary quadratic operads.*

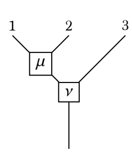
Definition 2.10. *An operadic quadratic data $(\mathcal{M}, \mathcal{R})$ is said to be* binary *if $\mathcal{M}$ is concentrated in arity 2.*

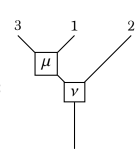
We then denote $\mathcal{M} := (0, 0, M, 0, \dots)$ where M is endowed with a right $\mathbb{K}[\mathfrak{S}_2]$-module structure and $\mathcal{T}(M) := \mathcal{T}(\mathcal{M})$. We have in particular that $\mathcal{T}(M)(0) = 0$, $\mathcal{T}(M)(1) = 0$, $\mathcal{T}(M)(2) = M$ and that $\mathcal{T}(M)(n) = \mathcal{T}(M)^{(n-1)}(n)$ for $n \geqslant 1$. Let us explicitly compute $\mathcal{T}(M)(3) = \mathcal{T}(M)^{(2)}(3)$.

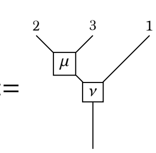
For μ and ν two arity 2 operations in M we denote

$$\nu \circ_I \mu := \nu \circ_1 \mu \qquad \nu \circ_{II} \mu := (\nu \circ_1 \mu)^{(123)} \qquad \nu \circ_{III} \mu := (\nu \circ_1 \mu)^{(321)}$$

or equivalently

$\nu \circ_I \mu :=$ 

$\nu \circ_{II} \mu :=$ 

$\nu \circ_{III} \mu :=$ 

.

We then compute that

$$\mathcal{T}(M)(3) = (M \otimes M)_I \oplus (M \otimes M)_{II} \oplus (M \otimes M)_{III}$$

where the operations of a summand $(M \otimes M)_u$ are linear combinations of operations $\nu \circ_u \mu$ for $u = I, II, III$. The $\mathfrak{S}$-module $\mathcal{T}(M)^{(2)}$ is concentrated in arity 3, and for $\mathcal{R} \subset \mathcal{T}(M)^{(2)}$ we will thereby denote $R := \mathcal{R}(3) \subset \mathcal{T}(M)(3) = \mathcal{T}(M)^{(2)}(3)$ the corresponding right $\mathfrak{S}_3$-module.

Example 2.11. (i) The operad $\mathcal{C}om$ is presented by the binary quadratic data

$$(\mathbb{K}\mu, \langle \mu \circ_I \mu - \mu \circ_{II} \mu \ , \ \mu \circ_{II} \mu - \mu \circ_{III} \mu \rangle)$$

where $\mu \cdot (12) = \mu$. We point out that the vector space $\langle \mu \circ_I \mu - \mu \circ_{II} \mu \ , \ \mu \circ_{II} \mu - \mu \circ_{III} \mu \rangle$ is exactly the right $\mathfrak{S}_3$-module generated by the element $\mu \circ_1 \mu - \mu \circ_2 \mu$.

(ii) The operad $\mathcal{A}ss$ is presented by the following binary quadratic data: the space of generating operations is $M = \mathbb{K}\mu \oplus \mathbb{K}\lambda$ where $\mu \cdot (12) = \lambda$ and the vector space of relators R is generated as a vector space by the six relators

$$\begin{array}{lll} \mu \circ_I \mu - \lambda \circ_{III} \mu & \lambda \circ_{II} \lambda - \mu \circ_I \lambda & \mu \circ_{III} \mu - \lambda \circ_{II} \mu \\ \lambda \circ_I \lambda - \mu \circ_{III} \lambda & \mu \circ_{II} \mu - \lambda \circ_I \mu & \lambda \circ_{III} \lambda - \mu \circ_{II} \lambda \ . \end{array}$$

(iii) The binary quadratic presentation of the operad $\mathcal{L}ie$ was already given in Example 2.14.

2.2.2. *Koszul dual operad of a binary quadratic operad.* In this section, we let M be a right $\mathbb{K}[\mathfrak{S}_2]$-module which is finite-dimensional as a vector space.

Definition 2.12. *The* signed graded dual *of M is defined to be the graded dual vector space M° endowed with the right $\mathfrak{S}_2$-action*

$$(f \cdot \sigma)(m) = \mathrm{sgn}(\sigma) f(m \cdot \sigma^{-1}) \ .$$

Proposition 2.13. *The following pairing on $\mathcal{T}(M^\circ)(3) \otimes \mathcal{T}(M)(3)$ is non-degenerate:*

$$\langle \alpha_1 \circ_u \alpha_2, \mu_1 \circ_v \mu_2 \rangle = \begin{cases} \alpha_1(\mu_1)\alpha_2(\mu_2) & \text{if } u = v, \\ 0 & \text{otherwise,} \end{cases}$$

where $u, v \in \{I, II, III\}$.

For $R \subset \mathcal{T}(M)(3)$ we will then denote

$$R^\perp := \{x \in \mathcal{T}(M^\circ)(3), \ \langle x, R \rangle = 0\} \ .$$

Theorem 17. *Let (M, R) be a finite-dimensional binary operadic quadratic data. Then the Koszul dual operad of $\mathcal{P}(M, R)$ is the quadratic operad*

$$\mathcal{P}(M, R)^! = \mathcal{P}(M^\circ, R^\perp) \ .$$

We check in particular that under the assumptions of Theorem 17 $(\mathcal{P}^!)^! = \mathcal{P}$.

Example 2.14. We can now apply Theorem 17 to the binary quadratic operads of Example 2.11.

(i) The Koszul dual operad of $\mathcal{C}om$ is generated by the space $(\mathbb{K}\mu)^\vee = \mathbb{K}\mu^\vee$ where $\mu^\vee \cdot (12) = -\mu^\vee$. The orthogonal $R^\perp$ of its vector space of relators R is moreover generated as a vector space by the relator $\mu^\vee \circ_I \mu^\vee + \mu^\vee \circ_{II} \mu^\vee + \mu^\vee \circ_{III} \mu^\vee$. In other words $\mathcal{C}om^! = \mathcal{L}ie$.

(ii) The Koszul dual operad of $\mathcal{A}ss$ is generated by the space $\mathbb{K}\mu^\vee \oplus \mathbb{K}\lambda^\vee$ where $\mu^\vee \cdot (12) = -\lambda^\vee$. The orthogonal $R^\perp$ of its vector space of relators R is moreover generated as a vector space by the six relators

$$\begin{array}{lll} \mu^\vee \circ_I \mu^\vee + \lambda^\vee \circ_{III} \mu^\vee & \lambda^\vee \circ_{II} \lambda^\vee + \mu^\vee \circ_I \lambda^\vee & \mu^\vee \circ_{III} \mu^\vee + \lambda^\vee \circ_{II} \mu^\vee \\ \lambda^\vee \circ_I \lambda^\vee + \mu^\vee \circ_{III} \lambda^\vee & \mu^\vee \circ_{II} \mu^\vee + \lambda^\vee \circ_I \mu^\vee & \lambda^\vee \circ_{III} \lambda^\vee + \mu^\vee \circ_{II} \lambda^\vee \ . \end{array}$$

The morphism of operads $\mathcal{A}ss^! \to \mathcal{A}ss$ mapping $\mu^\vee$ to μ and $\lambda^\vee$ to $-\lambda$ is then an isomorphism i.e. $\mathcal{A}ss^! \simeq \mathcal{A}ss$.

2.3. Koszul operads.

2.3.1. *Koszul complex of an operadic quadratic data.* For an operadic quadratic data $(\mathcal{M}, \mathcal{R})$ and $\mathcal{P} := \mathcal{P}(\mathcal{M}, \mathcal{R})$ we define the morphism of $\mathfrak{S}$-modules

$$\kappa : \mathcal{P}^{\text{¡}} = \mathcal{C}(s\mathcal{M}, s^2\mathcal{R}) \twoheadrightarrow s\mathcal{M} \xrightarrow{s^{-1}} \mathcal{M} \hookrightarrow \mathcal{P} \ .$$

Proposition 2.15. *The morphism of $\mathfrak{S}$-modules $\kappa : \mathcal{P}^{i} \to \mathcal{P}$ is a twisting morphism, where the graded cooperad $\mathcal{P}^{i}$ and the graded operad $\mathcal{P}$ are endowed with the null differential.*

Proof. The proof proceeds exactly as the proof of Proposition 1.11. □

The twisted composite products $\mathcal{P}^{\text{¡}} \circ_\kappa \mathcal{P}$ and $\mathcal{P}_\kappa \circ \mathcal{P}^{\text{¡}}$ are then respectively called the *left* and *right Koszul complexes* of the operadic quadratic data $(\mathcal{M}, \mathcal{R})$. Under the bijection of Proposition 3.12, the twisting morphism κ is moreover in correspondence with the natural inclusion $\mathcal{P}^{\text{¡}} \hookrightarrow B\mathcal{P}$ and the natural projection $\Omega\mathcal{P}^{\text{¡}} \twoheadrightarrow \mathcal{P}$.

We now assume that $(\mathcal{M}, \mathcal{R})$ is an operadic quadratic data concentrated in nonnegative degree. Then the quadratic operad $\mathcal{P}(\mathcal{M}, \mathcal{R})$ and the quadratic cooperad $\mathcal{P}(\mathcal{M}, \mathcal{R})^{\text{¡}}$ are wgd connected. Theorem 14 then implies the following criterion:

Theorem 18 (Koszul criterion). *Let $(\mathcal{M}, \mathcal{R})$ be an operadic quadratic data. Then the following conditions are equivalent:*

(1) the left Koszul complex $\mathcal{P}^{i} \circ_\kappa \mathcal{P}$ is acyclic,
(2) the right Koszul complex $\mathcal{P}_\kappa \circ \mathcal{P}^{i}$ is acyclic,
(3) the projection $\Omega\mathcal{P}^{i} \twoheadrightarrow \mathcal{P}$ is a quasi-isomorphism of operads,
(4) the inclusion $\mathcal{P}^{i} \hookrightarrow B\mathcal{P}$ is a quasi-isomorphism of operads.

A quadratic operad $\mathcal{P}$ satisfying one of the four equivalent conditions of Theorem 18 is said to be *Koszul*. We recover in fact the Koszul duality theory of associative algebras when the operadic quadratic data is concentrated in arity 1.

Theorem 19. *(i) The quadratic operads $\mathcal{A}ss$, $\mathcal{L}ie$ and $\mathcal{C}om$ are Koszul.*
(ii) The ns quadratic operad $\mathcal{A}s$ is Koszul.

Proof. We will prove Item (ii) in Section 2.4. □

We say that a $\mathfrak{S}$-module $\mathcal{M}$ is *reduced* if $\mathcal{M}(0) = 0$.

Proposition 2.16. *Let $(\mathcal{M}, \mathcal{R})$ be an operadic quadratic data where $\mathcal{M}$ is a reduced and arity-wise finite-dimensional $\mathfrak{S}$-module. Then the operad $\mathcal{P}$ is Koszul if and only if the operad $\mathcal{P}^!$ is Koszul.*

Proposition 2.17. *If a quadratic operad $\mathcal{P}$ is Koszul then the quasi-isomorphism $\Omega\mathcal{P}^{¡} \twoheadrightarrow \mathcal{P}$ provides a minimal model for the operad $\mathcal{P}$.*

We point out that in the previous proposition the quadratic operad $\mathcal{P}$ is seen as a dg operad with null differential.

2.3.2. *Homotopy $\mathcal{P}$-algebras.* Let $\mathcal{P}$ be a Koszul quadratic operad. The minimal operad $\Omega\mathcal{P}^{¡}$ is then usually denoted $\mathcal{P}_\infty := \Omega\mathcal{P}^{¡}$. The goal of Chapter 6 will be to prove that $\mathcal{P}_\infty$-algebras yield a satisfactory notion of ***homotopy $\mathcal{P}$-algebras*** or ***$\mathcal{P}$-algebras up to homotopy***, and provide an extensive study of their properties. We point out that every $\mathcal{P}$-algebra is in particular a $\mathcal{P}_\infty$-algebra by the morphism of operad $\mathcal{P}_\infty \to \mathcal{P}$.

2.3.3. *Inhomogeneous operadic Koszul duality.* We say that the operadic data $(\mathcal{M}, \mathcal{R})$ is ***quadratic linear*** if $\mathcal{R} \subset \mathcal{M} \oplus \mathcal{T}(\mathcal{M})^{(2)}$. There still exists a notion of operadic Koszul duality under this assumption, called ***inhomogeneous Koszul duality***. The operad $\mathcal{BV}$ encoding BV-algebras then admits a presentation as a quadratic linear operad and is moreover Koszul.

2.4. Rewriting method for binary quadratic ns operads. We now describe a rewriting method to prove that a finitely generated binary quadratic ns operad $\mathcal{P} = \mathcal{P}(M, R)$ is Kozsul. It is the operadic generalization of the method introduced in Section 1.3. We illustrate it on the ns operad

$$\mathcal{A}s = \mathcal{P}(\mu; \mu \circ_1 \mu - \mu \circ_2 \mu) \ .$$

Step 1. We choose an ordered basis $\mu_1 < \cdots < \mu_n$ of M, here μ.

Step 2. The arity 3 component of the free ns operad $\mathcal{T}_{ns}(M)(3)$ is spanned by the operations of the form $\nu \circ_1 \mu$ and $\nu \circ_2 \mu$. We order its basis $\mu_k \circ_i \mu_l$ as

$$\begin{cases} \mu_{k_1} \circ_1 \mu_{l_1} < \mu_{k_2} \circ_2 \mu_{l_2} & , \\ \mu_{k_1} \circ_i \mu_{l_1} < \mu_{k_2} \circ_i \mu_{l_2} & \text{if } (k_1, l_1) < (k_2, l_2) \ . \end{cases}$$

We then choose a normalized basis for $R \subset \mathcal{T}_{ns}(M)(3)$ as in Section 1.3

$$r = \mu_k \circ_i \mu_l - \sum_{k', i', l'} \lambda_{k', i', l'} \mu_{k'} \circ_{i'} \mu_{l'}$$

where $\mu_{k'} \circ_{i'} \mu_{l'} < \mu_k \circ_i \mu_l$ for all k', i', l', and call $\mu_k \circ_i \mu_l$ the *leading term* of r.

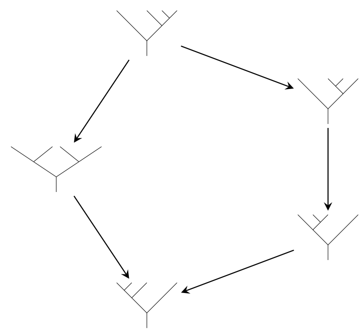
In the case of the ns operad $\mathcal{A}s$ the normalized basis is $\mu \circ_2 \mu - \mu \circ_1 \mu$.

Step 3. Each element of the basis provides a rewriting rule

$$\mu_k \circ_i \mu_l \mapsto \sum_{k', i', l'} \lambda_{k', i', l'} \mu_{k'} \circ_{i'} \mu_{l'} \ .$$

The arity 4 component of the free ns operad $\mathcal{T}_{ns}(M)(4)$ has five summands corresponding to the five binary planar trees of arity 4. A M-labeled binary tree of arity 4 is said to be *critical* if the two partial compositions appearing in this tree are leading terms. The successive applications of rewriting rules to a critical tree then determine a directed graph which is said to be *confluent* if it has only one terminal vertex.

In the case of the ns operad $\mathcal{A}s$, there is only one critical tree which gives rise to the following confluent graph



,

where each vertex is labeled by the operation μ.

THEOREM 20. *If every arity 4 critical tree of the operadic binary quadratic data (M, R) is confluent then the binary quadratic ns operad $\mathcal{P}(M, R)$ is Koszul.*

There also exists a rewriting method in the case of (symmetric) operads. It relies on the notion of *shuffle operads* that goes beyond the scope of this course. We will however mention that this method can be used to prove that the operads $\mathcal{A}ss$, $\mathcal{L}ie$ and $\mathcal{C}om$ are Koszul.

CHAPTER 6

Homotopy algebras

1. Homotopy algebras

We set a Koszul quadratic operad $\mathcal{P}$ throughout this section. We recall that a *homotopy $\mathcal{P}$-algebra* or *$\mathcal{P}_\infty$-algebra* has been defined to be an algebra over the operad $\mathcal{P}_\infty := \Omega\mathcal{P}^{¡}$.

1.1. The Rosetta stone.

The goal of this section is to prove the following theorem:

Theorem 21 (Rosetta stone). *Let A be a dg vector space. Then there are natural bijections*

$$\mathrm{Hom}_{\mathtt{dg\ Op}}(\Omega\mathcal{P}^{i}, \mathrm{End}_A) \simeq \mathrm{Tw}(\mathcal{P}^{i}, \mathrm{End}_A) \simeq \mathrm{Hom}_{\mathtt{conil\ dg\ Coop}}(\mathcal{P}^{i}, B\mathrm{End}_A) \simeq \mathrm{Codiff}(S^{\mathcal{P}^{i}}(A))\ .$$

We point out that:

(i) the cooperad $B\mathrm{End}_A$ denotes the cooperad $\mathcal{T}^c(s^{-1}\mathrm{End}_A)$, as the endomorphism operad End_A is not necessarily augmented.
(ii) a twisting morphism $\kappa : \mathcal{P}^{¡} \to \mathrm{End}_A$ satisfies the Maurer-Cartan equation and is such that $\kappa(\mathrm{id}_{\mathcal{P}^{¡}}) = 0$.
(iii) $\mathrm{Codiff}(S^{\mathcal{P}^{¡}}(A))$ denotes the set of coderivations on the cofree $\mathcal{P}^{¡}$-coalgebra $S^{\mathcal{P}^{¡}}(A)$ that extend the differential on A.

Proof. The first two bijections are given by Proposition 3.12. The third bijection will be proven in Section 1.1.2. □

1.1.1. *Codifferentials and cofree $\mathcal{C}$-coalgebras.* We refer to Definition 3.7 for the notion of coalgebras over cooperads.

Definition 1.1. *Let $\mathcal{C}$ be a graded cooperad and C be a graded $\mathcal{C}$-coalgebra. We define a* coderivation *on C to be a linear map $D : C \to C$ such that for every structure map $\Delta^n : C \to \mathcal{C}(n) \otimes C^{\otimes n}$ of C,*

$$\Delta^n D = \sum_{i=0}^{n-1} \mathrm{id}_{\mathcal{C}(n)} \otimes (\mathrm{id}_C^{\otimes i} \otimes D \otimes \mathrm{id}_C^{\otimes n-1-i})\Delta^n\ .$$

A *codifferential* on a $\mathcal{C}$-coalgebra C is then a degree -1 coderivation D on C such that $D^2 = 0$.

Definition 1.2. *Let $\mathcal{C}$ be a graded cooperad. The* cofree $\mathcal{C}$-coalgebra $S^{\mathcal{C}}(V)$ *on a graded vector space V is defined to be the cofree coalgebra over the comonad $S^{\mathcal{C}}$ generated by V.*

Let us compute the decomposition maps of $S^{\mathscr{C}}(V)$ when $\mathscr{C}$ is a ns cooperad. Consider an element of $S^{\mathscr{C}}(V)$ denoted as

$$\mu \otimes a_1 \cdots a_n \in \mathscr{C}(n) \otimes V^{\otimes n} \subset S^{\mathscr{C}}(V) = \bigoplus_{m \geqslant 0} \mathscr{C}(m) \otimes V^{\otimes m} .$$

The decomposition map $\Delta^k : S^{\mathscr{C}}(V) \to \mathscr{C}(k) \otimes (S^{\mathscr{C}}(V))^{\otimes k}$ evaluated on this element is then equal to

$$\sum_{i_1 + \cdots + i_k = n} \pm \nu \otimes (\nu_1 \otimes a_1 \cdots a_{i_1}) \otimes \cdots \otimes (\nu_k \otimes a_{i_1 + \cdots + i_{k-1} + 1} \cdots a_n) \in \mathscr{C}(k) \otimes (S^{\mathscr{C}}(V))^{\otimes k},$$

where $\pm$ is the sign obtained by applying the Koszul sign rule and we denote

$$\sum_{i_1 + \cdots + i_k = n} (\nu; \nu_1, \ldots, \nu_k) := \Delta^k_{\mathscr{C}}(\mu) = \sum_{i_1 + \cdots + i_k = n} \delta^{i_1, \ldots, i_k}_{\mathscr{C}}(\mu) \in \mathscr{C}(k) \otimes \bigoplus_{i_1 + \cdots + i_k = n} \mathscr{C}(i_1) \otimes \cdots \otimes \mathscr{C}(i_k) .$$

Example 1.3. We consider the ns cooperad $\mathscr{A}s^\vee$ defined as the arity-wise linear dual to the ns operad $\mathscr{A}s$. The cofree $\mathscr{A}s^\vee$-coalgebra generated by the graded vector space V is then exactly the *reduced cofree tensor coalgebra* on V

$$S^{\mathscr{A}s^\vee}(V) = \bar{T}^c(V) := V \oplus V^{\otimes 2} \oplus \cdots \oplus V^{\otimes n} \oplus \cdots$$

whose coproduct is the deconcatenation coproduct.

Proposition 1.4. *Let $\mathscr{C}$ be a graded cooperad. For a graded vector space V, there is a correspondence between coderivations of the cofree coalgebra $S^{\mathscr{C}}(V)$ and linear maps $S^{\mathscr{C}}(V) \to V$.*

Proof. We prove this result for a ns cooperad $\mathscr{C}$. We denote ε the counit of $\mathscr{C}$ and recall that for $\mu \in \mathscr{C}(n)$,

$$(\mathrm{id} \otimes \varepsilon^{\otimes k})\Delta^k_{\mathscr{C}}(\mu) = \mu \qquad (\varepsilon \otimes \mathrm{id})\Delta^1_{\mathscr{C}}(\mu) = \mu .$$

We moreover denote $\mathrm{proj}_V : S^{\mathscr{C}}(V) \to V$ the map defined as

$$\mu \otimes a_1 \ldots a_n \mapsto \begin{cases} \varepsilon(\mu) a_1 & \text{if } n = 1, \\ 0 & \text{otherwise} . \end{cases}$$

We will prove that the map $D \in \mathrm{Coder}(S^{\mathscr{C}}(V)) \mapsto \mathrm{proj}_V D \in \underline{\mathrm{Hom}}_{\mathtt{gr}\ \mathfrak{S}\text{-}\mathtt{mod}}(S^{\mathscr{C}}(V), V)$ is a bijection, where Coder denotes the graded vector space of graded coderivations.

We begin by pointing out that for $n \geqslant 1$ and $\mu \otimes a_1 \ldots a_n \in \mathscr{C}(n) \otimes V^{\otimes n}$,

$$\mu \otimes a_1 \ldots a_n = (\mathrm{id}_{\mathscr{C}(n)} \otimes \mathrm{proj}_V^{\otimes n})\Delta^n(\mu \otimes a_1 \ldots a_n) ,$$

where Δ^n is the decomposition map of $S^{\mathscr{C}}(V)$ defined previously. For $\mu \in \mathscr{C}(0) \subset S^{\mathscr{C}}(V)$ we also have that

$$\Delta^0(\mu) = \mu .$$

In other words, an element x of $S^{\mathscr{C}}(V)$ decomposes as

$$x = \Delta^0(x) + \sum_{k \geqslant 1} (\mathrm{id}_{\mathscr{C}(k)} \otimes \mathrm{proj}_V^{\otimes k})\Delta^k(x) ,$$

where the sum of the right-hand side is finite because $\mathscr{C}$ is a cooperad, see Definition 3.5.

For $\mu \otimes a_1 \dots a_n \in \mathscr{C}(n) \otimes V^{\otimes n}$ we can apply this formula to the element $D(\mu \otimes a_1 \dots a_n)$:

$$D(\mu \otimes a_1 \dots a_n) = \Delta^0(D(\mu \otimes a_1 \dots a_n)) + \sum_{k \geqslant 1} (\mathrm{id}_{\mathscr{C}(k)} \otimes \mathrm{proj}_V^{\otimes k}) \Delta^k (D(\mu \otimes a_1 \dots a_n)) \ .$$

The codifferential property of D implies that for $k \geqslant 1$

$$(\mathrm{id}_{\mathscr{C}(k)} \otimes \mathrm{proj}_V^{\otimes k}) \Delta^k D = \sum_{i=0}^{k-1} (\mathrm{id}_{\mathscr{C}(k)} \otimes \mathrm{proj}_V^{\otimes i} \otimes \mathrm{proj}_V D \otimes \mathrm{proj}_V^{\otimes k-1-i}) \Delta^k$$

and that $\Delta^0 D = 0$. In other words, the codifferential D is completely determined by its projection $\mathrm{proj}_V D$. Conversely, for a linear map $d : S^{\mathscr{C}}(V) \to V$ and $\mu \otimes a_1 \dots a_n \in \mathscr{C}(n) \otimes V^{\otimes n}$, the codifferential $D : S^{\mathscr{C}}(V) \to S^{\mathscr{C}}(V)$ associated to the linear map d is given by the formula

$$D(\mu \otimes a_1 \dots a_n) = \sum \mu^{(1)} \otimes a_1 \dots a_{i_1} d(\mu^{(2)} \otimes a_{i_1+1} \dots a_{i_1+i_2}) a_{i_1+i_2+1} \dots a_n$$

where $\sum \mu^{(1)} \otimes \mu^{(2)} := \Delta_{(1)}(\mu)$.

In the case of a (symmetric) operad, a similar proof using more advanced elements of operadic calculus shows that the codifferential D associated to the linear map d is again obtained using the infinitesimal decomposition $\Delta_{(1)}$ of the cooperad $\mathscr{C}$. □

Example 1.5. We recover in particular the following result proven in Exercise sheet 2 for Example 1.3:

$$\left\{ \begin{array}{c} \text{collections of linear maps} \\ m_n : V^{\otimes n} \to V \ , \ n \geqslant 1 \end{array} \right\} \longleftrightarrow \left\{ \text{ coderivations } D \text{ of } \bar{T}^c(V) \ \right\} \ .$$

1.1.2. *$\mathscr{P}_\infty$-algebras as codifferentials.* Proposition 1.4 implies the following bijections of graded vector spaces

$$\underline{\mathrm{Hom}}_{\mathtt{gr}\ \mathfrak{S}\text{-}\mathtt{mod}}(\mathscr{P}^{\mathrm{i}}, \mathrm{End}_V) \simeq \underline{\mathrm{Hom}}_{\mathtt{gr}\ \mathtt{Vect}}(S^{\mathscr{P}^{\mathrm{i}}}(V), V) \simeq \mathrm{Coder}(S^{\mathscr{P}^{\mathrm{i}}}(V)) \ ,$$

where V is a graded vector space. In order to prove Theorem 21, it remains to understand how the Maurer-Cartan equation satisfied by a twisting morphism $\mathscr{P}^{\mathrm{i}} \to \mathrm{End}_V$ is mapped under these two bijections. We begin by pointing out that if D_1 and D_2 are graded coderivations of a $\mathscr{C}$-coalgebra C then $[D_1, D_2] = D_1 D_2 - (-1)^{|D_1||D_2|} D_2 D_1$ is still a coderivation of degree $|D_1| + |D_2|$ of C.

Lemma 1.6. *The graded Lie algebras* $\underline{\mathrm{Hom}}_{\mathtt{gr}\ \mathfrak{S}\text{-}\mathtt{mod}}(\mathscr{P}^{i}, \mathrm{End}_V)$ *and* $\mathrm{Coder}(S^{\mathscr{P}^{i}}(V))$ *are isomorphic as graded Lie algebras, where* $\underline{\mathrm{Hom}}_{\mathtt{gr}\ \mathfrak{S}\text{-}\mathtt{mod}}(\mathscr{P}^{i}, \mathrm{End}_V)$ *is endowed with the Lie bracket associated to its convolution pre-Lie bracket defined in Proposition 3.4.*

Proof. From the description of the coderivation D_ϕ associated to a graded morphism of $\mathfrak{S}$-modules ϕ in Proposition 1.4, it is indeed straightforward to check that

$$D_{[\phi_1, \phi_2]} = [D_{\phi_1}, D_{\phi_2}] \ .$$

□

Let now A be a dg vector space and $\kappa : \mathscr{P}^{\text{i}} \to \mathrm{End}_A$ a morphism of $\mathfrak{S}$-modules such that $\kappa(\mathrm{id}_{\mathscr{P}^{\text{i}}}) = 0$. The associated coderivation D_κ of $S^{\mathscr{P}^{\text{i}}}(A)$ is equal to 0 on the summand $\mathbb{K}\mathrm{id}_{\mathscr{P}^{\text{i}}} \otimes A \subset S^{\mathscr{P}^{\text{i}}}(A)$. We moreover define $\partial_{S^{\mathscr{P}^{\text{i}}}(A)}$ to be the unique coderivation of $S^{\mathscr{P}^{\text{i}}}(A)$ equal to ∂_A on the summand $\mathbb{K}\mathrm{id}_{\mathscr{P}^{\text{i}}} \otimes A \subset S^{\mathscr{P}^{\text{i}}}(A)$ and equal to 0 everywhere else. We finally set

$$D_A := D_\phi + \partial_{S^{\mathscr{P}^{\text{i}}}(A)} \ .$$

Lemma 1.7. *Let $\kappa : \mathscr{P}^i \to \mathrm{End}_A$ be a degree -1 morphism of graded $\mathfrak{S}$-modules. Then κ is a twisting morphism if and only if $D_A^2 = 0$.*

Proof. The proof is again a straighforward computation using the map proj_A introduced in the proof of Proposition 1.4. □

This lemma concludes the proof of Theorem 21. The equation $D_A^2 = 0$ is moreover satisfied if and only if $\mathrm{proj}_A D_A^2 = 0$. In the ns case, this equation evaluated on an element $\mu \otimes a_1 \dots a_n$ reads exactly as

$$\sum \pm d_{\mu^{(1)}}(a_1 \dots a_{i_1} d_{\mu^{(2)}}(a_{i_1+1} \dots a_{i_1+i_2}) a_{i_1+i_2+1} \dots a_n) = 0 \ ,$$

where $\sum \mu^{(1)} \otimes \mu^{(2)} := \Delta_{(1)}(\mu)$ and $d_\mu(a_1 \dots a_n) := d(\mu \otimes a_1 \dots a_n)$. We refer to Definition 2.4 for an explicit example.

1.2. Homotopy theory of homotopy $\mathscr{P}$-algebras.

1.2.1. *$\mathscr{P}_\infty$-morphisms.*

Proposition 1.8. *Let $\mathscr{C}$ be a graded cooperad. For two graded vector space V and W, there is a correspondence between morphisms of $\mathscr{C}$-coalgebras $S^{\mathscr{C}}(V) \to S^{\mathscr{C}}(W)$ and linear maps $S^{\mathscr{C}}(V) \to W$.*

Proof. We check as in the proof of Proposition 1.4 that the map

$$F \in \underline{\mathrm{Hom}}_{\mathscr{C}\texttt{-cog}}(S^{\mathscr{C}}(V), S^{\mathscr{C}}(W)) \mapsto \mathrm{proj}_W F \in \underline{\mathrm{Hom}}_{\texttt{gr}\ \mathfrak{S}\texttt{-mod}}(S^{\mathscr{C}}(V) \to W)$$

is a bijection, where $\underline{\mathrm{Hom}}_{\mathscr{C}\texttt{-cog}}$ denotes the graded vector space of graded morphisms of $\mathscr{C}$-coalgebras. In the ns case, the morphism of $\mathscr{C}$-coalgebras $F : S^{\mathscr{C}}(V) \to S^{\mathscr{C}}(W)$ associated to the linear map $f : S^{\mathscr{C}}(V) \to W$ is then given by the formula

$$F(\mu \otimes a_1 \cdots a_n) = \sum_{i_1 + \cdots + i_k = n} \pm \nu' \otimes f(\nu_1 \otimes a_1 \cdots) \otimes \cdots \otimes f(\nu_k \otimes \cdots a_n) \ .$$

where $\Delta^k_{\mathscr{C}}(\mu) := \sum (\nu'; \nu_1, \dots, \nu_k)$. □

Definition 1.9. *A $\mathscr{P}_\infty$-*morphism *between two $\mathscr{P}_\infty$-algebras A and B is defined to be a morphism of dg $\mathscr{P}^i$-coalgebras $(S^{\mathscr{P}^i}(A), D_A) \to (S^{\mathscr{P}^i}(B), D_B)$.*

The equation $FD_A = D_BF$ is satisfied if and only if $\mathrm{proj}_B FD_A = \mathrm{proj}_B D_B F$. In the ns case, it evaluates on an element $\mu \otimes a_1 \ldots a_n$ as

$$\sum \pm f_{\mu^{(1)}}(a_1 \ldots a_{i_1} d_{\mu^{(2)}}(a_{i_1+1} \ldots a_{i_1+i_2}) a_{i_1+i_2+1} \ldots a_n)$$
$$= \sum_{\substack{i_1+\cdots+i_k=n \\ k \geqslant 1}} \pm d_{\nu'}(f_{\nu_1}(a_1 \cdots a_{i_1}) \cdots f_{\nu_k}(a_{i_1+\cdots+i_{k-1}+1} \cdots a_n))$$

where $f_\mu(a_1 \ldots a_n) := f(\mu \otimes a_1 \ldots a_n)$, $\Delta^k_{\mathcal{C}}(\mu) := \sum(\nu'; \nu_1, \ldots, \nu_k)$ and $\sum \mu^{(1)} \otimes \mu^{(2)} := \Delta_{(1)}(\mu)$.

Example 1.10. We will give explicit examples of $\mathcal{P}_\infty$-morphisms in the case of the ns operad A_∞ and the operad L_∞ in Sections 2 and 3.

Definition 1.11. *We define* ∞-$\mathcal{P}_\infty$-$\mathtt{alg}$ *to be the category whose objects are* $\mathcal{P}_\infty$*-algebras and whose morphisms are* $\mathcal{P}_\infty$*-morphisms.*

In other words the category ∞-$\mathcal{P}_\infty$-$\mathtt{alg}$ is the full subcategory of the category $\mathtt{dg}$ $\mathcal{P}^{\text{¡}}$-$\mathtt{cog}$ whose objects are quasi-cofree $\mathcal{P}^{\text{¡}}$-coalgebras.

Proposition 1.12. *A morphism of* $\mathcal{P}_\infty$*-algebras* $A \to B$ *in the sense of Definition 1.15 is exactly a* $\mathcal{P}_\infty$*-morphism equal to zero everywhere on* $S^{\mathcal{P}^i}(A)$ *except on the summand* $\mathbb{K}\mathrm{id}_{\mathcal{P}^i} \otimes A$.

Proof. The computation is straightforward from the explicit equations satisfied by $\mathcal{P}_\infty$-morphisms computed previously. □

The morphism of operads $\mathcal{P}_\infty \twoheadrightarrow \mathcal{P}$ as well as Proposition 1.12 imply the following inclusions of categories

$$\mathtt{dg}\ \mathcal{P}\text{-}\mathtt{alg} \subset \mathcal{P}_\infty\text{-}\mathtt{alg} \subset \infty\text{-}\mathcal{P}_\infty\text{-}\mathtt{alg} \subset \mathtt{dg}\ \mathcal{P}^{\text{¡}}\text{-}\mathtt{cog}\ .$$

The first two inclusions are however not full. We thereby denote $\mathcal{P}_\infty$-morphisms with a squiggly arrow $A \rightsquigarrow B$ and morphisms of $\mathcal{P}_\infty$-algebras with the usual arrow $A \to B$.

Proposition 1.13. *Let* $F : A \rightsquigarrow B$ *be a* $\mathcal{P}_\infty$*-morphism. Then* $F_{\mathrm{id}_{\mathcal{P}^i}} : A \to B$ *is a chain map.*

We will say that a $\mathcal{P}_\infty$-morphism $F : A \rightsquigarrow B$ *extends* a chain map $f : A \to B$ if $F_{\mathrm{id}_{\mathcal{P}^{\text{¡}}}} = f$.

Definition 1.14. *(i) A* $\mathcal{P}_\infty$-isomorphism $F : A \rightsquigarrow B$ *is defined to be a* $\mathcal{P}_\infty$*-morphism such that the chain map* $F_{\mathrm{id}_{\mathcal{P}^i}}$ *is an isomorphism.*

(ii) A $\mathcal{P}_\infty$-quasi-isomorphism $F : A \rightsquigarrow B$ *is defined to be a* $\mathcal{P}_\infty$*-morphism such that the chain map* $F_{\mathrm{id}_{\mathcal{P}^i}}$ *is a quasi-isomorphism.*

We will now illustrate in Sections 1.2.2 and 1.2.3 how the category ∞-$\mathcal{P}_\infty$-$\mathtt{alg}$ provides indeed a satisfactory framework to study the homotopy theory of these homotopy $\mathcal{P}$-algebras.

1.2.2. *Homotopy transfer theorem.*

Definition 1.15. *A* deformation retract diagram *is defined to be the data of two dg vector spaces* A *and* H *that fit into the following diagram*

$$\begin{tikzcd} A \arrow[loop left, "h"] \arrow[r, shift left, "p"] & H \arrow[l, shift left, "i"] \end{tikzcd}\ ,$$

where p and i are chain maps, $\mathrm{id} - ip = [\partial, h]$ *and i is a quasi-isomorphism.*

The dg vector spaces A and H are then in particular quasi-isomorphic, and the chain map p is a quasi-isomorphism inverse to i in homology. A deformation retract diagram hence provides a chain level homotopy equivalence between two dg vector spaces.

EXAMPLE 1.16. (i) In topology, a deformation retract of a space X into a subspace $A \subset X$ is defined to be a map $F : X \times [0,1] \to X$ such that $F(x,0) = x$, $F(x,1) \in A$ and $F(a,1) = a$. It gives rises to a deformation retract diagram

$$\begin{tikzcd} C_*(X) \arrow[loop left, "h"] \arrow[r, shift left, "p"] & C_*(A) \arrow[l, shift left, "i"] \end{tikzcd} ,$$

where h is the homotopy induced by $F(\cdot,\cdot)$, and i and p are the chain maps respectively induced by the continuous maps inc_A and $F(\cdot, 1)$.

(ii) The homology $H_*(A)$ of a dg vector space A is a deformation retract of A as proven in Exercise sheet 6.

Assume that the dg vector space A is a $\mathcal{P}$-algebra. A natural question to ask is whether this $\mathcal{P}$-algebra structure can be transferred under a deformation retract diagram, i.e. whether H can be made into a $\mathcal{P}$-algebra such that i and p extend to morphisms of $\mathcal{P}$-algebras. The answer is negative in general, but positive in the context of homotopy $\mathcal{P}$-algebras:

THEOREM 22 (Homotopy transfer theorem). *If A and H fit into a deformation retract diagram and A is endowed with a $\mathcal{P}_\infty$-algebra structure, then H can be endowed with a $\mathcal{P}_\infty$-algebra structure such that i extends to a $\mathcal{P}_\infty$-quasi-isomorphism $H \rightsquigarrow A$.*

PROOF. We give a sketch of the proof of the existence of the $\mathcal{P}_\infty$-algebra structure on H. A complete proof of the theorem in the case of A_∞-algebras will be given in Exercise sheet 6. The deformation retract diagram allows us to construct a morphism of dg cooperads $B\mathrm{End}_A \to B\mathrm{End}_H$ where we recall that $B\mathrm{End}_A := \mathcal{T}^c(s^{-1}\mathrm{End}_A)$. Under the bijection of Theorem 21,

$$\mathrm{Hom}_{\mathtt{dg\ Op}}(\Omega\mathcal{P}^{¡}, \mathrm{End}_A) \simeq \mathrm{Hom}_{\mathtt{conil\ dg\ Coop}}(\mathcal{P}^{¡}, B\mathrm{End}_A)$$

the $\mathcal{P}_\infty$-algebra structure on H is given by the morphism of cooperads

$$\mathcal{P}^{¡} \to B\mathrm{End}_A \to B\mathrm{End}_H \ .$$

□

Theorem 22 implies in particular that if A is a dg $\mathcal{P}$-algebra, H can be endowed with a homotopy $\mathcal{P}$-algebra structure.

EXAMPLE 1.17. In the case of the deformation retract of a dg vector space A onto its homology $H_*(A)$, the operations of the induced $\mathcal{P}_\infty$-algebra structure on the homology $H_*(A)$ are called the *operadic Massey products.* Both i and p can moreover be extended to $\mathcal{P}_\infty$-morphisms, while one can only extend i to a $\mathcal{P}_\infty$-morphism in the most general version of the homotopy transfer theorem.

1.2.3. *Inverses of isomorphisms and quasi-isomorphisms.*

THEOREM 23. *Every $\mathcal{P}_\infty$-isomorphism $F : A \rightsquigarrow B$ admits an inverse, i.e. a $\mathcal{P}_\infty$-morphism $G : B \rightsquigarrow A$ such that $FG = \mathrm{id}_B$ and $GF = \mathrm{id}_A$.*

In the theorem, id_A and id_B denote the morphisms of $\mathcal{P}_\infty$-algebras $A \to A$ and $B \to B$ seen as $\mathcal{P}_\infty$-morphisms, following Proposition 1.12. We will not give a proof of Theorem 23 but simply mention that it is possible to write an explicit formula for G using the viewpoint of Definition 1.9.

THEOREM 24. *Let $F : A \rightsquigarrow B$ be a $\mathcal{P}_\infty$-quasi-isomorphism. Then there exists a $\mathcal{P}_\infty$-quasi-isomorphism $G : B \rightsquigarrow A$ which is the inverse of $H_*(F) : H_*(A) \to H_*(B)$ on the level of homology.*

PROOF. Consider $i_\infty^A : H_*(A) \rightsquigarrow A$ and $p_\infty^B : B \rightsquigarrow H_*(B)$ the $\mathcal{P}_\infty$-quasi-isomorphisms of Example 1.17. The following composite of $\mathcal{P}_\infty$-morphisms is then a $\mathcal{P}_\infty$-isomorphism

$$H_*(A) \overset{i_\infty^A}{\rightsquigarrow} A \overset{F}{\rightsquigarrow} B \overset{p_\infty^B}{\rightsquigarrow} H_*(B) \ .$$

Applying Theorem 23, we get an inverse $\mathcal{P}_\infty$-isomorphism

$$\tilde{G} : H_*(B) \rightsquigarrow H_*(A) \ .$$

The composite $\mathcal{P}_\infty$-morphism

$$B \overset{p_\infty^B}{\rightsquigarrow} H_*(B) \overset{\tilde{G}}{\rightsquigarrow} H_*(A) \overset{i_\infty^A}{\rightsquigarrow} A$$

finally yields the $\mathcal{P}_\infty$-quasi-isomorphism $G : B \rightsquigarrow A$ inverting F on the level of homology. $\square$

2. A_∞-algebras and A_∞-categories

2.1. A_∞-algebras and A_∞-morphisms.

2.1.1. *Definitions.* We proved using the rewriting method in Section 2.4 that the quadratic binary ns operad $\mathcal{A}s$ is Koszul.

DEFINITION 2.1. *We define $\mathrm{A}_\infty := \Omega\mathcal{A}s^{i}$ to be the Koszul minimal model of $\mathcal{A}s$.*

A_∞-algebras are sometimes called *(strongly) homotopy associative algebras.* We will prove the two following lemmas in Exercise sheet 6.

Lemma 2.2. *The Koszul dual cooperad $\mathcal{A}s^{i}$ is given by the arity n space of operations*

$$\mathcal{A}s^{i}(n) := \mathbb{K}\mu_n^c$$

for $n \geqslant 1$ with $|\mu_n^c| = n-1$ and $\mu_1 = \mathrm{id}$, and with decomposition maps

$$\Delta^k_{\mathcal{A}s^{i}}(\mu_n^c) = \sum_{i_1+\cdots+i_k=n} \pm(\mu_k^c; \mu_{i_1}^c, \ldots, \mu_{i_k}^c) \ .$$

Lemma 2.3. *The cofree $\mathcal{A}s^{i}$-coalgebras on a graded vector space V is the cofree coalgebra $s^{-1}\bar{T}^c(sV)$.*

We now compute the explicit equations satisfied by a codifferential D_A on the cofree coalgebra $\bar{T}^c(sA)$ (which is equivalent to the datum of a codifferential on the cofree $\mathcal{A}s^{¡}$-coalgebra $s^{-1}\bar{T}^c(sA)$). From Example 1.5 we know that there is a correspondence

$$\begin{array}{ccc} \left\{ \begin{array}{c} \text{collections of morphisms of degree } n-2 \\ m_n : A^{\otimes n} \to A \ , \ n \geqslant 1 \end{array} \right\} & \longleftrightarrow & \left\{ \begin{array}{c} \text{collections of morphisms of degree } -1 \\ b_n : (sA)^{\otimes n} \to sA \ , \ n \geqslant 1 \end{array} \right\} \\ & & \updownarrow \\ & & \left\{ \text{ coderivations } D \text{ of degree } -1 \text{ of } \bar{T}^c(sA) \right\} \end{array} ,$$

The equation $D_A^2 = 0$ is then equivalent under this correspondence to the equation

$$\sum_{i_1+i_2+i_3=n} \pm m_{i_1+1+i_3}(\mathrm{id}^{\otimes i_1} \otimes m_{i_2} \otimes \mathrm{id}^{\otimes i_3}) = 0$$

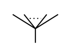
which can be rewritten as

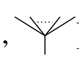
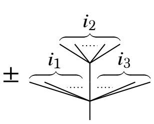
$$[m_1, m_n] = \sum_{\substack{i_1+i_2+i_3=n \\ 2\leqslant i_2 \leqslant n-1}} \pm m_{i_1+1+i_3}(\mathrm{id}^{\otimes i_1} \otimes m_{i_2} \otimes \mathrm{id}^{\otimes i_3}) \ .$$

We thereby get the following equivalent definition of an A_∞-algebra:

DEFINITION 2.4. *Let A be a dg vector space. An* A_∞-algebra *structure on A corresponds to the data of linear maps $m_n : A^{\otimes n} \to A$ of degree $n-2$ for $n \geqslant 1$ such that $m_1 = \partial_A$ and such that*

$$[m_1, m_n] = \sum_{\substack{i_1+i_2+i_3=n \\ 2\leqslant i_2 \leqslant n-1}} \pm m_{i_1+1+i_3}(\mathrm{id}^{\otimes i_1} \otimes m_{i_2} \otimes \mathrm{id}^{\otimes i_3}) \ .$$

The explicit signs will be computed in Exercise sheet 6. Representing m_n as  a corolla of arity n, these equations can be written as

$[m_1,$  $] = \sum_{\substack{i_1+i_2+i_3=n \\ 2\leqslant i_2 \leqslant n-1}} \pm$  .

We moreover check that

$$\begin{aligned} [m_1, m_2] &= 0 \ , \\ [m_1, m_3] &= m_2(\mathrm{id} \otimes m_2 - m_2 \otimes \mathrm{id}) \ , \end{aligned}$$

implying that m_2 descends to an associative product on $H_*(A)$. The operations m_n can then be interpreted as the higher coherent homotopies keeping track of the fact that the product is associative up to homotopy.

DEFINITION 2.5. *An* A_∞-morphism $A \rightsquigarrow B$ *between two A_∞-algebras A and B is defined to be a morphism of dg coalgebras $(\bar{T}^c(sA), D_A) \to (\bar{T}^c(sB), D_B)$.*

There is again a one-one correspondence

$$\left\{ \begin{array}{c} \text{collections of morphisms of degree } n-1 \\ f_n : A^{\otimes n} \to B \ , \ n \geqslant 1, \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{morphisms of graded coalgebras} \\ F : \bar{T}^c(sA) \to \bar{T}^c(sB) \end{array} \right\} \ .$$

The equation $FD_A = D_B F$ under this correspondence then reads as

$$\sum_{i_1+i_2+i_3=n} \pm f_{i_1+1+i_3}(\mathrm{id}^{\otimes i_1} \otimes m^A_{i_2} \otimes \mathrm{id}^{\otimes i_3}) = \sum_{i_1+\cdots+i_s=n} \pm m^B_s(f_{i_1} \otimes \cdots \otimes f_{i_s})$$

and can be rewritten as

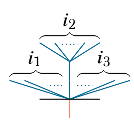
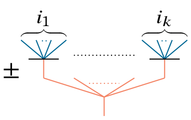
$$[m_1, f_n] = \sum_{\substack{i_1+i_2+i_3=n\\ i_2\geqslant 2}} \pm f_{i_1+1+i_3}(\mathrm{id}^{\otimes i_1} \otimes m^A_{i_2} \otimes \mathrm{id}^{\otimes i_3}) + \sum_{\substack{i_1+\cdots+i_s=n\\ s\geqslant 2}} \pm m^B_s(f_{i_1} \otimes \cdots \otimes f_{i_s}) .$$

We thereby get the following equivalent definition of an A$_\infty$-morphism:

Definition 2.6. *Let A and B be two* A$_\infty$*-algebras. An* A$_\infty$-morphism $A \rightsquigarrow B$ *is defined to be a collection of linear maps $f_n : A^{\otimes n} \to B$ of degree $n-1$ for $n \geqslant 1$ such that*

$$[m_1, f_n] = \sum_{\substack{i_1+i_2+i_3=n\\ i_2\geqslant 2}} \pm f_{i_1+1+i_3}(\mathrm{id}^{\otimes i_1} \otimes m^A_{i_2} \otimes \mathrm{id}^{\otimes i_3}) + \sum_{\substack{i_1+\cdots+i_k=n\\ k\geqslant 2}} \pm m^B_k(f_{i_1} \otimes \cdots \otimes f_{i_k}) .$$

The explicit signs will again be computed in Exercise sheet 6. Representing the operations f_n as , the operations m^A_n in blue and the operations m^B_n in red, these equations read as

$\Big[m_1,$  $\Big] = \sum_{\substack{i_1+i_2+i_3=n\\ i_2\geqslant 2}} \pm$  $+ \sum_{\substack{i_1+\cdots+i_k=n\\ k\geqslant 2}} \pm$  .

We check again that

$$[m_1, f_1] = 0 ,$$
$$[m_1, f_2] = f_1 m^A_2 - m^B_2(f_1 \otimes f_1) .$$

implying that an A$_\infty$-morphism between A$_\infty$-algebras induces a morphism of graded associative algebras $H_*(A) \to H_*(B)$.

2.1.2. *Associahedra, multiplihedra and* A$_\infty$*-spaces.* We denote `CW` the category whose objects are CW complexes and whose morphisms are cellular maps. The functor $C^{cell}_* : \mathtt{CW} \to \mathtt{dg\ Vect}$ is then strong monoidal.

Theorem 25. *There exists a collection of polytopes, called the* associahedra *and denoted K_n, which encode the operad* A$_\infty$.

More precisely, seeing the polytopes K_n as CW-complexes, they form an operad in `CW` whose image under the functor C^{cell}_* is exactly the operad A$_\infty$. We represent in Figure 1 the polytopes K_2, K_3 and K_4. In a similar fashion, there exists a collection of polytopes called the *multiplihedra* and denoted J_n, which encode the notion of A$_\infty$-morphisms. We represent in Figure 2 the polytopes J_1, J_2 and J_3.

Theorem 26. *Let (X, x_0) be a pointed topological space. Then the singular chains on the based loop space $C_*(\Omega X)$ form an* A$_\infty$*-algebra.*

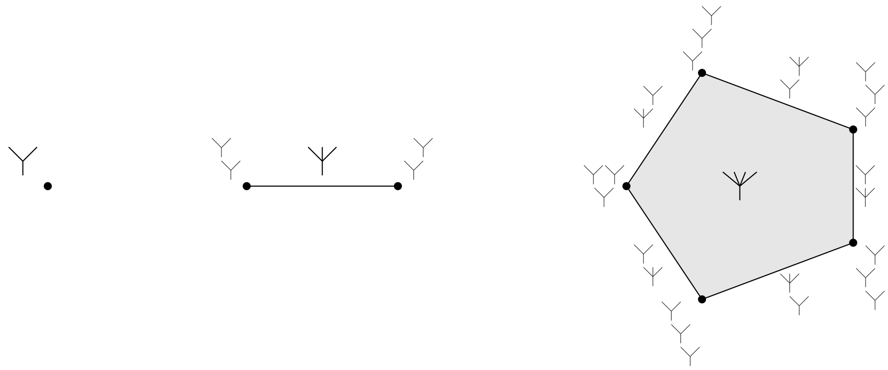


FIGURE 1. The associahedra K_2, K_3 and K_4, with faces labeled by the operations they define

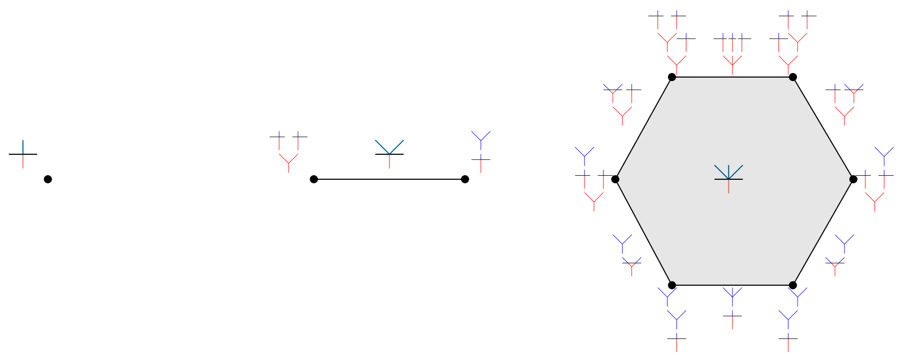


FIGURE 2. The multiplihedra J_1, J_2 and J_3

PROOF. Stasheff proves in his seminal paper [**Sta63**] that the based loop space ΩX can be endowed with an A_∞-space structure: a collection of continuous maps $K_n \times (\Omega X)^{\times n} \to \Omega X$ that satisfy A_∞-like equations. The theorem is then the chain-level version of this topological result. □

2.2. A_∞-categories.

DEFINITION 2.7. *An* A_∞-category $\mathtt{C}$ *corresponds to the following data:*

(1) A set of objects $\mathrm{Ob}(\mathtt{C})$.
(2) For each pair of objects $X, Y \in \mathrm{Ob}(\mathtt{C})$ *a dg vector space* $\mathtt{C}(X, Y)$.
(3) For each sequence of object $X_0, \ldots, X_n$ *of* $\mathtt{C}$*, a degree* $n-2$ *linear map*

$$m_n : \mathtt{C}(X_0, X_1) \otimes \cdots \otimes \mathtt{C}(X_{n-1}, X_n) \to \mathtt{C}(X_0, X_n) \ ,$$

such that

$$[\partial, m_n] = \sum_{\substack{i_1+i_2+i_3=n \\ 2 \leqslant i_2 \leqslant n-1}} \pm m_{i_1+1+i_3}(\mathrm{id}^{\otimes i_1} \otimes m_{i_2} \otimes \mathrm{id}^{\otimes i_3}) \ ,$$

where we do not write the dependence in $C_0, \ldots, C_n$ *in the equation.*

Remark 2.8. An A_∞-category $\mathtt{C}$ with one object $*$ corresponds exactly to an A_∞-algebra $\mathtt{C}(*, *)$.

We then check that the homology of an A_∞-category $H_*(\mathtt{C})$ is a *semicategory* enriched in $\mathtt{gr\ Vect}$, or *non-unital category*, i.e. a category without identities. In other words, an A_∞-category is a semicategory whose composition is associative only up to homotopy. We mention that there are several ways to define a notion of identities in an A_∞-category, that we will not cover in this course. The notion of A_∞-functor between A_∞-categories can moreover be defined in a similar fashion, and it induces a functor between semicategories in homology.

Example 2.9. A *symplectic manifold* is defined to be the data of a smooth manifold X together with a smooth 2-form ω which is closed ($d\omega = 0$) and non-degenerate. A symplectic manifold always has even dimension, $\dim(X) = 2n$. One way to understand the geometry of a symplectic manifold is to study its *Lagrangian submanifolds*, i.e. the n-dimensional submanifolds $L \subset X$ such that $\omega|_L = 0$. The most famous example of an A_∞-category is then the *Fukaya category* of a *symplectic manifold* (X, ω) whose objects are the Lagrangian submanifolds of X, whose dg vector spaces of morphisms are the *Floer chain complexes* and the higher compositions are defined by counting *pseudo-holomorphic disks* with boundary conditions on Lagrangian submanifolds. We refer to [**Aur14**] for an extensive introduction to the construction of this A_∞-category. Its study lies in fact at the crossroads of many areas of mathematics: microlocal analysis, sheaf theory, mirror symmetry, algebraic geometry, homological algebra and dynamical systems to cite a few.

3. L_∞-algebras

We assume that $\mathrm{char}(\mathbb{K}) = 0$ in this section.

3.1. L_∞-algebras and L_∞-morphisms.

Definition 3.1. *Let $i_1, \ldots, i_k \geqslant 1$. We define a $(i_1, \ldots, i_k)$-*shuffle *to be a permutation $\sigma \in \mathfrak{S}_{i_1+\cdots+i_k}$ such that*

$$\sigma(1) < \cdots < \sigma(i_1), \qquad \ldots, \qquad \sigma(i_1 + \cdots + i_{k-1} + 1) < \cdots < \sigma(i_1 + \cdots + i_k)\ .$$

The set of $(i_1, \ldots, i_k)$-shuffles *is then denoted* $\mathrm{Sh}(i_1, \ldots, i_k) \subset \mathfrak{S}_{i_1+\cdots+i_k}$.

A $(i_1, \ldots, i_k)$*-unshuffle* is a permutation $\sigma \in \mathfrak{S}_{i_1+\cdots+i_k}$ such that σ^{-1} is a $(i_1, \ldots, i_k)$-shuffle. The *set of* $(i_1, \ldots, i_k)$*-unshuffles* is then denoted $\mathrm{Sh}^{-1}(i_1, \ldots, i_k)$. Let V be a graded vector space. We say that a linear map $f : V^{\otimes n} \to V$ is *skew-symmetric* if $f^\sigma = (-1)^\sigma f$ for every $\sigma \in \mathfrak{S}_n$.

Definition 3.2. *Let A be a dg vector space. A* L_∞-algebra *structure on A corresponds to the data of skew-symmetric linear maps $l_n : A^{\otimes n} \to A$ of degree $n-2$ for $n \geqslant 1$ such that $l_1 = \partial_A$ and that satisfy*

$$[l_1, l_n] = \sum_{\substack{p+q=n \\ p,q>1}} \ \sum_{\sigma \in \mathrm{Sh}^{-1}(q,p-1)} \pm (l_p \circ_1 l_q)^\sigma\ .$$

We check that $l_2(x,y) = -(-1)^{|x||y|} l_2(y,x)$ and that

$$[l_1, l_2] = 0 \ ,$$
$$[l_1, l_3] = l_2 \circ_1 l_2 + (l_2 \circ_1 l_2)^{(123)} + (l_2 \circ_1 l_2)^{(321)} \ ,$$

implying that l_2 defines a graded Lie algebra structure on $H_*(A)$.

Definition 3.3. *Let A and B be two L_∞-algebras. A* L_∞-morphism *$A \rightsquigarrow B$ corresponds to the data of linear maps $f_n : A^{\otimes n} \to B$ of degree $n-1$ for $n \geqslant 1$ such that*

$$[l_1, f_n] = \sum_{\substack{p+q=n \\ q>1}} \sum_{\sigma \in \mathrm{Sh}^{-1}(q,p-1)} \pm (f_p \circ_1 l_q^A)^\sigma + \sum_{\substack{i_1+\cdots+i_k=n \\ k\geqslant 2}} \sum_{\sigma \in \mathrm{Sh}^{-1}(i_1,\ldots,i_k)} \pm l_k^B (f_{i_1} \otimes \cdots \otimes f_{i_k})^\sigma \ .$$

We check again that

$$[l_1, f_1] = 0 \ ,$$
$$[l_1, f_2] = f_1 l_2^A - l_2^B (f_1 \otimes f_1) \ .$$

implying that a L_∞-morphism between L_∞-algebras induces a morphism of graded Lie algebras $H_*(A) \to H_*(B)$.

We will prove in Exercise sheet 6 that a L_∞-algebra structure on a dg vector space A corresponds exactly to a $\Omega\mathscr{L}ie^{\text{¡}}$-algebra structure on A, and can equivalently be defined as a codifferential on the noncounital cofree graded cocommutative coalgebra $\bar{\Lambda}^c(sV)$. A L_∞*-morphism* between two L_∞-algebras A and B will then correspond to a morphism of $\mathscr{L}ie^{\text{¡}}$-coalgebras $(\bar{\Lambda}^c(sA), D_A) \to (\bar{\Lambda}^c(sB), D_B)$.

3.2. Deformation quantization of Poisson manifolds.

3.2.1. *Moduli space of formal deformations of Maurer-Cartan elements of a dg Lie algebra.*

Definition 3.4. *Let V be a vector space. We define $V[[t]]$ to be the vector space of formal power series in t with coefficients in V.*

In other words, an element of $V[[t]]$ can be written as an infinite sum

$$\alpha_0 + \alpha_1 t + \alpha_2 t^2 + \cdots \in V[[t]]$$

where $\alpha_n \in V$ for $n \geqslant 0$. We moreover denote $V((t)) \subset V[[t]]$ the subspace of $V[[t]]$ whose elements are formal series with null zero-th order coefficient. Let $\mathfrak{g}$ be a dg Lie algebra. The vector space $\mathfrak{g}((t))$ inherits a natural structure of dg Lie algebra whose differential is given by

$$\partial(\sum_{n\geqslant 1} \alpha_n t^n) = \sum_{n\geqslant 1} \partial(\alpha_n) t^n$$

and whose Lie bracket is given by

$$[\sum_{n\geqslant 1} \alpha_n t^n, \sum_{m\geqslant 1} \beta_m t^m] = \sum_{n\geqslant 1} \sum_{i+j=n} [\alpha_i, \beta_j] t^n \ .$$

DEFINITION 3.5. *Let $\mathfrak{g}$ be a dg Lie algebra. We define $\mathfrak{Def}(\mathfrak{g})$ the* moduli space of formal deformations of Maurer-Cartan elements *of $\mathfrak{g}$ to be the set of Maurer-Cartan elements of the dg Lie algebra $\mathfrak{g}((t))$ quotiented by the gauge group action.*

The *gauge group* of a Lie algebra $\mathfrak{g}$ is a group acting on the set $\mathrm{MC}(\mathfrak{g})$ that we will not define in these lectures.

Proposition 3.6. *Let $\mathfrak{g}$ and $\mathfrak{h}$ be two dg Lie algebras. Then every L_∞-quasi-isomorphism $\mathfrak{g} \rightsquigarrow \mathfrak{h}$ induces a bijection $\mathfrak{Def}(\mathfrak{g}) \simeq \mathfrak{Def}(\mathfrak{h})$.*

REMARK 3.7. In general, a *Maurer-Cartan element* α of a *complete filtered* L_∞-algebra $\mathfrak{g}$ is defined to be an element $\alpha \in \mathfrak{g}_1$ such that

$$\partial(\alpha) + \sum_{n \geqslant 2} \frac{1}{n!} l_n(\alpha, \ldots, \alpha) = 0 \ .$$

The assumption that $\mathfrak{g}$ is complete filtered is a technical assumption ensuring that the previous sum is well-defined. Proposition 3.6 then still holds for $\mathfrak{g}$ and $\mathfrak{h}$ two L_∞-algebras.

3.2.2. *Deformation quantization of Poisson manifolds.*

DEFINITION 3.8. *A* Poisson manifold *X is defined to be a smooth manifold X endowed with a bracket $\{\cdot,\cdot\} : \mathscr{C}^\infty(X) \times \mathscr{C}^\infty(X) \to \mathscr{C}^\infty(X)$ defining a commutative Poisson algebra structure on its algebra of smooth functions $\mathscr{C}^\infty(X)$.*

EXAMPLE 3.9. We proved in Exercise sheet 2 that every symplectic manifold is a Poisson manifold.

DEFINITION 3.10. *Let A be an associative algebra. An* associative deformation *of A is defined to be an associative algebra structure on $A[[t]]$ whose multiplication $*$ is $\mathbb{K}[[t]]$-bilinear and reads on two elements $a, b \in A$ as*

$$a * b = a \cdot b + B_1(a,b)t + B_2(a,b)t^2 + \cdots$$

where $\cdot$ is the multiplication of A.

Proposition 3.11. *We assume that A is a commutative algebra. Then for every associative deformation of A, the bracket*

$$[a,b] := \mathrm{B}_1(a,b) - \mathrm{B}_1(b,a)$$

defines a Lie bracket endowing A with a commutative Poisson algebra structure.

PROOF. See Exercise sheet 6. □

The Poisson algebra $P = (A, [\cdot,\cdot])$ obtained under an associative deformation is called its *classical limit.* The associative algebra $(A[[t]], *)$ is then called a *deformation quantization* of the Poisson algebra P.

Let X be a smooth manifold. We say that a bilinear map $\mathscr{C}^\infty(X) \times \mathscr{C}^\infty(X) \to \mathscr{C}^\infty(X)$ is a bidifferential operator if it is a differential operator with respect to both arguments, i.e. reads locally as a linear combination of partial derivatives.

Definition 3.12. *A* deformation quantization *of a Poisson manifold X is defined to be a deformation quantization of the Poisson algebra $\mathscr{C}^\infty(X)$ such that all B_n are bidifferential operators.*

Theorem 27. *Every Poisson manifold admits a deformation quantization.*

Proof. This theorem was proven by Kontsevich in [**Kon03**]. In his paper, he constructs a graded Lie algebra $\mathfrak{g}$ and a dg Lie algebra $\mathfrak{h}$ such that $\mathfrak{Def}(\mathfrak{g})$ contains the set of equivalence classes of Poisson structures on the manifold X and such that $\mathfrak{Def}(\mathfrak{h})$ is the set of equivalence classes of associative deformations of $\mathscr{C}^\infty(X)$ such that each B_n is a bidifferential operator. He then constructs a L_∞-quasi-isomorphism $\mathfrak{g} \rightsquigarrow \mathfrak{h}$ and concludes using Proposition 3.6. □

In classical mechanics, the manifold X is a cotangent bundle T^*M whose symplectic structure induces its Poisson manifold structure. The space of smooth functions $\mathscr{C}^\infty(T^*M)$ is to be understood as the space of observables on the phase space T^*M. A deformation quantization of $\mathscr{C}^\infty(T^*M)$ should then be interpreted as the process of forming a quantum mechanical system from the classical mechanical system associated to M.

Bibliography

[Ada56] J. F. Adams. On the cobar construction. *Proc. Natl. Acad. Sci. USA*, 42:409–412, 1956.

[Aur14] Denis Auroux. A beginner's introduction to Fukaya categories. In *Contact and symplectic topology*, volume 26 of *Bolyai Soc. Math. Stud.*, pages 85–136. János Bolyai Math. Soc., Budapest, 2014.

[Bro59] Edgar H. jun. Brown. Twisted tensor products. I. *Ann. Math. (2)*, 69:223–246, 1959.

[BV73] J. M. Boardman and R. M. Vogt. *Homotopy invariant algebraic structures on topological spaces*, volume 347 of *Lect. Notes Math.* Springer, Cham, 1973.

[CHV06] Ralph L. Cohen, Kathryn Hess, and Alexander A. Voronov. *String topology and cyclic homology*. Adv. Courses in Math. – CRM Barc. Basel: Birkhäuser, 2006.

[CJ02] Ralph L. Cohen and John D. S. Jones. A homotopy theoretic realization of string topology. *Math. Ann.*, 324(4):773–798, 2002.

[CS99] Moira Chas and Denis Sullivan. String topology, 1999. arXiv:9911159.

[DMZ07] M. Doubek, M. Markl, and P. Zima. Deformation theory (lecture notes), 2007. arXiv:0705.3719.

[DSV18] Vladimir Dotsenko, Sergey Shadrin, and Bruno Vallette. The twisting procedure, 2018. arXiv:1810.02941.

[FHT01] Yves Felix, Stephen Halperin, and Jean-Claude Thomas. *Rational homotopy theory*, volume 205 of *Grad. Texts Math.* New York, NY: Springer, 2001.

[For08] Stefan Forcey. Convex hull realizations of the multiplihedra. *Topology Appl.*, 156(2):326–347, 2008.

[FT08] Yves Félix and Jean-Claude Thomas. Rational BV-algebra in string topology. *Bull. Soc. Math. Fr.*, 136(2):311–327, 2008.

[Gin18] Grégory Ginot. Introduction à l'homotopie, 2018. Lecture notes available at `https://www.math.univ-paris13.fr/%7Eginot/Homotopie/Ginot-homotopie2019.pdf`.

[Hes07] Kathryn Hess. Rational homotopy theory: a brief introduction. In *Interactions between homotopy theory and algebra. Summer school, University of Chicago, IL, USA, July 26–August 6, 2004*, pages 175–202. Providence, RI: American Mathematical Society (AMS), 2007.

[Hur55] Witold Hurewicz. On the concept of fiber space. *Proc. Natl. Acad. Sci. USA*, 41:956–961, 1955.

[Idr22] Najib Idrissi. Théorie de l'homotopie, 2022. Lecture notes available at `https://idrissi.eu/class/homotopie.pdf`.

[Kar] Gizem Karaali. Deformation quantization – A brief survey.

[Koc04] Joachim Kock. *Frobenius algebras and 2D topological quantum field theories*, volume 59 of *Lond. Math. Soc. Stud. Texts.* Cambridge: Cambridge University Press, 2004.

[Kon03] Maxim Kontsevich. Deformation quantization of Poisson manifolds. *Lett. Math. Phys.*, 66(3):157–216, 2003.

[KS22] Andreas Kraft and Jonas Schnitzer. An introduction to L_∞-algebras and their homotopy theory, 2022. arXiv:2207.01861.

[LAM22] Guillaume Laplante-Anfossi and Thibaut Mazuir. The diagonal of the multiplihedra and the tensor product of A_∞-morphisms, 2022. arXiv:2206.05566.

[Lee13] John M. Lee. *Introduction to smooth manifolds*, volume 218 of *Grad. Texts Math.* New York, NY: Springer, 2nd revised ed edition, 2013.

[Liv17] Muriel Livernet. Topologie algébrique, 2017. Lecture notes.

[LO15] Janko Latschev and Alexandru Oancea, editors. *Free loop spaces in geometry and topology. Including the monograph Symplectic cohomology and Viterbo's theorem by Mohammed Abouzaid*, volume 24 of *IRMA Lect. Math. Theor. Phys.* Zürich: European Mathematical Society (EMS), 2015.

[Lod98] Jean-Louis Loday. *Cyclic homology.*, volume 301 of *Grundlehren Math. Wiss.* Berlin: Springer, 2nd ed. edition, 1998.

[LV12] Jean-Louis Loday and Bruno Vallette. *Algebraic operads*, volume 346 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences].* Springer-Verlag, Berlin, 2012.

[Mac65] S. MacLane. Categorical algebra. *Bull. Am. Math. Soc.*, 71:40–106, 1965.

[Mas] Naruki Masuda. Bar-cobar and koszul duality for algebraic operads. Notes available at `https://math.jhu.edu/~nmasuda2/bar-cobar-Koszul.pdf`.

[ML71] Saunders Mac Lane. *Categories for the working mathematician*, volume 5 of *Grad. Texts Math.* Springer, Cham, 1971.

[MSS02] Martin Markl, Steve Shnider, and James D. Stasheff. *Operads in algebra, topology and physics*, volume 96 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 2002.

[MTTV21] Naruki Masuda, Hugh Thomas, Andy Tonks, and Bruno Vallette. The diagonal of the associahedra. *J. Éc. Polytech., Math.*, 8:121–146, 2021.

[Riv22] Manuel Rivera. Adams' cobar construction revisited. *Doc. Math.*, 27:1213–1223, 2022.

[Sar] Maru Sarazola. Loop spaces and operads. Notes available at `https://drive.google.com/file/d/1_UWR66MmQWnxbYeZWB-eTQFJRbaS2eJw/view?usp=sharing`.

[Sei08] Paul Seidel. *Fukaya categories and Picard-Lefschetz theory.* Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2008.

[Sta63] James D. Stasheff. Homotopy associativity of H-spaces. I, II. *Trans. Amer. Math. Soc. 108 (1963), 275-292; ibid.*, 108:293–312, 1963.

[Val14] Bruno Vallette. Algebra + homotopy = operad. In *Symplectic, Poisson, and noncommutative geometry*, volume 62 of *Math. Sci. Res. Inst. Publ.*, pages 229–290. Cambridge Univ. Press, New York, 2014.

[Val17] Bruno Vallette. Algèbre+homotopie=opérades, 2017. Lecture notes.

[Val20] Bruno Vallette. Homotopy theory of homotopy algebras. *Ann. Inst. Fourier*, 70(2):683–738, 2020.

[Wei94] Charles A. Weibel. *An introduction to homological algebra*, volume 38 of *Camb. Stud. Adv. Math.* Cambridge: Cambridge University Press, 1994.