Week 8: Cardinality and combinatorics

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1 Important exercises

The solutions of the exercises which have not been solved in some group will be available on the course webpage.

Exercise 1. How many integers $1 \leq a, b, c \leq 100$ such that $a < b$ and $a < c$ are there?

Exercise 2. In how many ways is it possible to arrange in a line 7 girls and 3 boys in the following cases:

1) When the 3 boys follow each other.
2) When the first and last person are girls, and when all the 3 boys do not follow each other.

Exercise 3. Let $n \geq 2$ be an integer, and set $E = \{1, 2, \ldots, n\}$. Find the cardinalities of the following sets:

$$F = \{(i,j) \in E^2\}, \quad G = \{(i,j) \in E^2, i \neq j\}, \quad H = \{(i,j) \in E^2, i < j\}, \quad I = \{A \subseteq E, \mathrm{Card}(A) = 2\}.$$

Exercise 4. How many onto functions from $\{1, 2, \ldots, n\}$ to $\{1, 2, 3\}$ are there?

Exercise 5. Let E and F be finite sets *having the same cardinality*, and let $f : E \to F$ be a function. Show that the following three assertions are equivalent:

(1) f is onto;
(2) f is one-to-one;
(3) f is a bijection.

2 Homework exercise

You have to individually hand in the written solution of the next exercise to your TA on Monday, November 25th.

Exercise 6.

1) How many three-digit numbers abc have exactly one digit equal to 9? Justify your answer.
2) How many three-digit numbers abc have the property that $a \neq b$ or $b \neq c$? Justify your answer.
3) How many three-digit numbers abc have the property that $b > c$? Justify your answer.

Note. A three-digit numbers cannot start with a "0", for instance 011 is not a three-digit number.

3 More involved exercises (optional)

The solution of these exercises will be available on the course webpage at the end of week 8.

Exercise 7. Fix an integer $n \geq 1$ and set $E = \{1, 2, \ldots, n\}$. A function $f : E \to E$ is an *involution* if $f(f(x)) = x$ for every $x \in E$. Let u_n be the number of involutions of E.

1) Compute u_1 and u_2.
2) Show that for every $n \geq 1$, $u_{n+2} = u_{n+1} + (n+1)u_n$.

Exercise 8. (Shephard lemma or black sheep lemma) Let E and F be two finite sets and $f : E \to F$ a function. Assume that there exists an integer $p \geq 1$ such that for every $y \in F$, $\# f^{-1}(\{y\}) = p$. Show that $\# E = p \cdot \# F$.

Exercise 9. (Inclusion-exclusion formula) Fix an integer $n \geq 2$ and let $A_1, \ldots, A_n$ be sets. Show that

$$\#\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{\substack{I \subseteq \{1,2,\ldots,n\} \\ I \neq \emptyset}} (-1)^{-1+|I|} \, \#\left(\bigcap_{i \in I} A_i\right).$$

Exercise 10. Fix an integer $n \geq 1$. A permutation $\{x_1, x_2, \ldots, x_{2n}\}$ of the elements $1, 2, \ldots, 2n$ is a rearrangement of these $2n$ numbers in a different order. It is said to have property T if $|x_i - x_{i+1}| = n$ for at least one i in $\{1, 2, \ldots, 2n-1\}$. Show that there are more permutations with property T than without.

For example, for $n = 2$, the permutations which do not have the property T are

$$\{1234, 1432, 2143, 2341, 3214, 3412, 4123, 4321\}$$

and the permutations which have the property T are

$$\{1234, 1324, 1342, 1423, 2134, 2314, 2413, 2431, 3124, 3142, 3241, 4132, 4213, 4231, 4312\}.$$

Hint. If $(x_1, \ldots, x_{2n})$ is a permutation which does not have the property T, you may consider a function f defined by $f((x_1, \ldots, x_{2n})) = (x_2, x_3, \ldots, x_k, x_1, x_{k+1}, \ldots, x_{2n})$ where k is the unique index such that $|x_1 - x_k| = n$. For example, $f(4321) = 3241$.

4 Fun exercise (optional)

The solution of this exercise will be available on the course webpage at the end of week 8.

Exercise 11. Consider an equilateral triangle with side n, subdivised in small unit triangles as in Fig. 1. A capybara starts from the top triangle and wants to go down. He can only move to adjacent triangles, without going back to a visited triangle and cannot go upwards. He stops when reaching the bottom row. See Figure 1 for an example with $n = 5$. In how many ways can the capybara reach the bottom row when $n = 2017$?

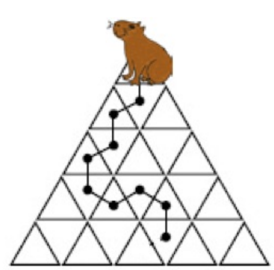

Figure 1: Example of a path reaching the bottom row .