Week 8 (Midterm exam): Tuesday, November 12th, 8am-10am

Very important:

- Please use different sheets of paper for different parts (or, in other words, use a new sheet of paper if you change parts).
- Please write your name on the sheets of paper.

All the exercises are independent. You may treat them in any order you want. The quality, the precision and the presentation of your mathematical writing will play a role in the appreciation of your work.

Advice. Use draft paper before writing your answers in the final form. Reread your work. Do not forget that what is graded is what is written, not what is in your head.

Part 1

Exercise 1.

1) Give an example of a function which is not onto (and explain why it is not onto) and an example of a function which is not one-to-one (and explain why it is not one-to-one).

2) Show that if $f : X \to Y$ and $g : Y \to Z$ are bijections, then $g \circ f$ is a bijection and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. (You may use without proof the fact that $g^{-1}(g(b)) = b$ for every $b \in Y$ and that $f^{-1}(f(a)) = a$ for every $a \in X$.)

Solution of exercise 1.

1) Set $E = \{1, 2\}$ and let $f : E \to E$ be defined by $f(1) = 1$ and $f(2) = 1$. The function f is not onto, because there is no $x \in E$ such that $f(x) = 2$. The function f is not one-to-one because $f(1) = f(2)$ but $1 \neq 2$.

2)

Step 1. We show that $g \circ f$ is one-to-one. Fix $x, y \in X$ such that $g \circ f(x) = g \circ f(y)$. We show that $x = y$. Since g is one-to-one, this implies that $f(x) = f(y)$. Since f is one-to-one, this implies that $x = y$. Hence $g \circ f$ is one-to-one.

Step 2. We show that $g \circ f$ is onto. Fix $z \in Z$. We show that z has a preimage by $g \circ f$. Since g is onto, there exists $y \in Y$ such that $g(y) = z$. Since f is onto, there exists $x \in X$ such that $f(x) = y$. Then $g \circ f(x) = g(y) = z$. Hence $g \circ f$ is onto.

Step 3. We show that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. Fix $z \in Z$ and let $x = (g \circ f)^{-1}(z)$. Then $g(f(x)) = z$. Hence $f(x) = g^{-1}(z)$. Hence $x = f^{-1}(g^{-1}(z))$. This shows that $(g \circ f)^{-1}(z) = f^{-1} \circ g^{-1}(z)$.

□

Part 2

Exercise 2. Let P, Q, R be three mathematical assertions.

1) Are the two assertions $P \implies (Q \wedge R)$ and $(P \implies Q) \wedge (P \implies R)$ logically equivalent? Justify your answer.

2) Are the two assertions $(Q \wedge R) \implies P$ and $(Q \implies P) \wedge (R \implies P)$ logically equivalent? Justify your answer.

Solution of exercise 2.

1) Yes, this can be shown by using a truth table. Alternatively, one may note that if P is false, then both assertions are true. And when P is true, the only case when $P \implies (Q \wedge R)$ is true is when Q and R are true, which is also the case for $(P \implies Q) \wedge (P \implies R)$.

2) No. For example, when Q is false, P is false and R is true, $(Q \wedge R) \implies P$ is true but $R \implies P$ is false, so $(Q \implies P) \wedge (R \implies P)$ is false. □

Exercise 3. Let A, B, C be three sets. Show that $C \subseteq A$ if and only if $(A \cap B) \cup C = A \cap (B \cup C)$.

Solution of exercise 3. We argue by double implication.

First assume that $(A \cap B) \cup C = A \cap (B \cup C)$. To show that $C \subseteq A$, fix $x \in C$. Then $x \in (A \cap B) \cup C$. Hence $x \in A \cap (B \cup C)$, which implies $x \in A$.

For the converse, assume that $C \subseteq A$. We show that $(A \cap B) \cup C = A \cap (B \cup C)$ by double inclusion.

– Take $x \in (A \cap B) \cup C$.

First case: $x \in C$. Then $x \in A$ because $C \subseteq A$. Also, clearly $x \in B \cup C$. Hence $x \in A \cap (B \cup C)$.

Second case: $x \in A \cap B$. Then $x \in A$ and $x \in B$, so $x \in B \cup C$. Hence $x \in A \cap (B \cup C)$.

– Take $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$.

First case: $x \in B$. Then $x \in A \cap B$, so $x \in (A \cap B) \cup C$.

Second case: $x \in C$. Then clearly $x \in (A \cap B) \cup C$.

□

Part 3

Exercise 4. Among the following assertions, which ones are true? Justify your answers.

a) $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, xy > 0$

b) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, xy > 0$

c) $\exists x \in \mathbb{R}, \forall y \in R, xy > 0$

d) $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, y^2 > x$

Solution of exercise 4.

a) is false. To show this, observe that its negation, which is $\exists x \in \mathbb{R}, \exists y \in \mathbb{R}, xy \leq 0$ is true because we can take $x = -1$ and $y = 1$.

b) is false. To show this, observe that its negation, which is $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, xy \leq 0$ is true because we can take $x = 0$ and then indeed for every $y \in \mathbb{R}$ we have $xy \leq 0$.

c) is false. To show this, observe that its negation, which is $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, xy \leq 0$ is true. Indeed, fix $x \in \mathbb{R}$. We take $y = -x$, and indeed $xy = -x^2 \leq 0$.

d) is true. Take $x = -1$. Then for every $y \in \mathbb{R}$ we have $y^2 > -1$. □

Exercise 5. Let E and F be two sets and let $f : E \to F$ be a function. Show that

$$f \text{ is one-to-one} \qquad \Longleftrightarrow \qquad \forall A, B \subseteq E, \quad f(A \cap B) = f(A) \cap f(B).$$

You may use without proof that if U, V are subsets of E such that $U \subseteq V$, then $f(U) \subseteq f(V)$.

Solution of exercise 5.

We argue by double implication.

First assume that f is one-to-one. Fix $A, B \subseteq E$. We show that $f(A \cap B) = f(A) \cap f(B)$ by double inclusion.

– Since $A \cap B \subseteq A$, we have $f(A \cap B) \subseteq f(A)$. Similarly, since $A \cap B \subseteq B$, we have $f(A \cap B) \subseteq f(B)$. it follows that $f(A \cap B) \subseteq f(A) \cap f(B)$.

– Take $y \in f(A) \cap f(B)$. We can write $y = f(a)$ and $y = f(b)$ with $a \in A$ and $b \in B$. Thus $f(a) = f(b)$. Since f is one-to-one, this implies that $a = b \in A \cap B$. Hence $y \in f(A \cap B)$.

For the converse, assume that $\forall A, B \subseteq E$ we have $f(A \cap B) = f(A) \cap f(B)$. We show that f is one-to-one. Fix $x, y \in E$ such that $f(x) = f(y)$. Take $A = \{x\}$ and $B = \{y\}$. Then $f(A) \cap f(B) = \{f(x)\}$. This imples that $A \cap B$ is nonempty, so $x = y$. □

Part 4

Exercise 6. Let $f : X \to Y$ and $g : Y \to Z$ be two functions.

1) Show that if $g \circ f$ is one-to-one and f is onto, then g is one-to-one.

2) Show that if $g \circ f$ is onto and g one-to-one, then f is onto.

3) Is it always true that if $g \circ f$ is bijective, then g or f is bijective? Justify your answer.

Solution of exercise 6.

1) Assume that $g \circ f$ is one-to-one and that f is onto. We show that g is one-to-one. Fix $a, b \in Y$ such that $g(a) = g(b)$. Since f is onto, there exists $a', b' \in X$ such that $a = f(a')$ and $b = f(b')$. Hence $g(f(a')) = g(f(b'))$. Since $g \circ f$ is one-to-one, if follows that $a = b$. Therefore g is one-to-one.

2) Assume that $g \circ f$ is onto and that g one-to-one. Fix $y \in Y$. We show that y has a preimage by f. Since $g \circ f$ is onto, there exists $x \in X$ such that $g \circ f(x) = g(y)$. Since g is one-to-one, this implies that $f(x) = y$.

3) No. Take $X = \{1, 2\}$, $Y = \{1, 2, 3\}$ and $Z = \{1, 2\}$, set $f(1) = 1$ and $f(2) = 2$, $g(1) = 1$, $g(2) = 2$, $g(3) = 2$. Then $g \circ f(x) = x$ for $x = 1, 2$ so $g \circ f$ is bijective. But f is not onto and g is not one-to-one, so they are not bijective.

□

Part 5 (optional)

This part is optional and does not count in the grading. Please go beyond only if you have solved all the previous exercises.

Exercise 7. Let E be a set.

1) Show that E has infinitely many elements if and only if for every function $f : E \to E$ there exists a set $A \subseteq E$ with $A \neq \varnothing$, $A \neq E$ such that $f(A) \subseteq A$.

2) Is it true that E has infinitely many elements if and only if for every function $f : E \to E$ there exists a set $A \subseteq E$ with $A \neq \varnothing$, $A \neq E$ such that $f(A) = A$?

Solution of exercise 7.

1) For convenience, we say that a set B is stable by f if $f(B) \subseteq B$.

We argue by double implication.

∗ First assume that E has infinitely many elements. We shall find a set $A \subseteq E$ with $A \neq \varnothing$, $A \neq E$ such that $f(A) \subseteq A$. To this end, the idea is to consider $a \in E$ and define

$$A = \bigcup_{n=1}^{\infty} \{f^{(n)}(a)\}$$

where $f^{(n)}(a) = f \circ f \circ \cdots \circ f(a)$ with f written n times. By construction, A is stable by f and is non empty. We shall check that $A \neq E$.

First case: $a \notin A$. Then $A \neq E$, so A satisfies the desired conditions.

Second case: $a \in A$. Then there exists an integer $n_0 \geq 1$ such that $a = f^{(n_0)}(a)$. This implies that

$$A \subseteq \{a, f(a), \ldots, f^{(n_0-1)}(a)\}.$$

The latter is a finite set, so A satisfies the desired conditions.

∗ To show the converse, we argue by contraposition. Assume that E is a finite set. We show that there exists a function $f : E \to E$ such that for every set $A \subseteq E$, if $A \neq \varnothing$ and $f(A) \subseteq A$ then $A = E$. To this end, write $E = \{a_1, a_2, \ldots, a_n\}$ and define f as $f(a_i) = a_{i+1}$ for $1 \leq i \leq n-1$ and $f(a_n) = a_1$. Now consider a set $A \subseteq E$, such that $A \neq \varnothing$ and $f(A) \subseteq A$. Let us show that $A = E$. To this end, since A is nonempty, there exists $1 \leq i \leq n$ such that $a_i \in A$. Then $f(a_i) \in f(A) \subseteq A$. But $f(a_i) = a_{i+1}$ if $i \leq n-1$ and $f(a_n) = a_1$. By composing again by f $n-1$ times, we see that $a_j \in A$ for every $1 \leq j \leq n$. This shows that $A = E$ and completes the proof.

2) The converse is true by the first question (indeed $f(A) = A$ implies $f(A) \subseteq A$).

However, the direct implication is not always true. For example, consider $E = \mathbb{Z}$ and let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(n) = n + 1$ for every $n \in \mathbb{Z}$. Assume by contradiction there there exists a set $A \subseteq \mathbb{Z}$ with $A \neq \varnothing$ such that $f(A) = A$. We will show that $A = \mathbb{Z}$. Since A is nonempty, we may fix $a \in \mathbb{Z}$. Then $a+1 = f(a) \in f(A) = A$, so we also have $a+1 \in A$. By induction, we get that $b \in A$ for every $b \geq a$. But $f(a-1) = a \in A = f(A)$, so $f(a-1) \in f(A)$. Since f is one-to-one, this implies that $a-1 \in A$. By induction, we get $b \in A$ for every $b \leq a$. This entails $A = \mathbb{Z}$.

□

Exercise 8. Let E be a set. Recall that $\mathcal{P}(E)$ denotes the set of all its subsets. Let $f : \mathcal{P}(E) \to \mathcal{P}(E)$ be a function such that for every $A, B \in \mathcal{P}(E)$, $A \subseteq B \implies F(A) \subseteq F(B)$. Show that there exists $M \in \mathcal{P}(E)$ such that $F(M) = M$.

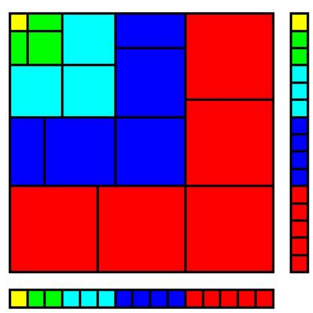
Hint. You may introduce the set $\mathcal{S} = \{A \in \mathcal{P}(E) : F(A) \subseteq A\}$ and define $M = \bigcap_{A \in \mathcal{S}} A$.

Solution of exercise 8. Note that $\mathcal{S}$ is nonempty as $E \in \mathcal{S}$. We show that $F(M) = M$ by double inclusion.

Step 1. We check that $F(M) \subseteq M$. To this end, we fix $A \in \mathcal{S}$. Then $M \subseteq A$, so $F(M) \subseteq F(A) \subseteq A$. Since this holds for every $A \in \mathcal{S}$, this implies that $F(M) \subseteq M$.

Step 2. We check that $M \subseteq F(M)$. By the first step, $F(F(M)) \subseteq F(M)$. Hence $F(M) \in \mathcal{S}$. Since M is the intersection of all the elements of $\mathcal{S}$, this yields that $M \subseteq F(M)$. □

Exercise 9. What does the following image prove?



Solution of exercise 9. This image proves that

$$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$$

for every $n \geq 1$. □