Introduction to algebraic operads

Exercise sheet 4: Twisting morphisms

Exercise 1 (The augmented bar construction is acyclic). *Let A be an augmented dg algebra. Our goal is to prove that the twisted tensor product $BA \otimes_\pi A$ is acyclic. We denote an element of $BA \otimes A$ as*

$$a_1|\dots|a_n|a_{n+1} := sa_1 \otimes \cdots \otimes sa_n \otimes a_{n+1}$$

where $a_i \in \bar{A}$ for $1 \leqslant i \leqslant n$ and $a_{n+1} \in A$.

1. Compute the twisted differential on $BA \otimes_\pi A$.

2. Prove that $BA \otimes_\pi A$ is contractible and conclude the proof.

Exercise 2 (Hochschild homology and bar construction). *Let C_* be a chain complex. We define a* chain subcomplex *$D_* \subset C_*$ to be a collection of vector spaces $D_n \subset C_n$ such that $\partial_n^C(D_n) \subset D_{n-1}$ for all $n \in \mathbb{Z}$.*

1. Prove that if D_* is acyclic then the projection map $C_* \to C_*/D_*$ is a quasi-isomorphism.

Let A be a unital dg algebra and M be an A-bimodule. We set $\bar{A} := A/\mathbb{K}1_A$. Consider the Hochschild complex $C_n(A, M) := M \otimes A^{\otimes n}$ and define D_n the subspace of $C_n(A, M)$ generated by all elements $m|a_1|\dots|a_n$ such that at least one of the a_i is equal to 1_A.

2. Prove that $D_* \subset C_*(A, M)$ is a chain subcomplex which is acyclic.

We define the normalized Hochschild chains *as*

$$\bar{C}_*(A, M) := C_*(A, M)/D_* \ .$$

Following question 1. we then have that $\bar{H}_*(M, A) \simeq H_*(M, A)$.

3. Let A be an augmented associative algebra seen as an augmented dg algebra concentrated in degree 0. Prove that the Hochschild homology of A with coefficients in $\mathbb{K}$ is isomorphic to the homology of its bar construction BA.

Exercise 3 (The d_2 differential). *Let s be an element of degree $|s| = 1$. We introduce the degree -1 map $\mu_s : \mathbb{K}s \otimes \mathbb{K}s \to \mathbb{K}s$ defined as $\mu_s(s \otimes s) = s$. We also point out that the suspension sV of a dg vector space V then corresponds to the dg tensor product $\mathbb{K}s \otimes V$.*

1. Let A be an augmented dg algebra A. Prove that the differential d_2 on BA can be equivalently defined as the unique extension of the degree -1 composite linear map

$$T^c(sA) \twoheadrightarrow s\bar{A} \otimes s\bar{A} = \mathbb{K}s \otimes \bar{A} \otimes \mathbb{K}s \otimes \bar{A} \simeq \mathbb{K}s \otimes \mathbb{K}s \otimes \bar{A} \otimes \bar{A} \xrightarrow{\mu_s \otimes \mu_A} \mathbb{K}s \otimes \bar{A} = s\bar{A} \ .$$

We also introduce the degree -1 linear map $\Delta_s : \mathbb{K}s^{-1} \to \mathbb{K}s^{-1} \otimes \mathbb{K}s^{-1}$.

2. Let C be a coaugmented dg coalgebra. Prove that the differential d_2 on ΩC can be equivalently defined as the unique extension of the degree -1 composite linear map

$$s^{-1}C = \mathbb{K}s^{-1} \otimes C \xrightarrow{\Delta_s \otimes \Delta_C} \mathbb{K}s^{-1} \otimes \mathbb{K}s^{-1} \otimes C \otimes C \simeq \mathbb{K}s^{-1} \otimes C \otimes \mathbb{K}s^{-1} \otimes C = s^{-1}C \otimes s^{-1}C \hookrightarrow \Omega C \ .$$

3. Let $\mathscr{P}$ be an operad. Compute an explicit formula (with signs) for the differential d_2 on the bar construction $B\mathscr{P}$.

4. Let $\mathscr{C}$ be a cooperad. Compute an explicit formula (with signs) for the differential d_2 on the cobar construction $\Omega\mathscr{C}$.

Exercise 4 (Schur lemma). *Let* $\mathbb{K}$ *be a field with* $\mathrm{char}(\mathbb{K}) = 0$.

Prove that for every natural transformation $\lambda_V : V^{\otimes n} \to V^{\otimes n}$ there exists an element $\lambda \in \mathbb{K}[\mathfrak{S}_n]$ such that $\lambda_V(v_1, \ldots, v_n) = \lambda \cdot (v_1 \otimes \cdots \otimes v_n)$ for $v_1, \ldots, v_n \in V$.

Exercise 5 (Derivation). *For an operad* $\mathscr{P}$ *in* `Vect` *and a* $\mathscr{P}$*-algebra* A*, we define a derivation* $d : A \to A$ *to be a linear map such that for every* $\mu \in \mathscr{P}(n)$*,*

$$d(\mu(a_1, \ldots, a_n)) = \sum_{i=1}^{n} \mu(a_1, \ldots, d(a_i), \ldots, a_n) \ .$$

Prove that for a vector space V, any derivation on the free $\mathscr{P}$-algebra $d : S_{\mathscr{P}}(V) \to S_{\mathscr{P}}(V)$ is completely determined by its restriction $V \to S_{\mathscr{P}}(V)$.

Exercise 6 (bar-cobar resolution).

Prove that the resolution $\Omega B\mathscr{A}ss \to \mathscr{A}ss$ is not minimal.

Exercise 7 (dg pre-Lie algebra).

Let $\mathscr{P}$ be an operad in `Vect` with $\mathscr{P}(0) = 0$. For $\nu \in \mathscr{P}(n)$, $\mu \in \mathscr{P}(m)$, we define

$$\{\mu, \nu\} := \sum_{i=1}^{n} \nu \circ_i \mu \ .$$

1. Prove that this bracket defines a dg pre-Lie algebra structure on the dg vector space $\prod_{n \geqslant 1} \mathscr{P}(n)$.

2. Prove that for a dg cooperad $\mathscr{C}$ and a dg operad $\mathscr{P}$ such that $\mathscr{P}(0) = 0$ and $\mathscr{C}(0) = 0$, the dg pre-Lie algebra associated to the the convolution operad of $\mathscr{C}$ and $\mathscr{P}$ is exactly the operadic convolution pre-Lie algebra.

Exercise 8 (Coderivation).

Prove that there is a correspondence between coderivations of the cofree cooperad $\mathscr{T}^c(\mathscr{M})$ and morphisms of $\mathfrak{S}$-modules $\mathscr{T}^c(\mathscr{M}) \to \mathscr{M}$.

Exercise 9 (Hurewicz fibration). *Let $p : E \to B$ be a Hurewicz fibration and assume that B is connected.*

1. Prove that for $b_0, b_1 \in B$, the fibers $p^{-1}(b_0)$ and $p^{-1}(b_1)$ are homotopy equivalent.

2. Prove that there exists a lifting function for p.