Week 6: Notation $\sum$, $\prod$, induction

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1 Important exercises

The solutions of the exercises which have not been solved in some group will be available on the course webpage.

Exercise 1. Let $(a_i)_{0\leq i\leq n}$ be a sequence of non-zero real numbers with $a_0 = 0$.

a) What is the value of $\sum_{i=1}^{n}(a_i - a_{i-1})$?

b) What is the value of $\prod_{i=1}^{n-1}\frac{a_{i+1}}{a_i}$?

c) What is the value of $\prod_{i=2}^{n}(1-\frac{1}{i})$?

Exercise 2. Show that for every integer $n \geq 1$, $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

Exercise 3. Let $(u_n)_{n\geq 1}$ be the sequence defined by $u_1 = 1$ and for every $n \geq 1$, $u_{n+1} = \frac{u_1+u_2^2+\cdots+u_n^n}{n^n}$. Show that for every $n \geq 1$ we have $0 < u_n \leq 1$.

Exercise 4. What do you think of the following reasoning?

Let us show that all sheep in Scotland have the same color.

Basis step. In a set of only one sheep, there is only one color.

Induction step. Assume that within any set of n sheep, there is only one color. Now look at any set of $n+1$ sheep. Number them: $1, 2, \ldots, n+1$. Consider the sets $\{1, 2, 3, \ldots, n\}$ and $\{2, 3, 4, \ldots, n+1\}$. Each is a set of only n sheep, therefore within each there is only one color. But the two sets overlap, so there must be only one color among all $n+1$ sheep.

Exercise 5. Compute $\displaystyle\sum_{1\leq i,j\leq n} \min(i, j)$.

2 Homework exercise

You have to individually hand in the written solution of the next exercises to your TA on Monday, November 18th.

Exercise 6. Let $(b_n)_{n\geq 0}$ be the sequence such that $b_1 = 1$, $b_2 = 3$ and such that for every $n \geq 1$ we have $b_{n+2} = 3b_{n+1} - 2b_n$.

(1) Compute b_3, b_4, b_5.

(2) Propose a simple expression for b_n and prove it.

Exercise 7. Let $(a_n)_{n\geq 1}$ be the sequence of positive real numbers such that $a_1 = 1$ and such that for every $n \geq 2$,

$$a_n^2 = \sum_{k=1}^{n-1}\frac{a_k}{k}.$$

1) Find the smallest possible value of $c > 0$ such that for every $n \geq 2$, $a_n \leq cn$. Justify your answer.

2) [Optional question] What can you say about a_n as $n \to \infty$?

3 More involved exercises (optional)

The solution of these exercises will be available on the course webpage at the end of week 6.

Exercise 8. In Mathland, which is a land where there is $n \in \mathbb{N}$ cities, two cities are always connected either by plane, either by boat (in both directions). Show that it is possible to choose a means of transportation such that starting from any city it is possible to go to any other city by using only the chosen means of transportation.

Exercise 9. A chocolate bar consists of unit squares arranged in an $m \times n$ rectangular grid. You may split the bar into individual unit squares, by breaking along the lines. What is the number of breaks required?

Exercise 10. Draw $n \geq 1$ circles in the plane so that two circles are never tangent. Show that using two colours only (for example crimson and teal) it is possible to colour the regions of the planes formed by the circles so that two regions separated by an arc always have different colours.

Exercise 11. 73 students travel in a bus with two ticket inspectors. At the beginning, nobody has a ticket, and a passenger only buys a ticket after the third time she is asked to buy a ticket inspector. The ticket inspectors can choose any passenger without a ticket and ask her to buy a ticket. This procedure continues until everyone has a ticket. How many tickets is the first ticket inspector (who is always the one asking first to buy a ticket) sure to sell ?

Exercise 12. Recall that $\mathbb{Z}_+ = \{0, 1, 2 \ldots\}$. Let $A \subset \mathbb{Z}_+$ be such that the following two properties hold:

$$1)\ 0 \in A \qquad\qquad 2)\ \forall n \in \mathbb{Z}_+, n \in A \implies n+1 \in A.$$

Show that $A = \mathbb{Z}_+$.

Hint. You may use the (*least integer principle*) following property of $\mathbb{Z}_+$: if $B \subset \mathbb{Z}_+$ is a nonempty subset of $\mathbb{Z}_+$, then B has a smallest element.

4 Fun exercise (optional)

The solution of these exercises will be available on the course webpage at the end of week 6.

Exercise 13. Suppose you are informed by your teacher that you will have a test next week, and it will take you by surprise. Then the test can never occur.

Indeed, let us induct backwards.

▹ If it doesn't happen by Thursday, then it must happen on Friday. But that will not be a surprise. So it must happen by Thursday.

▹ If it doesn't happen by Wednesday, then it must happen on Thursday. But that will not be a surprise. So it must happen by Wednesday.

▹ If it doesn't happen by Tuesday, then it must happen on Wednesday. But that will not be a surprise. So it must happen by Tuesday.

▹ If it doesn't happen by Monday, then it must happen on Tuesday. But that will not be a surprise. So it must happen by Monday.

▹ If it happens on Monday, you already predicted it and are not surprised. Hence, the test can never occur.

Do you agree?