Solution to the exercise

A forest $\underline{f}$ with k trees is a sequence $\underline{f} = (\underline{t}_1, \dots, \underline{t}_k)$ of k forests.

Recall the bijection

$$\phi_n : \Pi_n \to \overline{S}_n^{(1)}$$
$$\underline{t} \longmapsto (k_{u_0}(\underline{t})-1, \dots, k_{u_{|t|-1}}(\underline{t})-1)$$

Denote by $\mathbb{F}_n^{(k)}$ the set of all forests with k trees and n vertices.
Then set

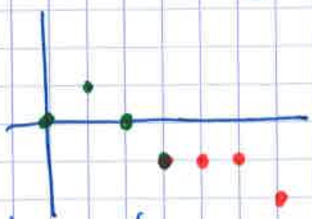
$$\Phi : \mathbb{F}_n^{(k)} \longmapsto \overline{S}_n^{(k)}$$
$$(\underline{t}_1, \dots, \underline{t}_k) \longmapsto \Phi_{|\underline{t}_1|}(\underline{t}_1) \cdot \; \dots \; \cdot \phi_{|\underline{t}_k|}(\underline{t}_k)$$

concatenation

For example, if $\underline{f} = \{$  $\}$.

Then $\Phi(\underline{f}) = \{1, -1, -1, 0, 0, -1\}$

And the associated Lukasiewicz path is



Then Φ is a bijection (similar proof)

Hence $\quad |\mathbb{F}_n^{(k)}| = |\overline{S}_n^{(k)}| = \binom{2n-k-1}{n-1} \times \frac{k}{n}$