Week 11: Modelling with graphs: the Tower of Hanoi

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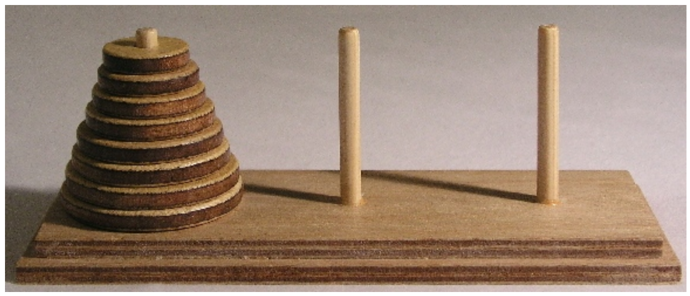


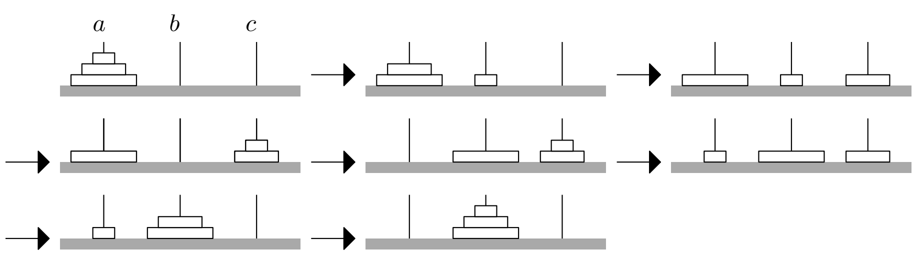
the Tower of Hanoi with 8 disks. (Credits: Wikipedia)

This exercise session is devoted to the study of the Tower of Hanoi, which is a puzzle invented by Édouard Lucas in 1883.

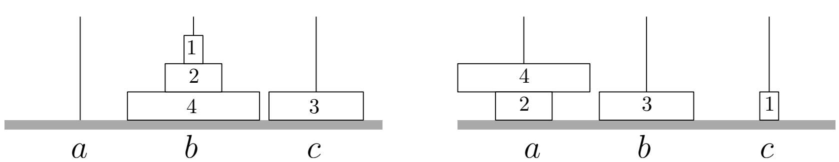
1 The puzzle

We are given a stack of n disks arranged from largest on the bottom to smallest on top placed on a rod a, together with two empty rods b, c. The Tower of Hanoi puzzle asks for the minimum number of moves required to move the entire stack, one disk at a time, from rod a to another (b or c). A move is allowed only if it moves a smaller disk on top of a larger one.

Here is an example which shows that the Tower of Hanoi with $n = 3$ disks is solvable in 7 moves:



Exercise 1. **The graph point of view: $n = 1, 2, 3, 4$.** We say that a configuration of n disks on the three rods a, b, c is *admissible* if on every rod, disks are arranged from largest to smallest.



Left: An admissible configuration of 4 disks. Right: A configuration which is not admissible.

Let $\mathcal{H}_n$ be the set of admissible configurations of n disks. Let F be the function defined by

$$\begin{array}{rccc} F: & \mathcal{H}_n & \rightarrow & \{a,b,c\}^n \\ & H & \mapsto & x_1 x_2 \cdots x_n, \end{array}$$

where $x_i \in \{a,b,c\}$ denotes the rod of the i-th smallest disk in configuration H. For example, if H is the left example on the figure above, then $F(H) = bbcb$.

1. Prove that F is a bijection. Deduce the cardinality of $\mathcal{H}_n$.

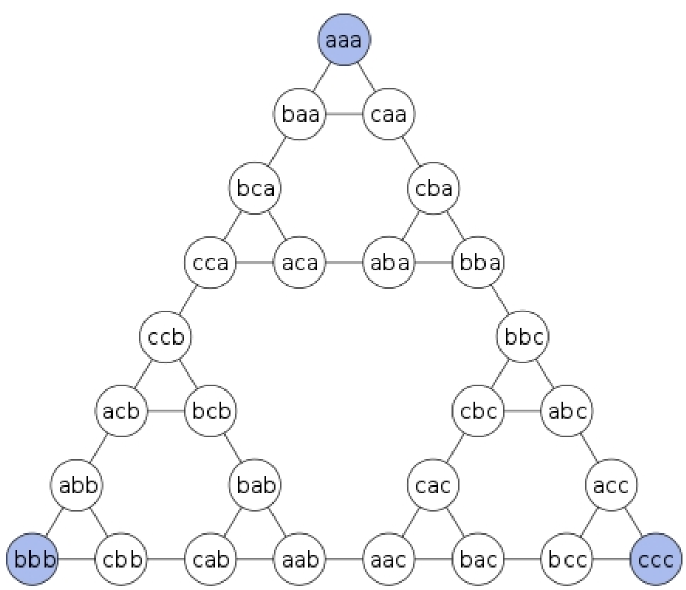
We define a graph $\texttt{Hanoi}(n)$ as follows:

- The vertices of $\texttt{Hanoi}(n)$ are given by all the admissible configurations of $\mathcal{H}_n$.
- We put an edge between H and H' if it is possible to go from configuration H to H' with exactly one (allowed) move.

2. Draw the graphs $\texttt{Hanoi}(1)$ and $\texttt{Hanoi}(2)$.

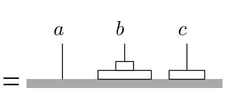
3. Justify briefly that, for every n, the construction of the graph $\texttt{Hanoi}(n)$ is symmetric, it the sense that if it is possible to go from configuration H to H' with exactly one (allowed) move, then it is possible to go from configuration H' to H with exactly one (allowed) move.

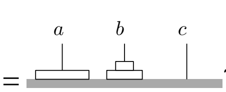
Here is a drawing of $\texttt{Hanoi}(3)$: (credits: `Wikipedia`)



4. For $n = 3$, is it possible to go from every configuration H to every configuration H'? Justify your answer with the graph.

5. For $n = 3$, what is the quickest way to solve the Tower of Hanoi? Justify your answer with the graph.

6. How many moves are needed to go from configuration $H_1 =$ 

to $H_2 =$ 

?

7. For $n = 3$, what are the most distant configurations? (*i.e.* the pairs H, H' for which the minimal number of moves to go from H to H' is the greatest).

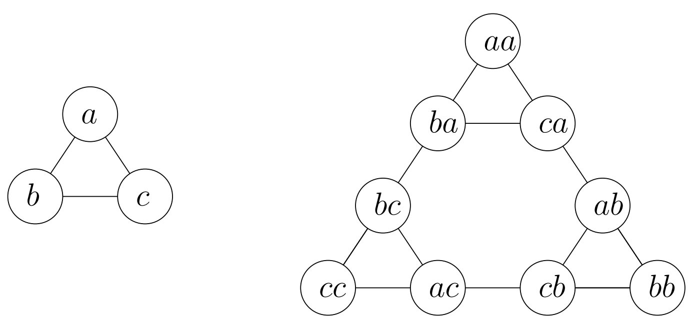
8. We return to the general case of $n \geq 1$ disks. For a vertex $H \in \{a,b,c\}^n$, degree(H) denotes the number of edges starting from H. Prove that degree$(H) \in \{2,3\}$ and that degree$(H) = 2$ if and only if $H = a^n, b^n$ or c^n.

9. Can you explain why Hanoi(3) is composed of three copies of Hanoi(2)? Deduce from this observation a rough sketch of Hanoi(4).

Solution of exercise 1.

1. F **is onto.** For every given element in $\{a,b,c\}^n$, we can order the disks on rod a (resp. b, c) from the largest to the smallest. This gives an admissible configuration.
   F **is one-to-one.** For an element in $\{a,b,c\}^n$ there is a unique way to order the disks on rod a (resp.b, c) from the largest to the smallest.
   Finally,
   $$\mathrm{card}(\mathcal{H}_n) = \mathrm{card}(\{a,b,c\}^n) = 3^n.$$

2.

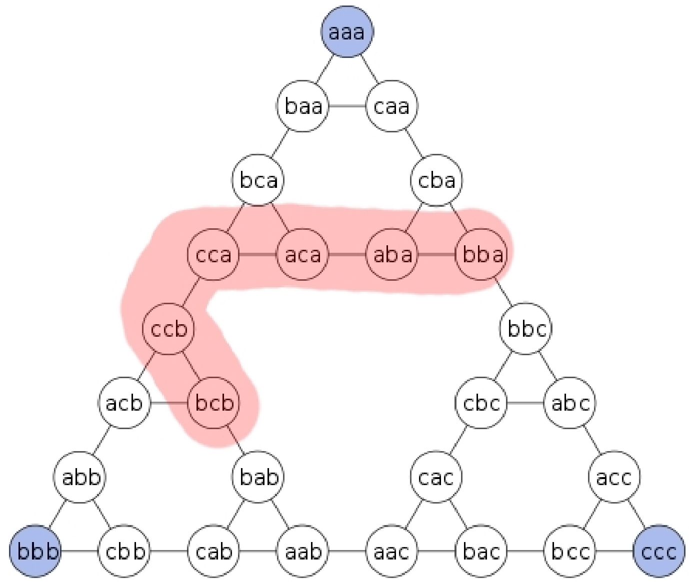


3. If it is allowed to move disk k from rod x to y, then it is allowed to move the disk k back to rod x.

4. We see that the graph is *connected*: every vertex H is connected to every vertex H', with a suitable sequence of edges.

5. The shortest path of allowed moves from aaa to bbb (or aaa to ccc) is 7 edges long. Therefore, the Tower of Hanoi with $n = 3$ disks is solvable in 7 moves but no less.

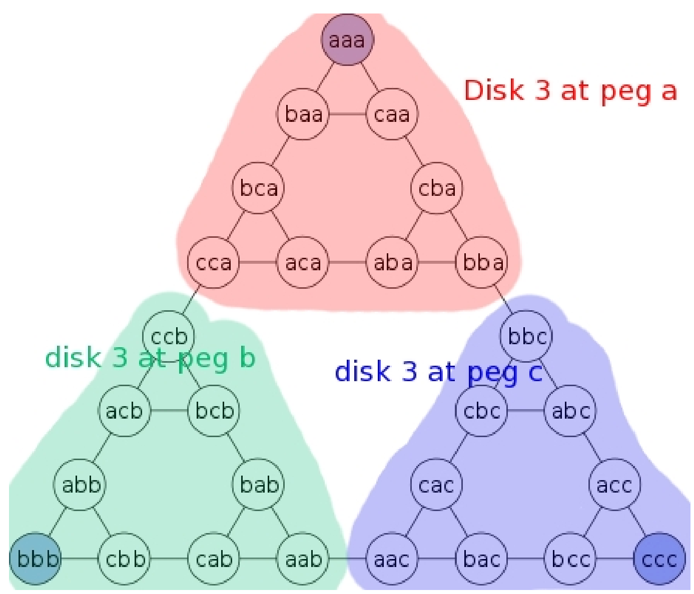
6. On the graph:



7. Many pairs H, H' are at distance 7 in this graph (for instance, *aaa* and *ccc* or *bca* and *ccc*).

8. There are always at least two legal moves: the disk 1 can be moved at either of the two remaining rods. If another disk $k > 1$ is at the top of one rod (*i.e.* if $H \neq a^n, b^n, c^n$) then this second largest available disk can be moved at one rod.

9. We have the following picture:



The graph $\texttt{Hanoi}(3)$ is composed of three components:

- A subgraph given by configurations $\star\star a$ for which the disk 3 is at rod a. If we only move disks $1, 2$ on these configurations, we recover the legal moves in $\texttt{Hanoi}(2)$. This is why this subgraph is identical to $\texttt{Hanoi}(2)$ (we say *isomorphic*).

- A subgraph given by configurations $\star\star b$ for which the disk 3 is at rod b.
- A subgraph given by configurations $\star\star c$ for which the disk 3 is at rod c.

Now, there is only one way to move from $\star\star a$ to $\star\star b$: the disk 3 has to be available, and the rod b has to be empty. This is the configuration cca. Therefore, there is only one edge $(cca \to ccb)$ to go from $\star\star a$ to $\star\star b$.
Similarly, there is only one edge $\star\star a$ to $\star\star c$ and $\star\star b$ to $\star\star c$.

All these arguments remain true for every n and allow us to build Hanoi(4) from Hanoi(3):

- Draw 3 graphs isomorphic to Hanoi(3).
- Draw three edges $ccca \to cccb$, $aaab \to aaac$ and $bbba \to bbbc$.

$\square$

Exercise 2. **Solving the puzzle: the general case.** We will now prove general results regarding the Tower of Hanoi, without using graphs.

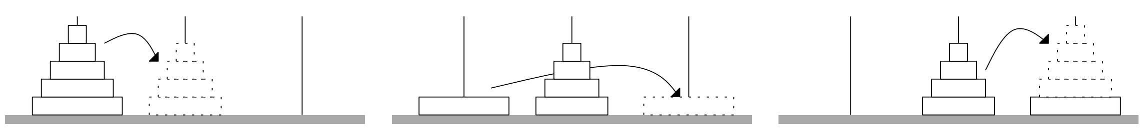
1. Prove by induction that, for every n, the Tower of Hanoi with n disks is solvable.
2. Let m_n be the number of moves needed to solve the Tower of Hanoi with n disks, using this recursive strategy. Prove that $m_1 = 1$ and that for every $n \geq 1$,

$$m_{n+1} = 2m_n + 1.$$

3. Compute m_2, m_3, m_4, m_5, m_6. Guess and prove the general formula for m_n.

Solution of exercise 2.

1. For $n = 1$ the property P_1 = "the Tower of Hanoi with n disks is solvable" is true. To prove that $P_n \Rightarrow P_{n+1}$, first look at the following picture:



The strategy is the following:

(a) Move disks 1 to n from rod a to rod b (this is doable thanks to the induction hypothesis P_n).

(b) Move disk $n+1$ from rod a to rod c.

(c) Move disk 1 to n from rod b to rod c (this is doable thanks to the induction hypothesis P_n).

2. In the previous strategy, each steps uses respectively $m_n, 1, m_n$ moves. Therefore $m_{n+1} = 2m_n + 1$.

3. We find

n	1	2	3	4	5	6
m_n	1	3	7	15	31	63

We guess $m_n = 2^n - 1$, which is correct for $n \leq 6$. We prove our guess by induction:

$$m_{n+1} = 2m_n + 1 = 2(2^n - 1) + 1 = 2^{n+1} - 1.$$

□

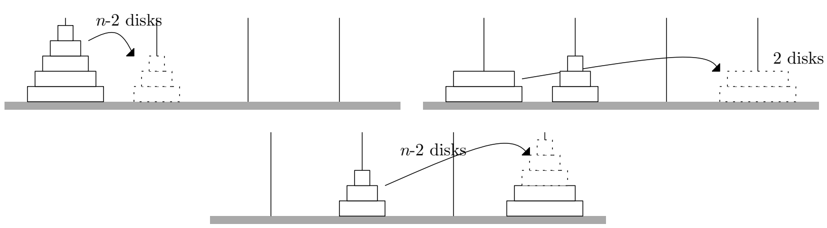
2 Homework exercises

There are no homework exercises this time ☹.

3 Fun exercise (optional)!

The solution of these exercises will be available on the course webpage at the end of week 11.

Exercise 3. We want to evaluate the number of moves of an algorithm to solve the n-disks Tower of Hanoi with 4 rods a, b, c, d. This algorithm is given as follows. If $n = 1$ or $n = 2$, use the algorithm of Exercise 2 to move the disk(s) to rod d. If $n \geq 3$, use the following recursion:



Let f_n be the number of moves required by this algorithm. Prove that $f_1 = 1, f_2 = 3$, and $f_n = 2f_{n-2} + 3$. Prove by induction that

$$f_n = \begin{cases} 2\sqrt{2}^{n+1} - 3 & \text{if } n \text{ is odd,} \\ 3\sqrt{2}^{n} - 3 & \text{if } n \text{ is even.} \end{cases}$$

Solution of exercise 3. **The relation $f_n = 2f_{n-2} + 3$.** In the picture above, the three steps require respectively

- f_{n-2} moves (we use rods a, b, c, d to move the $n-2$ disks),
- 3 moves ($a \to c$, $a \to d$, $c \to d$) to move the two disks from a to d,
- f_{n-2} moves (we use rods a, b, c, d to move the $n-2$ disks).

Therefore this strategy requires $2f_{n-2}+3$ moves.

The formula. We prove the formula by induction. Since f_n depends on f_{n-2} we have to prove the formula for even and odd integers separately.

- **The formula is true for n odd.** For $n=1$ we have

$$2\sqrt{2}^{n+1}-3=2\sqrt{2}^{1+1}-3=2\times 2-3=1=f_1.$$

Now we prove that for odd n, {formula true for n} $\Rightarrow$ {formula true for $n+2$}.

$$\begin{aligned} f_{n+2} &= 2f_n+3 \\ &= 2\times\left(2\sqrt{2}^{n+1}-3\right)+3 \\ &= 2\sqrt{2}^{(n+2)+1}-3, \end{aligned}$$

and the formula is true for $n+2$.

- **The formula is true for n even.** For $n=2$ we have

$$3\sqrt{2}^{2}-3=6-3=3=f_2.$$

Now we prove that for even n, {formula true for n} $\Rightarrow$ {formula true for $n+2$}.

$$\begin{aligned} f_{n+2} &= 2f_n+3 \\ &= 2\times\left(3\sqrt{2}^{n}-3\right)+3 \\ &= 3\sqrt{2}^{n+2}-3 \end{aligned}$$

and the formula is true for $n+2$.

$\square$