Introduction to algebraic operads

Exercise sheet 6: Homotopy algebras

Exercise 1 (The cooperad $\mathcal{As}^{¡}$). *We work over the field $\mathbb{Z}/2\mathbb{Z}$ in this exercise. We recall that the ns operad $\mathcal{As}$ is binary quadratic: it is generated by one arity 2 and degree 0 operation denoted μ satisfying $\mu \circ_1 \mu - \mu \circ_2 \mu = 0$. We then define the following elements of the cofree cooperad $\mathcal{T}^c_{ns}(\mathbb{K}s\mu)$:*

$$\mu_1^c = \mathrm{id} \qquad \mu_2^c = s\mu \qquad \mu_n^c = \sum_{t \in \mathrm{PBT}_n} t_\mu$$

where PBT_n *denotes the* set of planar binary ribbon trees *of arity n and the sum corresponds to planar binary ribbon trees all of whose vertices are labeled by $s\mu$.*

1. Prove that for every $n \geqslant 1$

$$\Delta^k_{\mathcal{T}^c_{ns}(\mathbb{K}s\mu)}(\mu_n^c) = \sum_{i_1+\cdots+i_k=n} (\mu_k^c; \mu_{i_1}^c, \ldots, \mu_{i_k}^c) \ .$$

2. Prove that for every $n \geqslant 1$

$$\mathcal{As}^{¡}(n) = \mathbb{K}\mu_n^c \subset \mathcal{T}^c_{ns}(\mathbb{K}s\mu) \ .$$

3. Prove that for a graded vector space V, the cofree $\mathcal{As}^{¡}$-coalgebra generated by V is the cofree coalgebra $s^{-1}\bar{T}^c(sV)$.

Exercise 2 (A_∞-equations). *Let A a graded vector space.*

1. Prove that a codifferential on the cofree coalgebra $\bar{T}^c(sA)$ is equivalent to a collection of degree -1 linear maps $b_n : (sA)^{\otimes n} \to sA$ for $n \geqslant 1$ satisfying

$$\sum_{i_1+i_2+i_3=n} b_{i_1+1+i_3}(\mathrm{id}^{\otimes i_1} \otimes b_{i_2} \otimes \mathrm{id}^{\otimes i_3}) = 0 \ .$$

2. Prove that the previous data is equivalent to a collection of maps $m_n : A^{\otimes n} \to A$ for $n \geqslant 1$ satisfying

$$\sum_{i_1+i_2+i_3=n} (-1)^{i_1+i_2 i_3} m_{i_1+1+i_3}(\mathrm{id}^{\otimes i_1} \otimes m_{i_2} \otimes \mathrm{id}^{\otimes i_3}) = 0 \ .$$

3. Prove that a morphism of dg coalgebras $(\bar{T}^c(sA), D_A) \to (\bar{T}^c(sB), D_B)$ is equivalent to a collection of maps $f_n : A^{\otimes n} \to B$ satisfying

$$\sum_{i_1+i_2+i_3=n} (-1)^{i_1+i_2 i_3} f_{i_1+1+i_3}(\mathrm{id}^{\otimes i_1} \otimes m_{i_2} \otimes \mathrm{id}^{\otimes i_3}) = \sum_{i_1+\cdots+i_s=n} (-1)^{\epsilon} m_s(f_{i_1} \otimes \cdots \otimes f_{i_s})$$

where $\epsilon = \sum_{u=1}^{s} (s-u)(1-i_u)$.

Exercise 3 (Homotopy transfer theorem for A_∞-algebras).

We work over the field $\mathbb{Z}/2\mathbb{Z}$. Consider a deformation retract diagram

$$h \circlearrowright (A, \partial_A) \underset{i}{\overset{p}{\rightleftarrows}} (H, \partial_H) \ .$$

Our goal is to prove the homotopy transfer theorem *for A_∞-algebras: if A is endowed with an A_∞-algebra structure, then H can be made into an A_∞-algebra such that i and p extend to A_∞-morphisms.*

We denote PT_n *the* set of planar trees *with n vertices. For $t \in \mathrm{PT}_n$ we then denote $t^H_{h,i,p} : H^{\otimes n} \to H$ the linear map defined by labeling the arity k vertices of t by the operation m_k, its internal edges by h, its incoming edges by i and its outgoind edge by p.*

1. Prove that the operations

$$m_n^H := \sum_{t \in \mathrm{PT}_n} t^H_{h,i,p} : H^{\otimes n} \to H$$

define an A_∞-algebra structure on H.

2. Prove that i and p can be extended to A_∞-morphisms using similar formulae.

Exercise 4 (Deformation retract diagram).

Let A be a dg vector space. For $n \in \mathbb{Z}$ we denote $H_n := H_n(A)$ and $B_n := \mathrm{Im}(\partial^A_{n+1})$.

1. Prove that $A_n \simeq B_n \oplus H_n \oplus B_{n-1}$.

2. Prove that the homology $H_*(A)$ with trivial differential is a deformation retract of A.

Exercise 5 (Homotopy Gerstenhaber algebra).

*We work over the field $\mathbb{Z}/2\mathbb{Z}$. Let A be a dg vector space. We define a G_∞-*algebra *structure on A to be a collection of maps $m_n : A^{\otimes n} \to A$ of degree $n-2$ for $n \geqslant 1$ and $m_{k,l} : A^{\otimes k} \otimes A^{\otimes l} \to A$ of degree $k+l-1$ for $k, l \geqslant 0$ that satisfy the following three families of equations:*

$$\sum_{i_1+i_2+i_3=n} m_{i_1+1+i_3}(a_1, \ldots, a_{i_1}, m_{i_2}(a_{i_1+1}, \ldots, a_{i_1+i_2}), a_{i_1+i_2+1}, \ldots, a_n) = 0 \ ,$$

$$\sum_{\substack{i_1+\cdots+i_s=l \\ j_1+\cdots+j_s=m}} m_{k,s}(a_1, \ldots, a_k; m_{i_1,j_1}(b_1, \ldots, b_{i_1}; c_1, \ldots, c_{j_1}), \ldots, m_{i_s,j_s}(b_{i_1+\cdots+i_{s-1}+1}, \ldots, b_l; c_{j_1+\cdots+j_{s-1}+1}, \ldots, c_m))$$
$$= \sum_{\substack{i_1+\cdots+i_s=k \\ j_1+\cdots+j_s=l}} m_{s,m}(m_{i_1,j_1}(a_1, \ldots, a_{i_1}; b_1, \ldots, b_{j_1}), \ldots, m_{i_s,j_s}(a_{i_1+\cdots+i_{s-1}+1}, \ldots, a_k; b_{j_1+\cdots+j_{s-1}+1}, \ldots, b_l); c_1, \ldots, c_m) \ ,$$

$$\sum_{\substack{i_1+\cdots+i_s=k\\ j_1+\cdots+j_s=l}} m_s(m_{i_1,j_1}(a_1,\ldots,a_{i_1};b_1,\ldots,b_{j_1}),\ldots,m_{i_s,j_s}(a_{i_1+\cdots+i_{s-1}+1},\ldots,a_k;b_{j_1+\cdots+j_{s-1}+1},\ldots,b_l))$$

$$= \sum_{i_1+i_2+i_3=k} m_{i_1+1+i_3,l}(a_1,\ldots,a_{i_1},m_{i_2}(a_{i_1+1},\ldots,a_{i_1+i_2}),a_{i_1+i_2+1},\ldots,a_k;b_1,\ldots,b_l)$$

$$+ \sum_{j_1+j_2+j_3=k} m_{k,j_1+1+j_3}(a_1,\ldots,a_k;b_1,\ldots,b_{j_1},m_{j_2}(b_{j_1+1},\ldots,b_{j_1+j_2}),b_{j_1+j_2+1},\ldots,b_l)\ ,$$

where $m_1 = \partial_A$, $m_{1,0} = m_{0,1} = \mathrm{id}$ *and* $m_{k,0} = m_{0,k} = 0$ *for* $k \neq 1$.

1. Prove that the homology of a G_∞-algebra is a Gerstenhaber algebra $H_*(A)$.

2. Prove that a G_∞-algebra structure on A is equivalent to a dg bialgebra structure on $T(sA)$ whose coproduct is the deconcatenation coproduct, whose unit is the inclusion $\mathbb{K} \hookrightarrow T(sA)$ and whose counit is the projection $T(sA) \twoheadrightarrow \mathbb{K}$.

Exercise 6 (Massey products).

Def arity 3 Massey product / Show that m2 and m3 correspond / Borromean ring

Exercise 7 (Curved A$_\infty$-algebra).