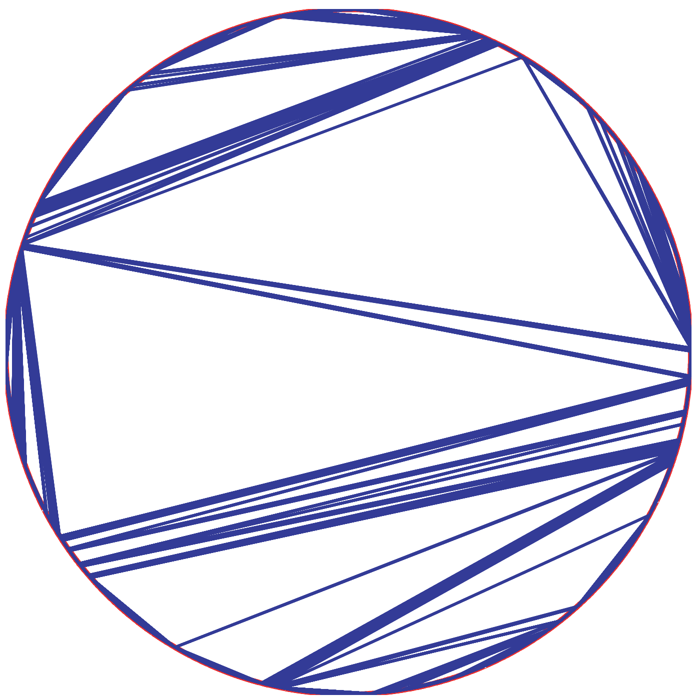
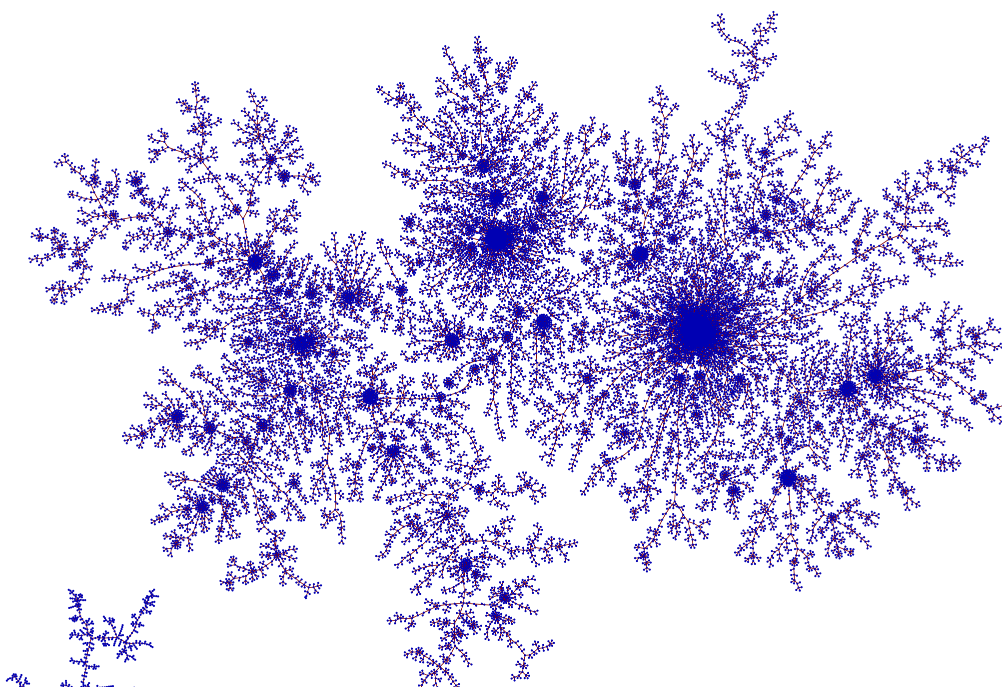
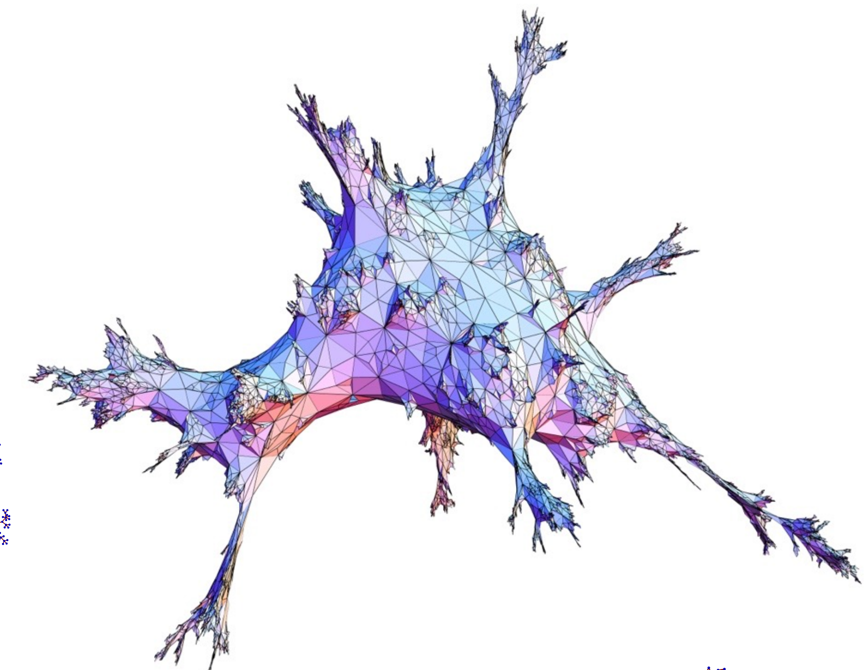
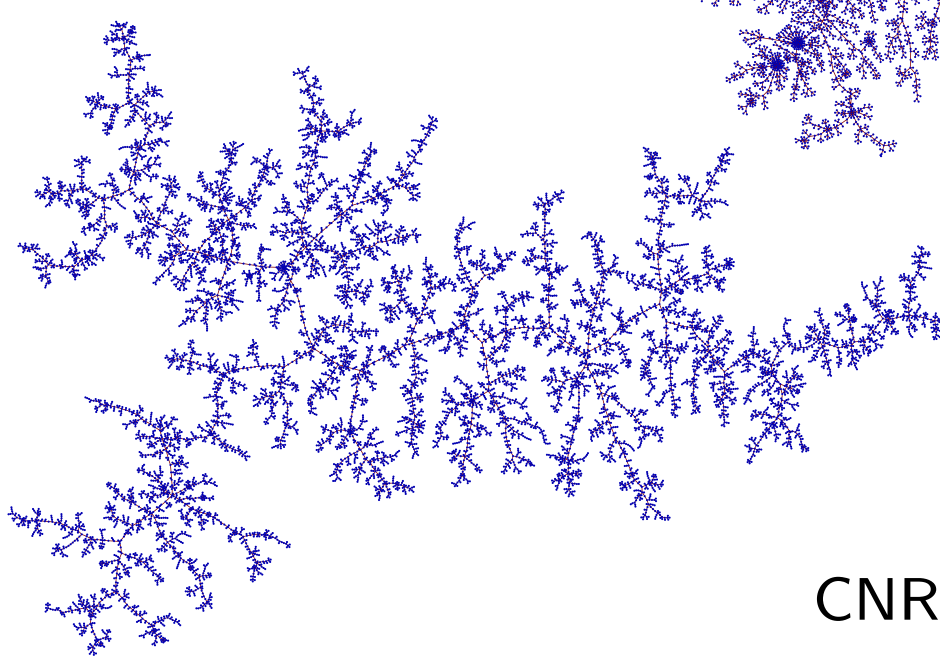
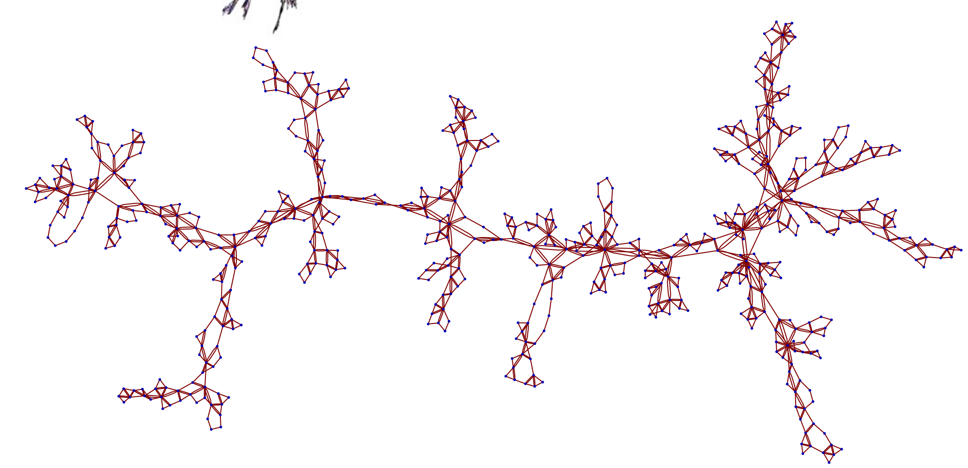
Scaling limits of large random discret structures











Igor Kortchemski

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- Understand the typical properties of $\mathcal{X}_n$. Let X_n be an element of $\mathcal{X}_n$ chosen *uniformly at random.* What can be said of X_n?
- A possibility to study X_n is to find a continuous object X such that $X_n \to X$ as $n \to \infty$.

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- *From the discrete to the continuous world:* if a property $\mathcal{P}$ is satisfied by all the X_n and passes to the limit, then X satisfies $\mathcal{P}$.

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- *Universality:* if $(Y_n)_{n \geqslant 1}$ is another sequence of objects converging towards X, then X_n and Y_n share approximately the same properties for n large.

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- *In what space do the objects live?* Here, a metric space (Z, d) which will be complete and separable (there exists a dense countable subset).
- *What is the sense of the convergence when the objects are random?* Here, convergence in distribution:

$$\mathbb{E}\left[F(X_n)\right] \underset{n \to \infty}{\longrightarrow} \mathbb{E}\left[F(X)\right]$$

for every continous bounded function $F : Z \to \mathbb{R}$.

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I. Random walks and Brownian motion (1951)

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Let $(X_n)_{n \geqslant 1}$ *be i.i.d. (independent identically distributed) random variables with* $\mathbb{E}[X_1] = 0$ *and* $\sigma^2 = \mathbb{E}\left[{X_1}^2\right] \in]0, \infty[$.

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$\rightsquigarrow$ Consequence: for every $a < b$,

$$\mathbb{P}\left(a < \frac{S_n}{\sigma\sqrt{n}} < b\right) \quad \xrightarrow[n\to\infty]{} \quad \int_a^b dx \, \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

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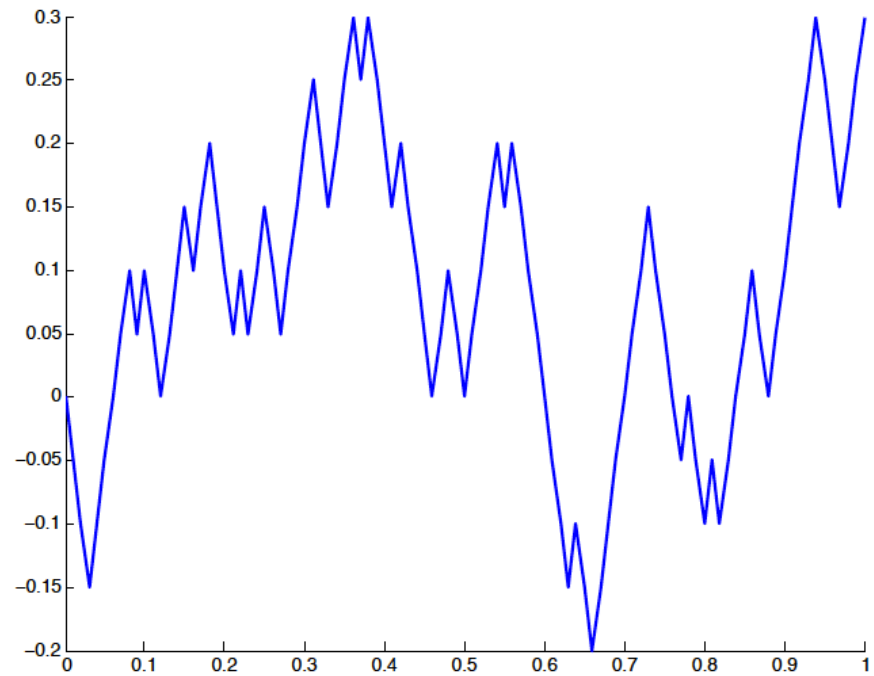
for $n = 100$:



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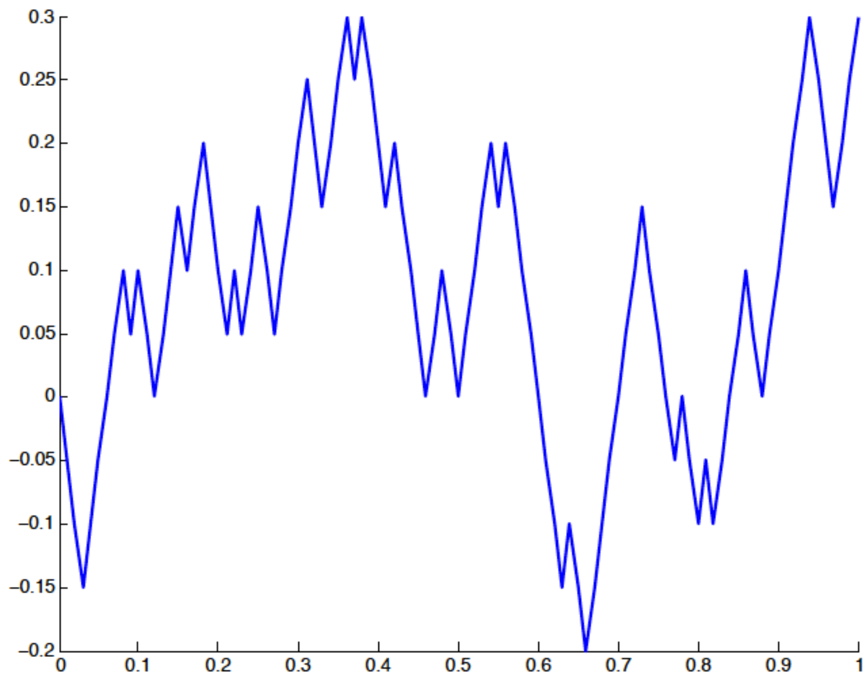

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Here the metric space (Z, d) is $\mathcal{C}([0,1], \mathbb{R})$, the space of $\mathbb{R}$-valued continuous functions on $[0,1]$, equiped with the topology of uniform convergence on $[0,1]$

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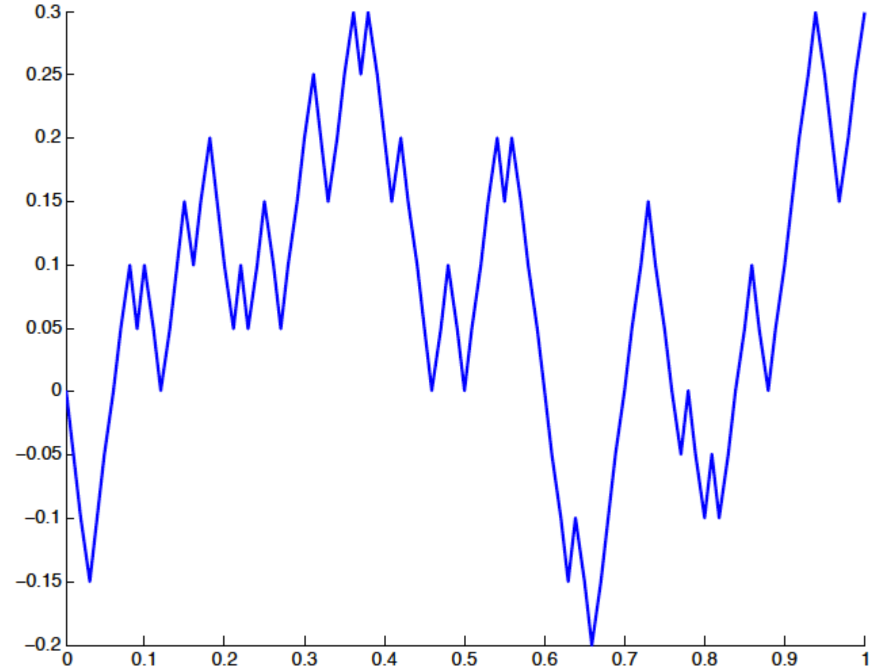

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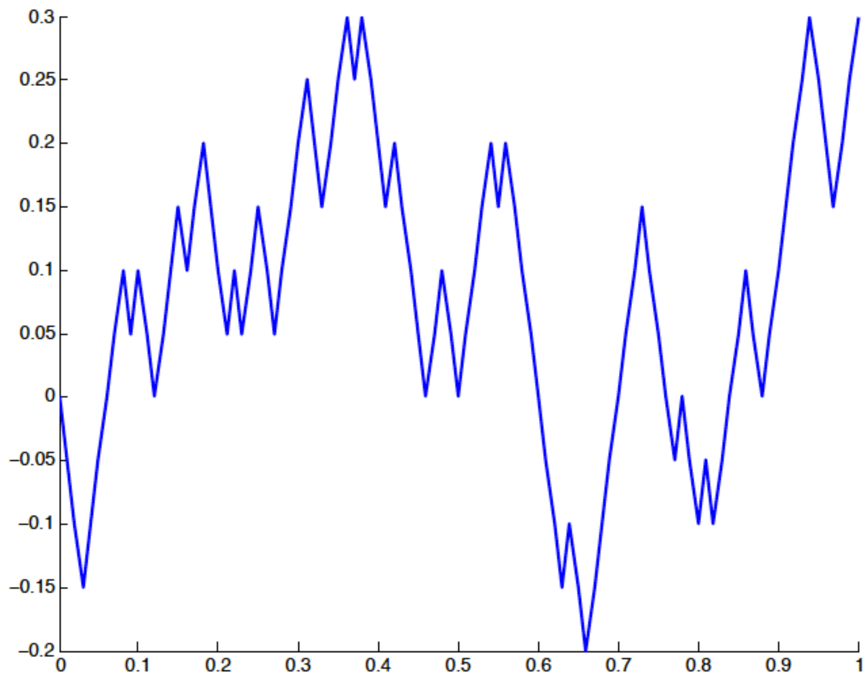

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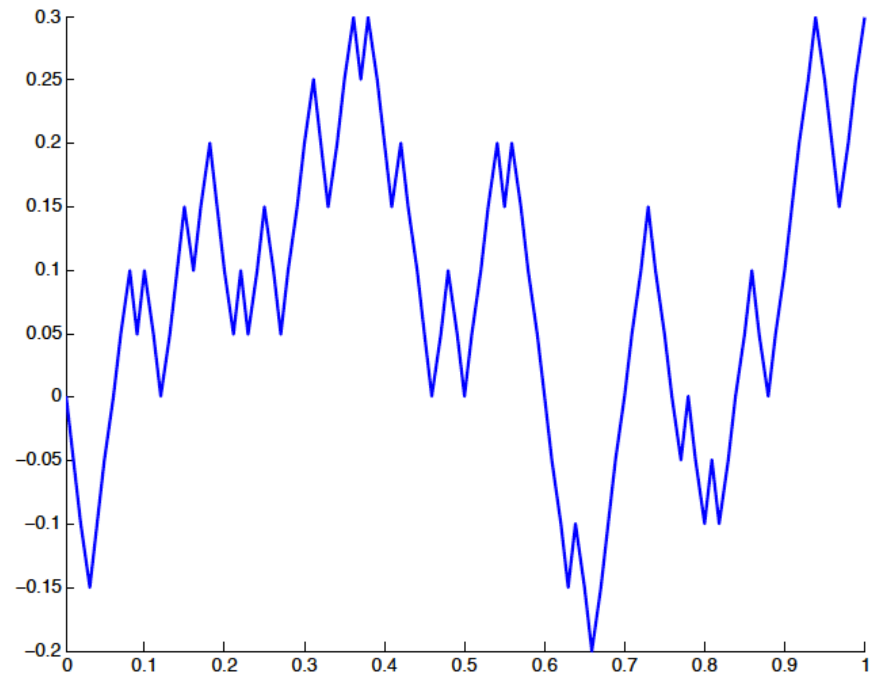

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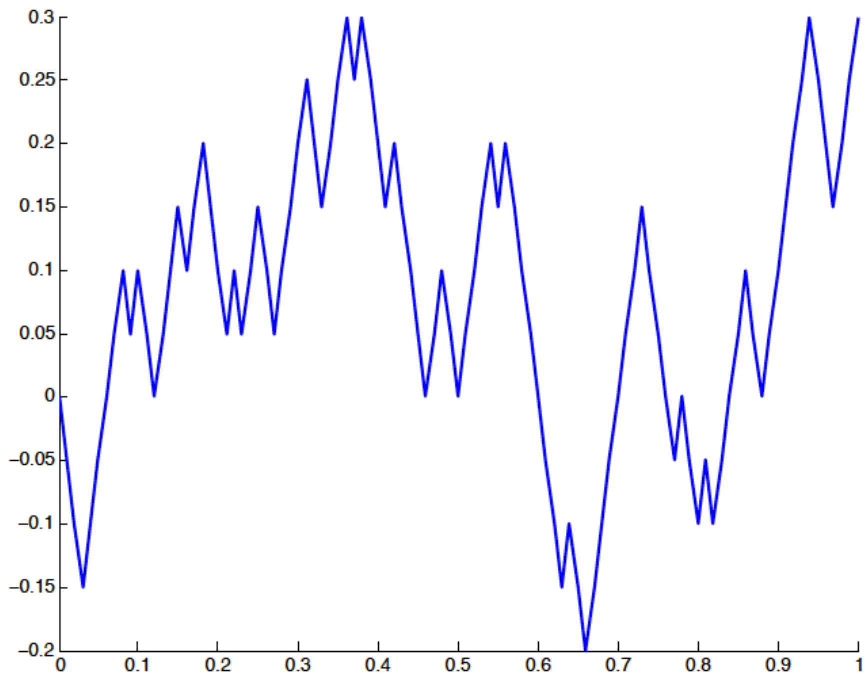

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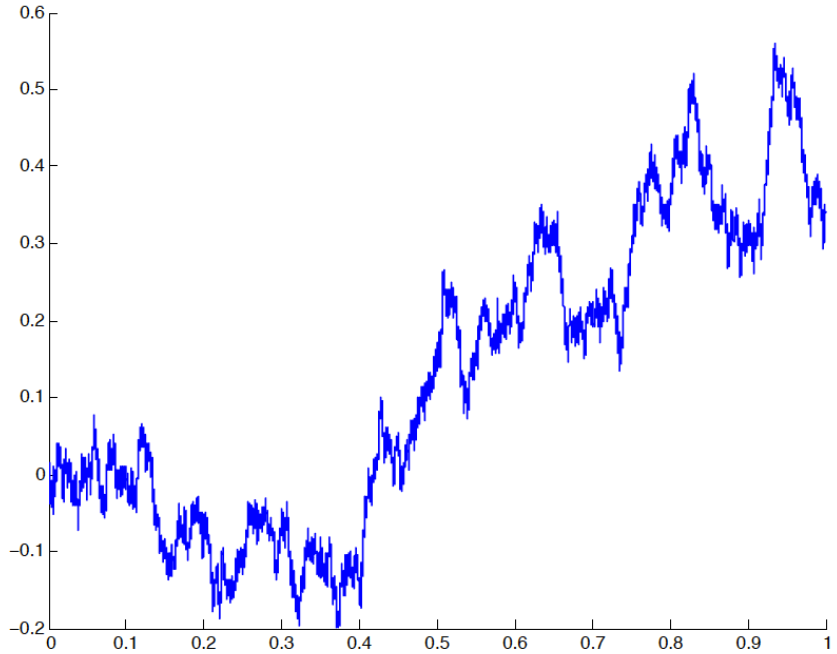

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⇝ Consequence: for every $a>0$,

$$\mathbb{P}\left(\sup_{0\leqslant t\leqslant 1}\frac{S_{nt}}{\sigma\sqrt{n}}>a\right) \quad \xrightarrow[n\to\infty]{} \quad \mathbb{P}\left(\sup_{0\leqslant t\leqslant 1} W_t>a\right)=2\int_a^\infty dx\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}.$$

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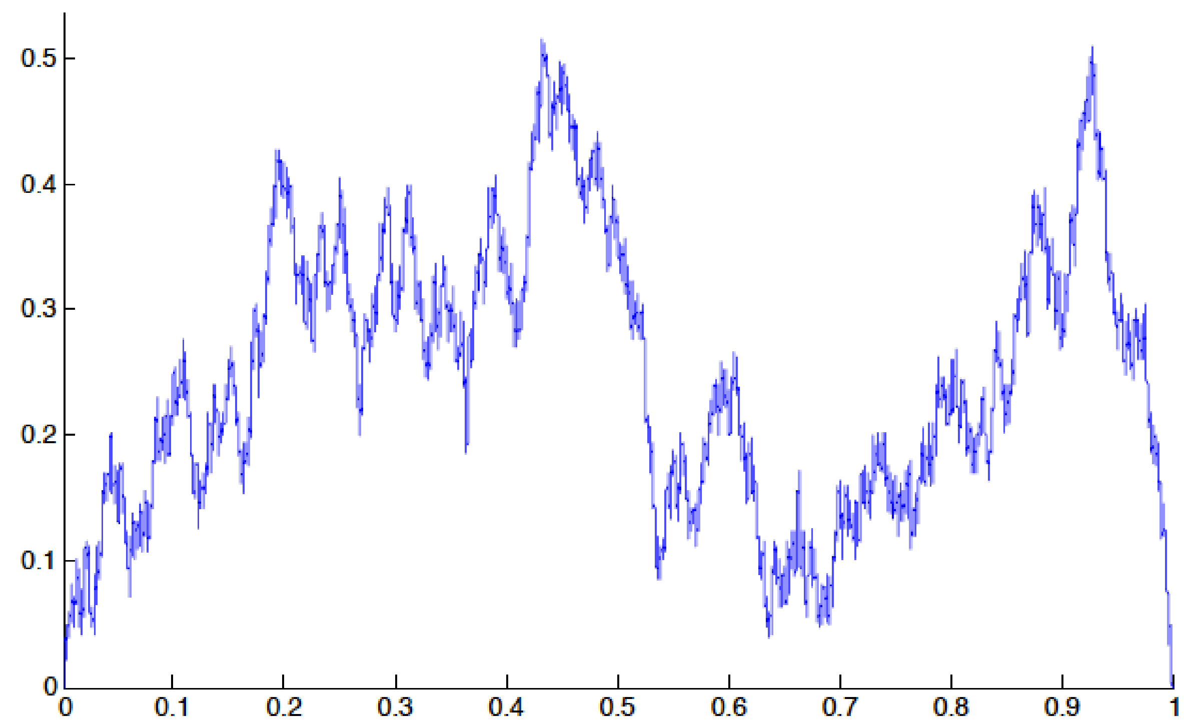

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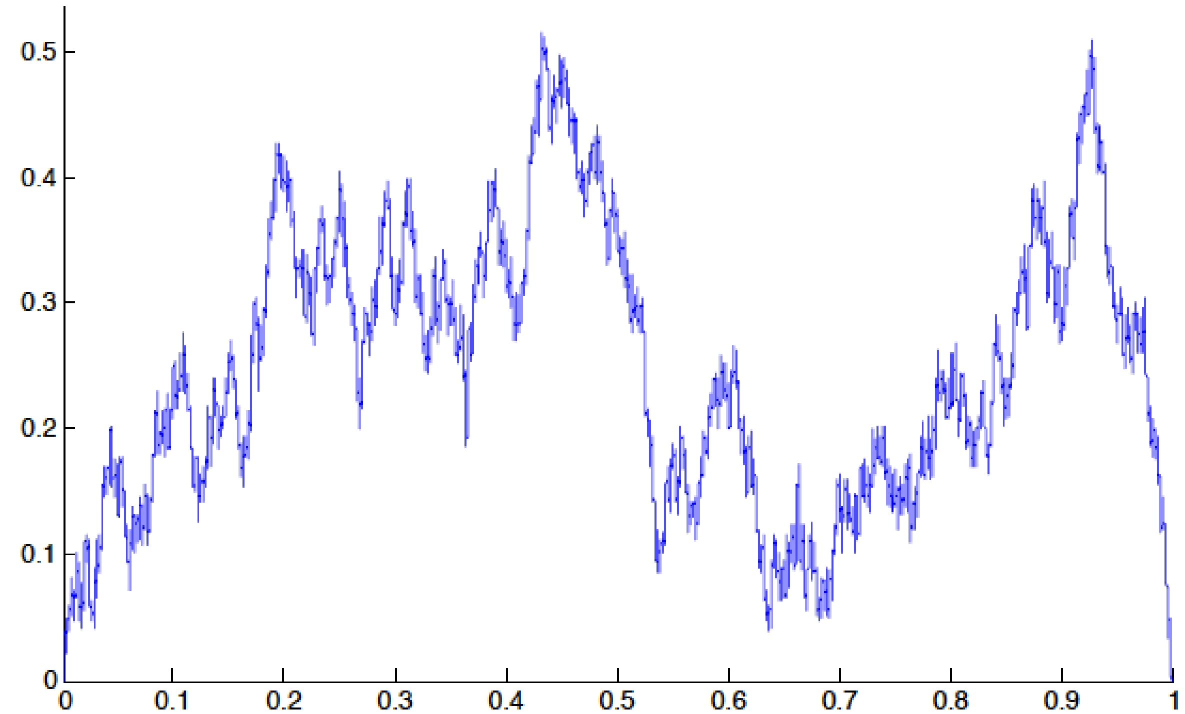
where $(\mathbb{e}_t)_{0\leqslant t\leqslant 1}$ *is a random variable with values in* $\mathcal{C}([0,1],\mathbb{R})$ *called the Brownian excursion.*



The Brownian excursion can be seen as Brownian motion $(W_t, 0 \leqslant t \leqslant 1)$ conditioned by the events $W_1 = 0$ and $W_t > 0$ for $t \in]0,1[$.

Theorem (Conditioned Donsker Theorem, Kaigh '75)

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I. Random walks and Brownian motion (1951)

II. Scaling limits of BGW trees (1991)

III. Plane non-crossing configurations (2012)

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Recall that in a Bienaymé–Galton–Watson tree, every individual has a random number of children (independently of each other) distributed according to μ (offspring distribution).

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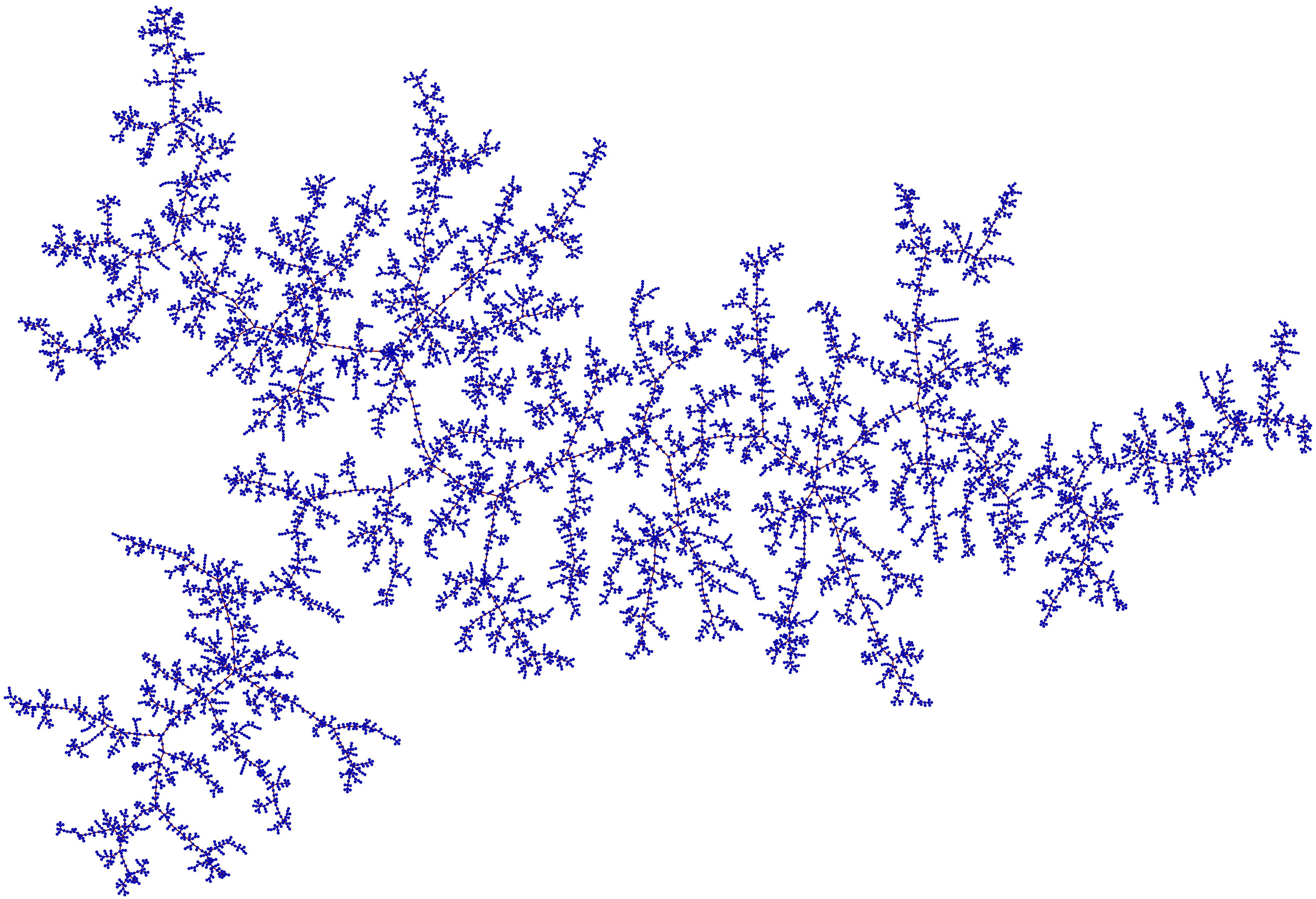
What does a large Bienaymé–Galton–Watson tree look like ?

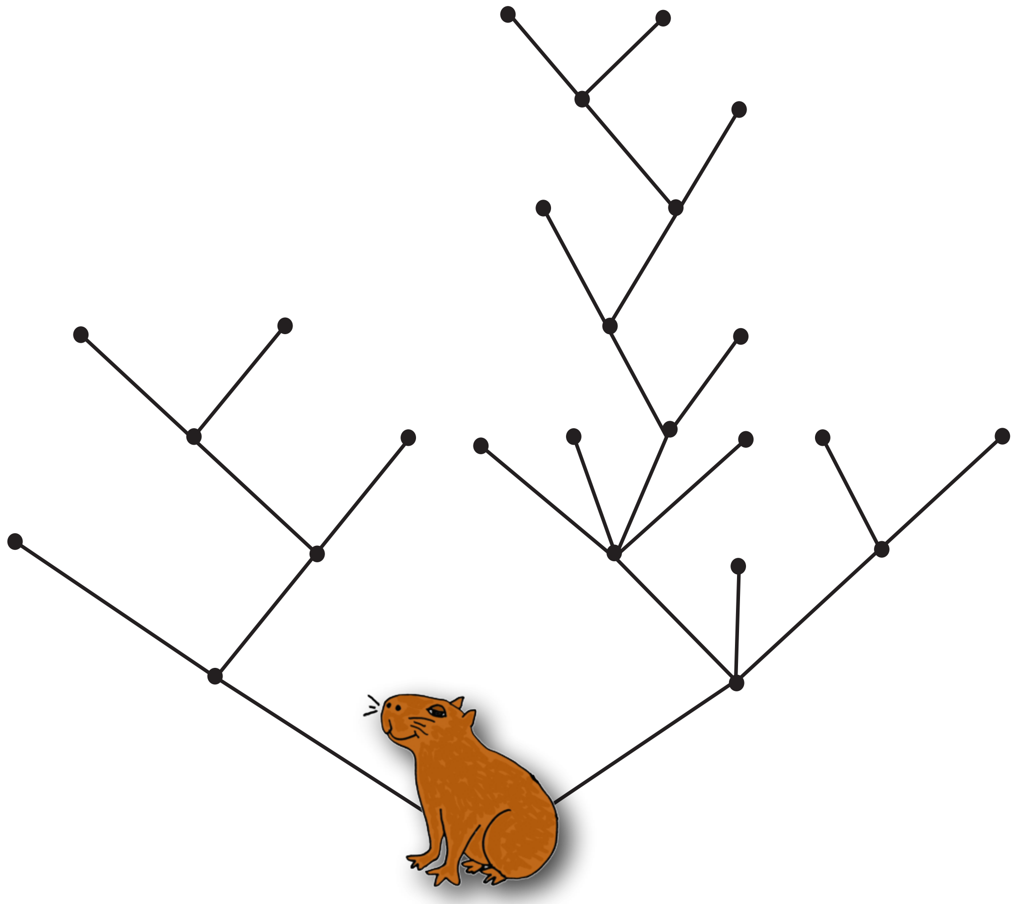
A simulation of a large random critical GW tree

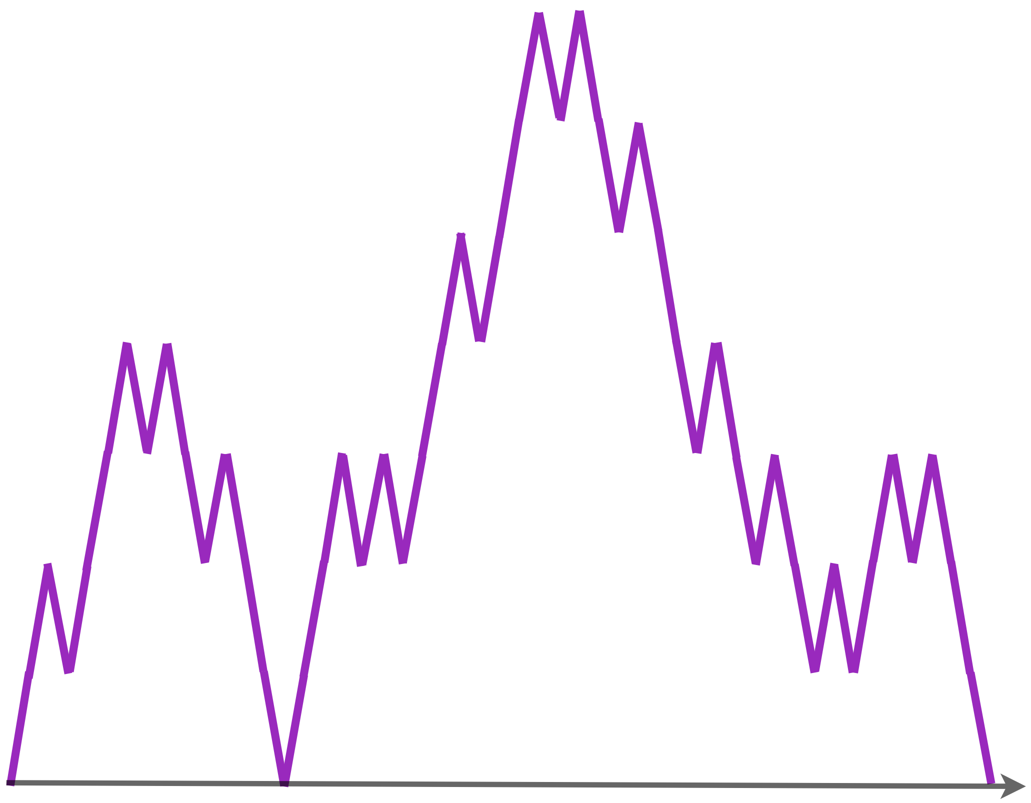


Coding trees by functions

Contour function of a tree

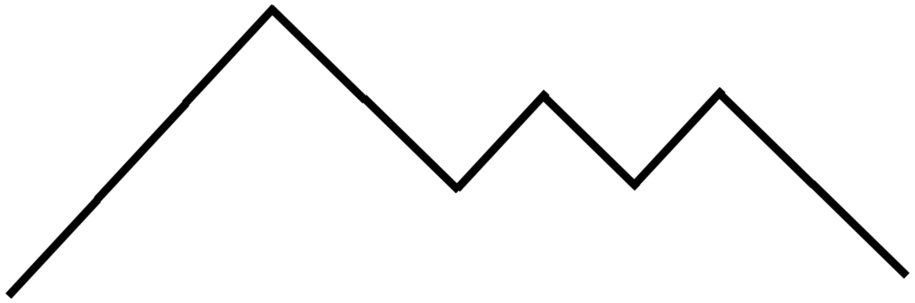
Define the contour function of a tree:





Coding trees by contour functions

Knowing the contour function, it is easy to recover the tree.



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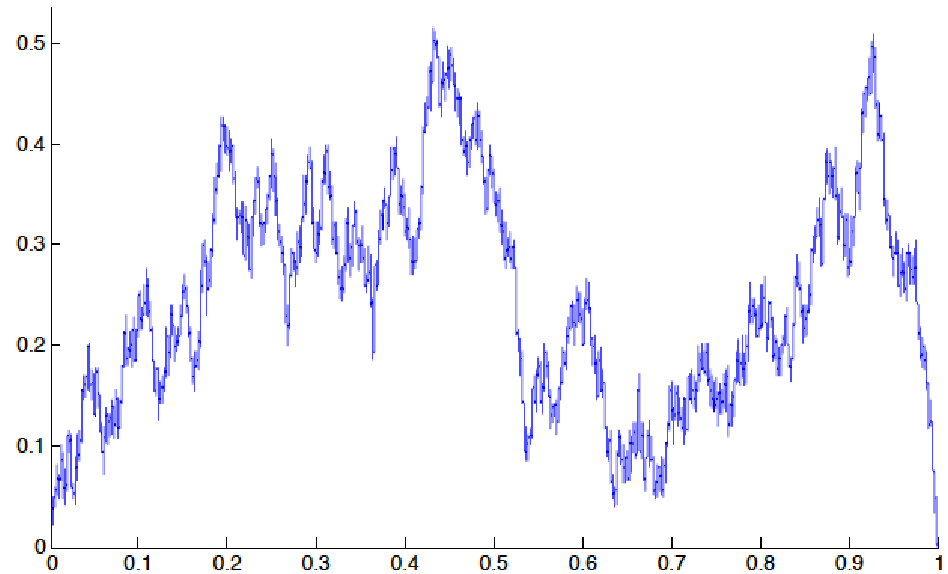

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- Go from the Lukasieiwicz path of $\mathcal{T}_n$ to its contour function.

Do the discrete trees converge to a continuous tree?

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Yes, if we view trees as compact metric spaces by equiping the vertices with the graph distance!

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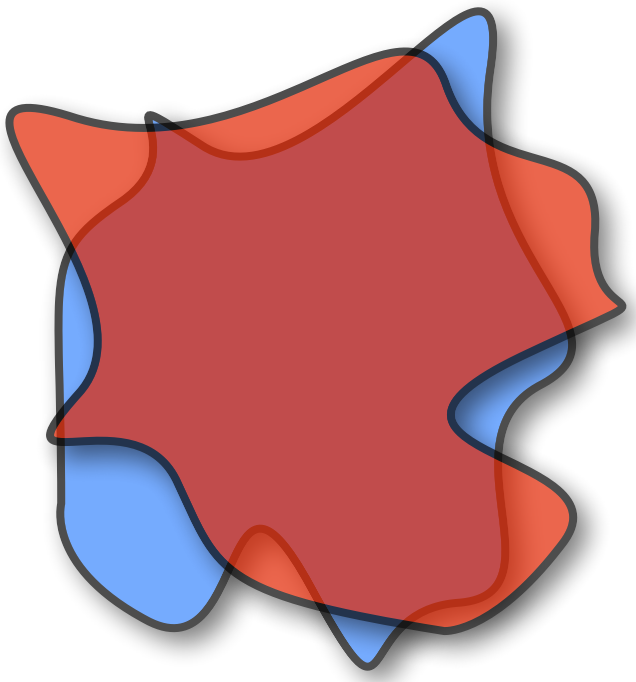

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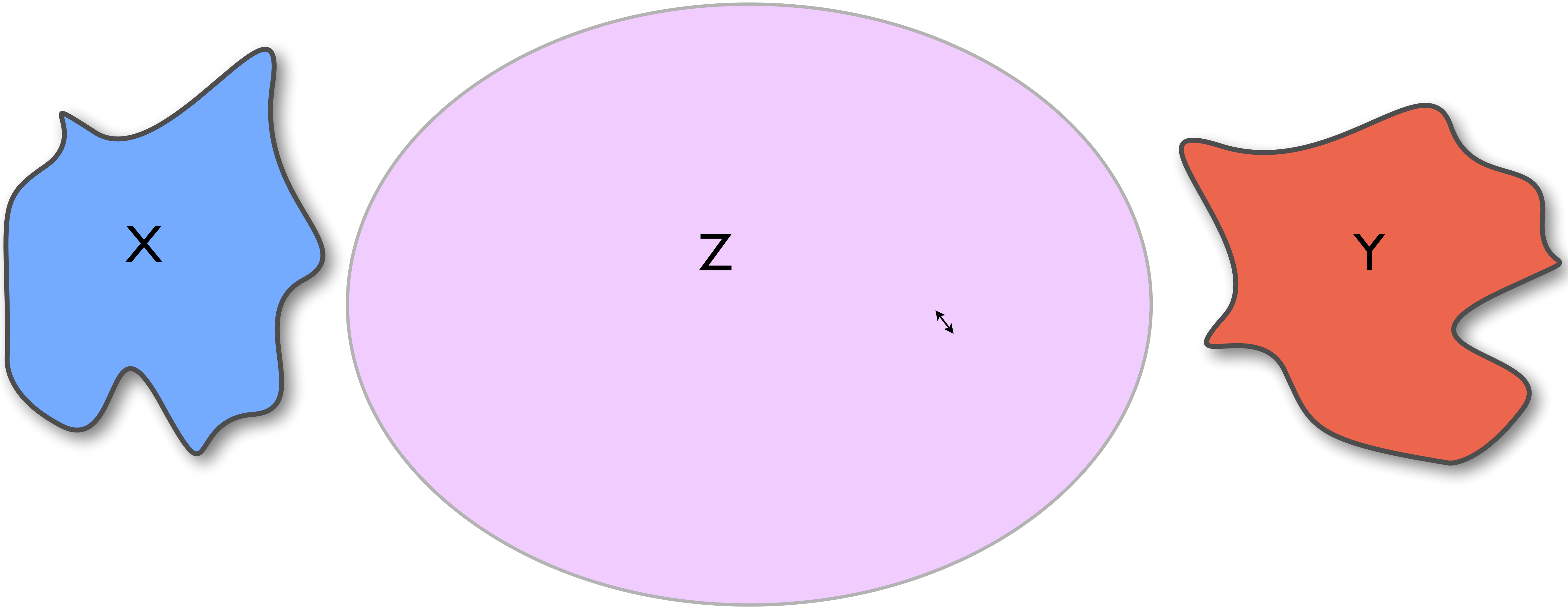

The Gromov–Hausdorff distance

Let X, Y be two compact metric spaces.

The Gromov–Hausdorff distance between X and Y is the smallest Hausdorff distance between all possible isometric embeddings of X and Y in a *same* metric space Z.

The Brownian tree

⇝ Consequence of Aldous' theorem (Duquesne, Le Gall): there exists a compact metric space such that the convergence

$$\frac{\sigma}{2\sqrt{n}} \cdot t_n \quad \xrightarrow[n\to\infty]{(d)} \quad \mathcal{T}_{\mathbb{e}},$$

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I. Random walks and Brownian motion (1951)

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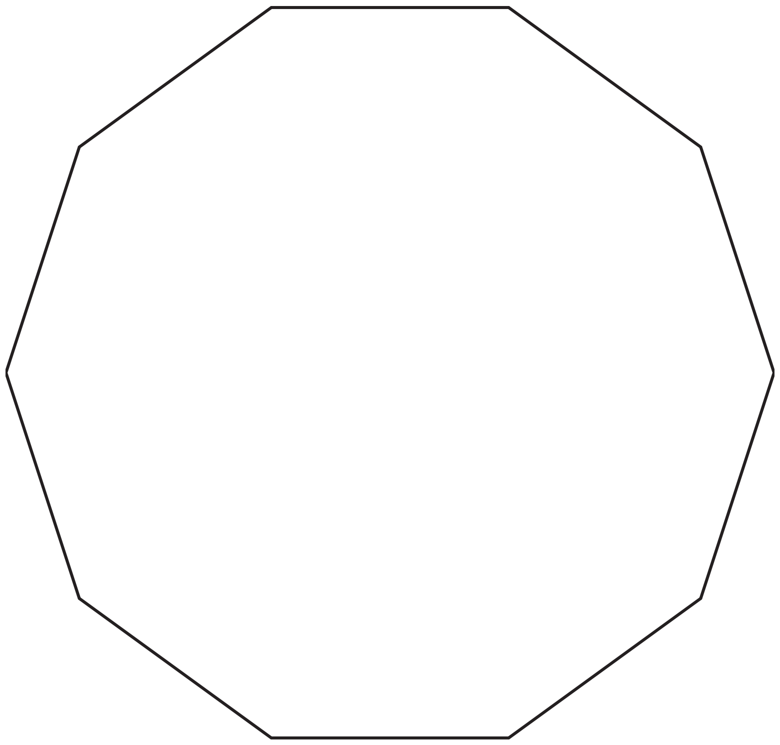
Let P_n be the polygon with vertices $e^{\frac{2i\pi j}{n}}$ $(j = 0, 1, \ldots, n-1)$.



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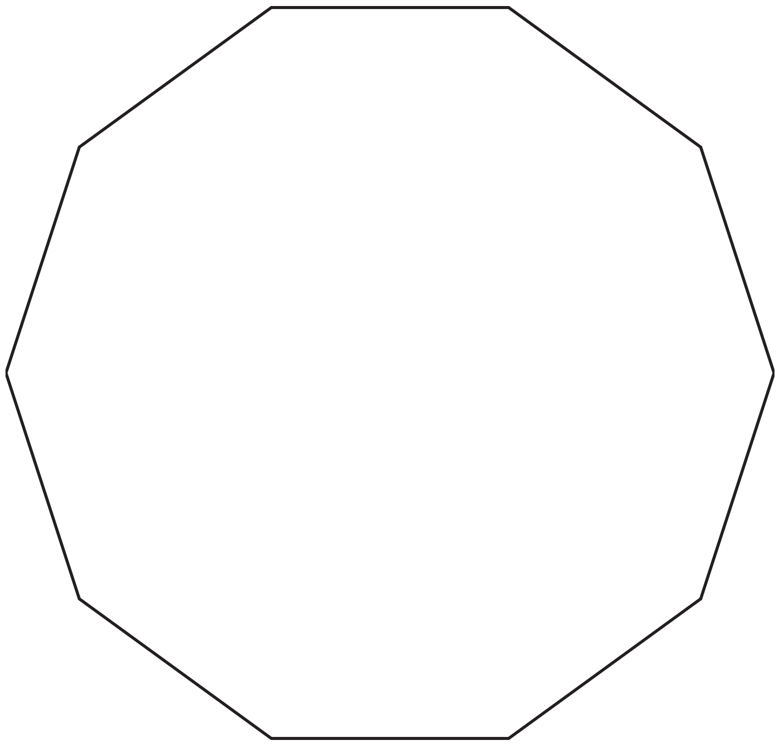

General philosophy: chose at random a non crossing configuration, obtained from the vertices of P_n by drawing diagonals which may not cross.

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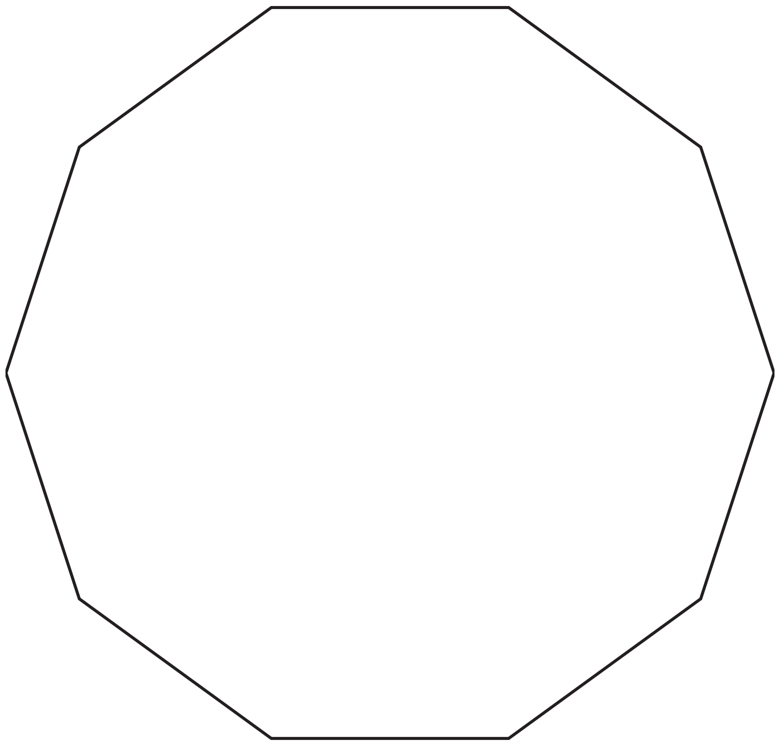

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What happens for n large?

Case of triangulations of P_n

Let $\mathcal{T}_n$ be a random triangulation, chosen uniformly among all triangulations of P_n. What does $\mathcal{T}_n$ look like when n is large?

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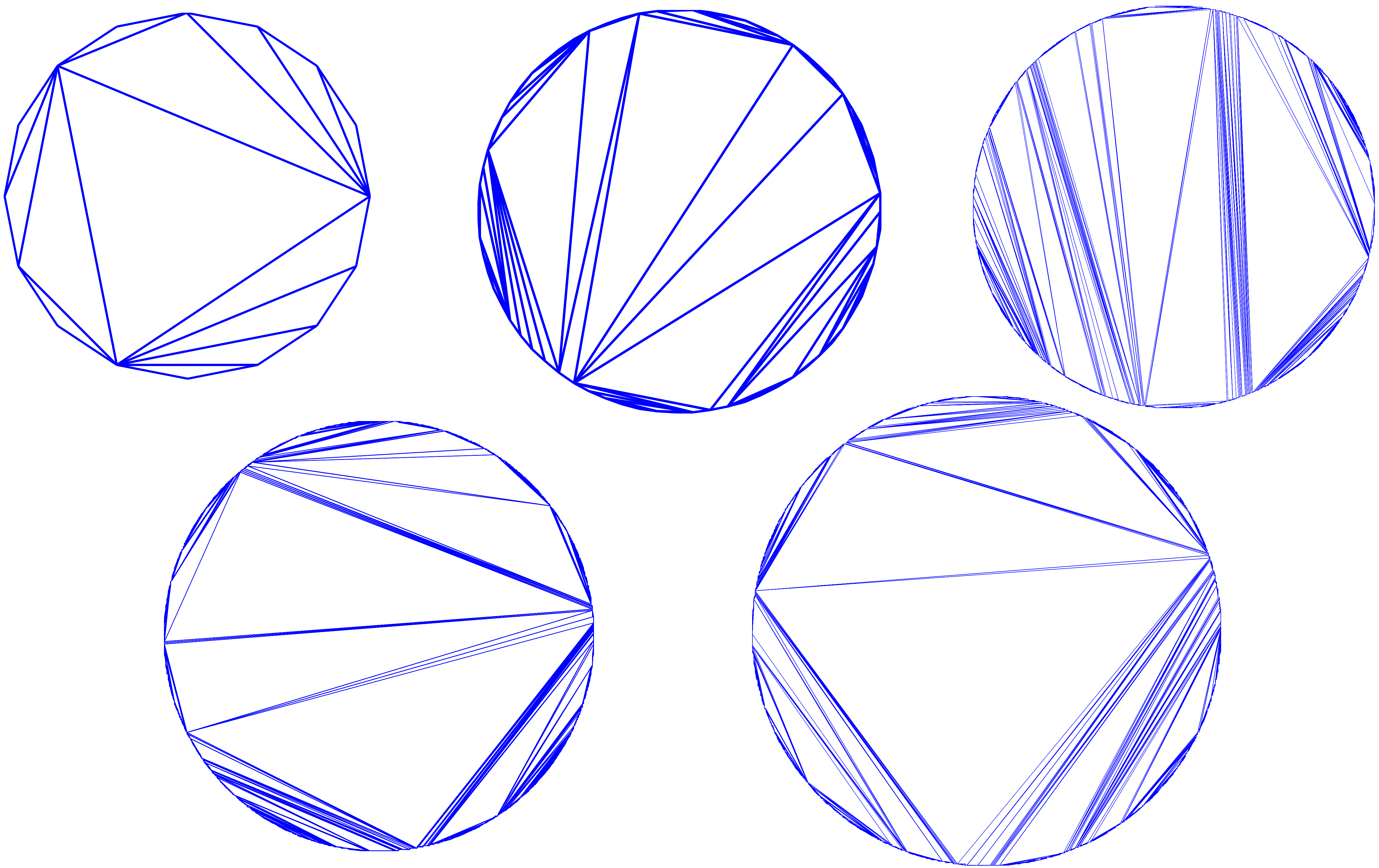

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$$\frac{1}{\pi}\frac{3x-1}{x^2(1-x)^2\sqrt{1-2x}}\mathbf{1}_{\frac{1}{3}\leqslant x\leqslant\frac{1}{2}}dx.$$

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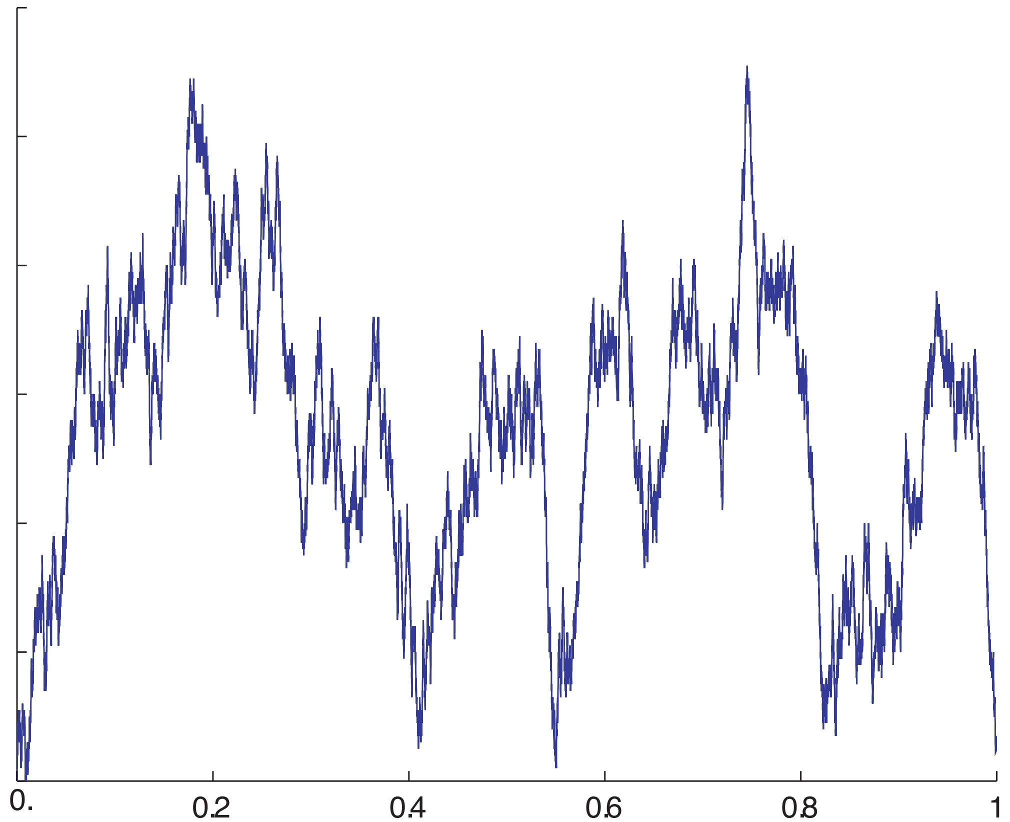

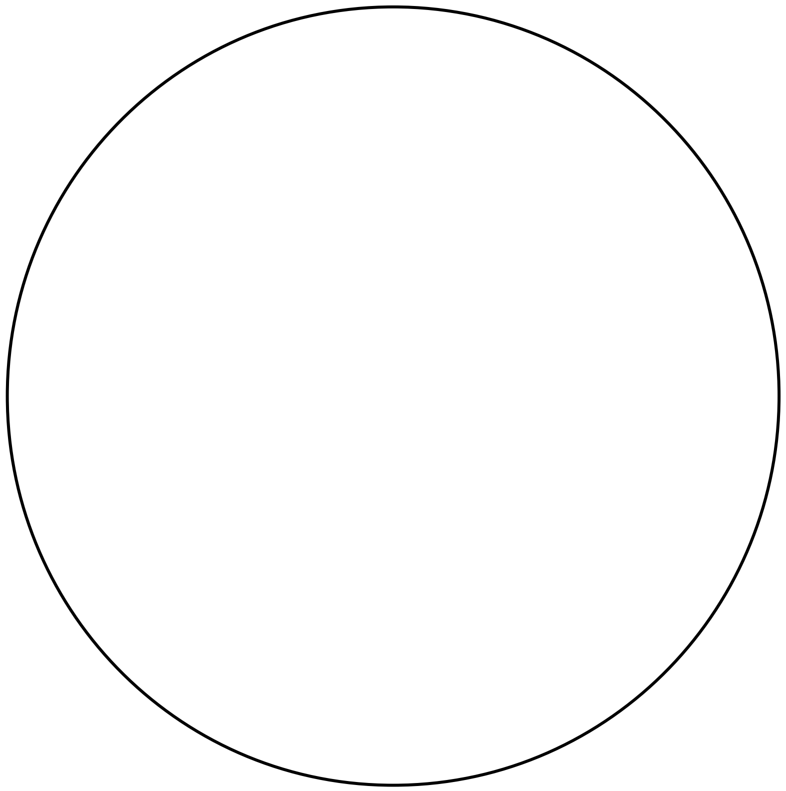

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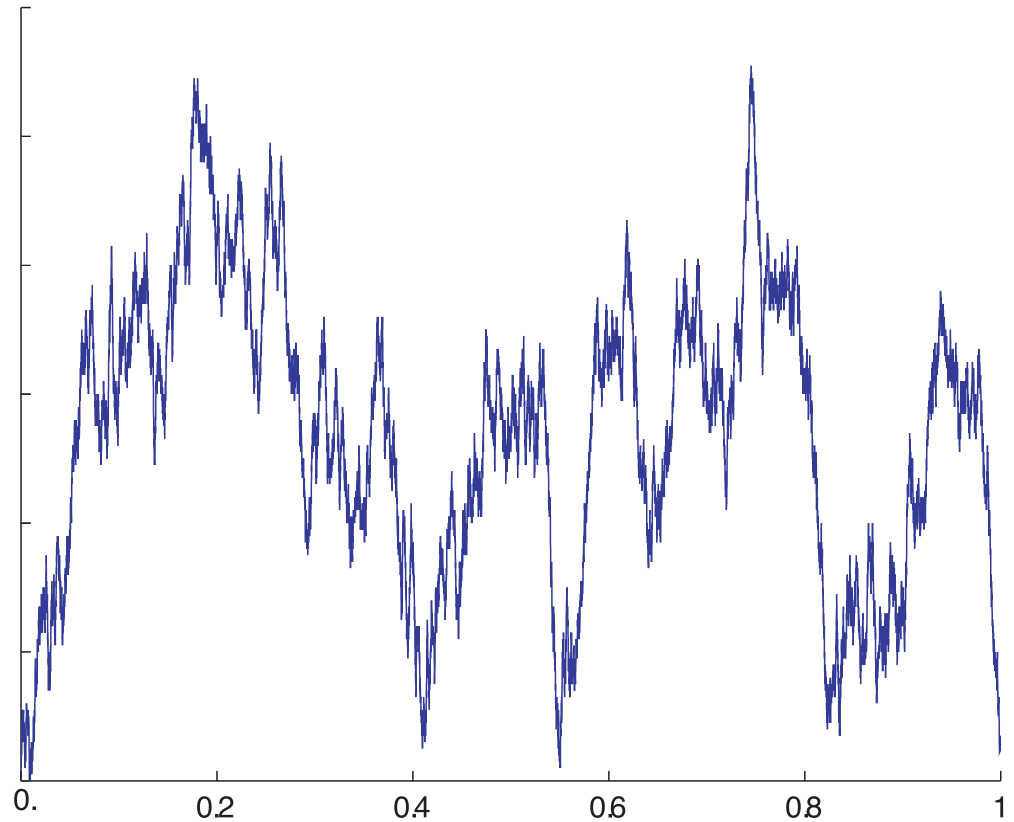

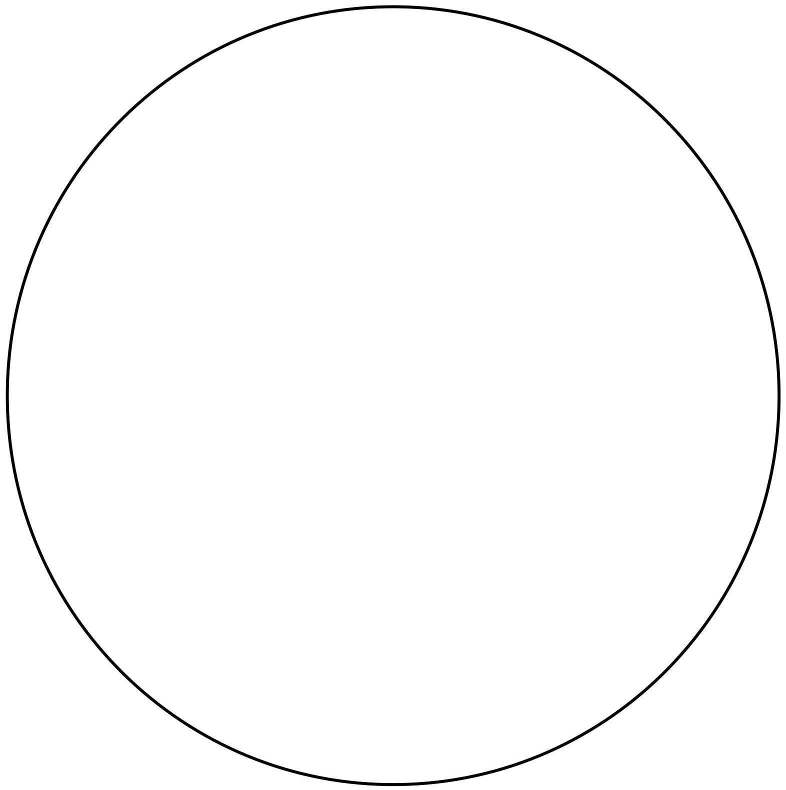

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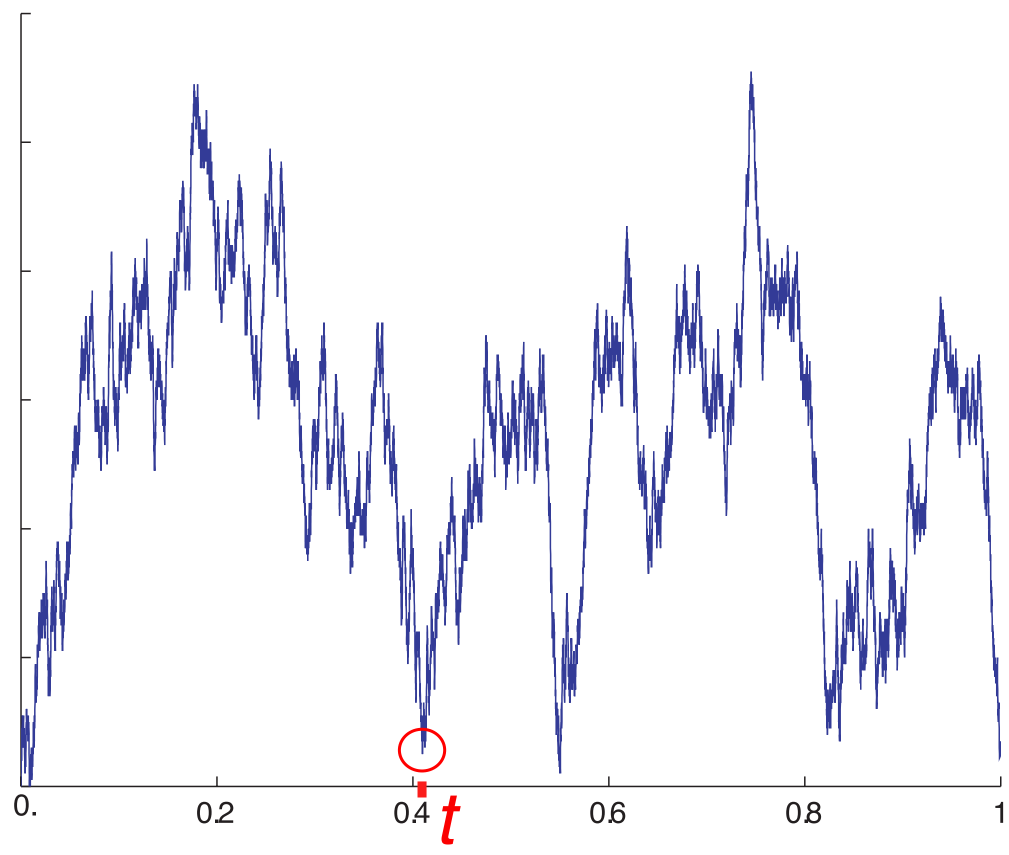

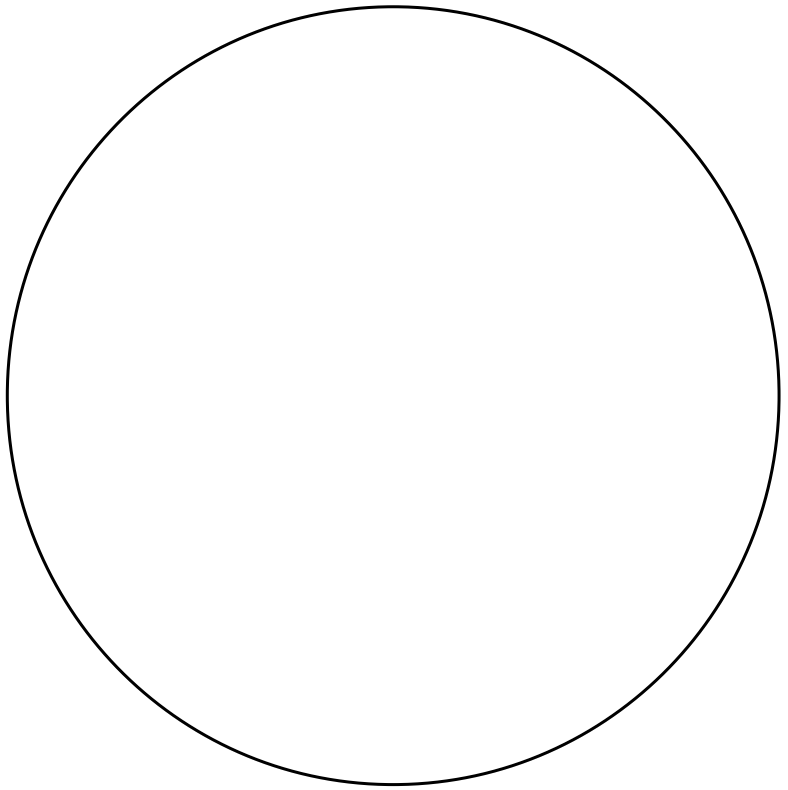

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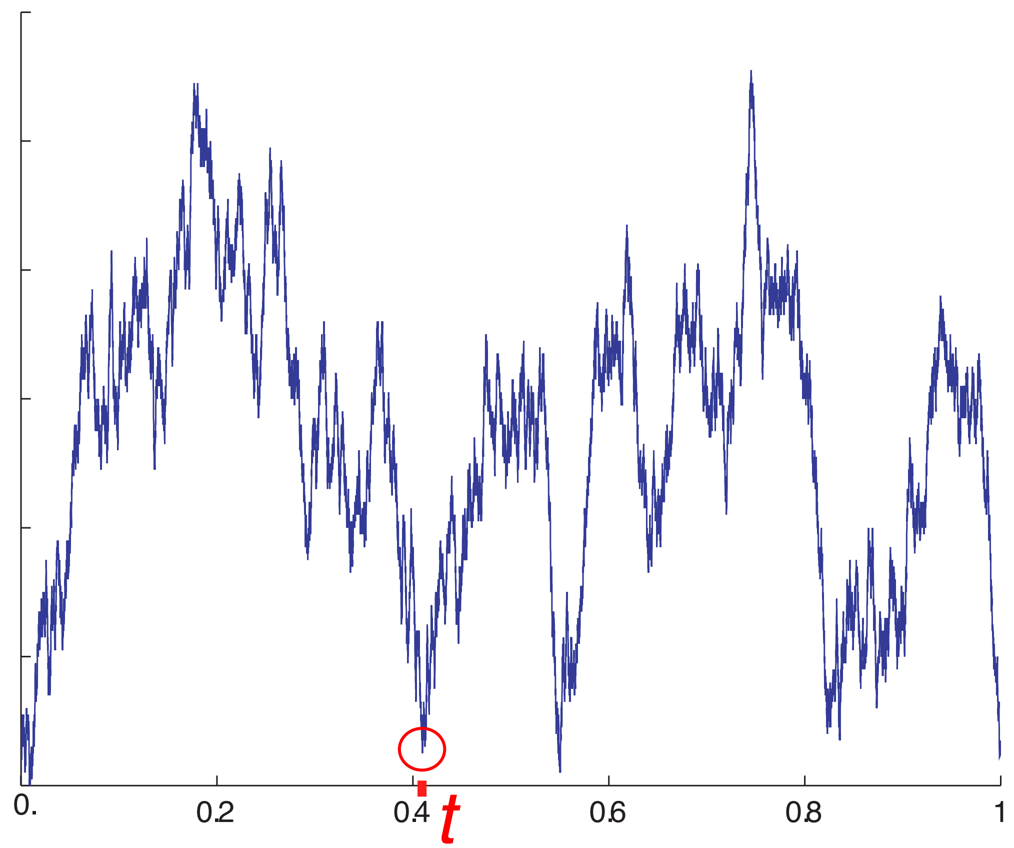

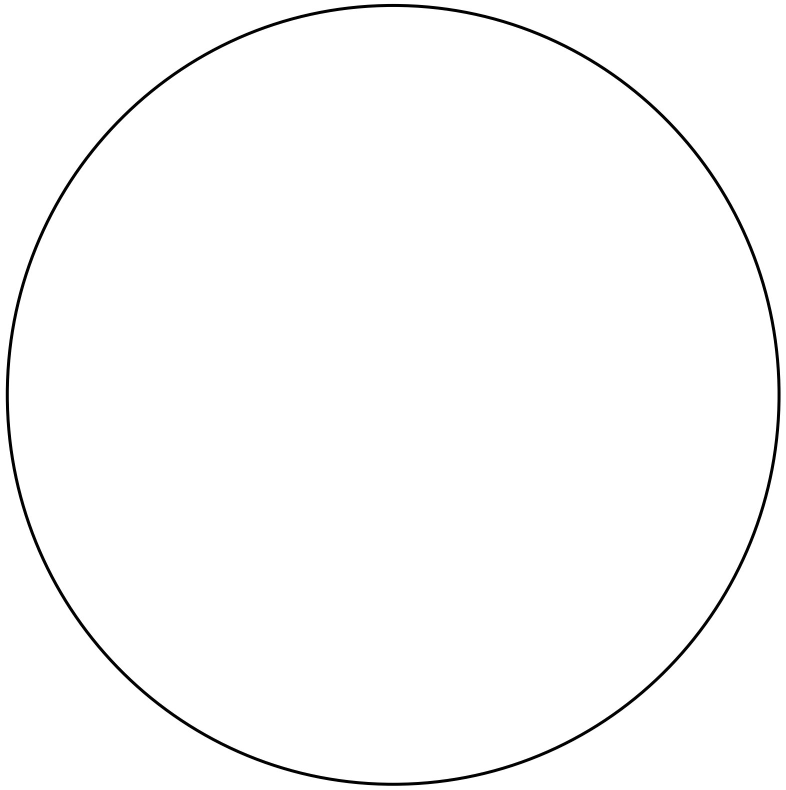

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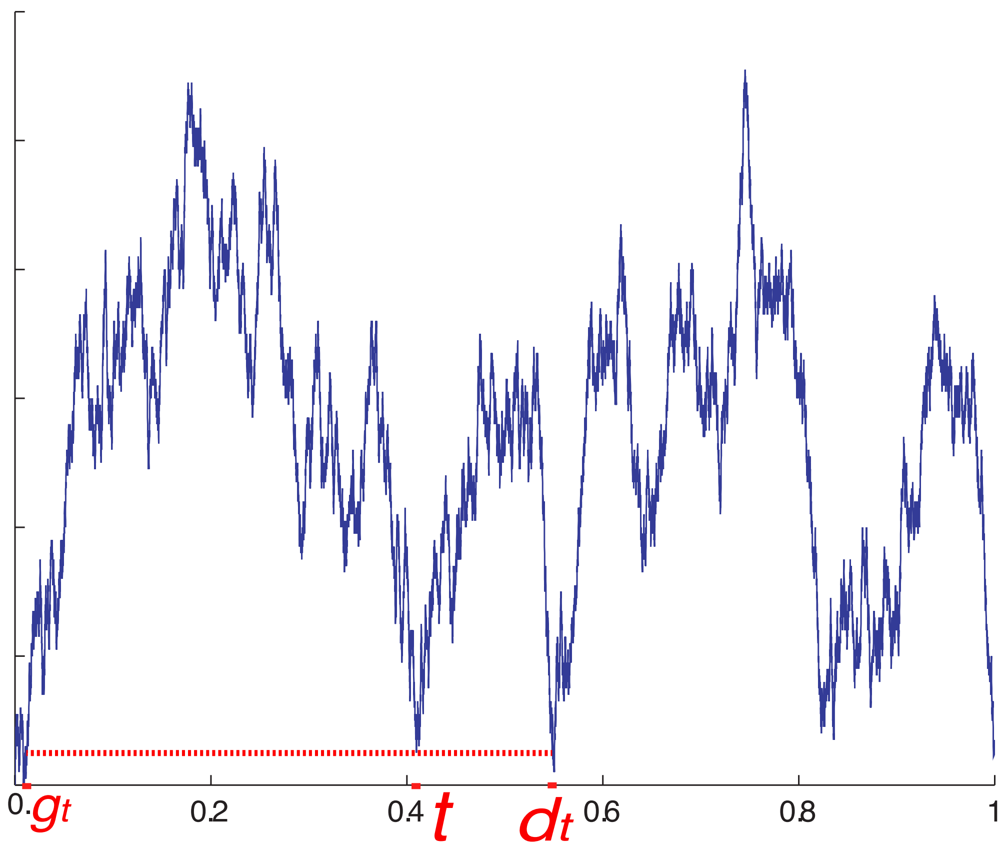

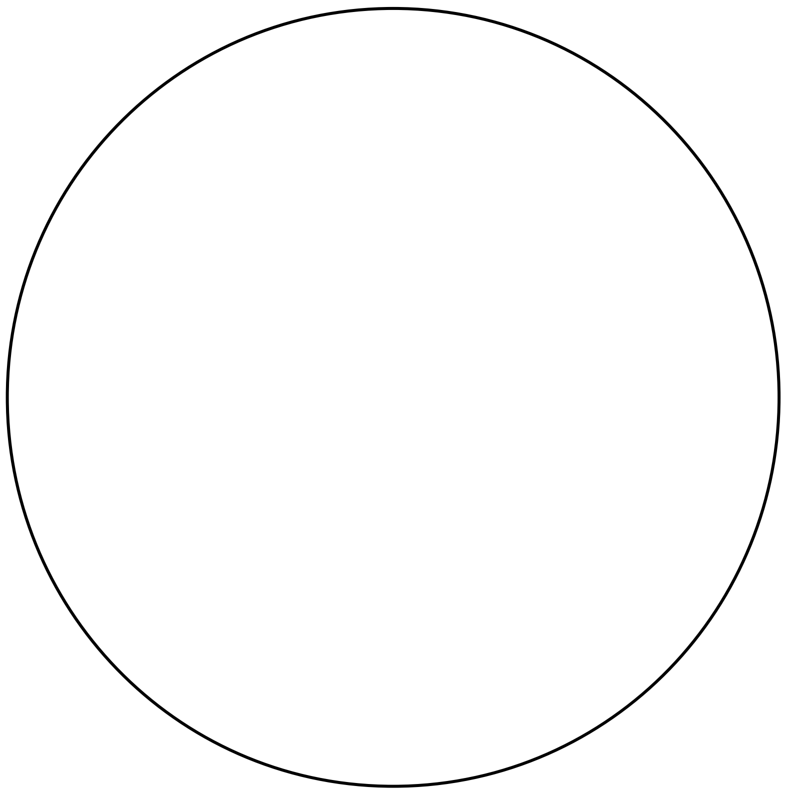

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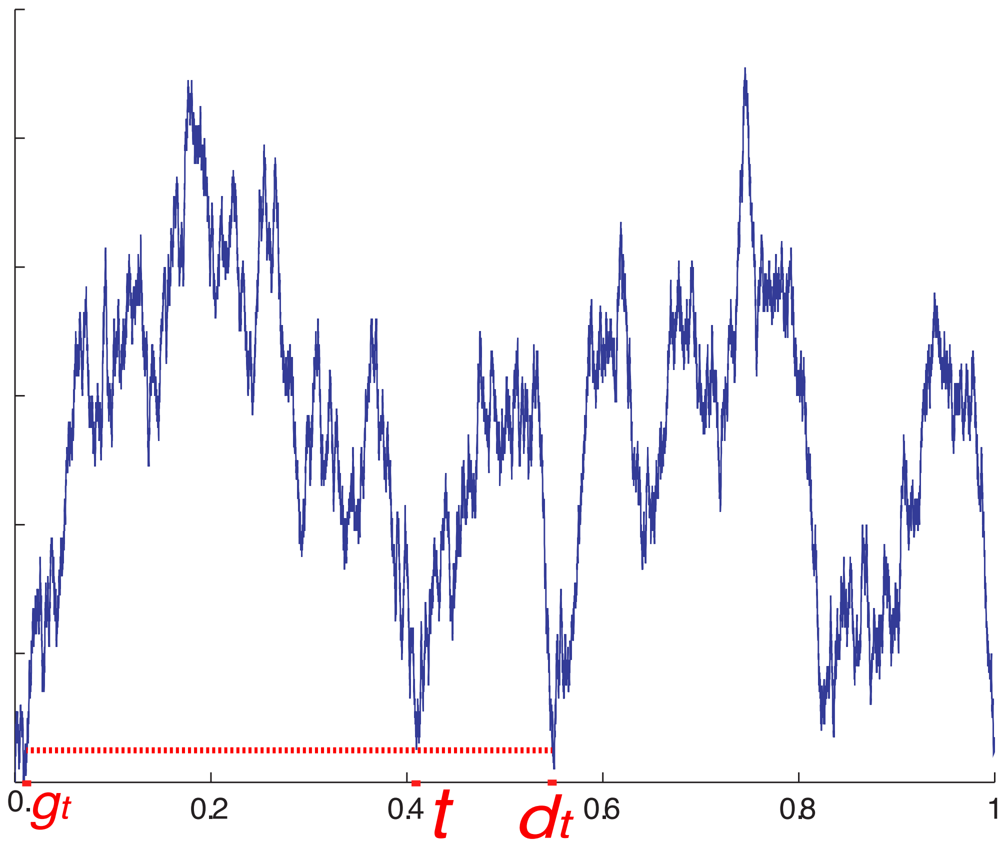

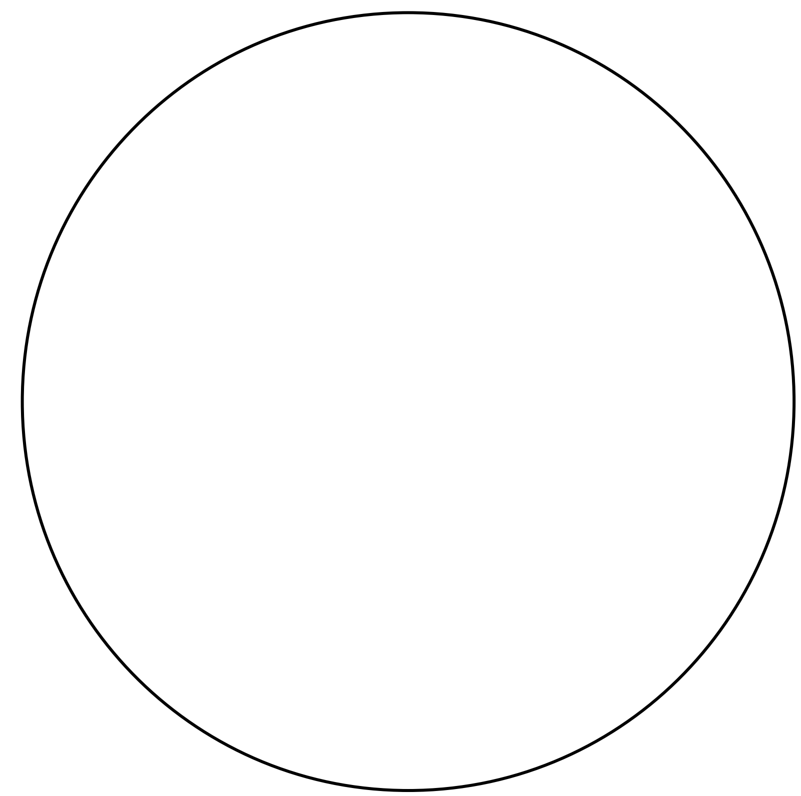

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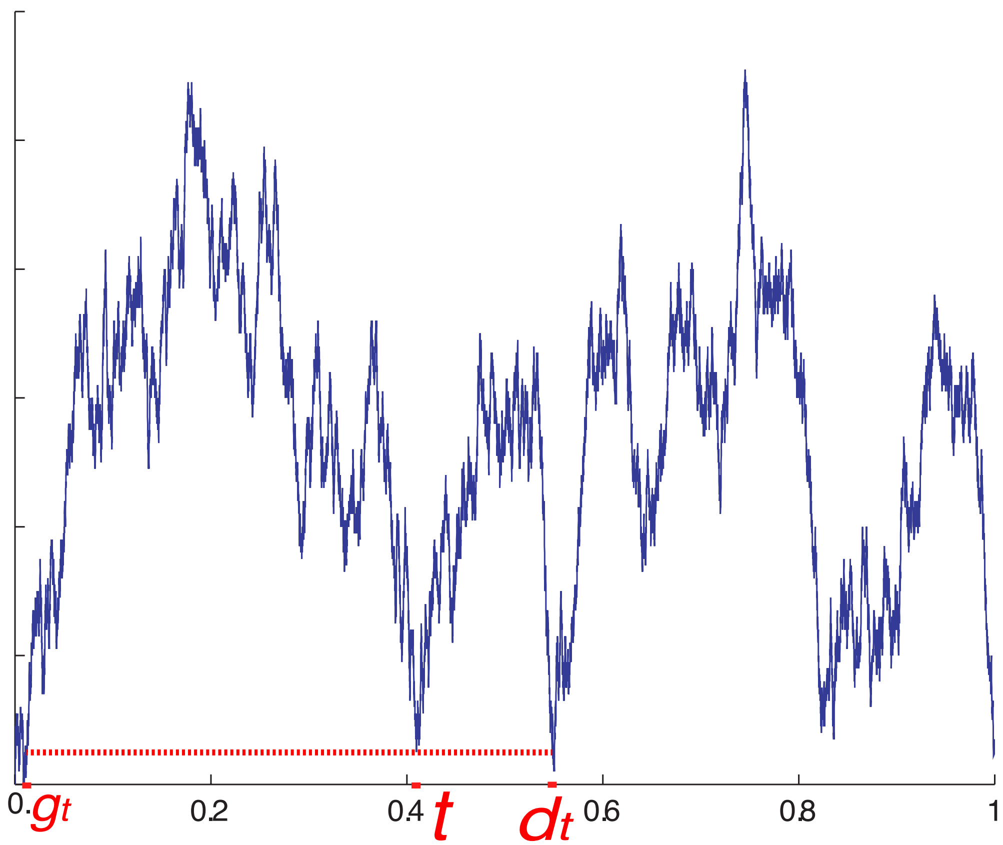

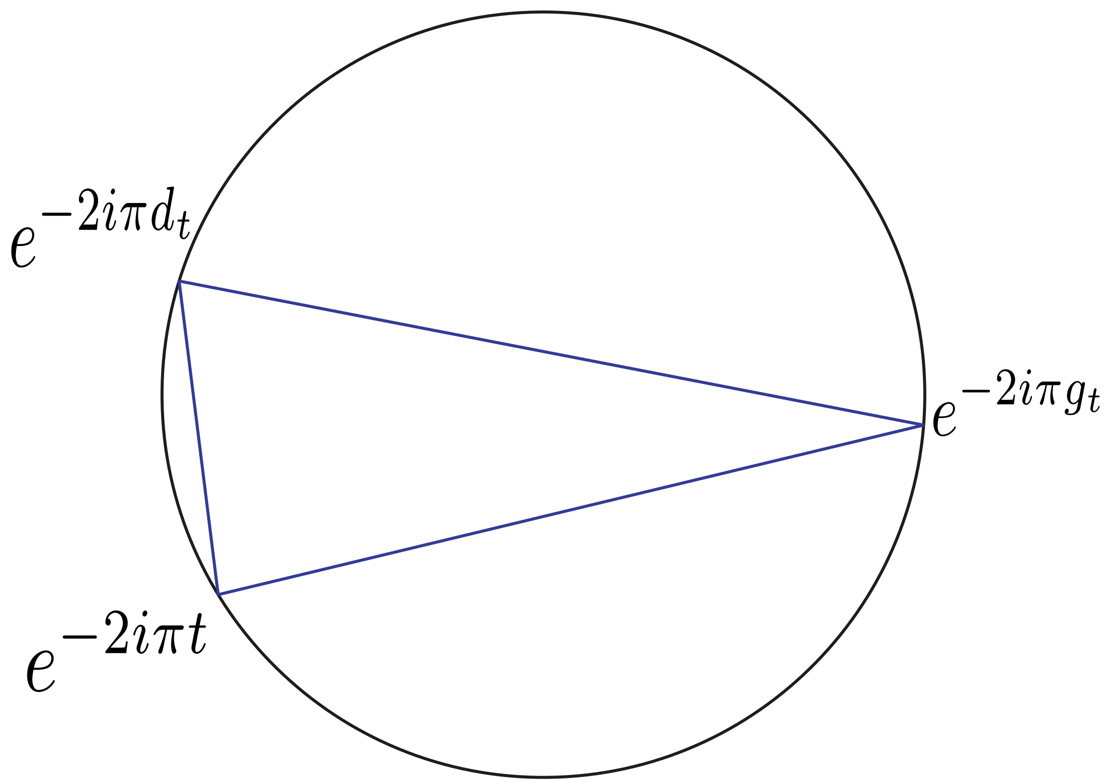

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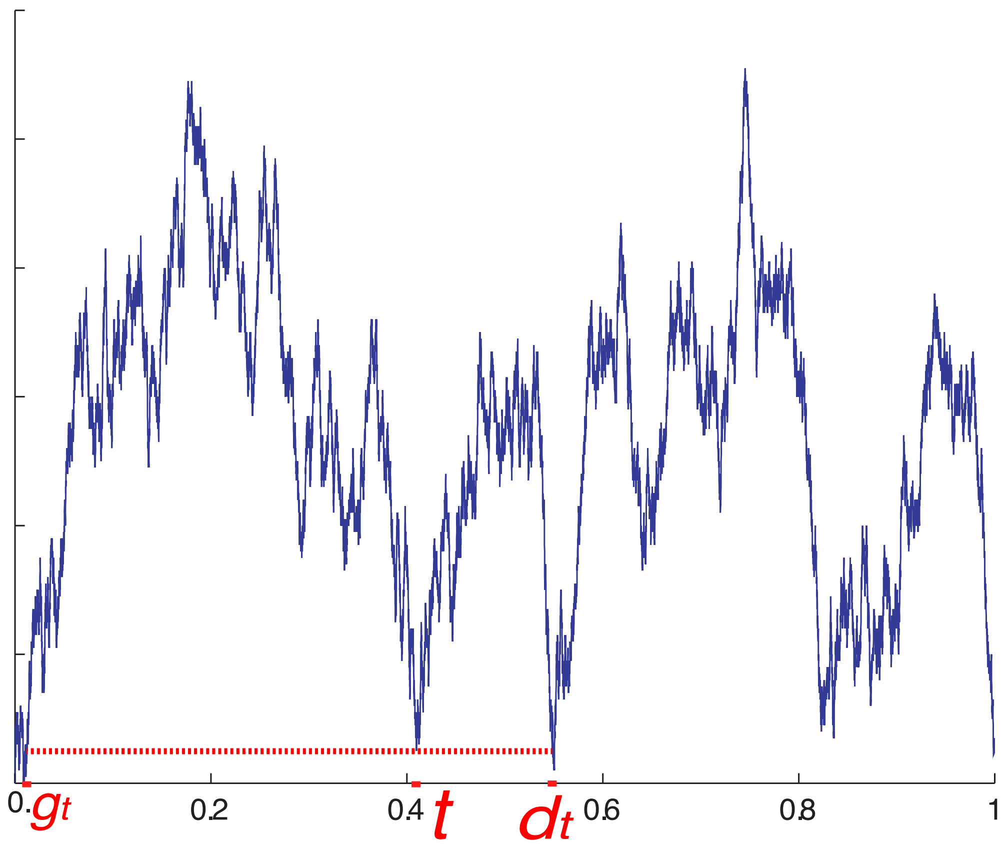

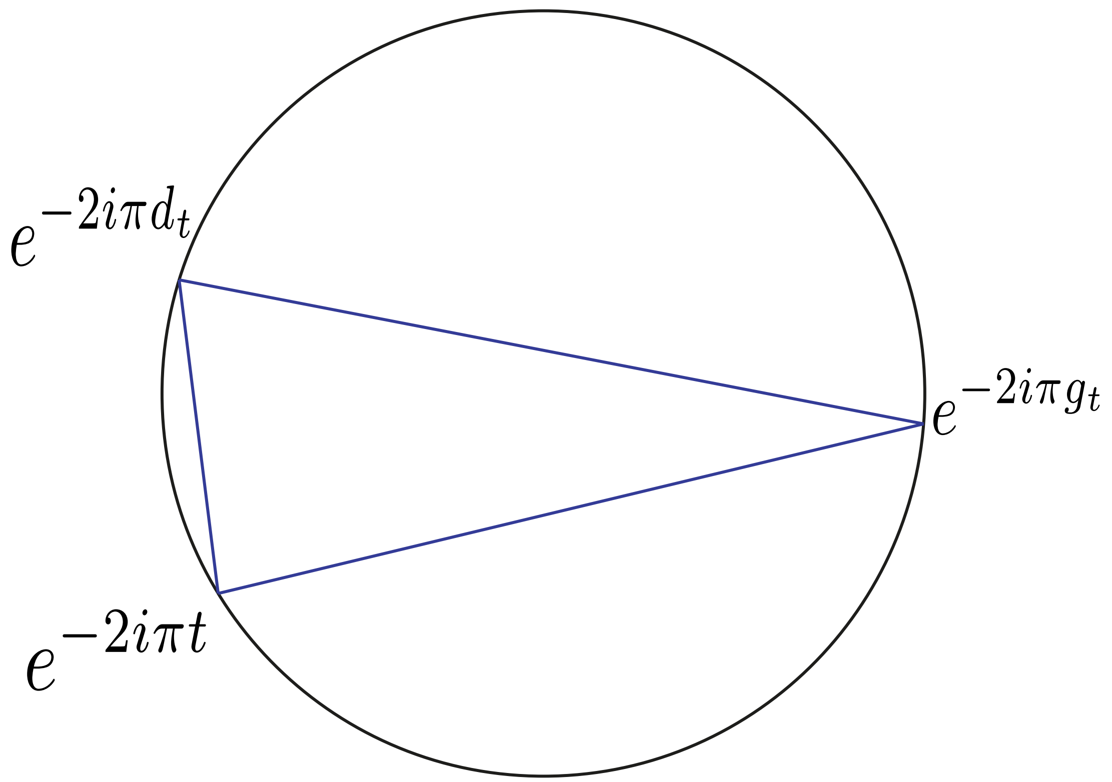

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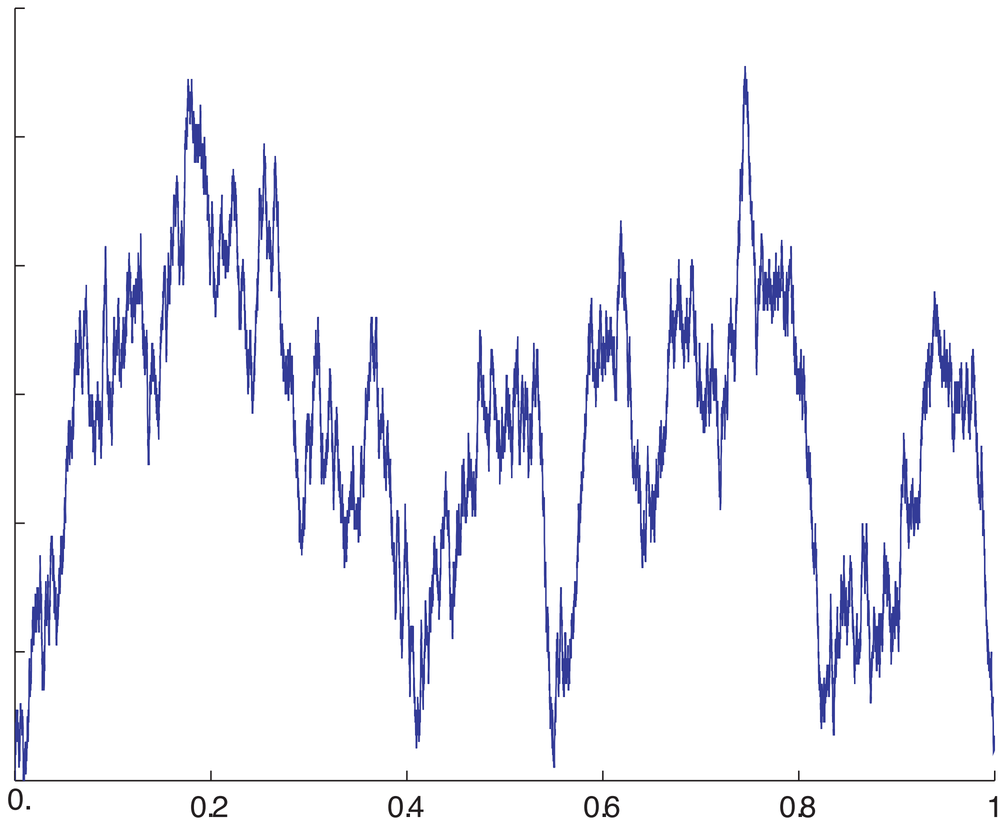

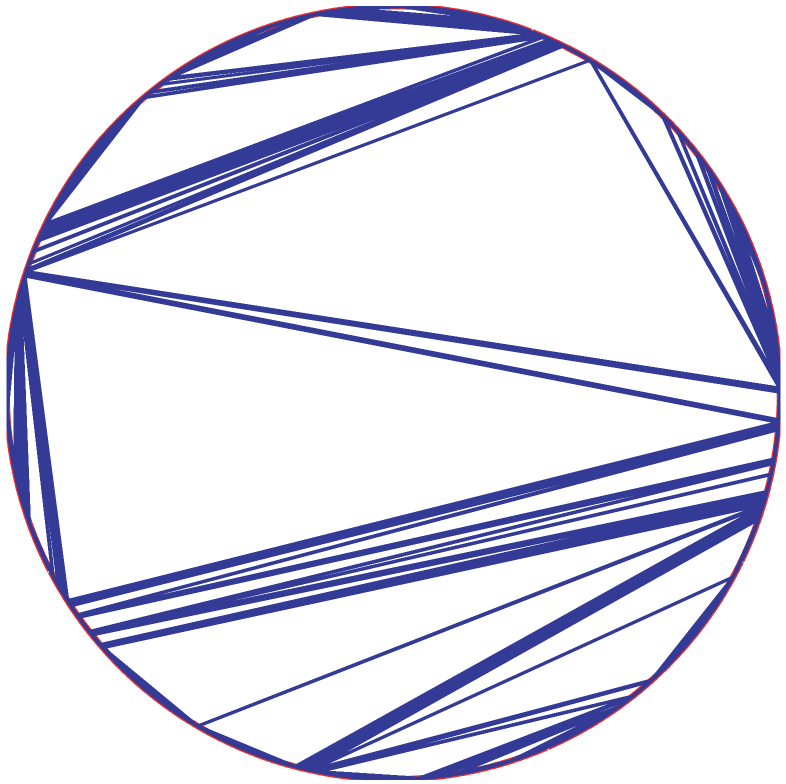

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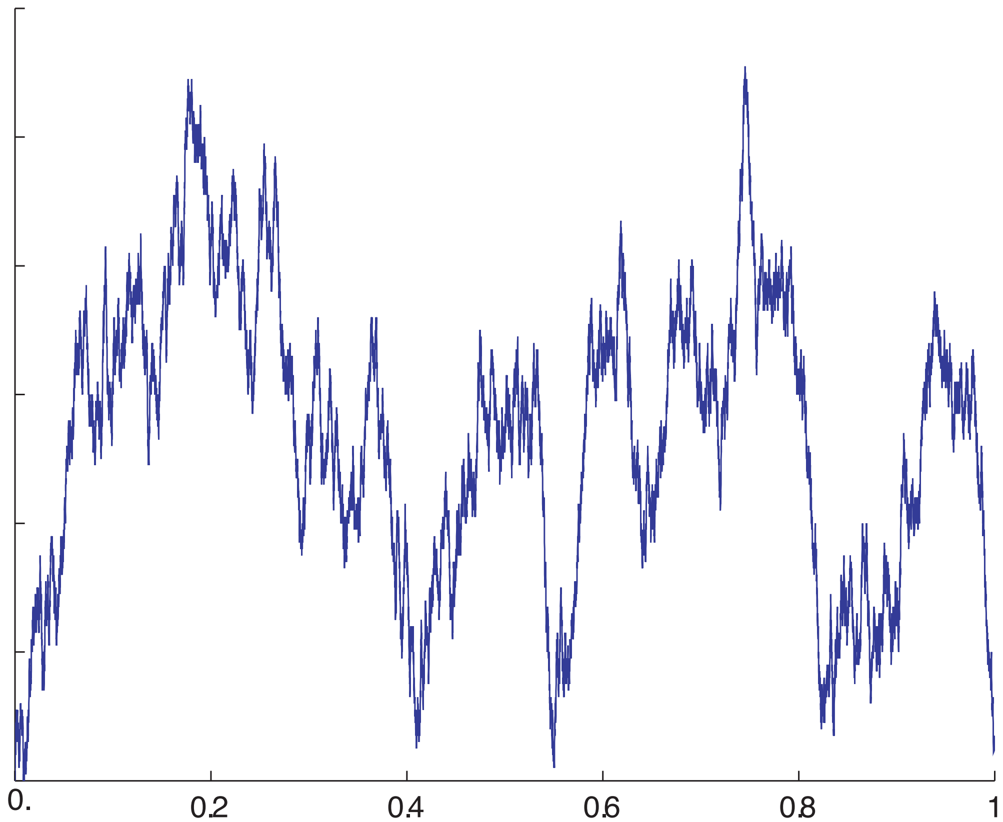

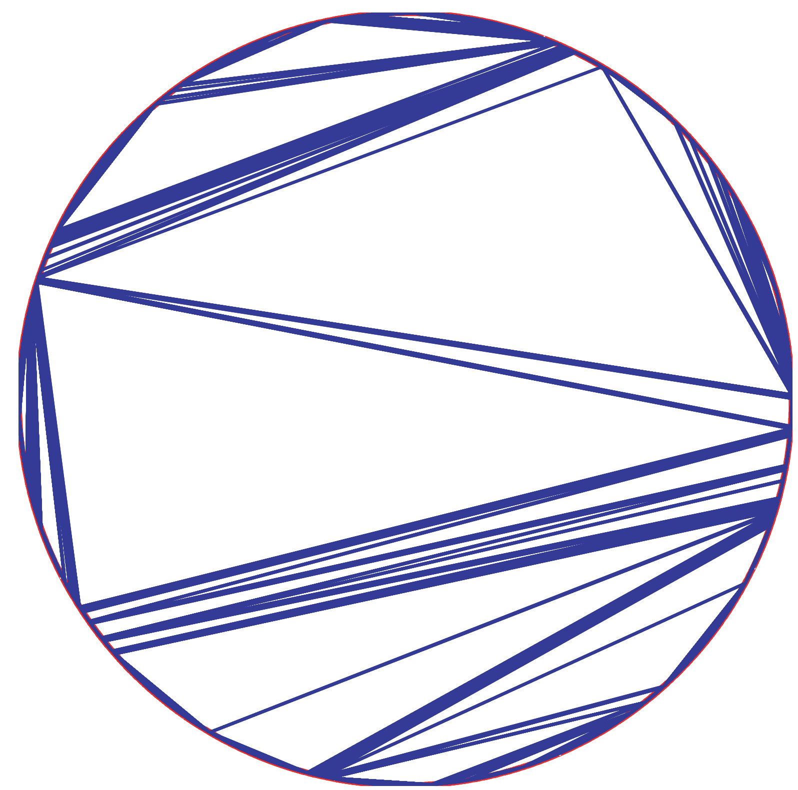

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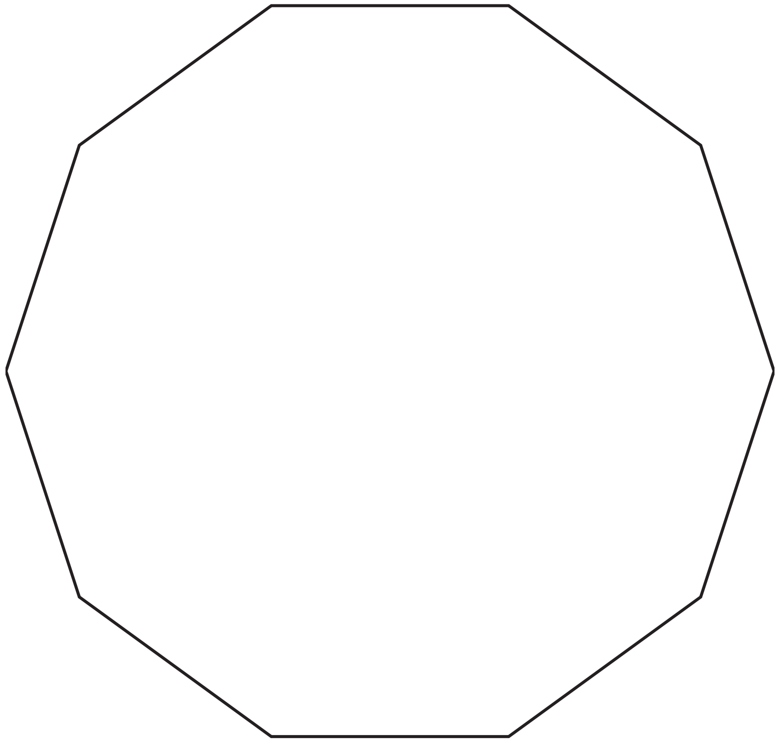
The closure of this object, denoted by $L(\mathbb{e})$, is called the Brownian triangulation.

Case of dissections of P_n

Dissections

Recall that P_n is the polygon with vertices $e^{\frac{2i\pi j}{n}}$ $(j = 0, 1, \ldots, n-1)$.



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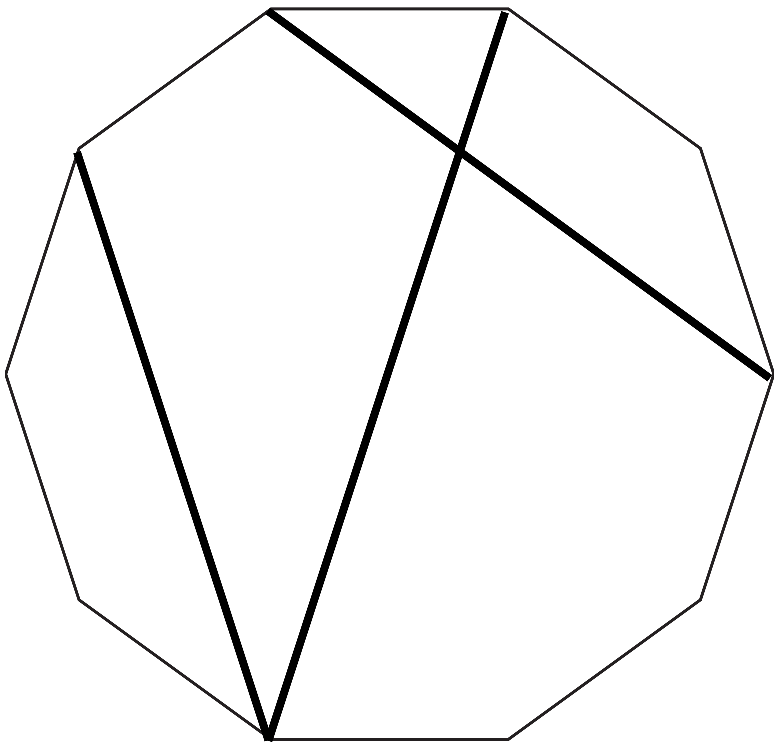

Not a dissection!

A dissection!

A ***dissection*** of P_n is the union of P_n with a collection of non-crossing diagonals.

Dissections

Let $\mathcal{D}_n$ be a random dissection, chosen uniformly at random among all dissections of P_n. What does $\mathcal{D}_n$ look like as $n \to \infty$?

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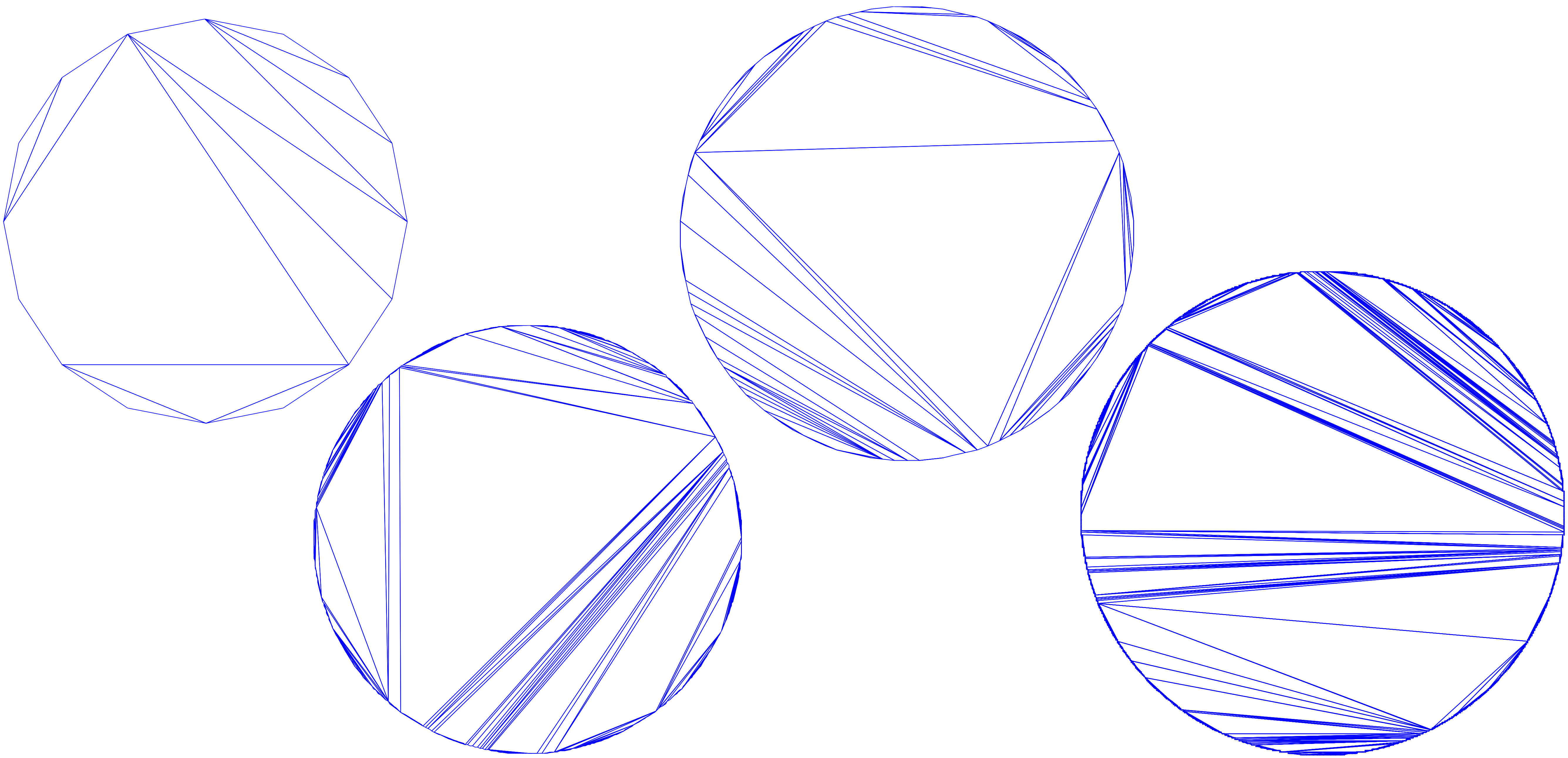

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$$\frac{1}{\pi}\frac{3x-1}{x^2(1-x)^2\sqrt{1-2x}}\mathbf{1}_{\frac{1}{3}\leqslant x\leqslant\frac{1}{2}}dx.$$

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Key point: these trees can be coded by BGW trees.

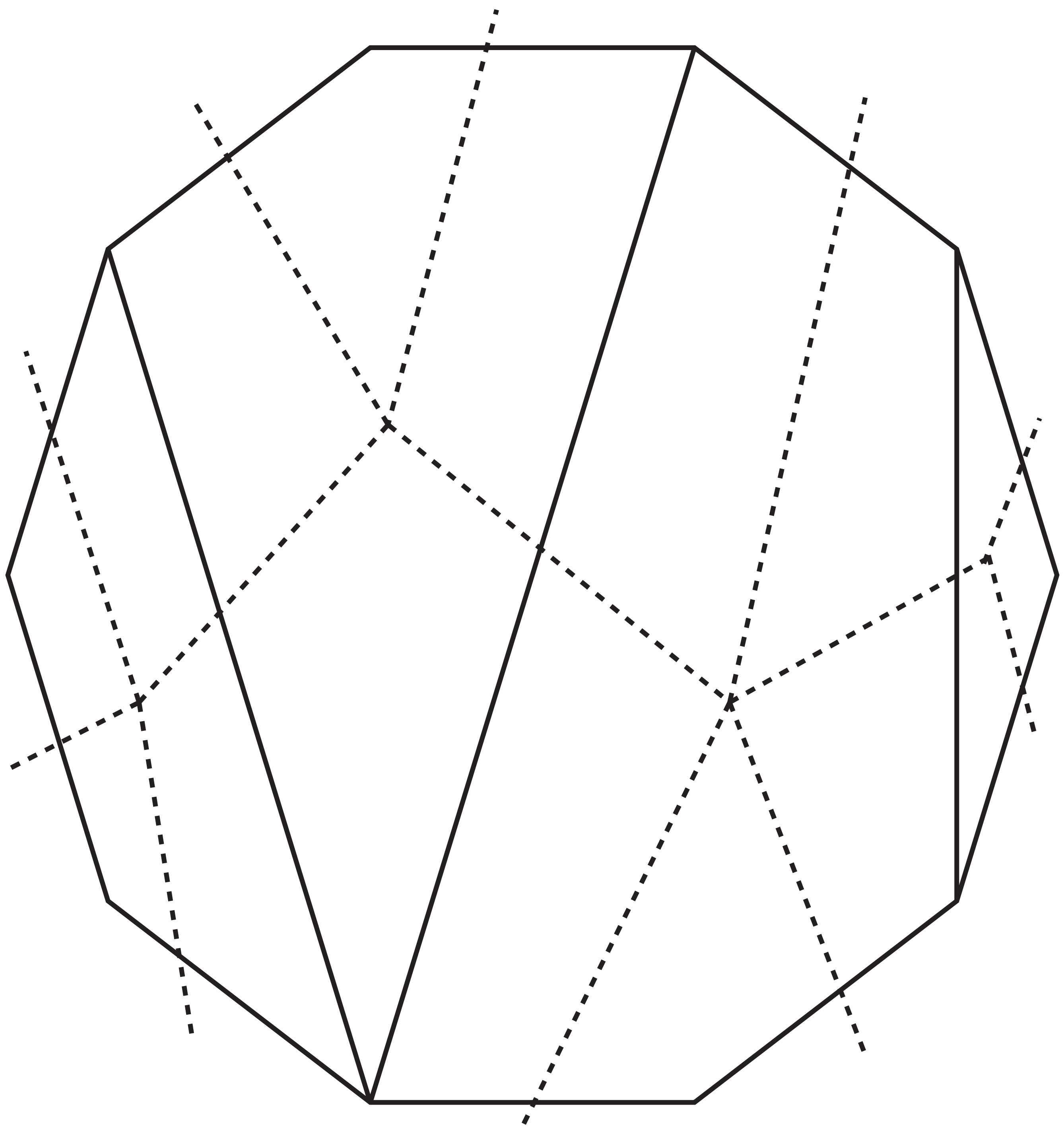


Figure: The dual tree of a dissection.

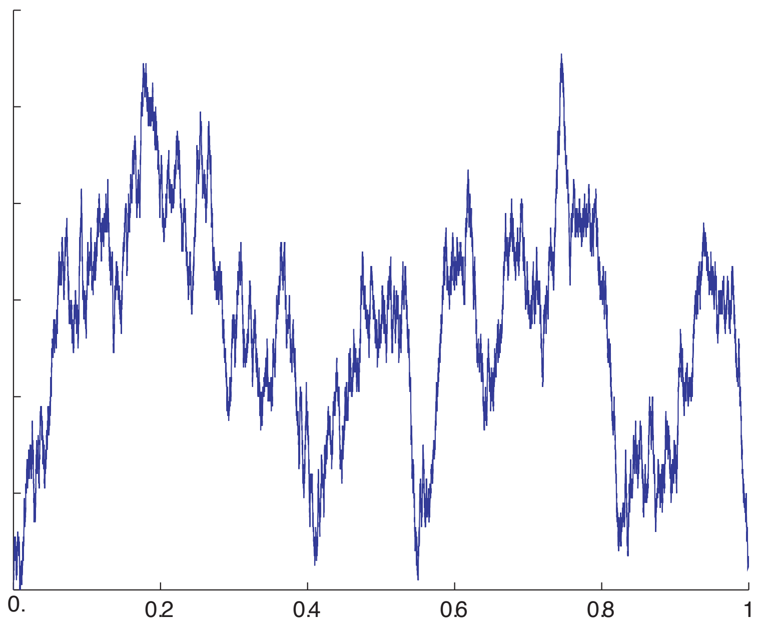
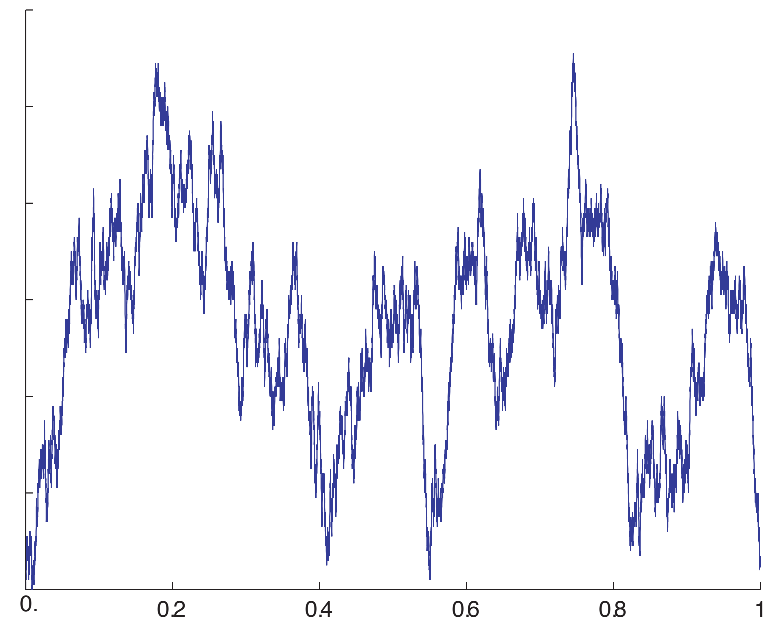


Figure: Normalized contour function of a large conditioned Bienaymé–Galton-Watson.



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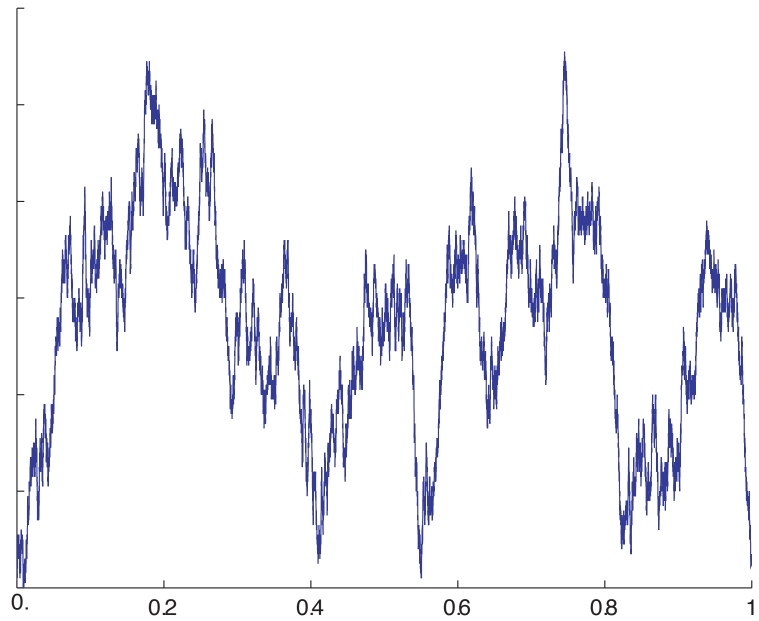

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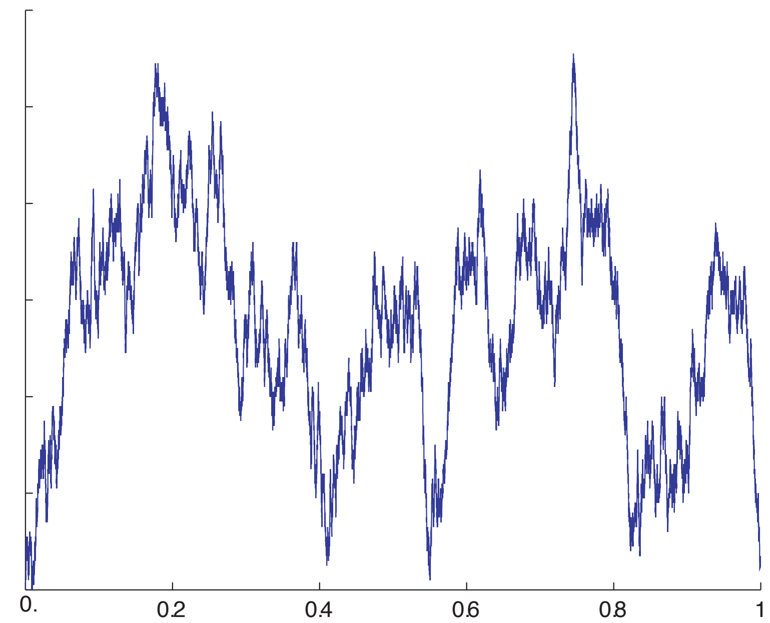

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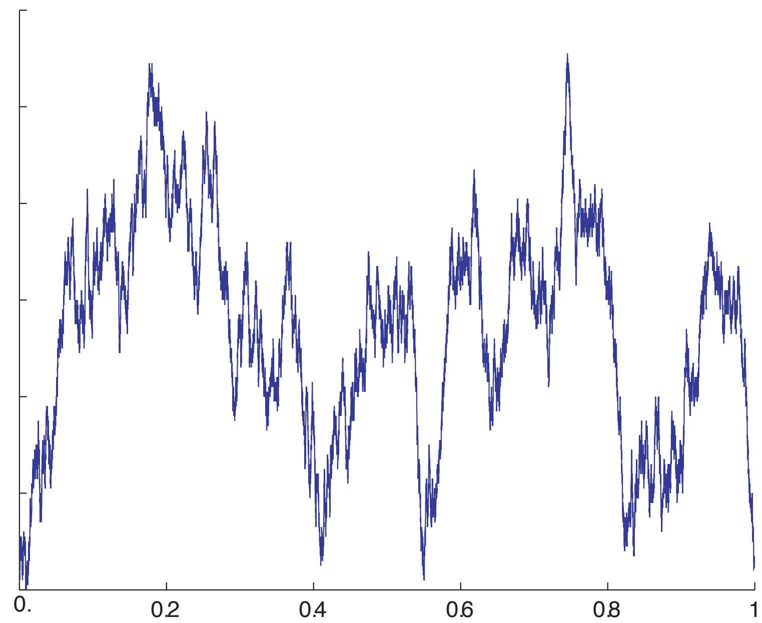

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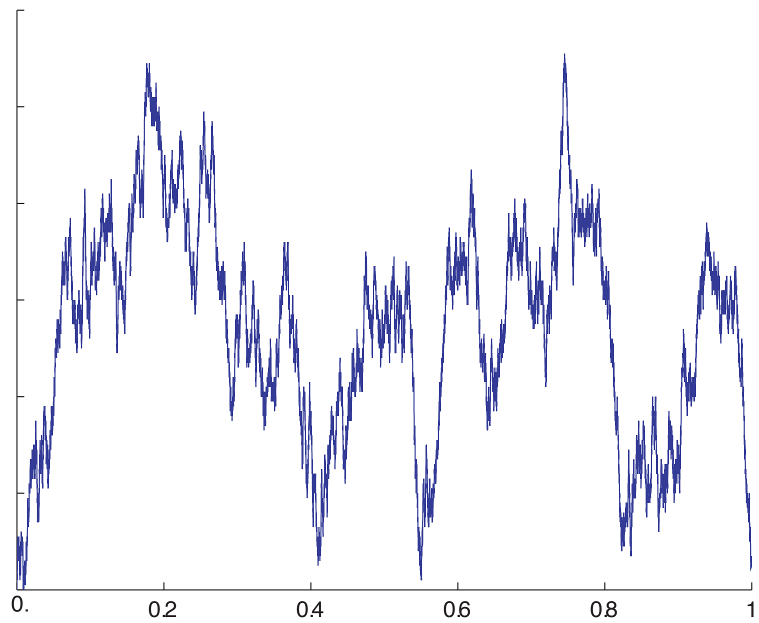

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Therefore these random plane non-crossing configurations converge to $\mathbb{L}(\mathbb{e})$.

What about dissections seen as compact metric spaces?

Dissections seen as compact metric spaces

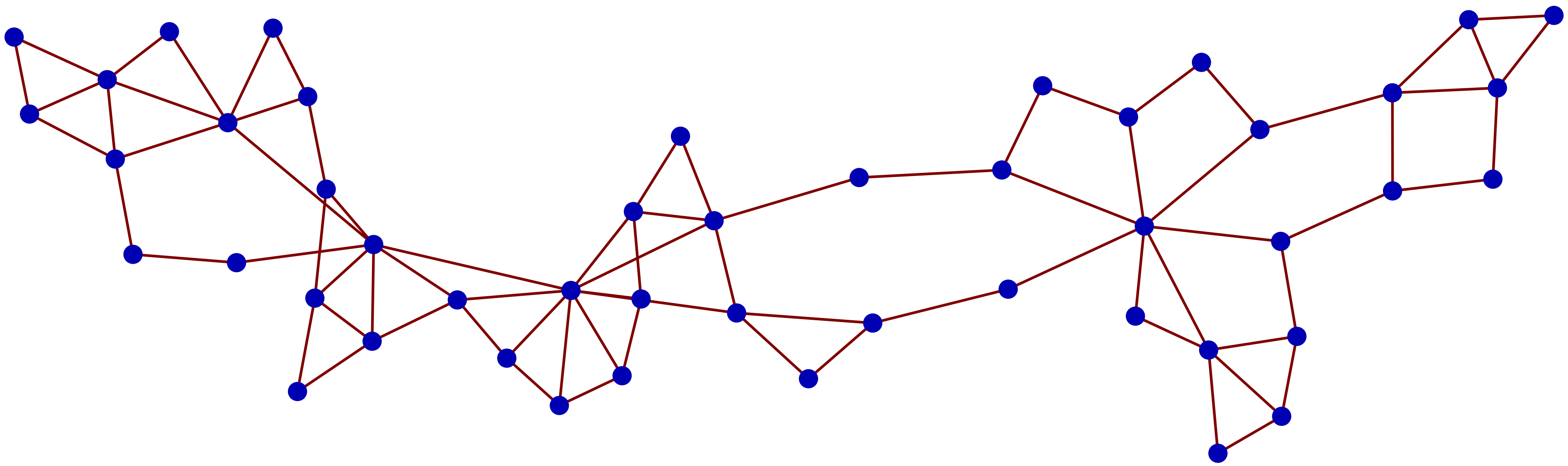
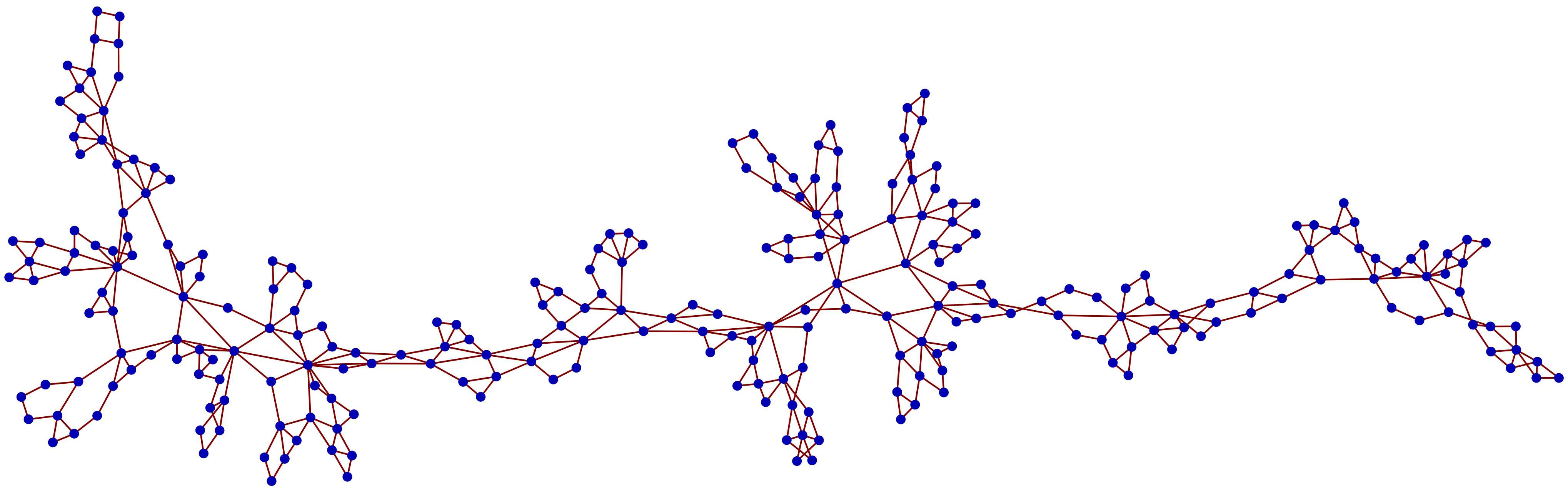


Figure: A uniform dissection of P_{45}.

Dissections seen as compact metric spaces



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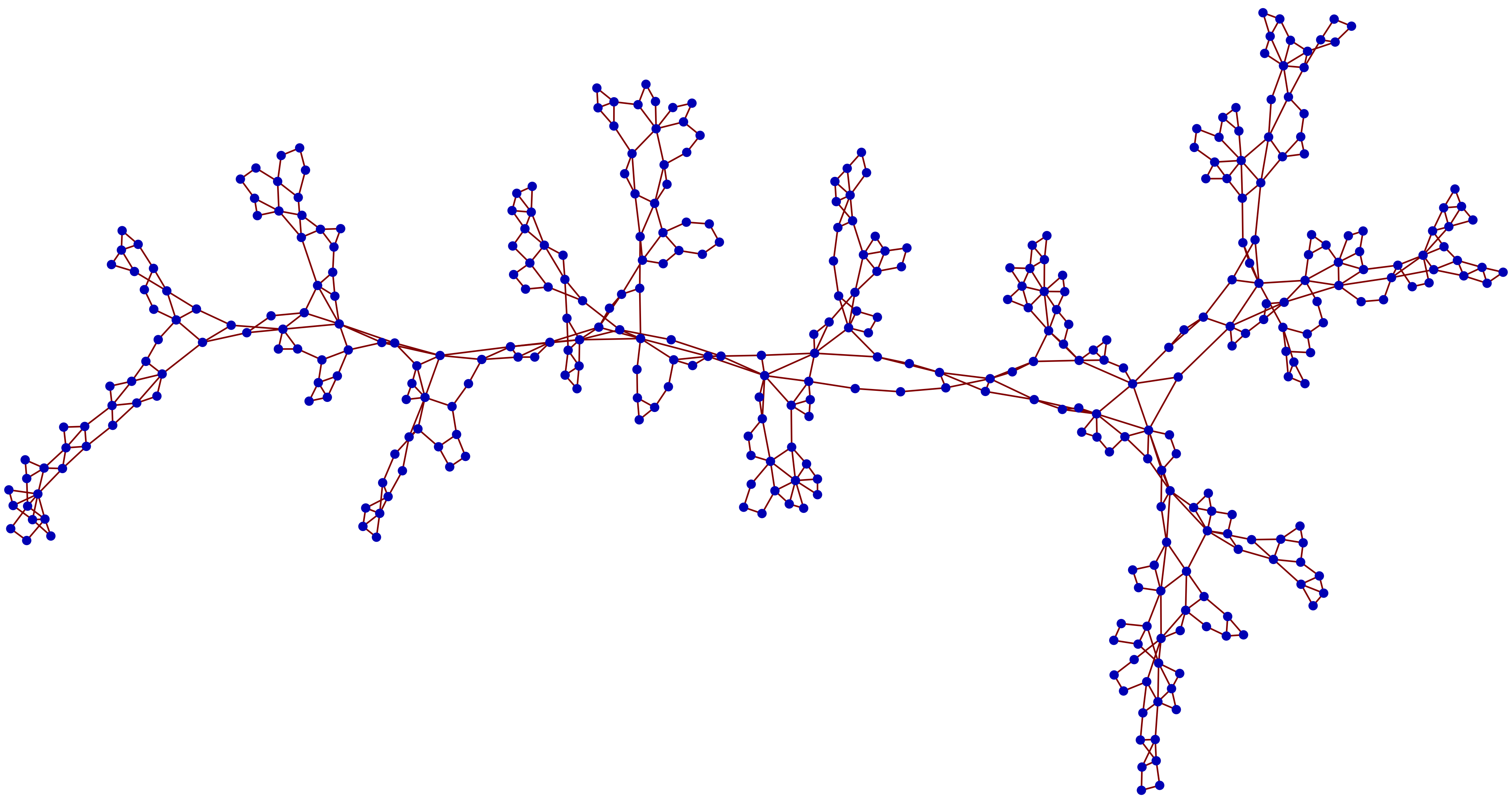

Figure: A uniform dissection of P_{387}.

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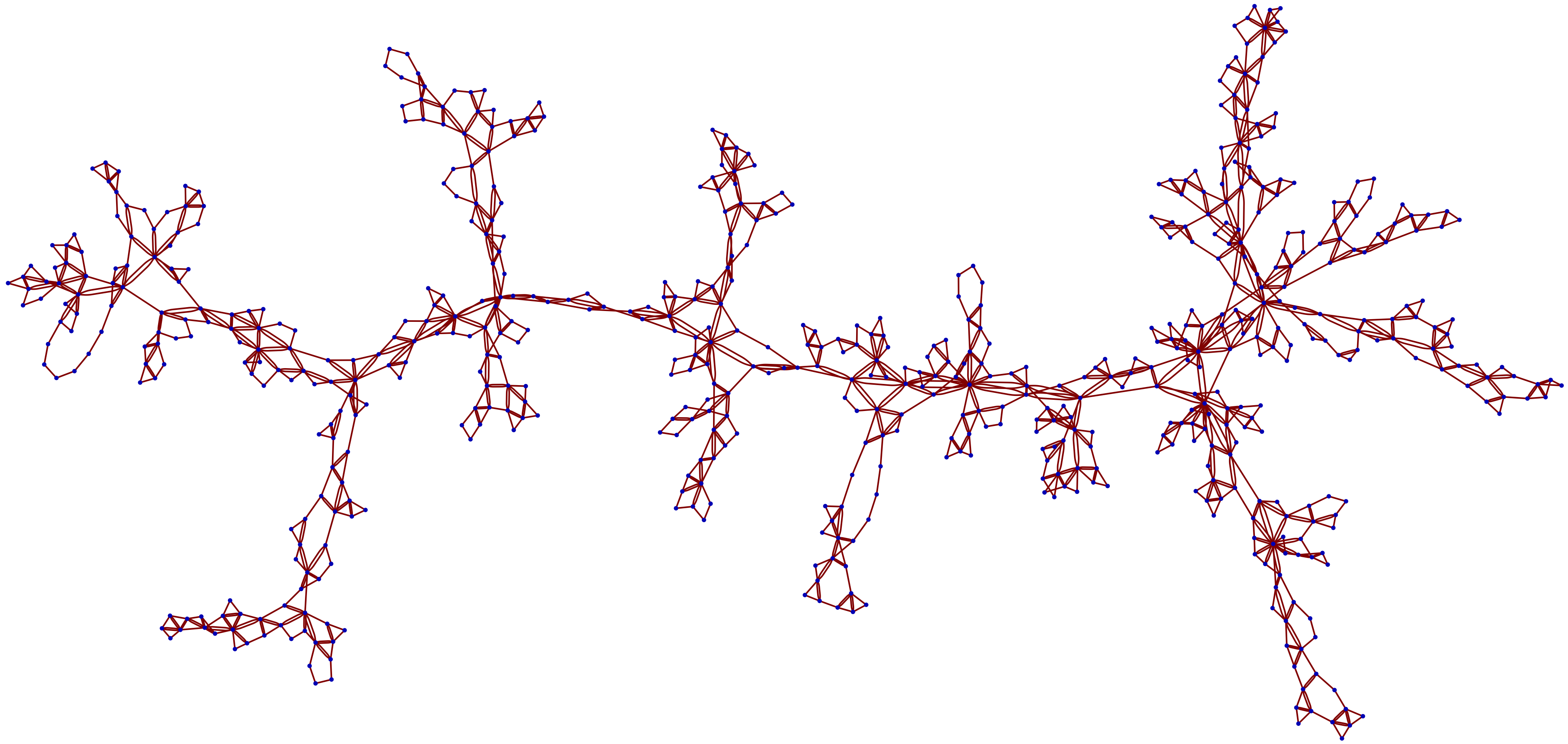
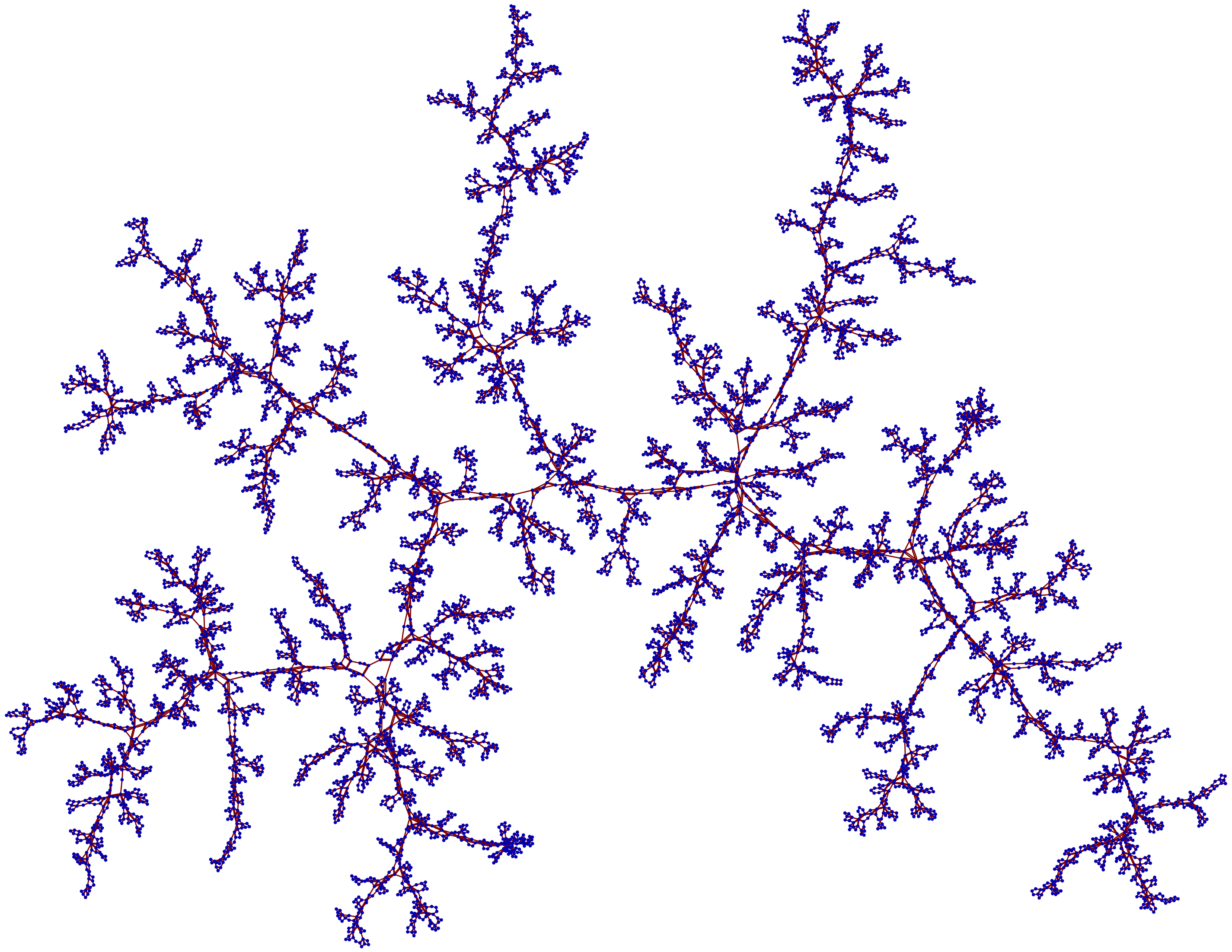


Figure: A uniform dissection of P_{637}.

Dissections seen as compact metric spaces



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I. Scaling limits of BGW trees (finite variance, 1991)

II. Scaling limits of BGW trees (infinite variance, 1998)

III. Plane non-crossing configurations (2012)

IV. Random maps (2004 – ?)

What does a "typical" random surface look like?

In dimension one

It is natural to view Brownian motion as a “typical” random path, describing the motion of a particle moving “uniformly at random”.

$\rightsquigarrow$ Idea: construct a (two-dimensional) random surface as a limit of random discrete surfaces.

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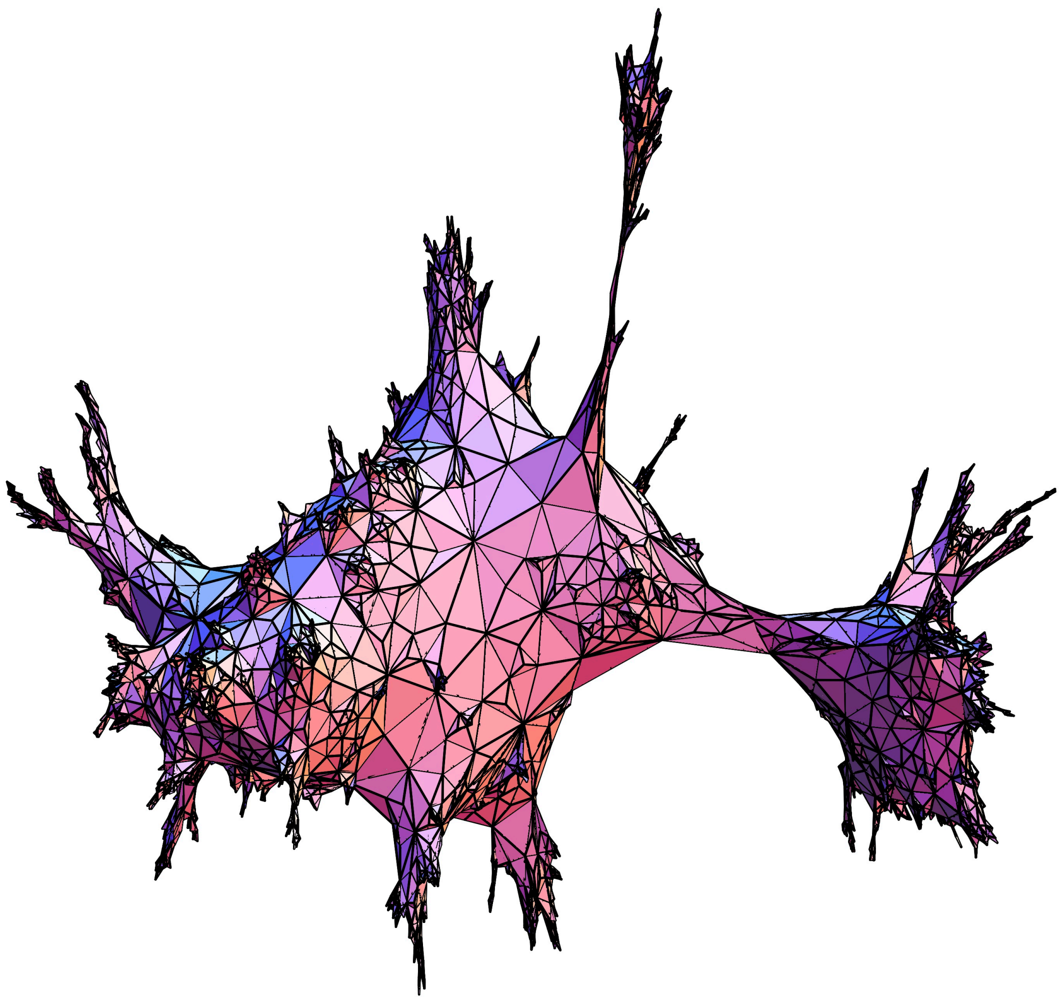

Figure: A large random triangulation (simulation by Nicolas Curien)

The Brownian map

Problem (Schramm at ICM '06): Let T_n be a random uniform triangulation of the sphere with n triangles.

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(see Le Gall's proceeding at ICM '14 for more information and references)

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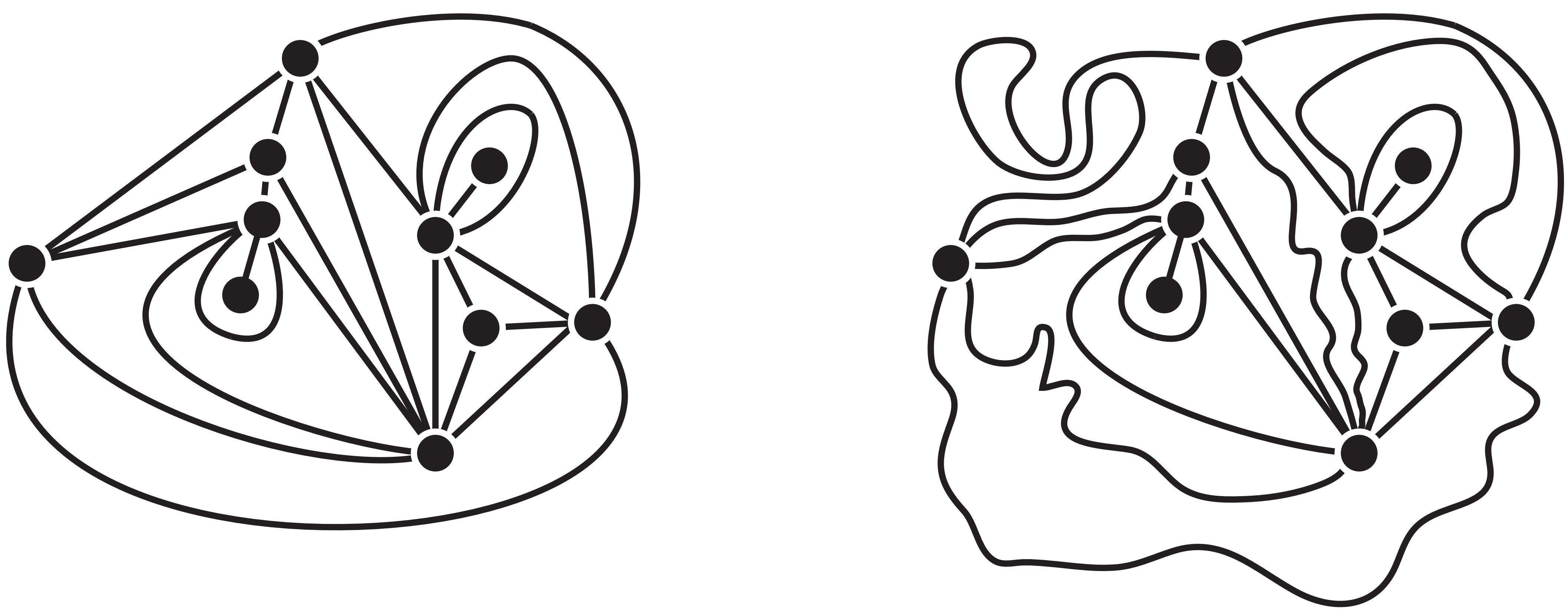

Figure: Two identical maps .

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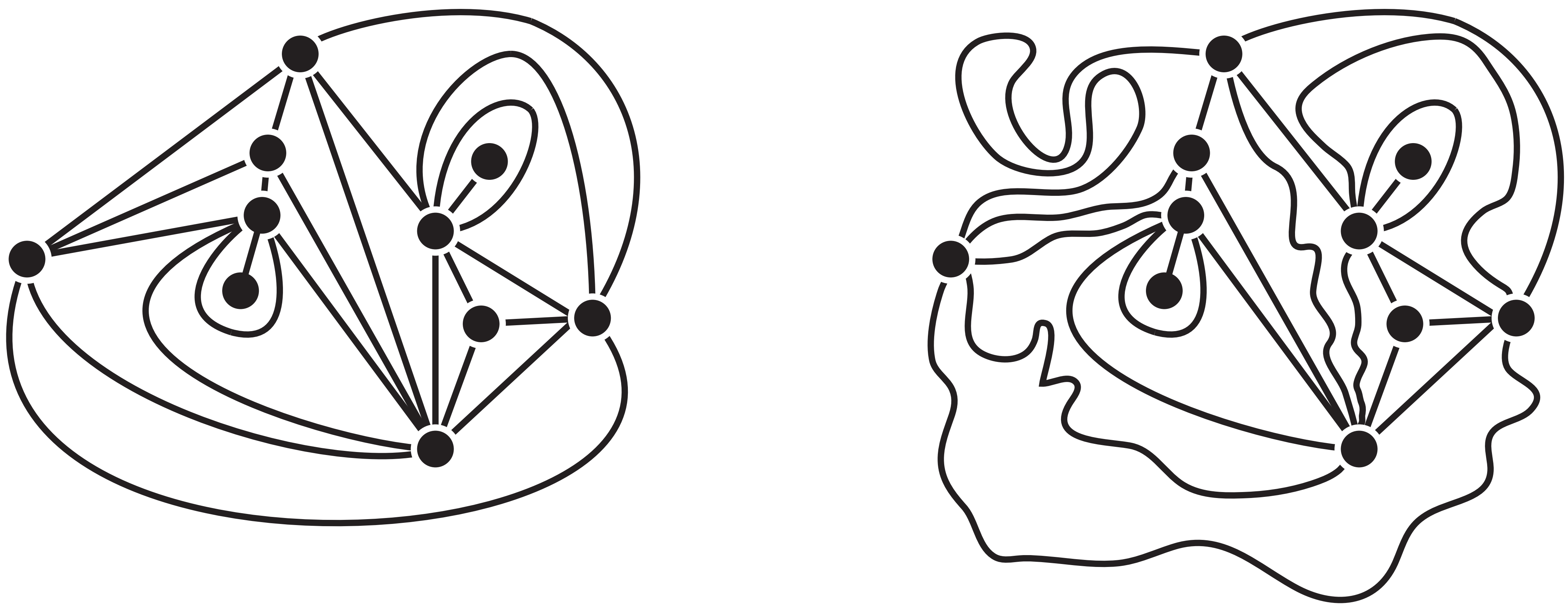

Figure: Two identical maps .

Definitions

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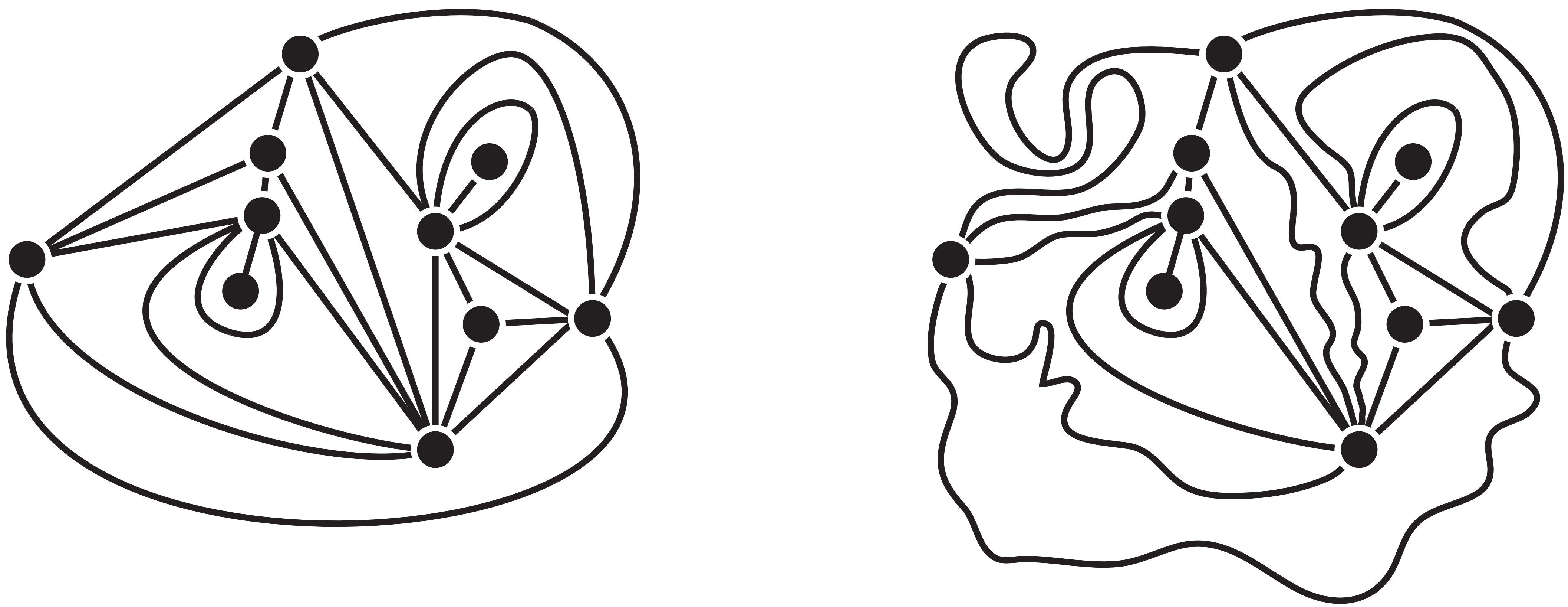

Figure: Two identical 3-angulations .

Why study maps?

- Combinatorics (Tutte starting in '60)
- Probability theory (model for a Brownian surface)
- Algebraix and geometric motivations Motivations (cf Lando–Zvonkine '04 *Graphs on surfaces and their applications*)
- Theoretical physics (connections with matrix integrals, 2D Liouville quantum gravity, KPZ formula.)

Scaling limits of large planar maps

Fix $p \geqslant 3$. Let M_n be a planar map, chosen uniformly at random among all p-angulations with n faces.

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Theorem (Le Gall ($p = 3$ or p odd), Miermont ($p = 4$), 2011)

There exists a constant $c_p > 0$ *and a random compact metric space* (m_∞, D^*), *called the Brownian map, such that the convergence*

$$\left(V(M_n), c_p n^{-1/4} d_{gr}\right) \quad \xrightarrow[n\to\infty]{(d)} \quad (m_\infty, D^*)$$

holds in distribution in the space of isometry classes of compact metric spaces equiped with the Gromov–Hausdorff distance.

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- Le Gall '08: almost surely, (m_∞, D^*) has Hausdorff dimension 4.