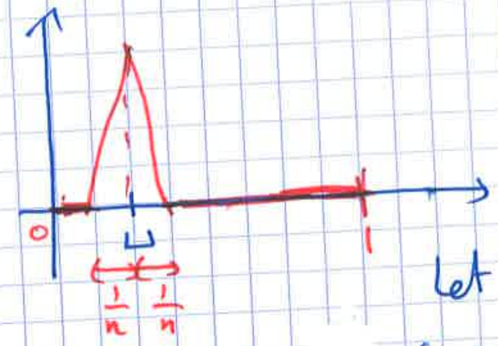
Corollary Let $(\mu_n), \mu \in \mathcal{M}_1(E)$.
If (μ_n) is tight and $\mu_n(A) \xrightarrow[n\to\infty]{} \mu(A)$ for every $A \in \mathcal{C}_{cyc}$, then $\mu_n \xrightarrow[n\to\infty]{\text{weakly}} \mu$

⚠ (μ_n) tight is an important hypothesis. For example, let U be a uniform random variable on $[0, \frac{1}{2}]$ and let μ_n be the law of the following random function:



Let $\mu = \delta_0 \leftarrow$ function $\equiv 0$

Then $\mu_n(A) \longrightarrow \mu(A) \quad \forall A \in \mathcal{C}_{cyc}$,
but $\mu_n \overset{\text{weakly}}{\underset{n\to\infty}{\not\longrightarrow}} \mu$.

Indeed, if $\mu_n \xrightarrow[n\to\infty]{\text{weakly}} \mu$, then $\mu_n(F) \xrightarrow[n\to\infty]{} \mu(F)$
for every $F : E \to \mathbb{R}$ continuous.
Clearly, here this does not hold with $F(f) = (\sup_{0\leq s\leq 1} f(s)) \wedge 1$

Thm [Tightness criterion in $C([0,1], \mathbb{R})$]
Let $(X_n)_{n\geq 1}$ be a sequence of random variables in E.
the laws of X_n are tight if and only if
(i) $(X_n(0))_{n\geq 1}$ is tight in $\mathbb{R}$
(ii) $\forall \varepsilon > 0, \forall \eta > 0, \exists \delta > 0$ such that

$$\limsup_{n\to\infty} \mathbb{P}\Big(\sup_{\substack{0\leq s,t\leq 1 \\ |s-t|\leq\delta}} |X_n(s) - X_n(t)| \geq \eta \Big) \leq \varepsilon.$$

4) Brownian motion and Donsker's invariance principle

We present Paul Lévy's construction of Brownian motion on $[0,1]$.
Set $\mathcal{D}_n = \{ \frac{k}{2^n} \ ; \ 0 \leq k \leq 2^n \}$ and $\mathcal{D} = \bigcup_{n \geq 0} \mathcal{D}_n$

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space s.t $(Z_i)_{i \in \mathcal{D}}$ are iid $\mathcal{N}(0,1)$ random variables on $(\Omega, \mathcal{F}, \mathbb{P})$.
Set $B_0 = 0$, $B_1 = Z$.

Idea: Construct recursively B on $(\mathcal{D}_n)$.
→ Assume that it is done at step $n-1$.
Then for every $t \in \mathcal{D}_n \setminus \mathcal{D}_{n-1}$ set

$$B_t = \frac{B_{t+2^{-n}} + B_{t-2^{-n}}}{2} + \frac{Z_t}{2^{\frac{n+1}{2}}}.$$

Exercise: $\left(B_{\frac{k+1}{2^n}} - B_{\frac{k}{2^n}} \ ; \ 0 \leq k \leq 2^n - 1 \right)$ are iid $\mathcal{N}(0, 2^{-n})$

Denote by $B^{(n)}$ the function which interpolates linearly $\{ B_t \ ; \ t \in \mathcal{D}_n \}$

Lemma $B^{(n)}$ converges almost surely to a random continuous function (for the $\|\cdot\|_\infty$ norm), which is denoted by B.

Proof By construction, $\|B^{(n)} - B^{(n-1)}\|_\infty = \frac{1}{2^{\frac{n+1}{2}}} \max_{t \in \mathcal{D}_n \setminus \mathcal{D}_{n-1}} Z_t$.
Hence, if $\alpha \in (\frac{1}{\sqrt{2}}, 1)$,

$$\mathbb{P}(\|B^{(n)} - B^{(n-1)}\|_\infty \geq \alpha^n) = \mathbb{P}\left(\max_{t \in \mathcal{D}_n \setminus \mathcal{D}_{n-1}} Z_t \geq \alpha^n 2^{\frac{n+1}{2}} \right)$$

$$\leq 2^n \, \mathbb{P}(|Z| \geq \alpha^n 2^{\frac{n+1}{2}}) \text{ where } Z = \mathcal{N}(0,1) \quad \text{(union bound)}$$

$$\leq C 2^n \exp(-(\alpha^2 2)^n) \quad \text{since } \mathbb{P}(|Z| \geq x) \leq C e^{-x^2/2}.$$

Hence $\sum_{n \geq 1} \mathbb{P}(\|B^{(n)} - B^{(n-1)}\|_\infty \geq \alpha^n) < \infty$.
the Borel-Cantelli lemma implies that a.s., for n large enough, $\|B^{(n)} - B^{(n-1)}\|_\infty \leq \alpha^n$
Hence a.s $\sum_{n \geq 1} \|B^{(n)} - B^{(n-1)}\|_\infty < \infty$.

This implies that the series $\sum_{n\geq 1} (B^{(n)} - B^{(n-1)})$ is absolutely convergent in the Banach space E. Hence $B^{(n)}$ converges a.s. $\boxed{\text{We now show that } B \text{ is a Brownian motion}}$

Def One-dimensional Brownian motion is a family of $\mathbb{R}$-valued random variables $(B_t ; t\geq 0)$ living on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ s.t.

(i) a.s, $B_0 = 0$

(ii) $\forall\, 0 = t_0 < t_1 < \dots < t_p$ the r.v $(B_{t_i} - B_{t_{i-1}} ; 1\leq i\leq p)$ are iid and

$$B_{t_i} - B_{t_{i-1}} \overset{(d)}{=} \mathcal{N}(0, t_i - t_{i-1})$$

(iii) $\begin{array}{l} \mathbb{R}^+ \longrightarrow \mathbb{R} \\ t \longmapsto B_t \end{array}$ is a.s continuous.

We check that B satisfies this (on $[0,1]$)

(i) and (iii) are clear by construction.

For (ii), fix $0 = t_0 < t_1 < \dots < t_p \leq 1$

And let $t_1^{(n)} < \dots < t_p^{(n)}$ be elements of D_n such that

$$t_i^{(n)} \xrightarrow[n\to\infty]{} t_i .$$

By construction, (i.e the exercise on the previous page) $(B_{t_{i+1}^{(n)}} - B_{t_i^{(n)}} ; 0\leq i\leq p-1)$ are iid with laws $\mathcal{N}(0, t_{i+1}^{(n)} - t_i^{(n)})$.

The result follows by passing to the limit $n\to\infty$

We have thus constructed Brownian motion on $[0,1]$.

Theorem (Donsker's invariance principle)

Let $(Z_i)_{i\geq 1}$ be a sequence of iid random variables such that $\mathbb{E}[Z_1]=0$ and $\sigma^2=\mathbb{E}[Z_1^2]<\infty$.

Set $S_n = Z_1+\dots+Z_n$ ($S_0=0$), and define $(S_t)_{t\geq 0}$ by linear interpolation. Then

$$\left(\frac{S_{nt}}{\sigma\sqrt{n}}\,;\,0\leq t\leq 1\right)\xrightarrow[n\to\infty]{(d)}(B_t\,;\,0\leq t\leq 1),$$

where the convergence holds in distribution in E.

Proof To simplify, assume $\boxed{\sigma=1}$ (otherwise consider Z_i/σ instead of Z_i)

Step 1 (Convergence of finite dimensional distributions)

Fix $0=t_0\leq t_1\leq\dots\leq t_p\leq 1$.

And let $(t_i^{(n)})$ be such that $nt_i^{(n)}\in\mathbb{Z}$ and $t_i^{(n)}\xrightarrow[n\to\infty]{} t_i$.

Then $\frac{S_{nt_1^{(n)}-nt_0^{(n)}}}{\sqrt{n}},\dots,\frac{S_{nt_p^{(n)}-nt_{p-1}^{(n)}}}{\sqrt{n}}$ are independent, and by the central limit theorem, the i-th random variable converges to $\mathcal{N}(0,t_i-t_{i-1})$

Hence $\left(\frac{S_{nt_0}}{\sqrt{n}},\dots,\frac{S_{nt_p}}{\sqrt{n}}\right)\xrightarrow[n\to\infty]{(d)}(0,N_1,N_1+N_2,\dots,N_1+\dots+N_p)$

where $N_1,\dots,N_p$ are $\perp\!\!\!\perp$ and $N_i\overset{(d)}{=}\mathcal{N}(0,t_i-t_{i-1})$.

Step 2 (Tightness)

We will use the following lemma, which is proved later.

Maximal inequality lemma: For every $\alpha>0$,

$$\mathbb{P}\left(\max_{1\leq k\leq n}|S_k|\geq\alpha\sqrt{n}\right)\leq 2\,\mathbb{P}\left(|S_n|>(\alpha-\sqrt{2})\sqrt{n}\right)$$

Now back to tightness. Fix $\varepsilon > 0, \eta > 0$. We have to estimate (for small δ)

$$\mathbb{P}\left(\sup_{\substack{0 \le s,t \le 1 \\ |s-t| \le \delta}} \left| \frac{S_{ns} - S_{nt}}{\sqrt{n}} \right| \geq \eta \right)$$

Observation: If $0 = t_0 < t_1 < \cdots < t_p = 1$ and

$$\min_{1 < i < p} (t_i - t_{i-1}) \geq \delta,$$

then $\sup_{\substack{0 \le s,t \le 1 \\ |s-t| \le \delta}} |f(s) - f(t)| \le 3 \max_{1 \le i \le p} \sup_{t_{i-1} \le s \le t_i} |f(s) - f(t_{i-1})|$

In particular, it is enough to show that

$\forall \varepsilon > 0, \eta > 0$, $\exists \delta \in (0,1)$, $\exists n_0$, s.t.

$\forall t \in [0,1]$, $\forall n \geq n_0$, $\frac{1}{\delta} \mathbb{P}\left(\sup_{t \le s \le t+\delta} \left| \frac{S_{ns} - S_{nt}}{\sigma \sqrt{n}} \right| \geq \eta \right) \le \varepsilon \quad (*)$

To simplify, we do everything as if $\delta n, ns, nt \in \mathbb{Z}$:

$$\mathbb{P}\left(\sup_{t \le s \le t+\delta} |S_{ns} - S_{nt}| \geq \eta \sqrt{n} \right)$$

$$= \mathbb{P}\left(\sup_{0 \le i \le \delta n} |S_i| \geq \frac{\eta}{\sqrt{\delta}} \sqrt{n\delta} \right)$$

$$\le 2\, \mathbb{P}\left(|S_{\delta n}| \geq \left(\frac{\eta}{\sqrt{\delta}} - \sqrt{2} \right) \sqrt{n\delta} \right)$$

By the central limit theorem, this converges to

$$2\, \mathbb{P}\left(|Z| \geq \frac{\eta}{\sqrt{\delta}} - \sqrt{2} \right), \text{ where } Z \sim N(0,1)$$

By Markov's inequality, this is $\le \dfrac{2\,\mathbb{E}[Z^4]}{\left(\frac{\eta}{\sqrt{\delta}} - \sqrt{2} \right)^4}$

Conclusion:

$$\limsup_{n \to \infty} \frac{1}{\delta} \mathbb{P}\left(\sup_{t \le s \le t+\delta} \left| \frac{S_{ns} - S_{nt}}{\sigma \sqrt{n}} \right| \geq \eta \right) \le \frac{2\,\mathbb{E}[Z^4]}{(\eta - \sqrt{2\delta})^4} \times \delta$$

This shows $(*)$ and completes the proof $\blacksquare$

Proof of the maximal inequality :

Let B_j be the event $B_j = \{ |S_1| \leq \lambda\sqrt{n}, \dots, |S_{j-1}| \leq \lambda\sqrt{n}, |S_j| > \lambda\sqrt{n} \}$

Then
$$\mathbb{P}\left(\max_{0 \leq i \leq n} |S_i| > \lambda\sqrt{n} \right) = \mathbb{P}(B_1 \cup B_2 \cup \dots \cup B_n)$$
$$= \mathbb{P}\left((B_1 \cup \dots \cup B_n) \cap \{ |S_n| > (\lambda - \sqrt{2})\sqrt{n} \} \right)$$
$$+ \mathbb{P}\left((B_1 \cup \dots \cup B_n) \cap \{ |S_n| < (\lambda - \sqrt{2})\sqrt{n} \} \right)$$
$$\leq \mathbb{P}\left(|S_n| > (\lambda - \sqrt{2})\sqrt{n} \right) + \sum_{j=1}^{n} \mathbb{P}\left(B_j, |S_n| < (\lambda - \sqrt{2})\sqrt{n} \right).$$

Note that by the triangular inequality, on the event $\{ B_j, |S_n| < (\lambda - \sqrt{2})\sqrt{n} \}$, we have
$$\underbrace{|S_n - S_j| > \sqrt{2n}}_{\text{this event is independent of } S_1, \dots, S_j \,!}.$$

Hence
$$\mathbb{P}\left(B_j, |S_n| < (\lambda - \sqrt{2})\sqrt{n} \right) \leq \mathbb{P}\left(B_j, |S_n - S_j| > \sqrt{2n} \right)$$
$$= \mathbb{P}(B_j)\, \mathbb{P}\left(|S_n - S_j| > \sqrt{2n} \right)$$
$$\leq \mathbb{P}(B_j)\, \frac{(n-j)\, \underbrace{\mathrm{Var}(S_1)}_{=1}}{2n} \quad \text{(Chebysheff inequality)}$$
$$\leq \frac{\mathbb{P}(B_j)}{2}$$

Hence
$$\mathbb{P}(B_1 \cup \dots \cup B_n) \leq \mathbb{P}\left(|S_n| > (\lambda - \sqrt{2})\sqrt{n} \right) + \frac{\mathbb{P}(B_1) + \dots + \mathbb{P}(B_n)}{2}$$
$$= \mathbb{P}\left(|S_n| > (\lambda - \sqrt{2})\sqrt{n} \right) + \frac{\mathbb{P}(B_1 \cup \dots \cup B_n)}{2}$$

because $B_1, \dots, B_n$ are disjoint $\boxtimes$

Application of Donsker's invariance principle :

If $X_n, X \in E$ are random s.t $X_n \xrightarrow{(d)} X$, and if $F : E \to E'$ is a measurable function (and E' a metric space), which is continuous at X almost surely, then $F(X_n) \xrightarrow{(d)} F(X)$.

As a consequence
$$\frac{\sup_{0 \leq i \leq n} S_i}{\sigma\sqrt{n}} = \sup_{0 \leq t \leq 1} \frac{S_{nt}}{\sigma\sqrt{n}} \xrightarrow[n \to \infty]{(d)} \underbrace{\sup_{0 \leq t \leq 1} B_t}_{\text{(this actually has an explicit law)}}.$$

Also, $\left(\frac{S_{nt}}{\sigma\sqrt{n}} - \inf_{0 \le s \le t} \frac{S_{ns}}{\sigma\sqrt{n}} ; 0 \le t \le 1 \right) \xrightarrow[n \to \infty]{(d)} \left(B_t - \inf_{0 \le s \le t} B_s ; 0 \le t \le 1 \right)$

and $\frac{S_n - I_n}{\sigma\sqrt{n}} \xrightarrow[n \to \infty]{(d)} B_1 - \inf_{0 \le s \le 1} B_s ,$

where $I_n = \inf_{0 \le i \le n} S_i$.