Mortensen observer for sub-differential dynamics

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Non-smooth dynamics

$$\begin{cases} \forall t > 0, \quad \dot{x}(t) + \partial g(x(t)) \ni f(x(t)), \\ x(0) \in \mathbb{R}_+, \end{cases}$$

for $g : \mathbb{R}^d \to \mathbb{R}$ convex.

Convex sub-differential

$$\partial g(x) := \{p \in \mathbb{R}^d, \forall y \in \mathbb{R}^d, \; g(y) \geq p \cdot (y - x) + g(x)\}.$$

A suitable model for

- Dry friction.
- Elasto-plasticty...

A 1D toy-model: the Skorokhod problem

Sub-differential description:

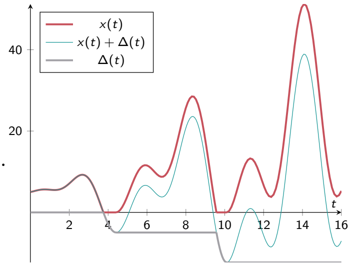
$$\forall t > 0, \quad \dot{x}_w(t) + \partial I_{\mathbb{R}_+}(x_w(t)) \ni w(t),$$

where

$$\partial I_{\mathbb{R}_+}(x) = \begin{cases} \{0\} \text{ if } x > 0, \\ \mathbb{R}_- \text{ if } x = 0. \end{cases}$$

$$\begin{cases} x_w(t) + \Delta(t) = x(0) + \int_0^t w(s)\,\mathrm{d}s, \\ \Delta(t) := \\ \min_{0 \le s \le t} \min\left(0\,,\, x(0) + \int_0^s w(\tau)\mathrm{d}\tau\right). \end{cases}$$

$\Rightarrow$ Reflection at 0.



A reflected trajectory.

Observer design

Measurement:

- Observation map $h : \mathbb{R}_+ \to \mathbb{R}$.
- Collected data $t \mapsto \dot{y}(t)$.

Objective: build a causal estimator [Kre98]

$$t \mapsto \hat{x}(t),$$

such that

$$\forall t > 0, \quad \dot{y}(t) \simeq h(\hat{x}(t)).$$

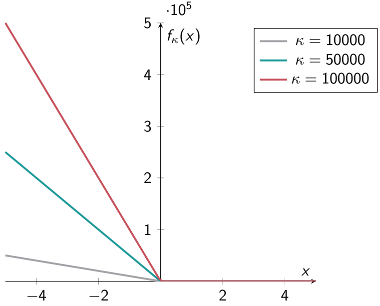
$\Rightarrow$ We follow Mortensen's approach [Mor68].

An approximated dynamics

The non-smooth dynamics is approached by

$$\begin{cases} \forall t > 0, \quad \dot{x}_w(t) + \kappa(x_w(t) - |x_w(t)|_+) = w(t), \\ x(0) \in \mathbb{R}_+, \end{cases}$$

as $\kappa \to +\infty$.



Graph of $f_\kappa(x) := \kappa(|x|_+ - x)$.

Estimation setting

Model dynamics

$$\begin{cases} \forall t > 0, \quad \dot{x}_w(t) + f_\kappa(x_w(t)) = w(t), \\ x_w(0) \in \mathbb{R}, \end{cases}$$

with

- $w(t)$: state disturbance.
- $x_w(0)$: initial condition disturbance.

Trade-off between observation and dynamics accuracy

$$\inf_{\substack{x_w(0)\in\mathbb{R} \\ w\in L^2(0,t)}} \psi(x_w(0)) + \int_0^t \frac{1}{2}|w(s)|^2 + |\dot{y}(s) - h(x_w(s))|^2 \mathrm{d}s.$$

Cost-to-come function

Cost-to-come function [Mor68]

$$V_\kappa(t,x) := \inf_{x_w(t)=x} \psi(x_w(0)) + \int_0^t \frac{1}{2}|w(s)|^2 + \frac{1}{2}\underbrace{|\dot{y}(s) - h(x_w(s))|^2}_{\text{observation adequacy}} \mathrm{d}s,$$

Mortensen's observer

$$\forall t \geq 0, \quad \boxed{\hat{x}(t) \in \operatorname{argmin}_x V_\kappa(t,x)}$$

- Linear case: V quadratic $\Rightarrow$ Kalman's filter.
- Fully convex case [RW00].
- Local well-posedness [BS23].

Linear case

When $f_\kappa(x) = -Ax$ and $h(x) = Cx$, Kalman's filter:

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{x}(t) = A\hat{x}(t) + P(t)C^\top[\dot{y}(t) - C\hat{x}(t)],$$

where the matrix P solves the Riccati equation

$$\frac{\mathrm{d}}{\mathrm{d}t}P(t) = \mathrm{Id} + AP(t) + P(t)A^\top - P(t)^\top CC^\top P(t).$$

In this case,

$$V(t,x) = \frac{1}{2}[x - \hat{x}(t)]^\top P(t)^{-1}[x - \hat{x}(t)] + \frac{1}{2}\int_0^t |\dot{y}(s) - C\hat{x}(s)|^2 \mathrm{d}s,$$

so that $D^2 V(t,\cdot) = P(t)^{-1}$.

A recursive estimator

If V is C^2 and strictly convex, writing that

$$\frac{\mathrm{d}}{\mathrm{d}t} DV(t,\hat{x}(t)) = 0,$$

we get [Fle97]

$$\dot{\hat{x}}(t) + f_\kappa(\hat{x}(t)) = \left[D^2 V_\kappa(t,\hat{x}(t))\right]^{-1} Dh(\hat{x}(t))\left[\dot{y}(t) - h(\hat{x}(t))\right].$$

⇒ Recursive formula.

⇒ Extends Kalman's filter.

Theorem (Dynamic programming)

$$V_\kappa(t,x) = \inf_{x_w(t)=x} V_\kappa(t-\tau, x_w(t-\tau)) + \int_{t-\tau}^{t} \frac{1}{2}|w(s)|^2 + \frac{1}{2}|\dot{y}(t) - h(x_w(s))|^2 \mathrm{d}s.$$

Theorem (Dynamic programming)

$$V_\kappa(t,x) = \inf_{x_w(t)=x} V_\kappa(t-\tau, x_w(t-\tau)) + \int_{t-\tau}^t \frac{1}{2}|w(s)|^2 + \frac{1}{2}|\dot{y}(t) - h(x_w(s))|^2 \mathrm{d}s.$$

Then for every $\tau > 0$,

$$\sup_{x_w(t)=x} \tau^{-1} \int_{t-\tau}^t \partial_s V_\kappa(s, x_w(s)) + [w(s) - f_\kappa(x_w(s))] \cdot DV_\kappa(s, x_w(s)) - \frac{1}{2}|w(s)|^2 - \frac{1}{2}|\dot{y}(t) - h(x_w(s))|^2 \mathrm{d}s = 0.$$

Bellman principle

Theorem (Dynamic programming)

$$V_\kappa(t,x) = \inf_{x_w(t)=x} V_\kappa(t-\tau, x_w(t-\tau)) + \int_{t-\tau}^{t} \frac{1}{2}|w(s)|^2 + \frac{1}{2}|\dot{y}(t) - h(x_w(s))|^2 \mathrm{d}s.$$

Then for every $\tau > 0$,

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Hamiltonian:

$$H_\kappa(t,x,p) := \sup_{w\in\mathbb{R}} [w - f_\kappa(x)] \cdot p - \frac{1}{2}|w|^2 - \frac{1}{2}|\dot{y}(t) - h(x)|^2.$$

HJB equation

Hamilton-Jacobi-Bellman equation

$$\begin{cases} \partial_t V_\kappa(t,x) + H_\kappa(t,x,DV_\kappa(t,x)) = 0, \\ V_\kappa(0,x) = \psi(x). \end{cases}$$

- Directly rigorous if V_κ is C^2.
- Sub-differential in the fully convex case [RW00].
- In general, viscosity solutions [JB88].
- Numerical schemes, discretised problem [Moi19].

Viscosity solution

Definition

$V_\kappa \in C([0, T], \mathbb{R})$ is a sub-solution if for every $(t, x) \in [0, T] \times \mathbb{R}$ and $\varphi \in C^1([0, T], \mathbb{R})$ such that $V_\kappa - \varphi$ has a maximum at (t, x),

$$\partial_t \varphi(t, x) + H_\kappa(t, x, D\varphi(t, x)) \leq 0.$$

V_κ is a super-solution if ...

Theorem

V_κ is a viscosity solution of the HJB equation.

Proof.

- For (t, x, φ) as required, and $w \in \mathbb{R}$, let $(\tilde{w}, x_{\tilde{w}})$ satisfy

$$\tilde{w} \in C([0,t],\mathbb{R}), \quad x_{\tilde{w}}(t) = x, \quad \tilde{w}(t) = w.$$

- Local maximum and Bellman principle for $\tau > 0$,

$$\varphi(t,x) - \varphi(t-\tau, x_{\tilde{w}}(t-\tau)) \leq V_\kappa(t,x) - V_\kappa(t-\tau, x_{\tilde{w}}(t-\tau))$$
$$\leq \int_{t-\tau}^{t} \tfrac{1}{2}|\tilde{w}(s)|^2 + \tfrac{1}{2}|\dot{y}(s) - h(x_{\tilde{w}}(s))|^2 \mathrm{d}s.$$

Proof.

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$$\varphi(t,x) - \varphi(t-\tau, x_{\tilde{w}}(t-\tau)) \leq V_\kappa(t,x) - V_\kappa(t-\tau, x_{\tilde{w}}(t-\tau)) \leq \int_{t-\tau}^{t} \tfrac{1}{2}|\tilde{w}(s)|^2 + \tfrac{1}{2}|\dot{y}(s) - h(x_{\tilde{w}}(s))|^2 \mathrm{d}s.$$

- Letting $\tau \to 0$,

$$\partial_t\varphi(t,x) + [w - f_\kappa(x)] \cdot D\varphi(t,x) - \tfrac{1}{2}|w|^2 - \tfrac{1}{2}|\dot{y}(s) - h(x)|^2 \leq 0.$$

- Maximising over $w \in \mathbb{R}$,

$$\partial_t\varphi(t,x) + H_\kappa(t,x,D\varphi(t,x)) \leq 0.$$

The filtering problem

Signal model

$$\mathrm{d}X_t^\varepsilon = -f_\kappa(X_t^\varepsilon)\mathrm{d}t + \sqrt{\varepsilon}\mathrm{d}B_t,$$

Observation

$$\mathrm{d}Y_t^\varepsilon = h(X_t^\varepsilon)\mathrm{d}t + \sqrt{\varepsilon}\mathrm{d}B'_t,$$

Filtering density

$$\forall \varphi \in C_b(\mathbb{R}), \quad \boxed{\langle \pi_t^\varepsilon, \varphi \rangle = \mathbb{E}\left[\varphi(X_t^\varepsilon) \middle| \sigma\left(Y_s^\varepsilon\right)_{0\leq s\leq t}\right],}$$

so that

$$\mathbb{E}|\langle \pi_t^\varepsilon, \varphi\rangle - \varphi(X_t^\varepsilon)|^2 = \inf_{Z \text{ is } \sigma(Y_s^\varepsilon)_{0\leq s\leq t}\text{-measurable}} \mathbb{E}|Z - \varphi(X_t^\varepsilon)|^2.$$

$\Rightarrow$ Random measure π_t^ε.

Small noise limit

Large deviations [JB88]

$$\pi_t^\varepsilon(\mathrm{d}x) = \exp\left[-\frac{1}{\varepsilon}\big(V_\kappa(t,x) + o(1)\big)\right]\mathrm{d}x.$$

π_t^ε concentrates on

$$\hat{x}(t) = \mathrm{argmin}_{x\in\mathbb{R}}\, V_\kappa(t,x).$$

$\Rightarrow$ Maximum likelihood procedure.

$\Rightarrow$ Recovers Mortensen's approach!

Girsanov transform:

- Provides $\rho_t(\mathrm{d}x) = q_t(x)\mathrm{d}x$ such that

$$\forall \varphi \in C_b(\mathbb{R}), \quad \langle \pi_t^\varepsilon, \varphi \rangle = \frac{\langle \rho_t^\varepsilon, \varphi \rangle}{\langle \rho_t^\varepsilon, 1 \rangle},$$

- Together with the Zakai equation [Par82]:

$$\boxed{\mathrm{d}q^\varepsilon(t,x) = \frac{\varepsilon}{2}\Delta q^\varepsilon(t,x)\mathrm{d}t + \mathrm{div}[q^\varepsilon(t,x) f_\kappa]\mathrm{d}t + \frac{q^\varepsilon(t,x)}{\varepsilon}\mathrm{d}Y_t^\varepsilon.}$$

$\Rightarrow$ A linear SPDE to be solved.

Robust formulation

The change of variable

$$p^\varepsilon(t,x) := \exp\left[-\frac{Y_t^\varepsilon h(x)}{\varepsilon}\right] q^\varepsilon(t,x),$$

gives the robust Zakai equation

$$\partial_t p^\varepsilon(t,x) + g^\varepsilon(t,x) D p^\varepsilon(t,x) + \frac{1}{\varepsilon} V^\varepsilon(t,x) p^\varepsilon(t,x) = \frac{\varepsilon}{2} \Delta p^\varepsilon(t,x)$$

with Y_t^ε-dependent coefficients.

$\Rightarrow$ Deterministic PDE given a C^1 realisation $(y(s))_{0\leq s\leq t}$ of $(Y_s^\varepsilon)_{0\leq s\leq t}$.

Small noise limit

Theorem

$$\lim_{\varepsilon \to 0} -\varepsilon \log p^{\varepsilon}(t,x) = V_{\kappa}(t,x) - y(t)h(x),$$

so that [JB88; Fle97]

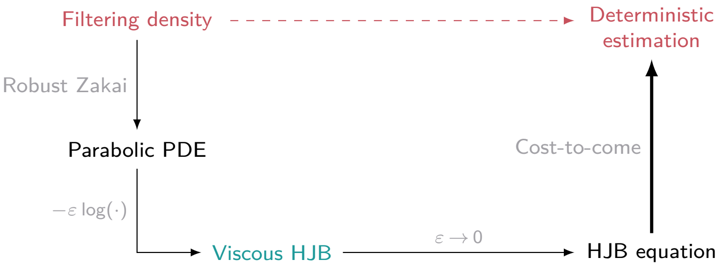
$$q^{\varepsilon}(t,x) = \exp\left[-\frac{1}{\varepsilon}\big(V_{\kappa}(t,x) + o(1)\big)\right],$$

as $\varepsilon \to 0$.

$$S^{\varepsilon} = -\varepsilon \log p^{\varepsilon} \;\Rightarrow\; \partial_t S^{\varepsilon}(t,x) + H^{\varepsilon}(t,x,DS^{\varepsilon}(t,x)) = \frac{\varepsilon}{2}\Delta S^{\varepsilon}(t,x).$$

$\Rightarrow$ Small noise limit in the HJB equation.

Summary


Filtering density
Deterministic estimation
Robust Zakai
Parabolic PDE
Cost-to-come
$-\varepsilon \log(\cdot)$
$\varepsilon \to 0$
Viscous HJB
HJB equation

Adapted cost-to-come

$$V(t,x) := \inf_{x_w(t)=x} \psi(x_w(0)) + \int_0^t \frac{1}{2}|w(s)|^2 + \frac{1}{2}|\dot{y}(s) - h(x_w(s))|^2 \mathrm{d}s,$$

where

$$\begin{cases} \forall t > 0, \quad \dot{x}_w(t) + \partial I_{\mathbb{R}_+}(x_w(t)) \ni w(t), \\ x_w(0) \in \mathbb{R}_+. \end{cases}$$

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Theorem

$$V_\kappa(t,x) \xrightarrow[\kappa \to +\infty]{} V(t,x).$$

$\Rightarrow$ Convergence of observers.

Heuristic limit equation

Penalised curve with $x_w^\kappa(t) = 0$

$$\dot{x}_w^\kappa(s) = -f_\kappa(x_w^\kappa(s)) + w(s), \quad 0 \leq s \leq t,$$

so that

$$f_\kappa(x_w^\kappa(t)) \xrightarrow[\kappa \to +\infty]{} |w(t)|_-.$$

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The HJB rewrites

$$\partial_t V_\kappa(t, x_w^\kappa(t)) - \frac{1}{2}|\dot{y}(t) - h(x_w^\kappa(t))|^2$$
$$+ \sup_{\substack{w(t) \\ x_w^\kappa(t) = x}} [w(t) - f_\kappa(x_w^\kappa(t)] \cdot DV_\kappa(t, x_w^\kappa(t)) - \frac{1}{2}|w(t)|^2 = 0.$$

Heuristic limit equation

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$$\dot{x}_w^\kappa(s) = -f_\kappa(x_w^\kappa(s)) + w(s), \quad 0 \leq s \leq t,$$

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$$+ \sup_{\substack{w(t) \\ x_w^\kappa(t)=x}} [w(t) - f_\kappa(x_w^\kappa(t)] \cdot DV_\kappa(t, x_w^\kappa(t)) - \frac{1}{2}|w(t)|^2 = 0.$$

As $\kappa \to +\infty$,

$$\text{“ } \partial_t V(t, 0) - \frac{1}{2}|\dot{y}(t) - h(0)|^2 = 0. \text{ ”}$$

Boundary condition

Theorem

V is a viscosity solution [Lio85; Lio05] of

$$\begin{cases} \partial_t V(t,x) + \frac{1}{2}|DV(t,x)|^2 - \frac{1}{2}|\dot{y}(t) - h(x)|^2 = 0, \quad x > 0, \\ V(0,x) = \psi(x), \\ \partial_t V(t,0) - \frac{1}{2}|\dot{y}(t) - h(0)|^2 = 0. \end{cases}$$

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Definition

$V \in C([0,T],\mathbb{R})$ is a sub-solution if for every $(t,x) \in [0,T] \times \mathbb{R}$ and $\varphi \in C^1([0,T],\mathbb{R})$ such that $V - \varphi$ has a maximum at (t,x),

- $\partial_t \varphi(t,x) + H(t,x,D\varphi(t,x)) \leq 0,$ if $x > 0,$
- $\min\ -D\varphi(t,0),\ \partial_t \varphi(t,0) + H(t,0,D\varphi(t,0)) \leq 0,$ if $x = 0.$

V is a super-solution if ...

Since

$$H(t,x,D\varphi(t,x)) = \sup_{w \in \mathbb{R}} w \cdot D\varphi(t,x) - \frac{1}{2}|w|^2 - \frac{1}{2}|\dot{y}(t) - h(x)|^2,$$

Notion of solution

Definition

$V \in C([0,T],\mathbb{R})$ is a sub-solution if for every $(t,x) \in [0,T] \times \mathbb{R}$ and $\varphi \in C^1([0,T],\mathbb{R})$ such that $V - \varphi$ has a maximum at (t,x),

- $\partial_t \varphi(t,x) + H(t,x,D\varphi(t,x)) \leq 0,$ if $x > 0$,
- $\min\left(-D\varphi(t,0),\ \partial_t \varphi(t,0) + H(t,0,D\varphi(t,0))\right) \leq 0,$ if $x = 0$.

V is a super-solution if ...

Since

$$H(t,x,D\varphi(t,x)) = \sup_{w \in \mathbb{R}} w \cdot D\varphi(t,x) - \frac{1}{2}|w|^2 - \frac{1}{2}|\dot{y}(t) - h(x)|^2,$$

we get

$$\partial_t \varphi(t,0) + \sup_{w \leq 0} w \cdot D\varphi(t,0) - \frac{1}{2}|w|^2 - \frac{1}{2}|\dot{y}(t) - h(0)|^2 \leq 0.$$

Proof.

- For $(t, 0, \varphi)$ as required, and $w \leq 0$, let $(\tilde{w}, x_{\tilde{w}})$ satisfy

$$\tilde{w} \in C([0,t], \mathbb{R}), \quad x_{\tilde{w}}(t) = 0, \quad \tilde{w}(t) = w.$$

- Local maximum and Bellman principle for $\tau > 0$,

$$\begin{aligned}\varphi(t,0) - \varphi(t-\tau, x_{\tilde{w}}(t-\tau)) &\leq V_\kappa(t,0) - V_\kappa(t-\tau, x_{\tilde{w}}(t-\tau)) \\ &\leq \int_{t-\tau}^{t} \tfrac{1}{2}|\tilde{w}(s)|^2 + \tfrac{1}{2}|\dot{y}(s) - h(x_{\tilde{w}}(s))|^2 \mathrm{d}s.\end{aligned}$$

Sub-solution at the boundary

Introduction

Penalised problem

Non-smooth problem

Boundary condition

Link with non-smooth filtering

Conclusion

References

Proof.

- For $(t, 0, \varphi)$ as required, and $w \leq 0$, let $(\tilde{w}, x_{\tilde{w}})$ satisfy

$$\tilde{w} \in C([0,t],\mathbb{R}), \quad x_{\tilde{w}}(t) = 0, \quad \tilde{w}(t) = w.$$

- Local maximum and Bellman principle for $\tau > 0$,

$$\begin{aligned}\varphi(t,0) - \varphi(t-\tau, x_{\tilde{w}}(t-\tau)) &\leq V_\kappa(t,0) - V_\kappa(t-\tau, x_{\tilde{w}}(t-\tau)) \\ &\leq \int_{t-\tau}^{t} \tfrac{1}{2}|\tilde{w}(s)|^2 + \tfrac{1}{2}|\dot{y}(s) - h(x_{\tilde{w}}(s))|^2 \mathrm{d}s.\end{aligned}$$

- Letting $\tau \to 0$,

$$\partial_t \varphi(t,0) + [w - f_\kappa(0)] \cdot D\varphi(t,0) - \tfrac{1}{2}|w|^2 - \tfrac{1}{2}|\dot{y}(s) - h(0)|^2 \leq 0.$$

- Maximising over $w \leq 0$,

$$\partial_t \varphi(t,0) + \sup_{w \leq 0} w \cdot D\varphi(t,0) - \tfrac{1}{2}|w|^2 - \tfrac{1}{2}|\dot{y}(t) - h(0)|^2 \leq 0.$$

Controlled dynamics

$$\begin{cases} \dot{z}_w(s) = w(s), \\ z_w(0) \in \mathbb{R}_+. \end{cases}$$

We notice that

$$V(t,x) = \inf_{\substack{z_w^\zeta(t)=x, \\ \forall s\in[0,t],\ z_w^\zeta(s)\geq 0}} \psi(z_w(0)) + \int_0^t \frac{1}{2}|w(s)|^2 + \frac{1}{2}|\dot{y}(s) - h(z_w(s))|^2 \mathrm{d}s.$$

Link with constrained control

Introduction

Penalised problem

Non-smooth problem

Boundary condition

Link with non-smooth filtering

Conclusion

References

Controlled dynamics

$$\begin{cases} \dot{z}_w(s) = w(s), \\ z_w(0) \in \mathbb{R}_+. \end{cases}$$

We notice that

$$V(t,x) = \inf_{\substack{z_w^\zeta(t)=x, \\ \forall s\in[0,t],\ z_w^\zeta(s)\geq 0}} \psi(z_w(0)) + \int_0^t \frac{1}{2}|w(s)|^2 + \frac{1}{2}|\dot{y}(s) - h(z_w(s))|^2 \mathrm{d}s.$$

This rewrites

$$\inf_{\substack{z_w(0)=x \\ \forall s\in[0,t],\ z_{-w}(s)\geq 0}} \psi(z_{-w}(0)) + \int_0^t \frac{1}{2}|w(s)|^2 + \frac{1}{2}|\dot{y}(s) - h(z_{-w}(s))|^2 \mathrm{d}s.$$

$\Rightarrow$ Constrained deterministic control.

Non-smooth filtering

Signal model

$$\mathrm{d}X_t^\varepsilon + \partial I_{\mathbb{R}_+}(X_t^\varepsilon) \ni \sqrt{\varepsilon}\mathrm{d}B_t,$$

Observation

$$\mathrm{d}Y_t^\varepsilon = h(X_t^\varepsilon)\mathrm{d}t + \sqrt{\varepsilon}\mathrm{d}B_t',$$

Filtering density

$$\forall \varphi \in C_b(\mathbb{R}_+, \mathbb{R}), \quad \boxed{\langle \pi_t^\varepsilon, \varphi \rangle = \mathbb{E}\left[\varphi(X_t^\varepsilon) \middle| \sigma\left(Y_s^\varepsilon\right)_{0 \le s \le t} \right],}$$

so that

$$\mathbb{E}|\langle \pi_t^\varepsilon, \varphi \rangle - \varphi(X_t^\varepsilon)|^2 = \inf_{Z \text{ is } \sigma(Y_s^\varepsilon)_{0 \le s \le t}\text{-measurable}} \mathbb{E}|Z - \varphi(X_t^\varepsilon)|^2.$$

$\Rightarrow$ Random measure π_t^ε.

Small noise limit

Theorem

Small noise limit [Cha+23]

$$\pi_t^\varepsilon(\mathrm{d}x) = \exp\left[-\frac{1}{\varepsilon}\big(V(t,x) + o(1)\big)\right]\mathrm{d}x.$$

π_t^ε concentrates on

$$\hat{x}(t) = \operatorname{argmin}_{x\in\mathbb{R}} V(t,x).$$

$\Rightarrow$ Maximum likelihood procedure.

$\Rightarrow$ Recovers Mortensen's approach.

Conclusion I

Introduction

Penalised problem

Non-smooth problem

Conclusion

References

- Mortensen's approach was extended to the non-smooth case.
- Adequate boundary condition.
- Compatibility with stochastic filtering.

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Conclusion II

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