Introduction to algebraic operads

Exercise sheet 5: Koszul duality

Exercise 1 (Koszul dual algebra).

Let (V, R) be a finite-dimensional quadratic data.

1. Prove that $A(V,R)^! = A(V^\circ, R^\perp)$ and that $A(V,R)$ is Koszul if and only if $A(V,R)^!$ is Koszul.

2. Prove that the algebra of polynomials with n variables $\mathbb{K}[x_1, \dots x_n]$ is quadratic and that its Koszul dual algebra is the exterior algebra $\Lambda(x_1^\vee, \dots, x_n^\vee)$.

Exercise 2 (Koszul algebras).

1. Let V be a graded vector space. Prove that the free graded algebra $T(V)$ is Koszul as a quadratic algebra.

2. Let V be a vector space concentrated in degree 0. Prove that the symmetric algebra $S(V)$ is Koszul.

Exercise 3 (Dual numbers algebra). *Let $A = \mathbb{K}[\varepsilon]/\langle \varepsilon^2 = 0\rangle$ be the quadratic algebra of dual numbers with $|\varepsilon| = 0$.*

Prove that as augmented dg algebras

$$\Omega A^{¡} \simeq \mathbb{K}\langle t_1, t_2, \dots, t_n, \dots \rangle \ ,$$

where $|t_n| = n-1$ and $\partial(t_n) = -\sum_{i+j=n}(-1)^i t_i t_j$.

Exercise 4 (Manin products). *For (V, R) and (W, S) two quadratic data we define the switching map*

$$\tau_{23} : V \otimes V \otimes W \otimes W \xrightarrow{\mathrm{id}\otimes\tau\mathrm{id}} V \otimes W \otimes V \otimes W \ .$$

We then introduce

(i) *The* white Manin product *defined as the quadratic data*

$$(V,R)\bigcirc(W,S) = (V \otimes W, \tau_{23}(V^{\otimes 2} \otimes S + R \otimes W^{\otimes 2})) \ ,$$

(ii) *the* black Manin product *defined as the quadratic data*

$$(V,R)\bullet(W,S) = (V \otimes W, \tau_{23}(R \otimes S)) \ .$$

1. Prove that the quadratic data $(\mathbb{K}x, 0)$ with $|x| = 0$ is the unit for the white product $\bigcirc$ and that the quadratic data $(\mathbb{K}x, (\mathbb{K}x)^{\otimes 2})$ with $|x| = 0$ is the unit for the black product $\bullet$.

We denote $A(V,R)\bigcirc A(W,S) := A((V,R)\bigcirc(W,S))$ and $A(V,R)\bullet A(W,S) := A((V,R)\bullet(W,S))$.

2. Prove that there exists a morphism of quadratic algebras

$$A(V,R)\bullet A(W,S) \to A(V,R)\bigcirc A(W,S)\ .$$

3. Prove that if V and W are finite-dimensional then

$$(A(V,R)\bigcirc A(W,S))^{!} = A(V,R)^{!}\bullet A(W,S)^{!}\ .$$

4. Prove that if (V_1,R_1), (V_2,R_2) and (V_3,R_3) are quadratic data with V_2 finite-dimensional then there is a bijection

$$\mathrm{Hom}_{\texttt{quadr data}}((V_1,R_1)\bigcirc(V_2,R_2),(V_3,R_3)) \simeq \mathrm{Hom}_{\texttt{quadr data}}((V_1,R_1),(V_2,R_2)^{!}\bullet(V_3,R_3))$$

where we denote $(V,R)^{!} = (V^{\circ},R^{\perp})$.

We finally mention that if two quadratic algebras are Koszul, then so are their white and black Manin products.

Exercise 5 (Rewriting method).

Using the rewriting method, prove that the following quadratic algebra is Koszul:

$$A = A(v_1,v_2,v_3; v_1v_2 - v_2v_3, v_3v_2 - v_2v_1, v_1v_3 - v_3v_1, v_2v_2 - v_3v_1)$$

where $|v_1| = |v_2| = |v_3| = 0$.

Exercise 6 (Operadic suspension).

1. Give an explicit description of an $\mathcal{S}$-algebra.

2. Let $\mathcal{P}$ be an operad and $\mathcal{C}$ be a cooperad. Prove that the vector space V is a $\mathcal{P}$-algebra if and only if the suspended vector space sV is a $\mathcal{S}\otimes\mathcal{P}$-algebra.

Exercise 7 (A morphism $\mathcal{L}ie \to \mathcal{P}\otimes\mathcal{P}^{!}$).

Let $\mathcal{P} := \mathcal{P}(M,R)$ be a finitely generated binary quadratic operad.

1. Prove that it is possible to choose a basis of the vector space M whose elements μ satisfy either $\mu^{(12)} = \mu$ or $\mu^{(12)} = -\mu$.

Let A be a $\mathcal{P}$-algebra and B be a $\mathcal{P}^{!}$-algebra. We choose a basis $\mathcal{B}$ of M as in question 1. . We define a bracket on $A\otimes B$ as

$$\{a_1\otimes b_1, a_2\otimes b_2\} := \sum_{\mu\in\mathcal{B}} \mu(a_1,a_2)\otimes\mu^{\vee}(b_1,b_2)$$

where we use the fact that $\mathcal{P}^{!} = \mathcal{P}(M^{\circ},R^{\perp})$.

2. Prove that the bracket $\{-,-\}$ defines a Lie algebra structure on $A\otimes B$.

Exercise 8 (Examples of Koszul dual operads).

1. Prove that the operad $\mathcal{P}ois$ encoding commutative Poisson algebras is binary quadratic and satisfies $\mathcal{P}ois^{!} = \mathcal{P}ois$.

We define a Leibniz algebra *to be a vector space A endowed with a linear map $[-,-] : A \otimes A \to A$ such that*

$$[[x,y],z] = [x,[y,z]] + [[x,z],y] \ .$$

We also define a Zinbiel algebra *to be a vector space A endowed with a linear map $\prec : A \otimes A \to A$ such that*

$$(x \prec y) \prec z) = x \prec (y \prec z + z \prec y) \ .$$

We write $\mathscr{L}eib$ and $\mathscr{Z}inb$ for the operads respectively encoding Leibniz algebra sand Ziebnel algebras.

2. Prove that $\mathscr{L}eib^! = \mathscr{Z}inb$.

Exercise 9 (Dendriform and diassociative algebras).

We define a dendriform algebra *to be a vector space A endowed with two linear maps*

$$\prec : A \otimes A \to A \qquad\qquad \succ : A \otimes A \to A$$

which satisfy the following relations

$$\begin{aligned} (x \prec y) \prec z &= x \prec (y \prec z) + x \prec (y \succ z) \\ (x \succ y) \prec z &= x \succ (y \prec z) \\ (x \prec y) \succ z + (x \succ y) \succ z &= x \succ (y \succ z) \ . \end{aligned}$$

1. Prove that the operation $x * y := x \prec y + x \succ y$ endows A with an associative algebra structure.

2. Using the rewriting method prove that the ns binary quadratic operad $\mathscr{D}end$ encoding dendriform algebras is Koszul.

We define a diassociative algebra *to be a vector space A endowed with two linear maps*

$$\dashv : A \otimes A \to A \qquad\qquad \vdash : A \otimes A \to A$$

which satisfy the following relations

$$\begin{aligned} &(x \dashv y) \dashv z = x \dashv (y \dashv z) = x \dashv (y \vdash z) \\ &(x \vdash y) \dashv z = x \vdash (y \dashv z) \\ &(x \vdash y) \vdash z = (x \dashv y) \vdash z = x \vdash (y \vdash z) \ . \end{aligned}$$

3. Writing $\mathscr{D}ias$ for the binary quadratic operad encoding diassociative algebras, prove that $\mathscr{D}ias^! = \mathscr{D}end$.