Introduction

Large deviations

Gibbs principle on path space

Stochastic control

References

Gibbs principle on path space and links to stochastic control

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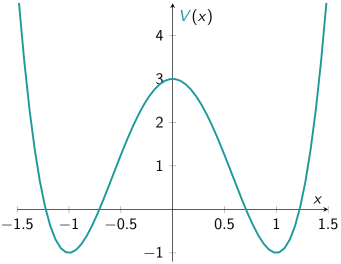
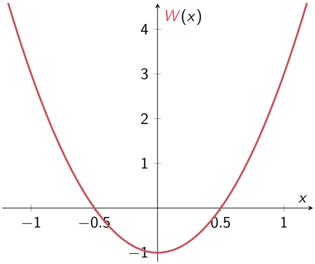
Conditioned particle system

Mean-field interaction

$$\mathrm{d}X_t^{i,N} = -\nabla V(X_t^{i,N})\mathrm{d}t - \frac{1}{N}\sum_{j=1}^{N} \nabla W(X_t^{i,N} - X_t^{j,N})\mathrm{d}t + \mathrm{d}B_t^{i,N}$$



Confinement potential V



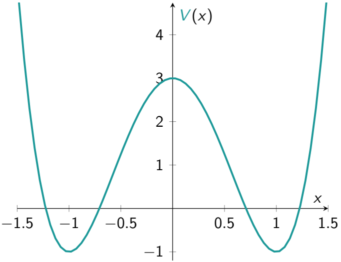
Interaction potential W

$\Rightarrow$ Metastability for large N?

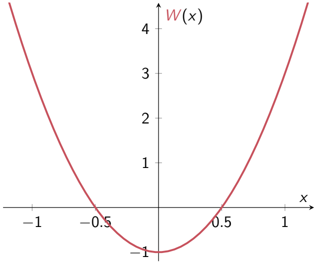
Conditioned particle system

Mean-field interaction

$$\mathrm{d}X_t^{i,N} = -\nabla V(X_t^{i,N})\mathrm{d}t - \frac{1}{N}\sum_{j=1}^{N} \nabla W(X_t^{i,N} - X_t^{j,N})\mathrm{d}t + \mathrm{d}B_t^{i,N}$$



Confinement potential V

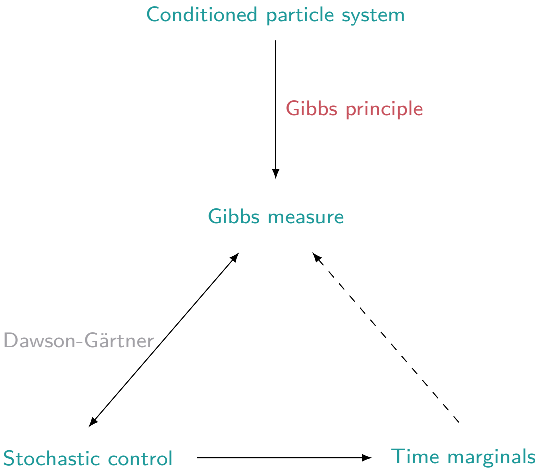


Interaction potential W

Mean-field conditioning: for every t in $[0, T]$,

$$\frac{1}{N}\sum_{i=1}^{N} V(X_t^{i,N}) \leq 0 \quad \text{or} \quad \frac{1}{N^2}\sum_{i,j=1}^{N} W(X_t^{i,N} - X_t^{j,N}) \leq 0.$$

From conditioning to control



The Freidlin-Wentzell framework

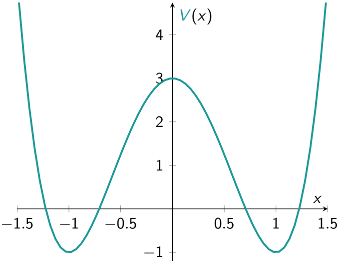
The random path

$$\mathrm{d}X_t^\varepsilon = -\nabla V(X_t^\varepsilon)\mathrm{d}t + \sqrt{\varepsilon}\mathrm{d}B_t,$$

converges towards the deterministic flow

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -\nabla V(x(t)),$$

as $\varepsilon \to 0$. Apparition of metastability.



Confinement potential V.

Definition (LDP)

A family $(\mu^\varepsilon)_\varepsilon \subset \mathcal{P}(E)$ satisfies a LDP if

$$\forall \text{ measurable } A, \begin{cases} -\inf_{\mathring{A}} I \leq \liminf_{\varepsilon\to 0} \varepsilon \log \mu^\varepsilon(A), \\ \limsup_{\varepsilon\to\infty} \varepsilon \log \mu^\varepsilon(A) \leq -\inf_{\overline{A}} I, \end{cases}$$

with $I \geq 0$ and lower semi-continuous.

If $X^\varepsilon \sim \mu^\varepsilon$, this means

$$\boxed{\mathbb{P}\big(X^\varepsilon \approx x\big) \approx_{\varepsilon\to 0} e^{-\varepsilon^{-1} I(x)},}$$

so that

$$X^\varepsilon \xrightarrow[\varepsilon\to 0]{} \operatorname{argmin}_{x\in E} I(x).$$

$\Rightarrow$ Variational principle.

Theorem (Freidlin-Wentzell)

$(\mathrm{Law}(X^{\varepsilon}_{[0,T]}))_{\varepsilon>0}$ *satisfies a LDP with rate function*

$$I((x_t)_{0\leq t\leq T}) = \begin{cases} \frac{1}{2}\int_0^T |\dot{x}_t + \nabla V(x_t)|^2 \mathrm{d}t \text{ if } (x_t)_{0\leq t\leq T} \text{ is AC}, \\ +\infty \text{ else.} \end{cases}$$

Theorem (Freidlin-Wentzell)

$(\mathrm{Law}(X^{\varepsilon}_{[0,T]}))_{\varepsilon>0}$ *satisfies a LDP with rate function*

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If $X^{\varepsilon}_0 = -1$ and

$$\tau := \inf\{t > 0\,,\, X^{\varepsilon}_t > 0\},$$

then $(\mathrm{Law}(X^{\varepsilon}_{[0,T]} | T < \tau))_{\varepsilon>0}$ satisfies a LDP with rate function

$$(x_t)_{0\leq t\leq T} \mapsto \begin{cases} I((x_t)_{0\leq t\leq T}) - C \text{ if } \forall t\in[0,T],\, x_t\leq 0, \\ +\infty \text{ else.} \end{cases}$$

$\Rightarrow$ Constrained variational principle: $C := \inf_{\substack{(x_t)_{0\leq t\leq T} \\ \forall t\in[0,T],\, x_t\leq 0}} I((x_t)_{0\leq t\leq T}).$

The case of i.i.d. particles

Let $(X^k_{[0,T]})_{k \geq 1}$ be independent $\nu_{[0,T]}$-distributed $C([0,T], \mathbb{R}^d)$-valued variables, and let

$$\Pi(\vec{X}^N_{[0,T]}) := \frac{1}{N} \sum_{i=1}^{N} \delta_{X^i_{[0,T]}}, \qquad \pi(\vec{X}^N_t) := \frac{1}{N} \sum_{i=1}^{N} \delta_{X^i_t},$$

denote the empirical measures.

The case of i.i.d. particles

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denote the empirical measures.

Theorem (Sanov)

$(\mathrm{Law}(\Pi(\vec{X}^N_{[0,T]})))_{N \geq 1} \subset \mathcal{P}(\mathcal{P}(C([0,T], \mathbb{R}^d))$ *satisfies a LDP with rate function*

$$H(\mu_{[0,T]} | \nu_{[0,T]}) := \begin{cases} \int_{C([0,T],\mathbb{R}^d)} \log \frac{\mathrm{d}\mu_{[0,T]}}{\mathrm{d}\nu_{[0,T]}} \mathrm{d}\mu_{[0,T]} & \text{if } \mu_{[0,T]} \ll \nu_{[0,T]}, \\ +\infty & \text{else.} \end{cases}$$

It is the relative entropy w.r.t. to $\nu_{[0,T]}$.

Given $\Psi : \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$ lower semi-continuous, define the stopping time

$$\tau_\Psi^N := \inf\{ t > 0 \,,\, \Psi(\pi(\vec{X}_t^N)) > 0 \}.$$

Example

When $\nu_{[0,T]}$ is the common law of the i.i.d. particles

$$\mathrm{d}X_t^i = -\nabla V(X_t^i)\mathrm{d}t + \mathrm{d}B_t^i,$$

the energy constraints

$$\Psi_V(\mu) = \langle \mu, V \rangle \quad \text{and} \quad \Psi_W(\mu) = \langle \mu, W \star \mu \rangle,$$

are respectively linear, and non-linear and non-convex in μ.

Conditioned particle system

Given $\Psi : \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$ lower semi-continuous, define the stopping time

$$\tau^N_\Psi := \inf\{t > 0\,,\ \Psi(\pi(\vec{X}^N_t)) > 0\}.$$

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$$\mathrm{d}X^i_t = -\nabla V(X^i_t)\mathrm{d}t + \mathrm{d}B^i_t,$$

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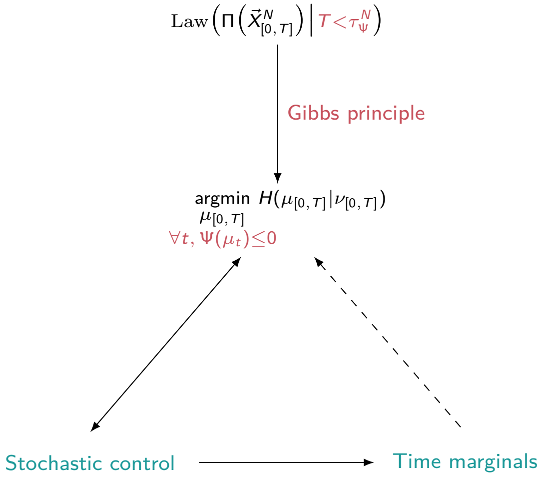
are respectively linear, and non-linear and non-convex in μ.

As $N \to +\infty$, $\mathrm{Law}(X^1_{[0,T]} | T < \tau^N_\Psi)$ is expected to converge towards

$$\underset{\substack{\mu_{[0,T]} \in \mathcal{P}(C([0,T],\mathbb{R}^d)) \\ \forall t \in [0,T],\, \Psi(\mu_t) \leq 0}}{\operatorname{argmin}} H(\mu_{[0,T]} | \nu_{[0,T]}).$$

$\Rightarrow$ Infinite dimensional constrained optimisation.

Gibbs principle on path space



Theorem (Gibbs variational principle)

Given ν in $\mathcal{P}(\mathbb{R}^d)$ and $E : \mathbb{R}^d \to \mathbb{R}$ continuous, bounded from below,

$$\inf_{\substack{\mu \in \mathcal{P}(\mathbb{R}^d) \\ \langle \mu, E \rangle = 0}} H(\mu|\nu)$$

is realised by a unique measure $\mu_{\overline{\beta}}$ for some $\overline{\beta} \in \mathbb{R}$, where

$$\frac{\mathrm{d}\mu_{\overline{\beta}}}{\mathrm{d}\nu}(x) = Z_{\overline{\beta}}^{-1} e^{-\overline{\beta} E(x)}.$$

$\hookrightarrow$ Single linear constraint [SZ91; DZ96].

Gibbs principle in statistical mechanics

Theorem (Gibbs variational principle)

Given ν in $\mathcal{P}(\mathbb{R}^d)$ and $E : \mathbb{R}^d \to \mathbb{R}$ continuous, bounded from below,

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↪ Single linear constraint [SZ91; DZ96].

↪ The Gibbs measure μ_β realises the free energy

$$G(\beta) = \inf_{\mu \in \mathcal{P}(\mathbb{R}^d)} H(\mu|\nu) + \beta \langle \mu, E \rangle.$$

↪ The Lagrange multiplier $\overline{\beta}$ maximises G.

Linear functional derivative

The map $\Psi : \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$ is assumed to be C^1 in the following sense: for any $\mu, \mu' \in \mathcal{P}(\mathbb{R}^d)$,

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} \Psi(\mu + \varepsilon(\mu' - \mu)) = \langle \mu - \mu', \tfrac{\delta \Psi}{\delta \mu}(\mu) \rangle,$$

for $\frac{\delta \Psi}{\delta \mu} : \mathcal{P}(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}$ continuous, with polynomial growth.

Constraint qualification assumption

Some $\tilde{\mu}_{[0,T]}$ with $H(\tilde{\mu}_{[0,T]} | \nu_{[0,T]}) < +\infty$ exists, satisfying

$$\forall t \in [0,T], \quad \Psi(\overline{\mu}_t) + \langle \tilde{\mu}_t - \overline{\mu}_t, \tfrac{\delta \Psi}{\delta \mu}(\overline{\mu}_t) \rangle < 0.$$

Assumptions

Linear functional derivative

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Example

In the linear case $\frac{\delta\Psi_V}{\delta\mu}(\mu) = V$, and a sufficient condition is

$$\nu_{[0,T]}(\{\forall t \in [0,T],\ V(x_t) \leq -\eta\}) > 0,$$

for some $\eta > 0$. Ψ_W would also work.

Gibbs principle on path space

Theorem (Gibbs principle)

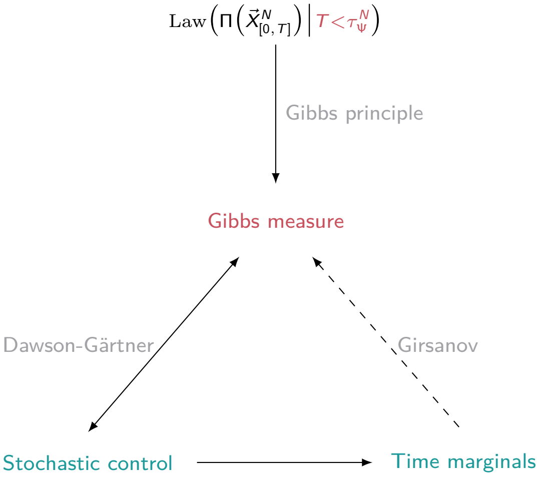
For any minimiser $\overline{\mu}_{[0,T]}$ of

$$\inf_{\substack{\mu \in \mathcal{P}(C([0,T],\mathbb{R}^d)) \\ \forall t \in [0,T],\, \Psi(\mu_t) \leq 0}} H(\mu_{[0,T]} | \nu_{[0,T]}).$$

some $\overline{\lambda}$ in $\mathcal{M}_+([0,T])$ exists s.t.

$$\frac{\mathrm{d}\overline{\mu}_{[0,T]}}{\mathrm{d}\nu_{[0,T]}}(x_{[0,T]}) = \left(Z_T^{\Psi}\right)^{-1} \exp\left[- \int_0^T \frac{\delta \Psi}{\delta \mu}(\overline{\mu}_t, x_t) \overline{\lambda}(\mathrm{d}t) \right],$$

with $\Psi(\overline{\mu}_t) = 0$ $\overline{\lambda}$-a.e. Sufficient condition in the convex case.


$\mathrm{Law}\left(\Pi\left(\vec{X}^N_{[0,T]}\right)\,\middle|\,T<\tau^N_\Psi\right)$
Gibbs principle
Gibbs measure
Dawson-Gärtner
Girsanov
Stochastic control
Time marginals

Controlled process

$$\begin{cases} \mathrm{d}X_s^{t,\mu,\alpha} = b(X_s^{t,\mu,\alpha})\mathrm{d}s + \alpha_s \mathrm{d}s + \mathrm{d}B_s, \quad t \leq s \leq T, \\ X_t^{t,\mu,\alpha} = X_t^{t,\mu}, \quad X_t^{t,\mu} \sim \mu, \end{cases}$$

for some adapted process $\alpha = (\alpha_s)_{t \leq s \leq T}$.

Value function

$$V(t,\mu) := \inf_{\substack{(\alpha_s)_{t\leq s\leq T} \\ \forall s \in [t,T],\ \Psi(\mathrm{Law}(X_s^{t,\mu,\alpha})) \leq 0}} \mathbb{E} \int_t^T \frac{1}{2} |\alpha_s|^2 \mathrm{d}s.$$

$\hookrightarrow$ One looks for $\overline{\alpha}$ which realises $V(0, \overline{\mu}_0)$.

$\Rightarrow$ Control problem with law constraints [Dau21].

HJB equation and verification

Theorem (Well-posedness)

If $(t,x) \mapsto \frac{\delta \Psi}{\delta \mu}(\overline{\mu}_t, x)$ *belongs to* $C([0,T], C_b^n(\mathbb{R}^d))$, $n \geq 2$, *and* $\lambda \in C^1([0,T], \mathbb{R}_+)$, *then*

$$\begin{cases} \partial_t \varphi(t,x) + \frac{1}{2}\partial_x^2 \varphi(t,x) - \frac{1}{2}|\partial_x \varphi(t,x)|^2 = -\lambda(t) \frac{\delta \Psi}{\delta \mu}(\overline{\mu}_t, x), \\ \varphi(T,x) = 0, \end{cases}$$

has a unique (mild) solution in $C([0,T], C_b^n(\mathbb{R}^d))$.

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Theorem (Verification)

$$\mathbb{E}\varphi\big(t, X_t^{t,\overline{\mu}_t}\big) = \inf_{(\alpha_s)_{t \leq s \leq T}} \mathbb{E}\int_t^T \frac{1}{2}|\alpha_s|^2 + \lambda(s)\frac{\delta \Psi}{\delta \mu}\big(\overline{\mu}_s, X_s^{t,\overline{\mu}_t,\alpha}\big)\,\mathrm{d}s$$
$$\leq V(t, \overline{\mu}_t),$$

and $\tilde{\alpha}_s := -\nabla\varphi\big(s, X_s^{t,\overline{\mu}_t,\tilde{\alpha}}\big)$ *is optimal.*

Pathwise law

Theorem (Girsanov)

The pathwise law of $X^{0,\nu_0,\tilde{\alpha}}_{[0,T]}$ *satisfies*

$$\frac{\mathrm{d}\tilde{\mu}_{[0,T]}}{\mathrm{d}\nu_{[0,T]}}(x_{[0,T]}) = Z_{\tilde{\alpha}}^{-1} \exp\left[- \int_0^T \frac{\delta \Psi}{\delta \mu}(\overline{\mu}_t, x_t) \lambda(t) \mathrm{d}t \right].$$

Theorem (Girsanov)

The pathwise law of $X^{0,\nu_0,\tilde{\alpha}}_{[0,T]}$ satisfies

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$\hookrightarrow$ *Approximating $\overline{\lambda}$ using λ recovers $\overline{\mu}_{[0,T]}$, and*

$$\inf_{\substack{\mu \in \mathcal{P}(C([0,T],\mathbb{R}^d)) \\ \forall t \in [0,T],\, \Psi(\mu_t) \leq 0}} H(\mu_{[0,T]}|\nu_{[0,T]}) = H(\overline{\mu}_0|\nu_0) + V(0,\overline{\mu}_0).$$

$\Rightarrow$ *This characterises time marginals.*

Potential mean-field game structure

$(\varphi, (\overline{\mu}_t)_t)$ is a solution of the MFG system

$$\begin{cases} -\partial \varphi(t,x) + \frac{1}{2} |\partial_x \varphi(t,x)|^2 - \partial_x^2 \varphi(t,x) = \lambda(t) \frac{\delta \Psi}{\delta \mu}(\overline{\mu}_t, x), \\ \partial_t \overline{\mu}_t - L^\star \overline{\mu}_t - \mathrm{div}(\overline{\mu}_t \, \partial_x \varphi(t,\cdot)) = 0, \\ \varphi(T,\cdot) = 0, \quad \overline{\mu}_0. \end{cases}$$

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Perspectives

- Add mean-field interactions.
- Long-time behaviour $T \to +\infty$.
- Particle approximation.
- ...

Summary

$$\mathrm{Law}\left(\Pi\left(\vec{X}^N_{[0,T]}\right) \,\middle|\, T < \tau^N_\Psi\right)$$

Gibbs principle

$$\underset{\substack{\mu_{[0,T]} \\ \forall t,\, \Psi(\mu_t) \leq 0}}{\mathrm{argmin}} \; H(\mu_{[0,T]} | \nu_{[0,T]})$$

Dawson-Gärtner

Girsanov

$$\inf_{\substack{(\alpha_t)_t \\ \forall t,\, \Psi(\mathrm{Law}(X^\alpha_t)) \leq 0}} \frac{1}{2} \int_0^T \mathbb{E}|\alpha_t|^2 \mathrm{d}t$$

HJB equation

MFG system on $(\overline{\mu}_t, V(t,\cdot))$

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[DZ96] A. Dembo and O. Zeitouni. "Refinements of the Gibbs conditioning principle". In: *Probability theory and related fields* 104.1 (1996), pp. 1–14.

[SZ91] D. W. Stroock and O. Zeitouni. "Microcanonical distributions, Gibbs states, and the equivalence of ensembles". In: *Random walks, Brownian motion, and interacting particle systems*. Springer, 1991, pp. 399–424.

Similarly, let

$$\pi(\vec{X}_t^N) := \frac{1}{N} \sum_{i=1}^{N} \delta_{X_t^i}$$

denote the pointwise empirical measure at time t.

Theorem (Contraction)

$(\mathrm{Law}((\pi(\vec{X}_t^N))_{0 \le t \le T}))_{N \ge 1} \subset \mathcal{P}(\mathcal{P}(C([0,T],\mathbb{R}^d))$ *satisfies a LDP with rate function*

$$S((\mu_t)_{0 \le t \le T}) := \inf_{\substack{\mu' \in \mathcal{P}(C([0,T],\mathbb{R}^d)) \\ \forall 0 \le t \le T,\, \mu'_t = \mu_t}} H(\mu'_{[0,T]} | \nu_{[0,T]}).$$

$((\pi(\vec{X}_t^N))_{0 \le t \le T})_{N \ge 1}$ *converges towards the deterministic flow* $(\nu_t)_{0 \le t \le T}$.

Example

When $\nu_{[0,T]}$ is the law of the diffusion

$$\mathrm{d}X_t = -\nabla V(X_t)\mathrm{d}t + \mathrm{d}B_t,$$

the marginal law ν_t at time t is defined by

$$\forall \varphi \in C_b(\mathbb{R}^d), \quad \mathbb{E}\varphi(X_t) = \int \varphi \mathrm{d}\nu_t =: \langle \nu_t, \varphi \rangle.$$

It satisfies

$$\boxed{\forall \varphi \in C_b^2(\mathbb{R}^d), \quad \frac{\mathrm{d}}{\mathrm{d}t}\langle \nu_t, \varphi \rangle = \langle \nu_t, L\varphi \rangle,}$$

where the infinitesimal generator is

$$L := -\nabla V \cdot \nabla + \frac{1}{2}\Delta.$$

Link with PDE control

Theorem (Dawson-Gärtner)

When μ_0 is imposed [DG87],

$$S((\mu_t)_{0\le t\le T}) = \begin{cases} \frac{1}{2}\int_0^T \|\partial_t \mu_t - L^\star \mu_t\|^2_{\mu_t} \mathrm{d}t \text{ if } (\mu_t)_{0\le t\le T} \text{ is AC,} \\ +\infty \text{ else.} \end{cases}$$

If $S((\mu_t)_{0\le t\le T}) < +\infty$, then $t \mapsto h_t$ exists such that

$$\forall \varphi \in C_b^2(\mathbb{R}^d), \quad \frac{\mathrm{d}}{\mathrm{d}t}\langle \mu_t, \varphi \rangle = \langle \mu_t, L\varphi + h_t \cdot \nabla \varphi \rangle,$$

and

$$\boxed{S((\mu_t)_{0\le t\le T}) = \frac{1}{2}\int_0^T \langle \mu_t, |h_t|^2 \rangle \mathrm{d}t.}$$

Law of large numbers

Ito's formula yields for φ in $C_b^2(\mathbb{R}^d)$

$$\underbrace{\langle \pi(\vec{X}_t^N), \varphi \rangle} = \langle \nu_0, \varphi \rangle + \int_0^t \langle \pi(\vec{X}_s^N), L\varphi \rangle \mathrm{d}s + \underbrace{\frac{1}{N} \sum_{i=1}^N \int_0^t \nabla \varphi(X_s^i) \cdot \mathrm{d}B_s^i},$$

$$X_t^\varepsilon = X_0^\varepsilon - \int_0^t \nabla V(X_s^\varepsilon) \mathrm{d}s + \sqrt{\varepsilon} B_t.$$

Rate functions

$$\frac{1}{2} \int_0^T \| \partial_t \mu_t - L^\star \mu_t \|_{\mu_t}^2 \mathrm{d}t \qquad \text{vs.} \qquad \frac{1}{2} \int_0^T |\dot{x}_t + \nabla V(x_t)|^2 \mathrm{d}t.$$

$\Rightarrow$ Conditioning can then be added as before.

From conditioning to constraints

Schrödinger bridge

Minimise $H(\mu_{[0,T]}|\nu_{[0,T]})$ with imposed endpoints $\mu_0,\ \mu_T$ yields

$$\inf_{\substack{(\mu_t)_{0\leq t\leq T}\in AC([0,T],\mathcal{P}(\mathbb{R}^d))\\ \mu_0,\mu_T \text{ imposed}}} \frac{1}{2}\int_0^T \|\partial_t \mu_t - L^\star \mu_t\|^2_{\mu_t}\mathrm{d}t.$$

Mean-field constraint

Given $\Psi : \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$, define the stopping time

$$\tau_\Psi^N := \inf\{t > 0\,,\ \Psi(\pi(\vec{X}_t^N)) > 0\},$$

Then $-\frac{1}{N}\log \mathbb{P}(T < \tau_\psi^N)$ converges to

$$\inf_{\substack{(\mu_t)_{0\leq t\leq T}\in AC([0,T],\mathcal{P}(\mathbb{R}^d))\\ \forall t\in[0,T],\ \Psi(\mu_t)\leq 0}} \frac{1}{2}\int_0^T \|\partial_t \mu_t - L^\star \mu_t\|^2_{\mu_t}\mathrm{d}t.$$

Example (Markov diffusion)

When $\nu_{[0,T]}$ is the law of the diffusion

$$\mathrm{d}X^i_t = b(X^i_t)\mathrm{d}t + \mathrm{d}B^i_t,$$

the Doob transform shows that $\vec{Z}^N_{[0,T]} := (Z^{1,N}_{[0,T]}, \dots, Z^{N,N}_{[0,T]})$,

$$\boxed{\mathrm{d}Z^{i,N}_t = b(Z^{i,N}_t)\mathrm{d}t + \nabla_{Z^i} \ln \mathbb{P}_{\vec{Z}^N_t}\big(\tau^N_\Psi > T - t\big)\mathrm{d}t + \mathrm{d}B^i_t,}$$

is exchangeable and $\mathrm{Law}(X^1_{[0,T]} | T < \tau^N_\Psi)$-distributed.

$\Rightarrow$ Mean-field limit for interacting particles.

Proof.

The Gibbs free energy

$$G(\beta) := \inf_{\mu \in \mathcal{P}(\mathbb{R}^d)} H(\mu|\nu) + \beta\langle \mu, E \rangle,$$

is uniquely realised by $\mu = \mu_\beta$, because

$$H(\mu|\nu) + \beta\langle \mu, E \rangle = H(\mu_\beta|\nu) + \beta\langle \mu_\beta, E \rangle + \overbrace{H(\mu|\mu_\beta)}^{\geq 0},$$

hence $G(\beta) = -\log Z_\beta$.

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hence $G(\beta) = -\log Z_\beta$. Moreover,

$$\inf_{\substack{\mu \in \mathcal{P}(\mathbb{R}^d) \\ \langle \mu, E\rangle = 0}} H(\mu|\nu) = \inf_{\substack{\mu \in \mathcal{P}(\mathbb{R}^d) \\ \langle \mu, E\rangle = 0}} \sup_{\beta \in \mathbb{R}} H(\mu|\nu) + \beta\langle \mu, E\rangle$$

$$\geq \sup_{\beta \in \mathbb{R}} G(\beta),$$

and $\beta \mapsto \langle \mu_\beta, E\rangle$ is continuous and decreasing... □

Existence for the multiplier and proof scheme

- For $(\lambda_k)_{k\geq 1}$ maximising G,

$$-1 \leq F(\lambda_k) \leq H(\tilde{\mu}_{[0,T]}|\nu_{[0,T]}) + \int_0^T \Psi(\tilde{\mu}_t)\lambda(\mathrm{d}t),$$

hence

$$\boxed{\forall k, \quad \lambda_k([0,T]) \leq \eta^{-1} C,}$$

and Prokhorov's theorem gives tightness for $(\frac{\lambda_k}{\lambda_k([0,T])})_k$.

- Linearisation: $\mu^\lambda_{[0,T]}$ is optimal for

$$\inf_{\mu_{[0,T]} \in \mathcal{P}(C([0,T],\mathbb{R}^d))} H(\mu_{[0,T]}|\nu_{[0,T]}) + \int_0^T \langle \mu_t, \frac{\delta\Psi}{\delta\mu}(\mu^\lambda_t, \cdot)\rangle \lambda(\mathrm{d}t),$$

hence $\mu^\lambda_{[0,T]}$ is a Gibbs measure.

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- Admissibility conditions:

$$\forall t_0 \in [0,T), \forall \varepsilon > 0, \quad F(\overline{\lambda} + \mathbb{1}_{[t_0,t_0+\varepsilon]}) \leq F(\overline{\lambda}) \dots$$

Propagation du Chaos

Un système de particules échangeables

$$\vec{X}_t^N = \left(X_t^{1,N}, \dots, X_t^{N,N}\right) \qquad \left(X_t^{1,N}, \dots, X_t^{k,N}\right) \sim f_t^{k,N},$$

est f_t-chaotique si

$$\forall k \geq 1, \qquad f_t^{k,N} \xrightarrow[N \to +\infty]{} f_t^{\otimes k}.$$

Le processus limite a pour loi marginale f_t.

Cela correspond à une loi des grands nombres

$$\mu_t^N := \frac{1}{N} \sum_{i=1}^{N} \delta_{X_t^{i,N}} \xrightarrow[N \to +\infty]{} f_t,$$

de l'aléatoire au déterministe !