Large mean-field systems conditioned by rare events

and links to stochastic control

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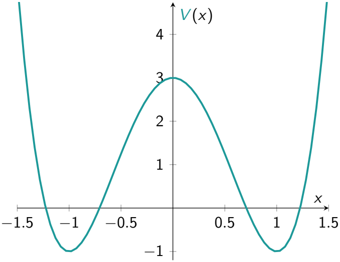
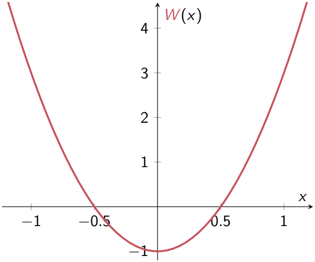
Particle system

Mean-field interaction

$$\mathrm{d}X_t^{i,N} = -\nabla V(X_t^{i,N})\mathrm{d}t - \frac{1}{N}\sum_{j=1}^{N} \nabla W(X_t^{i,N} - X_t^{j,N})\mathrm{d}t + \mathrm{d}B_t^{i,N}$$



Confinement potential V



Interaction potential W

Conditioning

$$\forall t \in [0, T], \quad \overline{X}_t^N := \frac{1}{N} \sum_{i=1}^{N} X_t^{i,N} \leq 0.$$

Mean-field limit

As $N \to +\infty$, this becomes

$$\forall t \in [0, T], \quad \mathbb{E}[\overline{X}_t] \leq 0,$$

where

$$\mathrm{d}\overline{X}_t = -\nabla V(\overline{X}_t)\mathrm{d}t - \nabla[W \star \mathrm{Law}(\overline{X}_t)](\overline{X}_t)\mathrm{d}t + \mathrm{d}B_t.$$

$\Rightarrow$ Deterministic constraint!

Mean-field conditioning

Conditioning

$$\forall t \in [0,T], \quad \overline{X}_t^N := \frac{1}{N}\sum_{i=1}^{N} X_t^{i,N} \leq 0.$$

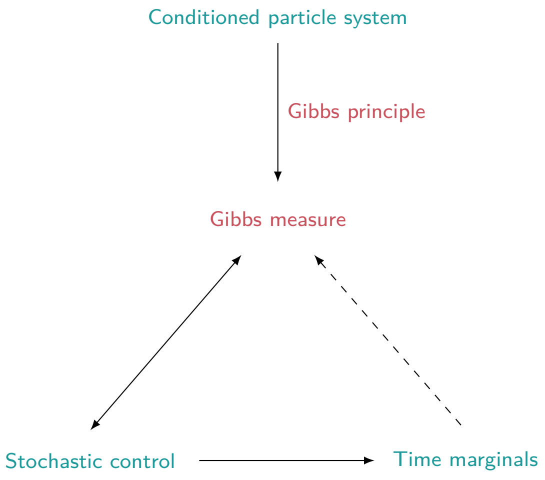
Rare event

Tricky situation when

$$\mathbb{P}\big(\forall t \in [0,T], \overline{X}_t^N \leq 0\big) \xrightarrow[N\to+\infty]{} 0.$$

$\Rightarrow$ What is the conditional law of particles as $N \to +\infty$?

From conditioning to control



Large deviation principle

Path empirical measure

$$\mu^N_{[0,T]} := \frac{1}{N}\sum_{i=1}^{N} \delta_{X^{i,N}_{[0,T]}}.$$

Large deviations

There exists $H_T : \mathcal{P}(C([0,T],\mathbb{R})) \to [0,+\infty]$ such that for any $A \subset \mathcal{P}(C([0,T],\mathbb{R}))$,

$$\mathbb{P}\Big(\mu^N_{[0,T]} \in A \,\Big|\, \forall t \in [0,T],\, \overline{X}^N_t \leq 0\Big) \asymp e^{-N[I_T(A) - \min I_T]},$$

where

$$I_T(A) := \inf_{\substack{\mu_{[0,T]} \in A \\ \forall t \in [0,T],\, \int x \mathrm{d}\mu_t(x) \leq 0}} H_T(\mu_{[0,T]}).$$

$\Rightarrow$ Constrained variational principle.

Related control problem

Controlled process

$$\begin{cases} \mathrm{d}X_s^{t,\mu,\alpha} = -\nabla V(X_s^{t,\mu,\alpha})\mathrm{d}s - \nabla[W \star \mathrm{Law}(X_s^{t,\mu,\alpha})](X_s^{t,\mu,\alpha})\mathrm{d}s \\ \qquad\qquad +\alpha(s, X_s^{t,\mu,\alpha})\mathrm{d}s + \mathrm{d}B_s, \quad t \leq s \leq T, \\ X_t^{t,\mu,\alpha} = X_t^{t,\mu}, \quad X_t^{t,\mu} \sim \mu, \end{cases}$$

for some control function $\alpha : [t, T] \times \mathbb{R} \to \mathbb{R}$.

Value function

$$F_T(t,\mu) := \inf_{\substack{\alpha \text{ measurable} \\ \forall s \in [t,T],\ \mathbb{E}[X_s^{t,\mu,\alpha}] \leq 0}} \mathbb{E} \int_t^T \frac{1}{2} |\alpha(s, X_s^{t,\mu,\alpha})|^2 \mathrm{d}s.$$

$\hookrightarrow$ One looks for $\overline{\alpha}$ which realises $F_T(0,\mu)$.

$\Rightarrow$ McKean-Vlasov control problem with law constraints.

Theorem

$$\inf_{\substack{\mu \in \mathcal{P}(C([0,T],\mathbb{R})) \\ \forall t \in [0,T],\, \int x \mathrm{d}\mu_t(x) \leq 0}} H_T(\mu_{[0,T]}) = \inf_{\substack{\mu_0 \in \mathcal{P}(\mathbb{R}) \\ \int x \mathrm{d}\mu_0(x) \leq 0}} H_0(\mu_0) + F_T(0,\mu_0),$$

and any minimiser $\overline{\mu}_{[0,T]}$ of the l.h.s. is the path-law of the optimally controlled process:

$$\begin{cases} \mathrm{d}\overline{X}_t = -\nabla V(\overline{X}_t)\mathrm{d}t - \nabla[W \star \mathrm{Law}(\overline{X}_t)](\overline{X}_t)\mathrm{d}t \\ \qquad\qquad + \overline{\alpha}(t,\overline{X}_t)\mathrm{d}t + \mathrm{d}B_t, \quad 0 \leq t \leq T, \\ \overline{X}_0 \sim \mu_0, \end{cases}$$

for $\overline{\alpha}$ which realises $F_T(0,\overline{\mu}_0)$.

HJB equation

Optimal control

Moreover, there exists $\lambda \in \mathcal{M}_+([0,T])$ such that $\overline{\alpha} = -\nabla\varphi$, where

$$\begin{cases} \partial_t \varphi - \nabla V \cdot \nabla \varphi - 2\nabla(W \star \overline{\mu}_t) \cdot \nabla\varphi + \frac{1}{2}\Delta\varphi \\ \qquad\qquad\qquad\qquad - \frac{1}{2}|\nabla\varphi|^2 = -\lambda(\mathrm{d}t)\mathrm{Id}_{\mathbb{R}}, \\ \varphi(T,\cdot) = 0. \end{cases}$$

Optimal control

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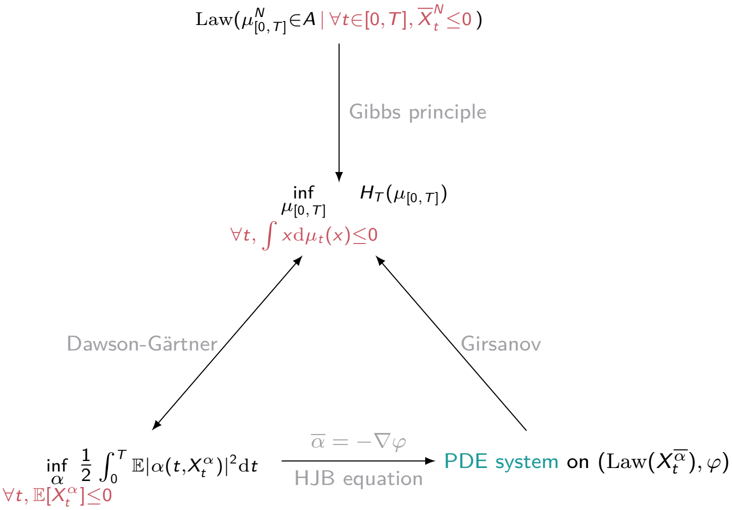
Long time perspectives

As $T \to +\infty$, expected convergence towards stationary solutions of

$$\inf_{\substack{\alpha \text{ measurable} \\ \forall t \in \mathbb{R}_+,\, \mathbb{E}[X_t^{0,\mu,\alpha}] \leq 0}} \limsup_{T \to +\infty} \frac{1}{2T} \mathbb{E} \int_0^T |\alpha(t, X_t^{0,\mu,\alpha})|^2 \mathrm{d}t.$$

Summary



$\mathrm{Law}(\mu^N_{[0,T]} \in A \mid \forall t \in [0,T], \overline{X}^N_t \leq 0)$
Gibbs principle
$\inf_{\mu_{[0,T]}} H_T(\mu_{[0,T]})$
$\forall t, \int x \mathrm{d}\mu_t(x) \leq 0$
Dawson-Gärtner
Girsanov
$\inf_{\alpha} \frac{1}{2} \int_0^T \mathbb{E}|\alpha(t, X^\alpha_t)|^2 \mathrm{d}t$
$\forall t, \mathbb{E}[X^\alpha_t] \leq 0$
$\overline{\alpha} = -\nabla\varphi$
HJB equation
PDE system on $(\mathrm{Law}(X^{\overline{\alpha}}_t), \varphi)$

Assumptions

Linear functional derivative

The map $\Psi : \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$ is assumed to be C^1 in the following sense: for any $\mu, \mu' \in \mathcal{P}(\mathbb{R}^d)$,

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} \Psi(\mu + \varepsilon(\mu' - \mu)) = \langle \mu - \mu', \tfrac{\delta \Psi}{\delta \mu}(\mu) \rangle,$$

for $\frac{\delta \Psi}{\delta \mu} : \mathcal{P}(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}$ continuous, with polynomial growth.

Constraint qualification assumption

Some $\tilde{\mu}_{[0,T]}$ with $H(\tilde{\mu}_{[0,T]} | \nu_{[0,T]}) < +\infty$ exists, satisfying

$$\forall t \in [0, T], \quad \Psi(\overline{\mu}_t) + \langle \tilde{\mu}_t - \overline{\mu}_t, \tfrac{\delta \Psi}{\delta \mu}(\overline{\mu}_t) \rangle < 0.$$

Example

In the linear case $\frac{\delta \Psi_V}{\delta \mu}(\mu) = V$, and a sufficient condition is

$$\nu_{[0,T]}(\{\forall t \in [0, T],\ V(x_t) \leq -\eta\}) > 0,$$

for some $\eta > 0$. Ψ_W would also work.

Mean-field constraint

Given $\Psi : \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$, define the stopping time

$$\tau_\Psi^N := \inf\{ t > 0 \,,\, \Psi(\pi(\vec{X}_t^N)) > 0 \},$$

Then $-\frac{1}{N} \log \mathbb{P}(T < \tau_\psi^N)$ converges to

$$\inf_{\substack{(\mu_t)_{0 \leq t \leq T} \in AC([0,T], \mathcal{P}(\mathbb{R}^d)) \\ \forall t \in [0,T],\, \Psi(\mu_t) \leq 0}} \frac{1}{2} \int_0^T \| \partial_t \mu_t - L^\star_{\mu_t} \mu_t \|^2_{\mu_t} \mathrm{d}t.$$

$\Rightarrow$ Describes large conditioned systems.

Example (Markov diffusion)

When $\nu_{[0,T]}$ is the law of the diffusion

$$\mathrm{d}X_t^i = b(X_t^i)\mathrm{d}t + \mathrm{d}B_t^i,$$

the Doob transform shows that $\vec{Z}^N_{[0,T]} := (Z^{1,N}_{[0,T]}, \dots, Z^{N,N}_{[0,T]})$,

$$\boxed{\mathrm{d}Z_t^{i,N} = b(Z_t^{i,N})\mathrm{d}t + \nabla_{Z^i} \ln \mathbb{P}_{\vec{Z}_t^N}\big(\tau_\Psi^N > T - t\big)\mathrm{d}t + \mathrm{d}B_t^i,}$$

is exchangeable and $\mathrm{Law}(X^1_{[0,T]} | T < \tau^N_\Psi)$-distributed.

$\Rightarrow$ Mean-field limit for interacting particles.