implies

<u>Step 2</u> If $F : E \to \mathbb{R}$ is bounded continuous, set $B = (B_s ; 0 \leq s \leq 1)$

Hence

$$\mathbb{E}[F(B) \mid |B_1| \leq \varepsilon) = \frac{1}{\mathbb{P}(|B_1| \leq \varepsilon)} \mathbb{E}[F(b_s + s B_1 ; 0 \leq s \leq 1) \mathbb{1}_{|B_1| \leq \varepsilon}]$$

$$= \frac{1}{\mathbb{P}(|B_1| \leq \varepsilon)} \int_{-\varepsilon}^{\varepsilon} p_1(x)\, dx\, \mathbb{E}[F(b_s + s x ; 0 \leq s \leq 1)]$$

where $p_1(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ is the density of B_1.

Hence

$$|\mathbb{E}[F(B) \mid |B_1| \leq \varepsilon] - \mathbb{E}[F(b)]| \leq \frac{1}{\mathbb{P}(|B_1| \leq \varepsilon)} \int_{-\varepsilon}^{\varepsilon} p_1(x)\, dx \Big| \mathbb{E}[F(b_s + s x) ; 0 \leq s \leq 1] - \mathbb{E}[F(b)] \Big| .$$

But by dominated convergence,

$$\mathbb{E}[F(b_s + s x ; 0 \leq s \leq 1)] \xrightarrow[x \to 0]{} \mathbb{E}[F(b)] .$$

This implies the desired result. ☒

Recall that $E = \{ f : [0,1] \to \mathbb{R} ;\ f \text{ continuous} \}$. In the sequel, we let $b = (b_s ; 0 \le s \le 1)$ be a random variable having the law of the Brownian bridge $B = (B_s ; 0 \le s \le 1)$ —— Brownian motion. (= BM)

We will use the following result (consequence of the definition of BM).

Markov property of BM For every $0 \le s < 1$, $(B_{s+t} - B_s ; 0 \le t \le 1-s)$ is independent of $(B_t ; 0 \le t \le s)$ and has the same law as $(B_t ; 0 \le t \le 1-s)$. In particular, we can write

$$B_{s+t} = B_s + \tilde{B}_t$$

were $(\tilde{B}_t ; 0 \le t \le 1-s)$ is ⫫ of $(B_t ; 0 \le t \le s)$ and has the same law as $(B_t ; 0 \le t \le 1-s)$.

Lemma For every $0 \le s < 1$ and every $F : \mathcal{C}([0,s], \mathbb{R}) \longrightarrow \mathbb{R}$ which is bounded and continuous, we have

$$\mathbb{E}[F(b_t ; 0 \le t \le s)] \overset{(*)}{=} \mathbb{E}\left[F(B_t ; 0 \le t \le s) \frac{p_{1-s}(-B_s)}{p_1(0)}\right],$$

where p_t is the density of B_t, i.e of a $\mathcal{N}(0,t)$ random variable.
If $(\tilde{b}_t ; 0 \le t \le 1)$ is another random continuous function such that (*) holds for every $0 \le s < 1$, then $(\tilde{b}_t ; 0 \le t \le 1) \overset{(d)}{=} (b_t ; 0 \le t \le 1)$

Proof: Write, using the Markov property and the notation $\tilde{B}$ introduced above:

$$\mathbb{E}[F(B_t ; 0 \le t \le s) \,|\, |B_1| \le \varepsilon] = \frac{1}{\mathbb{P}(|B_1| \le \varepsilon)} \mathbb{E}\left[F(B_t ; 0 \le t \le s) \mathbb{1}_{\{|B_s + \tilde{B}_{1-s}| \le \varepsilon\}}\right]$$

$$= \frac{1}{\mathbb{P}(|B_1| \le \varepsilon)} \mathbb{E}\left[F(B_t ; 0 \le t \le s) \mathbb{1}_{\tilde{B}_{1-s} \in [-B_s - \varepsilon ,\, B_s + \varepsilon]}\right]$$

$$= \frac{1}{\int_{-\varepsilon}^{\varepsilon} p_1(x)\,dx} \mathbb{E}\left[F(B_t ; 0 \le t \le s) \int_{-B_s-\varepsilon}^{-B_s+\varepsilon} p_{1-s}(x)\,dx\right]$$

But by continuity of p_1, $\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} p_1(x)\,dx \xrightarrow[\varepsilon\to 0]{} p_1(0)$

Similarly, $\frac{1}{2\varepsilon}\int_{-B_s-\varepsilon}^{-B_s+\varepsilon} p_{1-s}(x)\,dx \xrightarrow[\varepsilon\to 0]{} p_{1-s}(-B_s)$

By dominated convergence, we get that

$$\begin{tikzcd}
\mathbb{E}[F(B_t;0\leq t\leq s)\mid |B_1|\leq\varepsilon] \arrow[r, "\varepsilon\to 0"'] \arrow[d, "\varepsilon\to 0"] & \mathbb{E}\left[F(B_t;0\leq t\leq s)\cdot\frac{p_{1-s}(-B_s)}{p_1(0)}\right] \\
\mathbb{E}[F(b_t;0\leq t\leq s)] &
\end{tikzcd}$$

The second assertion follows from the fact that (*) characterizes the finite dimensional distributions on $[0,1[$, and by continuity at 1.

Remark: Since $B_t \overset{(d)}{=} N(0,t)$, we have $\frac{B_t}{\sqrt{t}} \overset{(d)}{=} N(0,1)$, so that $\boxed{p_1(x)=\sqrt{s}\,p_s(x\sqrt{s})}$ $\forall x\in\mathbb{R}, s\geq 0$.

Proposition We have $\left(\frac{W_{nt}}{\sigma\sqrt{n}};0\leq t\leq 1\right)$ under $\mathbb{P}(\cdot\mid W_n=-1)$

$$\xrightarrow[n\to\infty]{(d)} b.$$

Proof Step 0 Fix $0<s<1$. We show convergence on $[0,s]$. For simplicity, assume that $ns\in\mathbb{Z}$.
Let $F:\mathcal{C}([0,s],\mathbb{R})\to\mathbb{R}$ be bounded, continuous, and write

$$\mathbb{E}\left[F\left(\frac{W_{nt}}{\sigma\sqrt{n}};0\leq t\leq s\right)\middle| W_n=-1\right]=\mathbb{E}\left[F\left(\frac{W_{nt}}{\sigma\sqrt{n}};0\leq t\leq s\right)\frac{Q_{n-ns}(-1-W_{ns})}{Q_n(-1)}\right]$$

where $Q_r(k)=\mathbb{P}(W_r=k)$.

By the Skorokhod embedding thm (see below), we may assume that almost surely, $\left(\frac{W_{nt}}{\sigma\sqrt{n}};0\leq t\leq s\right)\xrightarrow[n\to\infty]{}(B_t;0\leq t\leq s)$.

Then, by the local limit theorem,

1) $\exists C>0$ such that a.s $\frac{Q_{n-ns}(-1-W_{ns})}{Q_n(-1)}\leq C$ $\forall n\geq 1$

2) a.s, $\frac{q_{n-ns}(-1-W_{ns})}{q_n(-1)} \xrightarrow[n\to\infty]{} \frac{p_1\left(-\frac{B_s}{\sqrt{1-s}}\right)}{\sqrt{1-s}\, p_1(0)} = \frac{p_{1-s}(-B_s)}{p_1(0)}$

Hence, by Dominated Convergence,

$$\mathbb{E}\left[F\left(\frac{W_{nt}}{\sigma\sqrt{n}}; 0\le t\le s\right) \Big| W_n=-1\right] \xrightarrow[n\to\infty]{} \mathbb{E}\left[F(B_t; 0\le t\le s) \cdot \frac{p_{1-s}(-B_s)}{p_1(0)}\right]$$

$$= \mathbb{E}[F(b_t; 0\le t\le s)]$$

Step 1 | Convergence of finite dimensional distributions.

→ this follows from Step 0

Step 2 | Tightness

Recall that by time-reversal, $\hat{S}^{(n)} = (S_n - S_{n-i}; 0\le i\le n) \overset{(d)}{=} (S_0, S_1, \dots, S_n)$

Hence we have that

under $|W_n=-1$, $\begin{cases} W^{1,(n)} = \left(\frac{W_{nt}}{\sigma\sqrt{n}}; 0\le t\le \frac{3}{4}\right) \xrightarrow[n\to\infty]{(d)} \left(b_t; 0\le t\le \frac{3}{4}\right) \\ W^{2,(n)}\left(\frac{W_{n(1-t)}}{\sigma\sqrt{n}}; 0\le t\le \frac{3}{4}\right) \xrightarrow[n\to\infty]{} \left(\tilde{b}_t; 0\le t\le \frac{3}{4}\right). \end{cases}$

Hence $(W^{1,(n)})_n$ is tight and $(W^{2,(n)})$ is tight.

It is then possible to deduce that $\left(\frac{W_{nt}}{\sigma\sqrt{n}}; 0\le t\le 1\right)_{n\ge 1}$ is tight (exercise).

Thm (Skorokhod Embedding) Let (X_n), $X\in E$ be such that $X_n \xrightarrow[n\to\infty]{(d)} X$. Then there exists a new $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and $\tilde{X}_n$, $\tilde{X}$ all defined on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ such that • $\forall n\ge 1$, $X_n \overset{(d)}{=} \tilde{X}_n$ • $X \overset{(d)}{=} \tilde{X}$ and $\tilde{X}_n \xrightarrow{a.s} \tilde{X}$.

3) The Brownian excursion

If $f:[0,1]\to\mathbb{R}$ is continuous and $f(0)=f(1)=0$, set $t_* = \inf\{t\geq 0;\ f(t)=\inf_{[0,1]} f\}$ and define Vf, the Vervaat transform of f by

$$Vf(t)=\begin{cases} f(t+t_*)-f(t_*) & \text{if } 0\le t\le 1-t_* \\ f(t+t_*-1)-f(t_*) & \text{if } 1-t_*<t\le 1 .\end{cases}$$

Definition For $0\le t\le 1$, Set $\mathbf{e}(t)=V\underset{\substack{\uparrow \\ \text{Brownian bridge}}}{b}(t)$

$\mathbf{e}$ is called the Brownian excursion.

Lemma (exercise) If $f\in E$, $f(0)=f(1)=0$ and $|\{t\geq 0;\ f(t)=\inf_{[0,1]} f\}|=1$, we say that f attains its minimum at a unique time. In this case, V is continuous at f.

Lemma (exercise) A.s. b attains its minimum at a unique time.

Idea: This is true for BM (Markov property) and hence for the B. Bridge by absolute continuity.

We can now show:

Prop (Conditioned Donsker's theorem). Recall that $\zeta_1=\inf\{i\geq 1;\ W_i=-1\}$

Under $\mathbb{P}(\cdot\mid \zeta_1=n)$, $\left(\frac{W_{nt}}{\sigma\sqrt{n}};\ 0\le t\le 1\right)\xrightarrow[n\to\infty]{(a)} \mathbf{e}$

Proof By the preceding remarks and the proposition,

Under $\mathbb{P}(\cdot\mid W_n=-1)$, $V\left(\frac{W_{nt}}{\sigma\sqrt{n}};\ 0\le t\le 1\right)\xrightarrow[n\to\infty]{(d)} V(b)=\mathbf{e}$

But we saw that under $\mathbb{P}(\cdot \mid W_n=-1)$ $V\left(\frac{W_{nt}}{\sigma\sqrt{n}}; 0\leq t\leq 1\right) \overset{(d)}{=} \left(\frac{W_{nt}}{\sigma\sqrt{n}}; 0\leq t\leq 1\right)$ under $\mathbb{P}(\cdot \mid \zeta_1=n)$

4) Proof of the main Theorem

Recall that
- $H_\cdot(T_n)$ is the height function of T_n
- $W_\cdot(T_n)$ — Lukasiewicz path of —
- $(W_n)_{n\geq 0}$ is a random walk, (H_n) its associated height function.

Goal: $\left(\frac{H_{nt}(T_n)}{\sqrt{n}}; 0\leq t\leq 1\right) \xrightarrow[n\to\infty]{(d)} \frac{2}{\sigma}\mathbf{e}$.

We saw a long time ago that

$$\left((W_0(T_n),\dots,W_n(T_n)),\ (H_0(T_n),\dots\dots,H_{n-1}(T_n))\right)$$
$$\overset{(d)}{=} \left((W_0,\dots,W_n),\ (H_0,\dots,H_n)\right) \text{ under } \mathbb{P}(\cdot \mid \zeta_1=n).$$

In particular, by Donsker's conditioned theorem, we have under $\mathbb{P}(\cdot \mid \zeta_1=n)$, $\left(\frac{W_{nt}}{\sqrt{n}}; 0\leq t\leq 1\right) \xrightarrow[n\to\infty]{(d)} \sigma\cdot\mathbf{e}$

Is therefore enough to show that

$$A_n=\mathbb{P}\left(\sup_{0\leq t\leq 1}\left|\frac{W_{nt}}{\sqrt{n}}-\frac{\sigma^2}{2}\frac{H_{nt}}{\sqrt{n}}\right|>n^{3/8} \,\Big|\, \zeta_1=n\right)=o(n).$$

But we know that $\mathbb{P}(\zeta_1=n)\sim\frac{c}{n^{3/2}}$ as $n\to\infty$.

Hence
$$A_n\leq\frac{\mathbb{P}\left(\sup_{0\leq t\leq 1}\left|\frac{W_{nt}}{\sqrt{n}}-\frac{\sigma^2}{2}\frac{H_{nt}}{\sqrt{n}}\right|\right)\overset{=o(n)}{}\text{(by last time)}}{\mathbb{P}(\zeta_1=n)}$$
$$=o(n).$$

This completes the proof.

Corollary ① $\frac{\text{Height}(T_n)}{\sqrt{n}} \xrightarrow[n\to\infty]{(d)} \frac{2}{\sigma} \max \mathbb{e}$

② Let U_n be a uniform vertex of T_n. Then

$$\frac{|U_n|}{\sqrt{n}} \xrightarrow[n\to\infty]{(d)} \frac{2}{\sigma} \mathbb{e}_U \,,$$ where U is uniform on $[0,1]$, $\perp\!\!\!\perp \mathbb{e}$.

NB it is possible to show that $\mathbb{P}(2\mathbb{e}_U \geq x) = e^{-x^2/2}$.

5) The Brownian tree

For $0 \leq s,t \leq 1$ set $d_{\mathbb{e}}(s,t) = \mathbb{e}_s + \mathbb{e}_t - 2 \min\limits_{[\min(s,t),\max(s,t)]} \mathbb{e}$

and set $s \sim t$ if $d_{\mathbb{e}}(s,t) = 0$.

Let $\mathcal{T}_{\mathbb{e}} = [0,1]/\sim$ be the associated quotient metric space.

It is possible to show that $\mathcal{T}_{\mathbb{e}}$ is a compact "tree like" metric space, and with the previous shown results, that

$$\frac{T_n}{\sqrt{n}} \xrightarrow[n\to\infty]{(d)} \frac{2}{\sigma} \cdot \mathcal{T}_{\mathbb{e}}$$

for a certain topology on compact metric spaces (called the Gromov-Hausdorff topology).