Week 5: Functions (images and preimages of sets)

Instructor: Igor Kortchemski (igor.kortchemski@polytechnique.edu)
Tutorial Assistants:
- Apolline Louvet (groups A&B, apolline.louvet@polytechnique.edu)
- Milica Tomasevic (groups C&E, milica.tomasevic@polytechnique.edu)
- Benoît Tran (groups D&F, benoit.tran@polytechnique.edu).

1 Important exercises

The solutions of the exercises which have not been solved in some group will be available on the course webpage.

Exercise 1. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = |x-2|$ for $x \in \mathbb{R}$.

1) Find $f\Big((1,6)\Big), f\Big((0,4]\Big), f\Big(\mathbb{Q}\Big)$.

2) Find $f^{-1}\Big(\{5\}\Big), f^{-1}\Big([1,+\infty)\Big)$.

Solution of exercise 1.

1) We have $f\Big((1,6)\Big) = [0,4), f\Big((0,4]\Big) = [0,2], f\Big(\mathbb{Q}\Big) = \mathbb{Q}_{\geq 0}$.

2) We have $f^{-1}\Big(\{5\}\Big) = \{-3,7\}, f^{-1}\Big([1,+\infty)\Big) = (-\infty,1] \cup [3,+\infty)$.

These equalities can be visualized on the graphical representation of f, and are established by double inclusion. □

Exercise 2. If U is a set, recall that card(U) denotes the number of its elements and that $\mathcal{P}(U)$ denotes the set of all subsets of U. Let f be the function defined by

$$f : \begin{array}{ccc} \mathcal{P}(\{1,2,3\}) & \longrightarrow & \{0,1,2,3\} \\ X & \longmapsto & \mathrm{card}(X) \end{array}$$

1) Is f onto? Is f one-to-one?

2) Find $f^{-1}(1)$, $f^{-1}(\{1\})$, $f^{-1}(\{0\})$ and $f^{-1}(\emptyset)$.

Solution of exercise 2. 1) f is onto. Indeed, $f(\varnothing) = 0$, $f(\{1\}) = 1$, $f(\{1,2\}) = 2$ and $f(\{1,2,3\}) = 3$. However, f is not one-to-one. Indeed, $f(\{1\}) = f(\{2\})$.

2) Since f is not bijective, the notation $f^{-1}(1)$ does not make any sense. We have $f^{-1}(\{1\}) = \{\{1\},\{2\},\{3\}\}$, $f^{-1}(\{0\}) = \{\varnothing\}$ and $f^{-1}(\emptyset) = \emptyset$. □

Exercise 3. Let $f : E \to F$ be a function and $B \subseteq F$.

1) Show that $f\left(f^{-1}(B)\right) \subseteq B$.

2) Do we always have $f\left(f^{-1}(B)\right) = B$?

3) Show that if f is onto, then $f\left(f^{-1}(B)\right) = B$.

Solution of exercise 3. 1) Fix $y \in f\left(f^{-1}(B)\right)$. We show that $y \in B$.

Since $y \in f\left(f^{-1}(B)\right)$, there exists $a \in f^{-1}(B)$ such that $f(a) = y$. Since $a \in f^{-1}(B)$, this means that $f(a) \in B$. Therefore $y = f(a) \in B$.

2) No, for exemple if $E = F = \{1,2\}$ and $f(x) = 1$ for $x = 1,2$. If $B = \{2\}$, then $f^{-1}(B) = \varnothing$ and

$f(f^{-1}(B)) = \varnothing$.

3) By 1), it is enough to show that $B \subseteq f\left(f^{-1}(B)\right)$. To this end, fix $y \in B$. We show that $y \in f(f^{-1}(B))$.

Since f is onto, there exists $x \in E$ such that $y = f(x)$. Since $f(x) \in B$, this means that $x \in f^{-1}(B)$. Therefore $y \in f(f^{-1}(B))$. (Indeed, more generally, if $x \in A$, then $f(x) \in f(A)$).

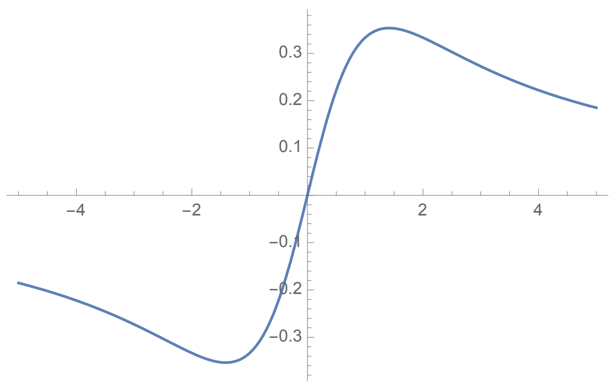
□

2 Homework exercises

You have to individually hand in the written solution of the next exercises to your TA on November, 4th.

Exercise 4. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \dfrac{x}{2+x^2}$. Find $f^{-1}(\mathbb{R})$ and $f(\mathbb{R})$.

Solution of exercise 4. Here is a plot of f:



First of all, by definition, $f^{-1}(\mathbb{R}) = \{x \in \mathbb{R} : f(x) \in \mathbb{R}\} = \mathbb{R}$.

Next, we show that $f(\mathbb{R}) = [-1/\sqrt{8}, 1/\sqrt{8}]$. To this end, let us solve the equation $\frac{x}{2+x^2} = t$ with $t \in \mathbb{R}$ fixed and x being the unknown. This equation is equivalent to $x = 2t + tx^2$, which is equivalent to $tx^2 - x + 2t = 0$.

The discriminant of this equation is $1 - 8t^2$ which is nonnegative if and only if $|t| \leq 1/\sqrt{8}$. This shows that there exists $x \in \mathbb{R}$ such that $f(x) = t$ if and only if $|t| \leq 1/\sqrt{8}$, hence the result. □

Exercise 5. Let $f : E \to F$ be a function. Let $A \subseteq E$.

1) Show that $A \subseteq f^{-1}(f(A))$.

2) Do we always have $f^{-1}(f(A)) \subseteq A$?

3) Show that if f is one-to-one, then $A = f^{-1}(f(A))$.

Solution of exercise 5.

1) Take $x \in A$. Then $f(x) \in f(A)$, which implies that $x \in f^{-1}(f(A))$ by definition of $f^{-1}(f(A))$.

2) No, for example if $E = F = \{1, 2\}$, set $f(1) = 1$ and $f(2) = 1$. Take $A = \{1\}$. Then $f(A) = \{1\}$ and $f^{-1}(f(A)) = \{1, 2\} \not\subseteq A$.

3) Assume that f is one-to-one. We show that $f^{-1}(f(A)) \subseteq A$. To this end, take $x \in f^{-1}(f(A))$. Therefore $f(x) \in f(A)$. Therefore there exists $a \in A$ such that $f(a) = f(x)$. Since f is one-to-one, this

implies that $a = x$. Hence $x \in A$. □

3 More involved exercises (optional)

The solution of these exercises will be available on the course webpage at the end of week 5.

Exercise 6. Let $f : E \to F$ be a function and $B \subseteq F$. Show that $f^{-1}(F \backslash B) = E \backslash f^{-1}(B)$.

Solution of exercise 6. We argue by double inclusion.

First fix $x \in f^{-1}(F \backslash B)$. Then $f(x) \in F \backslash B$, so that $f(x) \notin B$. Argue by contradiction and assume that $x \notin E \backslash f^{-1}(B)$, or, in other words, that $x \in f^{-1}(B)$. This implies that $f(x) \in B$, which is a contradiction.

Now fix $x \in E \backslash f^{-1}(B)$. Then $x \notin f^{-1}(B)$. This means that $f(x) \notin B$ (indeed, if $f(x) \in B$, then $x \in f^{-1}(B)$). Therefore $f(x) \in F \backslash B$. Therefore $x \in f^{-1}(F \backslash B)$. □

Exercise 7. Let X be a set and $f : X \to \mathcal{P}(X)$ a function, where we recall that $\mathcal{P}(X)$ denotes the set of all subsets of X. Show that f is not onto.

Hint. You may consider the set $A = \{x \in X : x \notin f(x)\}$.

Solution of exercise 7. We argue by contradiction and assume that f is onto. Set $A = \{x \in X : x \notin f(x)\}$. Then there exists $a \in X$ such that $f(a) = A$.

Case 1: $a \in A$. Then by definition of A we have $a \notin f(a) = A$, which is a contradiction.

Case 2: $a \notin A$. Then by definition of A we have $a \in f(a) = A$, which is a contradiction. □

Exercise 8. Let $K \geq 1$ be a fixed integer and let f be a one-to-one correspondence from $\{1, 2, \ldots, K\}$ to itself. We set $f^{(0)} = \mathrm{Id}$, where Id is the identity function defined by $\mathrm{Id}(x) = x$ for every $x \in \{1, 2, \ldots, K\}$, and, for every $n \geq 0$, $f^{(n+1)} = f \circ f^{(n)}$.

1) Explain briefly why $f^{(n)}$ is also a one-to-one correspondence.
2) How many one-to-one correspondences from $\{1, 2, \ldots, K\}$ to itself are there?
3) Prove that there exist two integers $i \neq j$ such that $f^{(i)} = f^{(j)}$.
4) Deduce from the above that there exists $n \geq 1$ such that $f^{(n)} = \mathrm{Id}$.

Solution of exercise 8.

1) We already have seen that the composition of two bijections is a bijection. Therefore, $f \circ f, f \circ f \circ f, \ldots$ are bijections.

2) We have K choices for $f(1)$, $K-1$ choices for $f(2)$, and so on. There are $K(K-1)(K-2)\ldots 3 \times 2 \times 1 = K!$ bijections.

3) The infinite sequence $f^{(0)}, f^{(1)}, f^{(2)}, \ldots, f^{(n)}, \ldots$ only takes finitely many distinct values. Therefore there exist two integers $i \neq j$ such that $f^{(i)} = f^{(j)}$.

4) Let $i < j$ be such that $f^{(i)} = f^{(j)}$. We apply i times f^{-1}:

$$\underbrace{f^{-1} \circ f^{-1} \circ f^{-1} \circ \ldots f^{-1}}_{i \text{ times}} \circ f^{(i)} = \underbrace{f^{-1} \circ f^{-1} \circ f^{-1} \circ \ldots f^{-1}}_{i \text{ times}} \circ f^{(j)}$$
$$= \underbrace{f^{-1} \circ f^{-1} \circ f^{-1} \circ \ldots f^{-1}}_{i \text{ times}} \circ \underbrace{f \circ f \circ f \circ \ldots f}_{i \text{ times}} \circ f^{(j-i)}.$$

This gives

$$\mathrm{Id} = f^{(j-i)}.$$

□

Exercise 9. Let $f : X \to Y$, $g : Y \to Z$ be two functions.

1) Show that for every $A \subseteq X$, $g \circ f(A) = g(f(A))$.

2) Show that for every $B \subseteq Z$, $(g \circ f)^{-1}(B) = f^{-1} \circ g^{-1}(B)$.

Solution of exercise 9. 1) We argue by double inclusion. First take $z \in g \circ f(A)$. Then there exists $x \in A$ such that $z = g \circ f(x)$, so that $z = g(f(x))$, with $f(x) \in f(A)$. Therefore $z \in g(f(A))$.

Next take $z \in g(f(A))$. This means that we can write $z = g(y)$ with $y \in f(A)$. Therefore there exists $x \in A$ such that $y = f(x)$. Therefore $z = g(f(x)) = g \circ f(x)$ with $x \in A$. Thus $z \in g \circ f(A)$.

2) We argue by double inclusion. Take $x \in (g \circ f)^{-1}(B)$. This means that $g \circ f(x) \in B$. Therefore $g(f(x)) \in B$. Therefore $f(x) \in g^{-1}(B)$. Therefore $x \in f^{-1}(g^{-1}(B))$.

Next take $x \in f^{-1} \circ g^{-1}(B)$. Therefore $f(x) \in g^{-1}(B)$. Therefore $g(f(x)) \in B$. Thus $g \circ f(x) \in B$, so that $x \in g \circ f^{-1}(B)$. □

Exercise 10. Let E, F be sets and $f : E \to F$ a function. Show that for every $u, v \in F$, if $u \neq v$, then $f^{-1}(\{u\}) \cap f^{-1}(\{v\}) = \emptyset$.

Solution of exercise 10. Fix $u, v \in F$ and assume that $u \neq v$. We argue by contradiction and assume that $f^{-1}(\{u\}) \cap f^{-1}(\{v\}) \neq \emptyset$. We may therefore choose an element $x \in f^{-1}(\{u\}) \cap f^{-1}(\{v\})$. Then $x \in f^{-1}(\{u\})$, so that $f(x) \in \{u\}$, which implies that $f(x) = u$, and $x \in f^{-1}(\{v\})$, so that $f(x) \in \{v\}$, which implies that $f(x) = v$. Therefore $u = v$, which is a contradiction. □

Exercise 11. Let E, F be two sets and $f : E \to F$, $g : F \to E$ be two functions such that $f \circ g(x) = x$ for every $x \in F$. Show that $(g \circ f)(E) = g(F)$.

Solution of exercise 11. Since $f(g(x)) = x$ for every $x \in F$, this implies that f is onto. Therefore $f(E) = F$, so that, using Exercise **??** 1), we have $g \circ f(E) = g(f(E)) = g(F)$. □

Exercise 12. If A, B are two subsets of a set E, recall that $A \Delta B$ is the subset of E defined by $A \Delta B = (A \cap \overline{B}) \cup (\overline{A} \cap B)$, where, to simplify notation, we denote by $\overline{C}$ the complement of C in E. When $A', B' \subseteq F$, we define $A'\Delta B'$ in a similar way (and also denote by $\overline{C'}$ the complement of C' in F when $C' \subseteq F$). Let E, F be two sets and let $f : E \to F$ be a function.

1) Show that for every $A', B' \subseteq F$, we have $f^{-1}(A'\Delta B') = f^{-1}(A')\Delta f^{-1}(B')$.

2) Show that f is one-to-one if and only if for every $A, B \subseteq E$ we have $f(A\Delta B) = f(A)\Delta f(B)$.

Solution of exercise 12.

1) We use the fact that for every $U, V \subseteq F$ we have $f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$ and $f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$ (as seen in the course), and that $f^{-1}(\overline{U}) = \overline{f^{-1}(U)}$ (as seen in Exercise **??**). Specifically,

$$\begin{aligned} f^{-1}(A'\Delta B') &= f^{-1}((A' \cap \overline{B'}) \cup (\overline{A'} \cap B')) \\ &= f^{-1}(A' \cap \overline{B'}) \cup f^{-1}(\overline{A'} \cap B') \\ &= \left(f^{-1}(A') \cap f^{-1}(\overline{B'})\right) \cup \left(f^{-1}(\overline{A'}) \cap f^{-1}(B')\right) \\ &= \left(f^{-1}(A') \cap \overline{f^{-1}(B')}\right) \cup \left(\overline{f^{-1}(A')} \cap f^{-1}(B')\right) \\ &= f^{-1}(A')\Delta f^{-1}(B'). \end{aligned}$$

2) First assume that f is one-to-one. We argue by double inclusion.

If $U, V \subseteq E$, we already know from the course that $f(U \cup V) \subseteq f(U) \cup f(V)$, $f(U \cap V) \subseteq f(U) \cap f(V)$. Let us check that $f(\overline{U}) \subseteq \overline{f(U)}$.

If $y \in f(\overline{U})$, this means that $y = f(x)$ with $x \notin U$. Argue by contradiction and assume that $y \in f(U)$. Then $y = f(u)$ with $u \in U$. Then $f(x) = f(u)$. Since f is one-to-one, we have $x = u$ with $x \notin U$ and $u \in U$, which is a contradiction. Therefore $y \notin f(U)$, so that $y \in \overline{f(U)}$.

By using these three ingredients, exactly as in 1), we get that $f(A\Delta B) \subseteq f(A)\Delta f(B)$.

For the other inclusion, let us first show that $f(U) \cap \overline{f(V)} \subseteq f(U \cap \overline{V})$ for $U, V \subseteq E$. To this end, take $y \in f(U) \cap \overline{f(V)}$. Then $y = f(u)$ with $u \in U$. We claim that $u \notin V$ (indeed, if $u \in V$, then $y \in f(V)$, which is a contradiction). Therefore $u \in U \cap \overline{V}$ so that $y \in f(U \cap \overline{V})$.

In particular,

$$f(A) \cap \overline{f(B)} \subseteq f(A \cap \overline{B}) \qquad \text{and} \qquad \overline{f(A)} \cap f(B) \subseteq f(\overline{A} \cap B).$$

Therefore

$$\begin{aligned} f(A)\Delta f(B) &= (f(A) \cap \overline{f(B)}) \cup (\overline{f(A)} \cap f(B)) \\ &\subseteq f(A \cap \overline{B}) \cup f(\overline{A} \cap B) \\ &= f(A\Delta B), \end{aligned}$$

where we have used for the last equality the fact that $f(U \cup V) = f(U) \cup f(V)$ for every $U, V \subseteq E$.

Conversely, assume that for every $A, B \subseteq E$ we have $f(A\Delta B) = f(A)\Delta f(B)$. We argue by contradiction and assume that f is not one-to-one. Then there exist $x, y \in E$ with $x \neq y$ such that $f(x) = f(y)$. We take $A = \{x\}$ and $B = \{y\}$. Then

$$f(A\Delta B) = f(\{x,y\}) = \{f(x)\}, \qquad f(A)\Delta f(B) = \{f(x)\}\Delta\{f(x)\} = \varnothing,$$

which is a contradiction. □

Exercise 13. Let E be a set. Let $A, B \subseteq E$ be two subsets of E. Let f be the function defined by

$$\begin{array}{cccc} f & : & \mathcal{P}(E) & \longrightarrow & \mathcal{P}(A)\times\mathcal{P}(B) \\ & & X & \longmapsto & (X\cap A, X\cap B). \end{array}$$

1) Find a necessary and sufficient condition on A and B for f to be one-to-one (recall that an assertion Q is a necessary and sufficient condition for P when $P \Leftrightarrow Q$ is true).

2) Find a necessary and sufficient condition on A and B for f to be onto.

3) Find a necessary and sufficient condition on A and B for f to be a bijection.

Solution of exercise 13.

1) We show that f is one-to-one if and only if $A\cup B = E$. We establish the double implication.

First assume that f is one-to-one. Then $f(A\cup B) = ((A\cup B)\cap A, (A\cup B)\cap B) = (A,B)$ and $f(E) = (E,\cap A, E\cap B) = (A,B)$, it follows that $A\cup B = E$.

Next, assume that $A\cup B = E$ and let $X, X' \subseteq E$ be such that $f(X) = f(X')$. Then

$$X = X\cap E = X\cap(A\cup B) = (X\cap A)\cup(X\cap B) = (X'\cap A)\cup(X'\cap B) = X'\cap(A\cup B) = X'\cap E = X',$$

so that f is one-to-one.

2) We show that f is onto if and only if $A\cap B = \varnothing$. We establish the double implication.

First assume that f is onto. Then there exists $C\subseteq E$ such that $f(C) = (\varnothing, B)$. Therefore $C\cap A = \varnothing$ and $C\cap B = B$. The first equality implies that $C\subseteq E\backslash A$ and the second one implies that $B\subseteq C$. Therefore $B\subseteq E\backslash A$, which shows that $A\cap B = \varnothing$.

Finally assume that $A\cap B = \varnothing$. Fix $Y\subseteq A$ and $Z\subseteq B$ and let us show that $Y\cup Z$ is a preimage of (Y,Z) by f. To this end, notice that since $Y\subseteq A$ and $Z\subseteq B$ and since $A\cap B=\varnothing$, we have $Y\cap A = Y$, $Y\cap B = \varnothing$, $Z\cap A = \varnothing$ and $Z\cap B = Z$. Therefore

$$f(Y\cup Z) = \big((Y\cup Z)\cap A, (Y\cup Z)\cap B\big) = \big((Y\cap A)\cup(Z\cap A), (Y\cap B)\cup(Z\cap B)\big) = (Y,Z),$$

which shows that f is onto.

3) By the previous questions, f is a bijection if and only if $A\cup B = E$ and $A\cap B = \varnothing$, or, in other words, if and only if B is the complement of A in E. □

4 Fun exercise (optional)

The solution of this exercise will be available on the course webpage at the end of week 5.

Exercise 14. Chicken McNuggets are sold by boxes of 4, 6, 9 or 20 pieces. We say that $n \geq 1$ is a *McNugget* number if one can make an order of exactly n McNuggets.

Find all the positive integers which are not McNugget numbers.

Solution of exercise 14. For small n, we check by hand that $1,2,3,5,7,11$ are not McNugget numbers, while $4,6,8,9,10,12$ are McNugget numbers.

Let us prove that any $n \geq 12$ is a McNugget number. Any such n can be written as

$$n = 4k + 12 + r,$$

where $k \geq 0$ and $r \in \{0,1,2,3\}$. Indeed, k is the quotient and r is the remainder of the division $(n-12)|4$.

Since $4k$ is a possible order of McNuggets, it suffices to prove that $12+r$ is a McNugget number for every $r \in \{0,1,2,3\}$:

$$\begin{aligned} 12 &= 6+6 \\ 12+1 &= 9+4 \\ 12+2 &= 6+4+4 \\ 12+3 &= 9+6. \end{aligned}$$

This completes the proof. □