# Geometry of random trees

Igor Kortchemski
igor.kortchemski@math.uzh.ch
Office K44

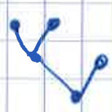
## I] Galton-Watson processes

Let $\mu = (\mu(i); i \geq 0)$ be a probability measure on $\mathbb{Z}_+ = \{0, 1, \dots\}$ such that $\mu_0 + \mu_1 < 1$.

Let $(X_j^{(n)})_{j, n \geq 0}$ be iid random variables with law $\mu$, defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

The Galton-Watson process $(Z_n)_{n \geq 0}$ is defined as follows

- $Z_0 = 1$
- $Z_{n+1} = \sum_{j=1}^{Z_n} X_j^{(n)}$ for $n \geq 1$.

Example :  $Z_0 = 1$, $Z_1 = 2$, $Z_3 = 2$, $Z_i = 0$ $(i \geq 4)$

Set $m = \sum_{i \geq 0} i \mu_i = \mathbb{E}[Z_1]$.

$$\boxed{\textbf{Thm} \quad m \leq 1 \iff \text{a.s., } \exists n \geq 0 \text{ s.t. } Z_n = 0.}$$

Preliminary observation:

$\forall n \geq 0$, $Z_n$ is $\sigma(X_k^{(j)}; j \leq n-1, k \geq 1)$ measurable (by induction on $n$).

**Proof** Set $\phi(s) = \sum_{n \geq 0} \mu(n) s^n$

For $0 \leq s < 1$, we have

$$\phi'(s) = \sum_{i=0}^{\infty} i \mu(i) s^{i-1} > 0$$

$$\phi''(s) = \sum_{i=2}^{\infty} i(i-1)\mu(i) s^{i-1} > 0$$

**Step 1** $\phi(1) = 1$, $\phi(0) = \mu_0$.

Hence $\phi$ and $\phi'$ are increasing on $[0, 1)$. (since $\mu(0) + \mu(1) < 1$)

**Step 2** Introduce $(0 \leq s \leq 1)$ $\phi_n(s) = \mathbb{E}[s^{Z_n}] = \sum_{k \geq 0} \mathbb{P}(Z_n = k) s^k$

Note that $\phi_1 = \phi$.

We claim that $\phi_{n+1}(s) = \phi_n(\phi(s)) \ \forall s \in [0, 1]$.

Indeed,

$$\phi_{n+1}(s) = \mathbb{E}\left[ s^{\sum_{j=1}^{Z_n} X_j^{(n)}} \right]$$

$$= \mathbb{E}\left[ s^{\sum_{j=1}^{Z_n} X_j^{(n)}} \left( \sum_{k \geq 0} \mathbb{1}_{Z_n = k} \right) \right]$$

$$= \sum_{k \geq 0} \mathbb{E}\left[ s^{\sum_{j=1}^{k} X_j^{(n)}} \mathbb{1}_{Z_n = k} \right]$$

$Z_n$ is $\sigma\left(X_k^{(j)}; \begin{matrix} j \leq n-1 \\ k \geq 1 \end{matrix}\right)$ measurable, hence

$$= \sum_{k \geq 0} \mathbb{P}(Z_n = k)\, \mathbb{E}\left[ s^{\sum_{j=1}^{k} X_j^{(n)}} \right] \quad \text{since } \perp\!\!\!\perp \left(X_1^{(n)}, \dots, X_k^{(n)}\right)$$

$$= \sum_{k=0}^{\infty} \mathbb{P}(Z_n = k)\, \phi(s)^k \quad \text{since } X_1^{(n)}, \dots, X_k^{(n)} \text{ are iid}$$

$$= \phi_n(\phi(s)).$$

$\boxed{\text{Step 3}}$ Set $q = \mathbb{P}(\exists n \geq 0;\ Z_n = 0)$

Then $q = \mathbb{P}\left( \bigcup_{n \geq 0} \{ Z_n = 0 \} \right) = \lim_{n \to \infty} \mathbb{P}(Z_n = 0)$

(increasing sets in $n$)

$= \lim_{n \to \infty} \phi_n(0)$.

$\boxed{\text{Step 4}}$ (Study of $\phi_n(0)$ as $n \to \infty$ and of the fixed points of $\phi$)

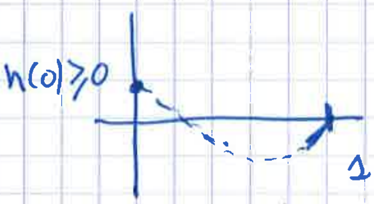
Set $h(s) = \phi(s) - s$.

If $m \leq 1$: $h'(s) = \phi'(s) - 1 < \phi'(1) - 1 \leq 0$.

Hence $h$ is decreasing on $[0,1]$ and $h(1) = 0$

Hence $\phi(t) > t$ on $[0,1)$

If $m > 1$: $h'(1) > 0$ and $h$ is convex (strictly, since $h'' > 0$ on $[0,1)$)




Since $h(0) \geq 0$, there exists a unique $r \in ]0,1[$ s.t $h(r) = 0$, ie $\phi(r) = r$. Indeed, since $h'(1) > 0$, $\exists \varepsilon > 0$ st $h(1-\varepsilon) < 0$. By $C^0$ of $h$, $\exists r \in [0, 1-\varepsilon]$ s.t $h(r) = 0$. Since $h$ is strictly convex, $h$ lies below $0$ on $(r,1)$ and above $0$ on $(0,r)$.

We know that $\phi_n(0)$ cv. to a fixed point of $\phi$.

But by induction, since $\phi$ is increasing, $\phi_n(0) \leq r \ \forall n \geq 0$

Hence $\phi_n(0) \longrightarrow r < 1$.

Indeed $0 \leq r$, and if $\phi_n(0) \leq r$, then
$\phi_{n+1}(0) = \phi(\phi_n(0))$
$\leq \phi(r)$
$= r$

Conclusion: $\mathbb{P}(\exists n;\ Z_n = 0) = 1$ if $m \leq 1$
$< 1$ if $m > 1$.

N.B. The assumption $\mu_0 + \mu_1 < 1$ is to avoid degenerate cases.

Further questions? $\mathbb{P}(Z_n > 0)$? What can be said about $\sum_{n \geq 0} Z_n$?

To answer these questions, we will study the genealogical tree of the process.

Exercise for next time

a) Calculate $\mathbb{E}[Z_n]$

b) Calculate $\mathbb{E}[Z_n^2]$.