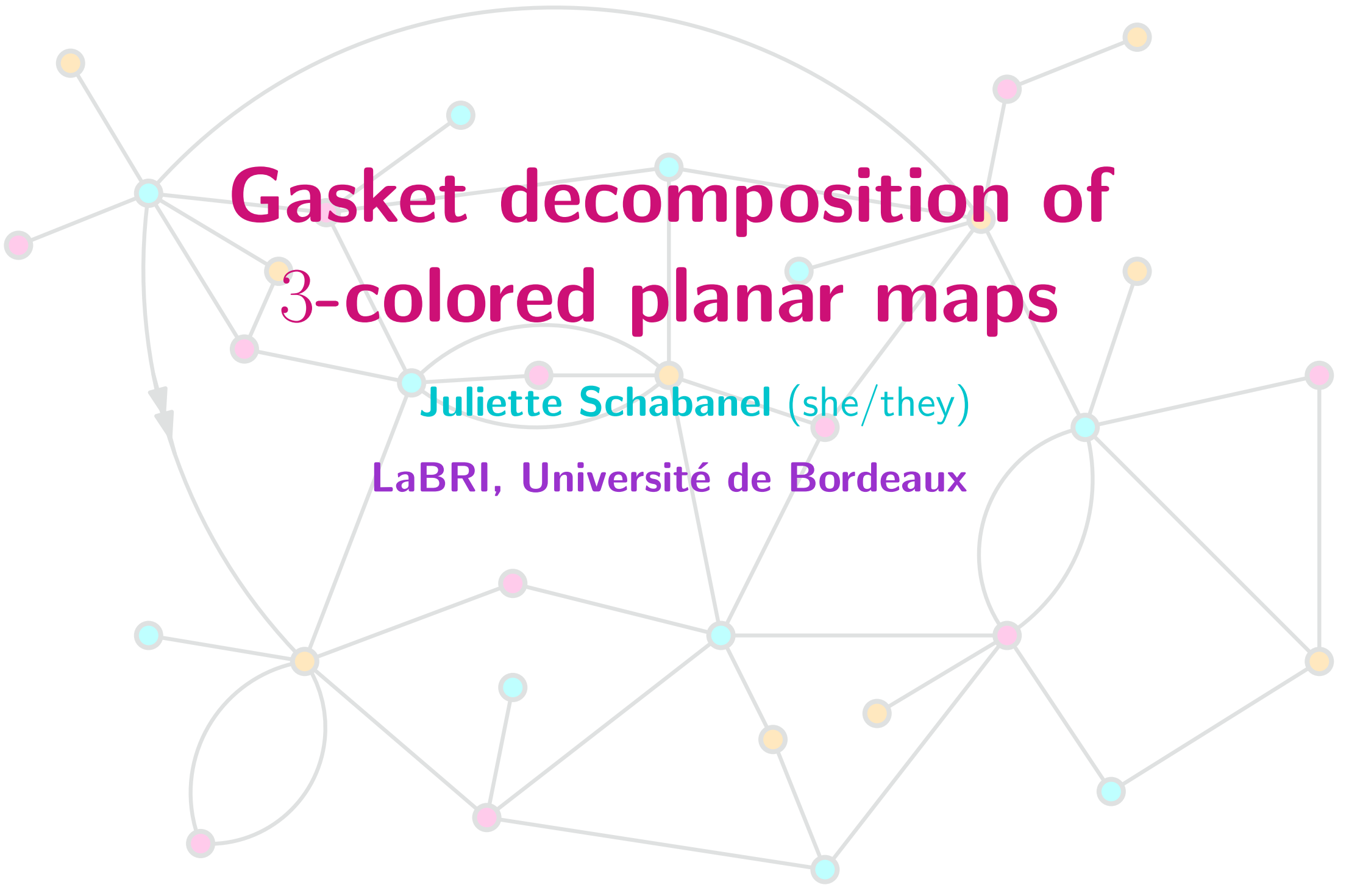
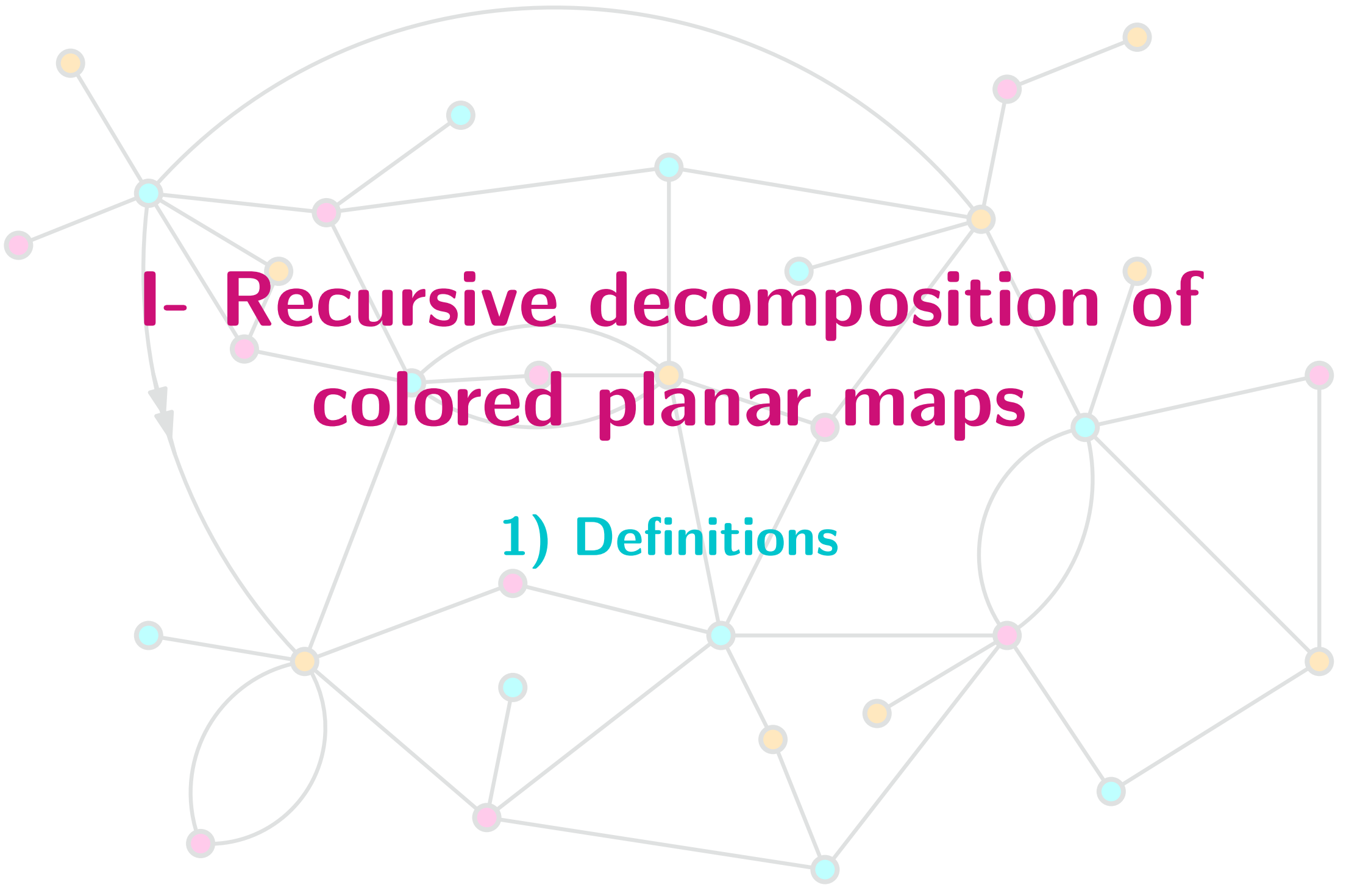




Gasket decomposition of 3-colored planar maps

Juliette Schabanel (she/they)

LaBRI, Université de Bordeaux

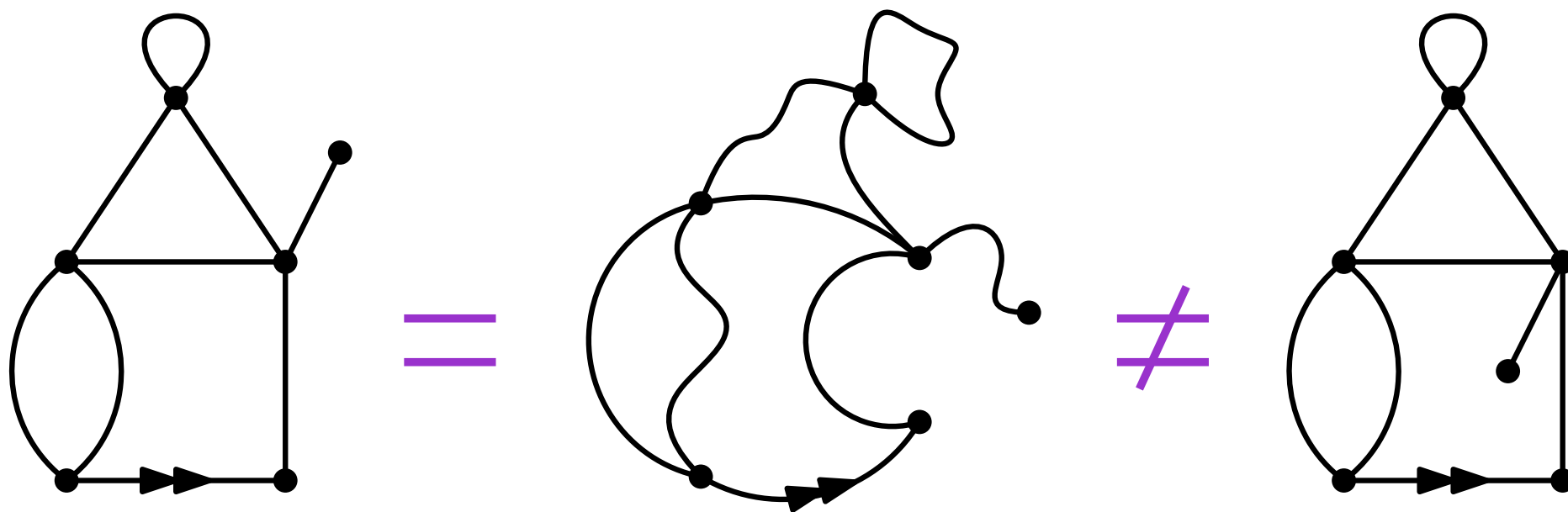


I- Recursive decomposition of colored planar maps

1) Definitions

Planar maps

A **planar map** is the embedding of a connected graph into the sphere, up to orientation preserving homeomorphism. Multi-edges and loops are allowed.



Planar map = planar graph + cyclic ordering of the edges around each vertex.

All maps are **rooted**, i.e. an oriented edge is marked.

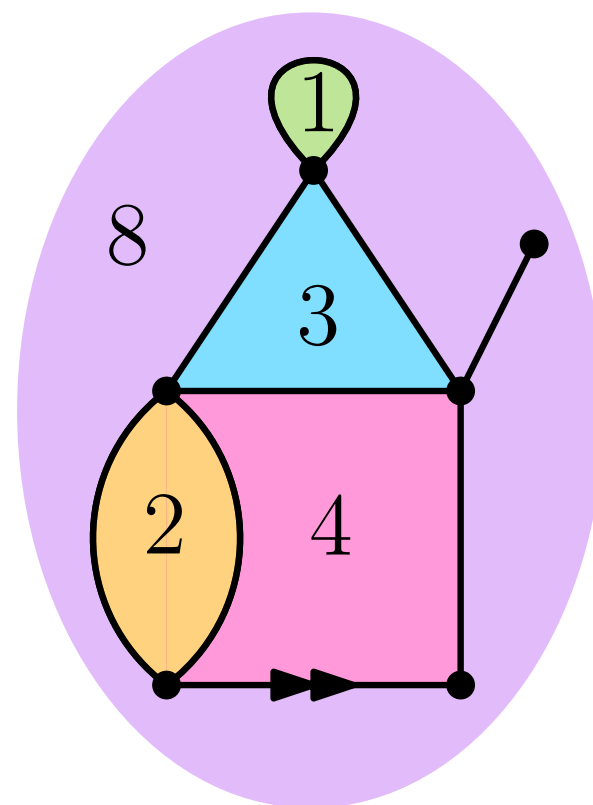
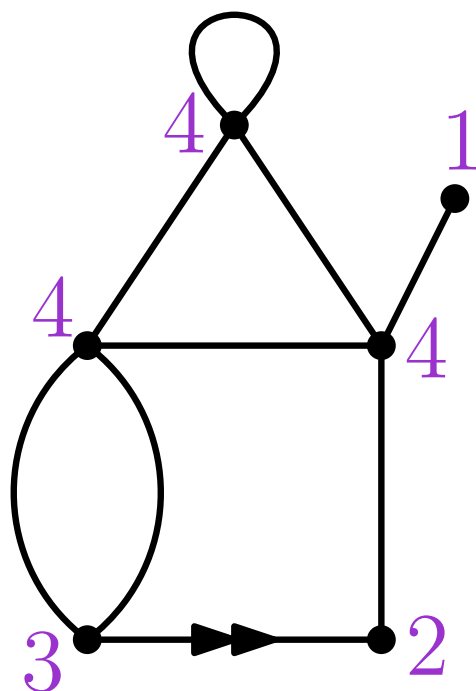
Planar maps

Planar map = planar graph + cyclic ordering of the edges around each vertex.

- **Vertices** and **edges** are inherited from the graph.
- **Faces** are the connected components of the sphere minus the map.

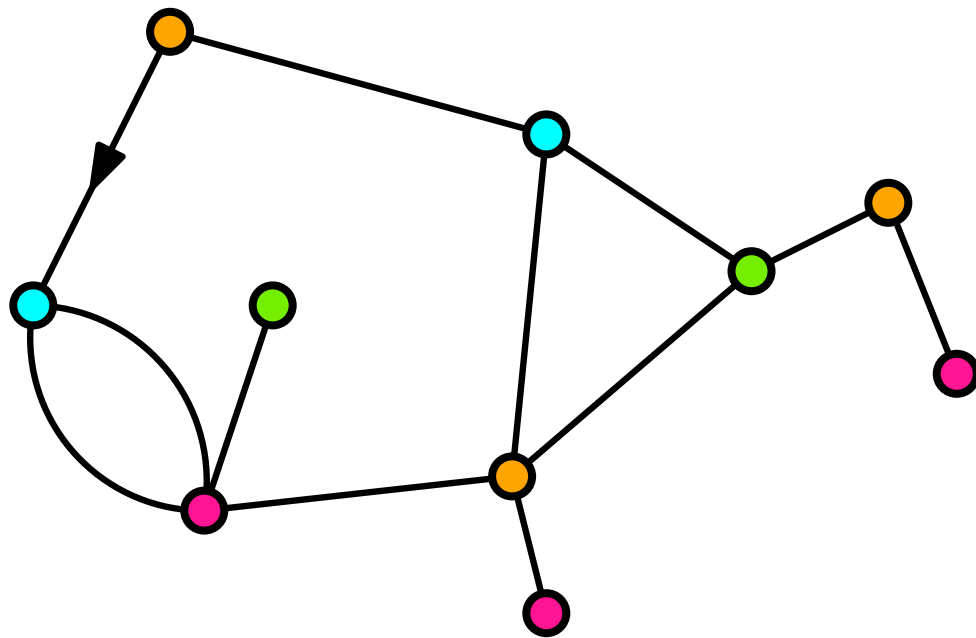
Size: number of edges n .

Degree (of a vertex or face) = number of incident half-edges.

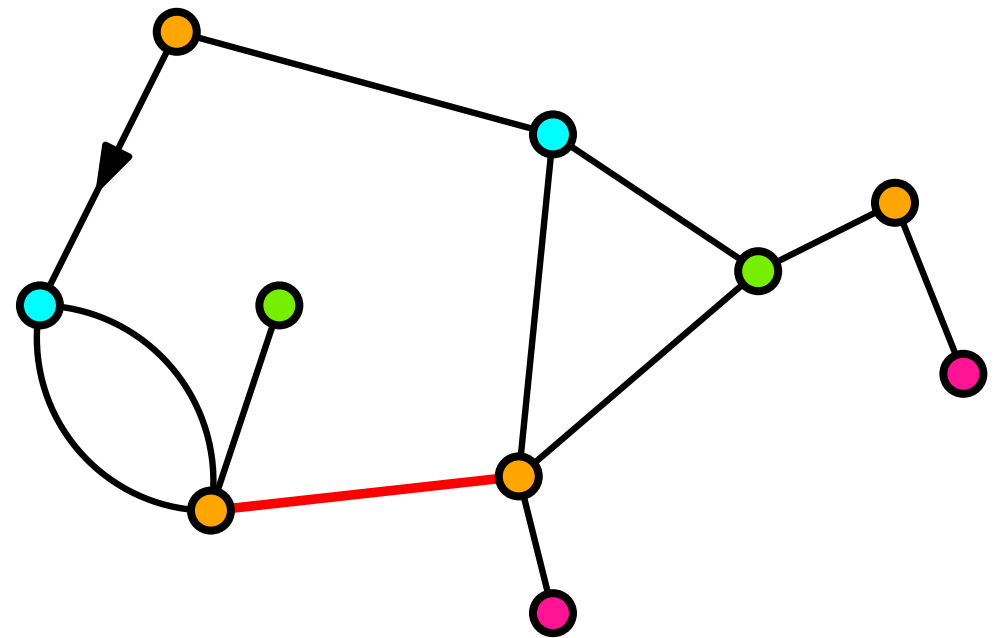


Colored maps

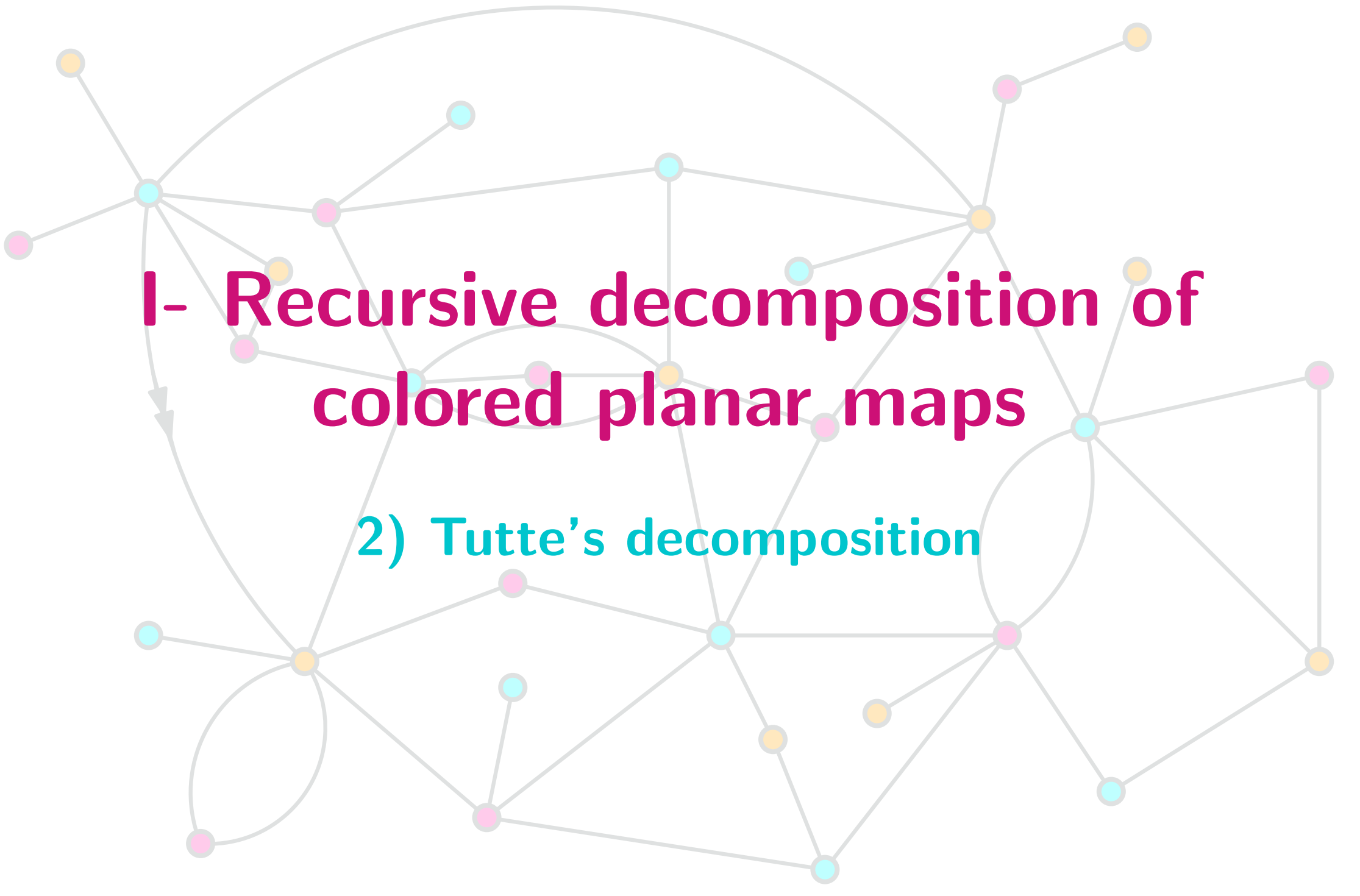
Colored map: assign a color to each vertex so that adjacent vertices receive different colors.



proper coloring



non-proper coloring



I- Recursive decomposition of colored planar maps

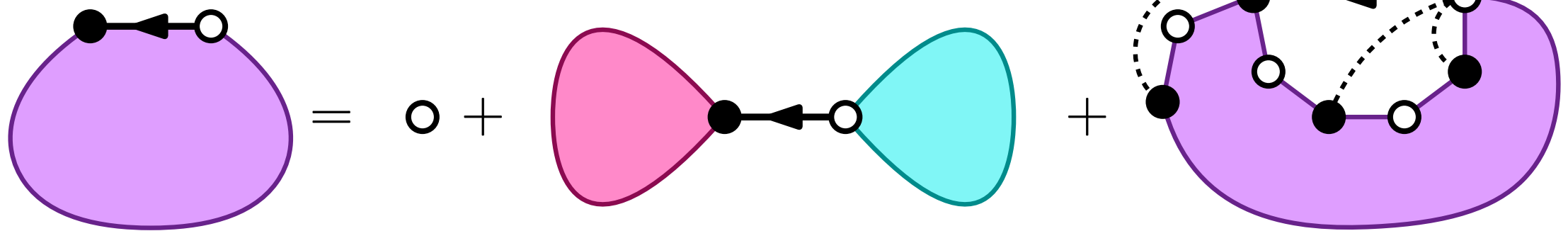
2) Tutte's decomposition

2-colored maps

Generating function of bicolored planar maps:

$$B(t, y) = \sum_{\mathfrak{b} \in \mathcal{B}} t^{e(\mathfrak{b})} y^{d(\mathfrak{b})}$$

with $d(\mathfrak{b})$ the half degree of the outer face.



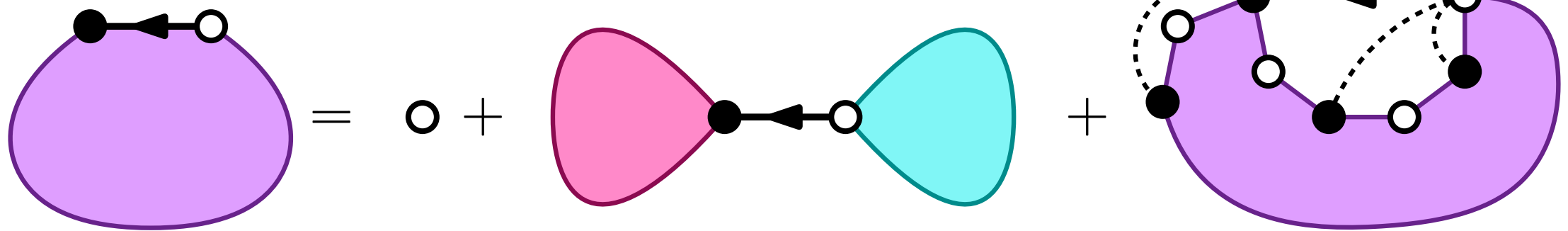
$$B(t, y) = 1 + tyB(t, y)^2 + ty \sum_{p \geq 1} \sum_{k=1}^p y^{p-k} [y^p] B(t, y)$$

2-colored maps

Generating function of bicolored planar maps:

$$B(t, y) = \sum_{\mathfrak{b} \in \mathcal{B}} t^{e(\mathfrak{b})} y^{d(\mathfrak{b})}$$

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$$B(t, y) = 1 + tyB(t, y)^2 + ty \frac{B(t, y) - B(t, 1)}{y - 1}$$

→ Can be solved with the quadratic method [Brown '60s].

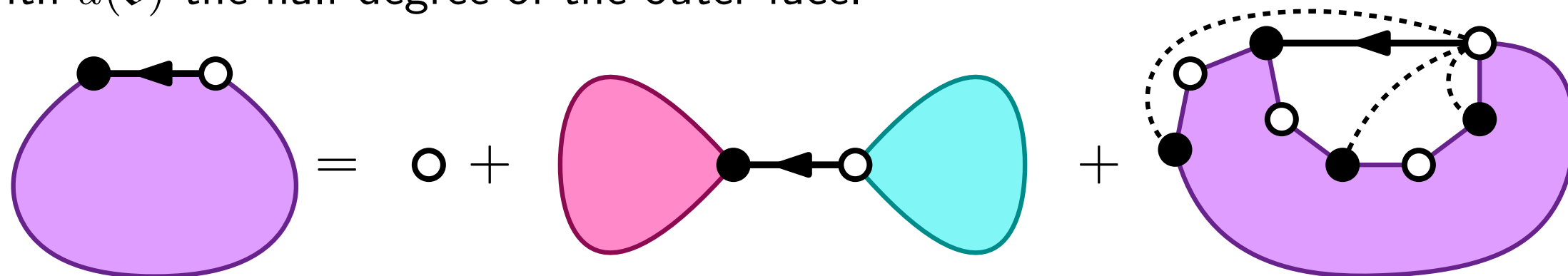
+ This method provides an algebraic equation of degree 4 for $B(t, y)$.

2-colored maps

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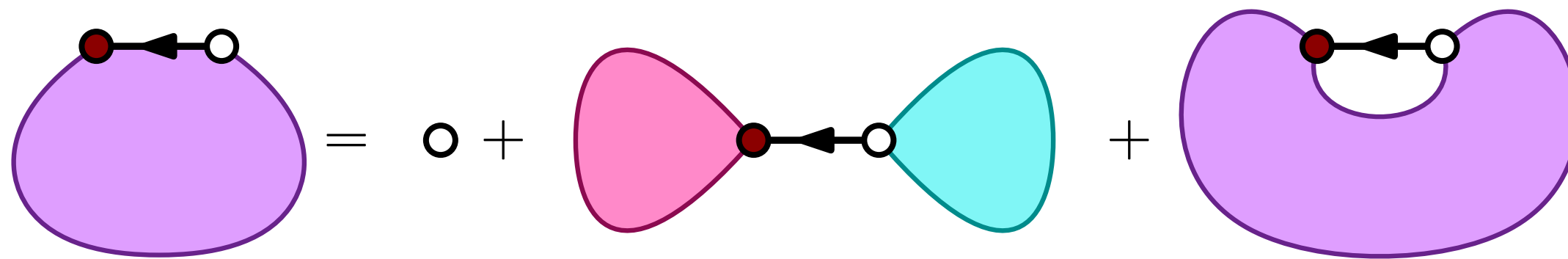


$$B(t, y) = 1 + tyB(t, y)^2 + ty \frac{B(t, y) - B(t, 1)}{y - 1}$$

- Can be solved with the quadratic method [Brown '60s].
- + This method provides an algebraic equation of degree 4 for $B(t, y)$.
- Explicit bijection with a families of trees [Schaeffer '97].

3-colored maps

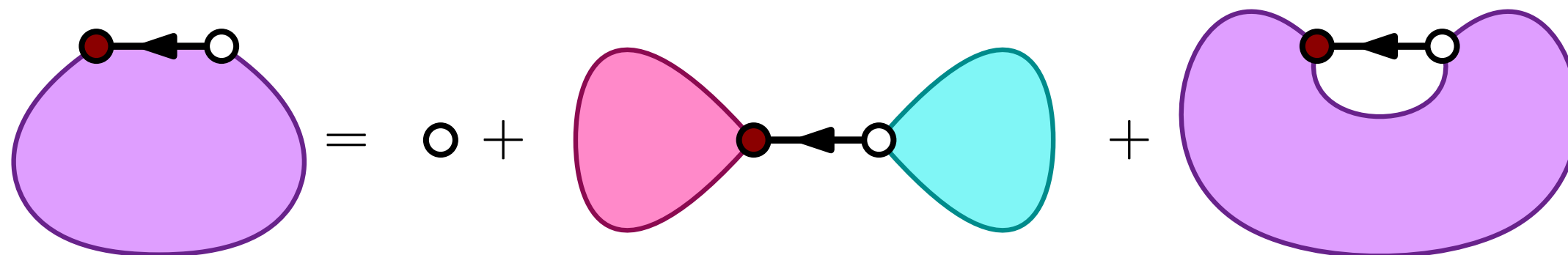
$M(t, y)$: generating function of 3-colored planar maps with $t \rightarrow$ number of edges and $y \rightarrow$ degree of the outer face.



$$M(t, y) = 1 + 2ty^2 M(t, y)^2 + ty???$$

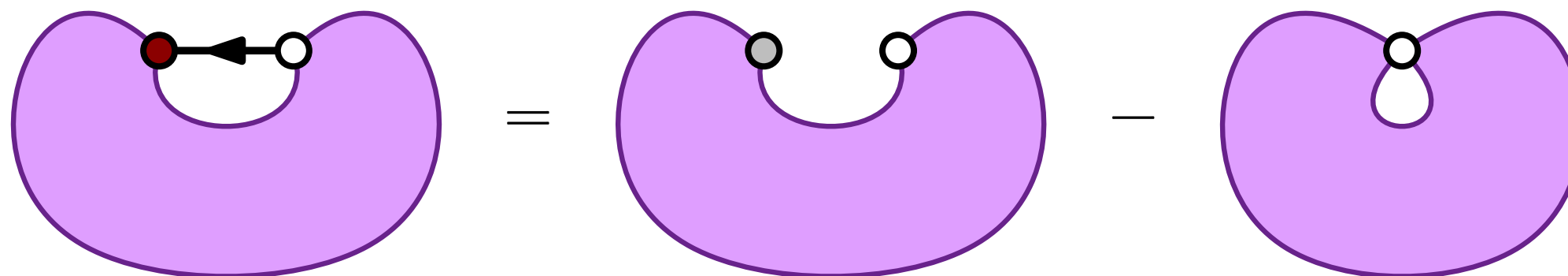
3-colored maps

$M(t, y)$: generating function of 3-colored planar maps with $t \rightarrow$ number of edges and $y \rightarrow$ degree of the outer face.



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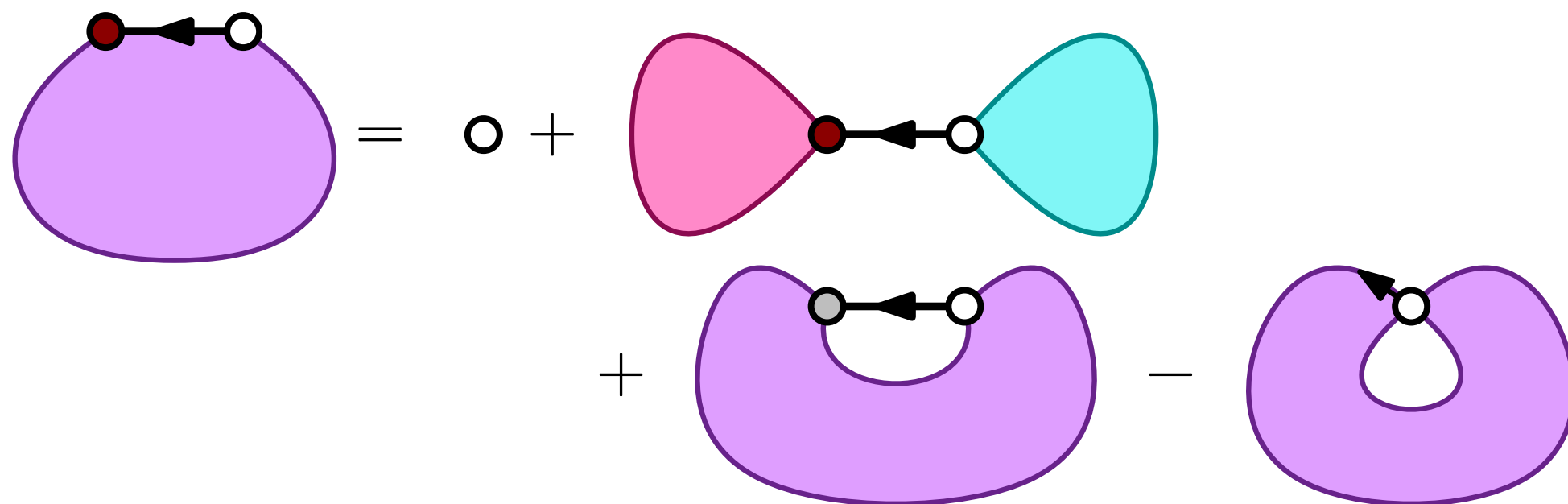
Tutte's solution:



$\Rightarrow$ need a second catalytic variable recording the degree of the root vertex.

3-colored maps

$M(t, x, y)$: generating function of 3-colored planar maps with $t \rightarrow$ number of edges, $y \rightarrow$ degree of the outer face and $x \rightarrow$ degree of the root vertex.



Theorem. [Tutte '68, Bernardi Bousquet-Mélou '11]

$$M(x, y) = 1 + xyt(2y - 1)M(x, y)M(1, y) - xytM(x, y)M(x, 1) - xyt \frac{xM(x, y) - M(1, y)}{x - 1} + xyt \frac{yM(x, y) - M(x, 1)}{y - 1}.$$

3-colored maps

Equation for 3-colored planar maps:

$$M(x, y) = 1 + xyt(2y - 1)M(x, y)M(1, y) - xytM(x, y)M(x, 1) - xyt \frac{xM(x, y) - M(1, y)}{x - 1} + xyt \frac{yM(x, y) - M(x, 1)}{y - 1}.$$

Not an algebraic/D-finite/... equation $\implies$ no toolbox?

2 catalytic variables $\implies$ no method to get an algebraic equation.

Negative coefficients $\implies$ cannot build a random generator.

3-colored maps

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Not an algebraic/D-finite/... equation $\implies$ no toolbox?

2 catalytic variables $\implies$ no method to get an algebraic equation.

Negative coefficients $\implies$ cannot build a random generator.

Theorem. [Bernardi Bousquet-Mélou '11] The generating function of 3-colored planar map is algebraic and is equal to

$$M(1, 1) = \frac{(1 + 2S)(1 - 2S^2 - 4S^3 - 4S^4)}{(1 - 2S^3)^2}$$

with S the solution of $t = \frac{S(1 - S^3)}{(1 + 2S)^3}$ with constant term 0.

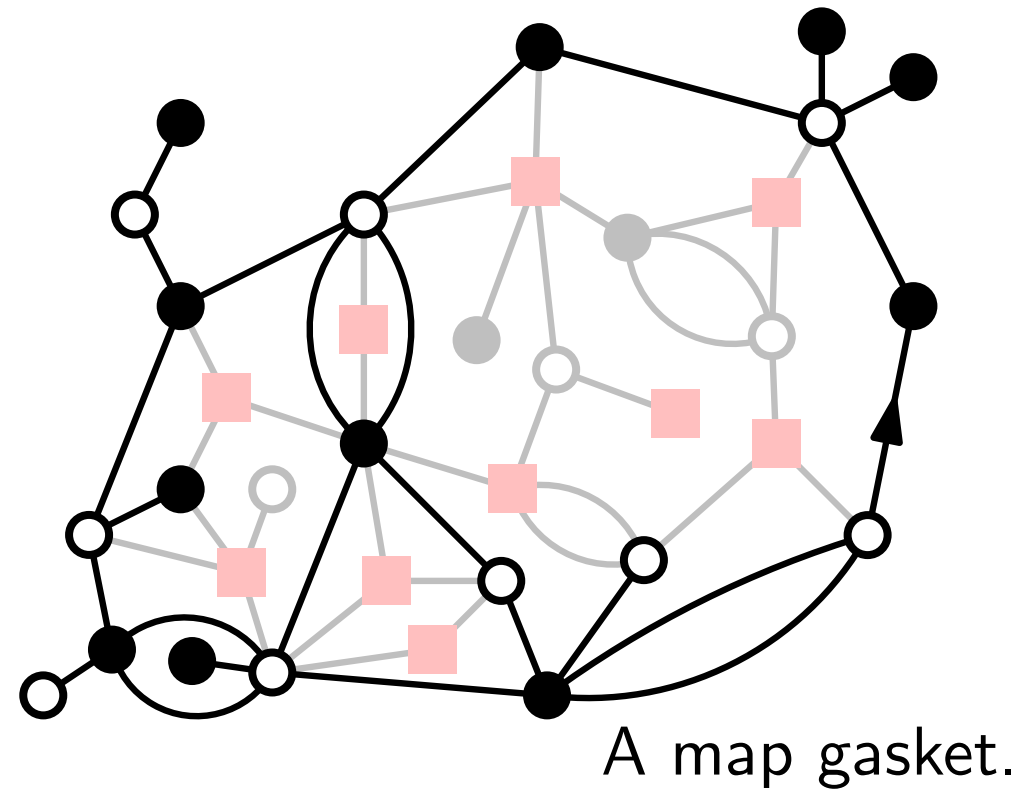
$\hookrightarrow$ Obtained via a reduction to 1-catalytic equation.

I- Recursive decomposition of colored planar maps

3) Gasket decomposition



A car gasket.

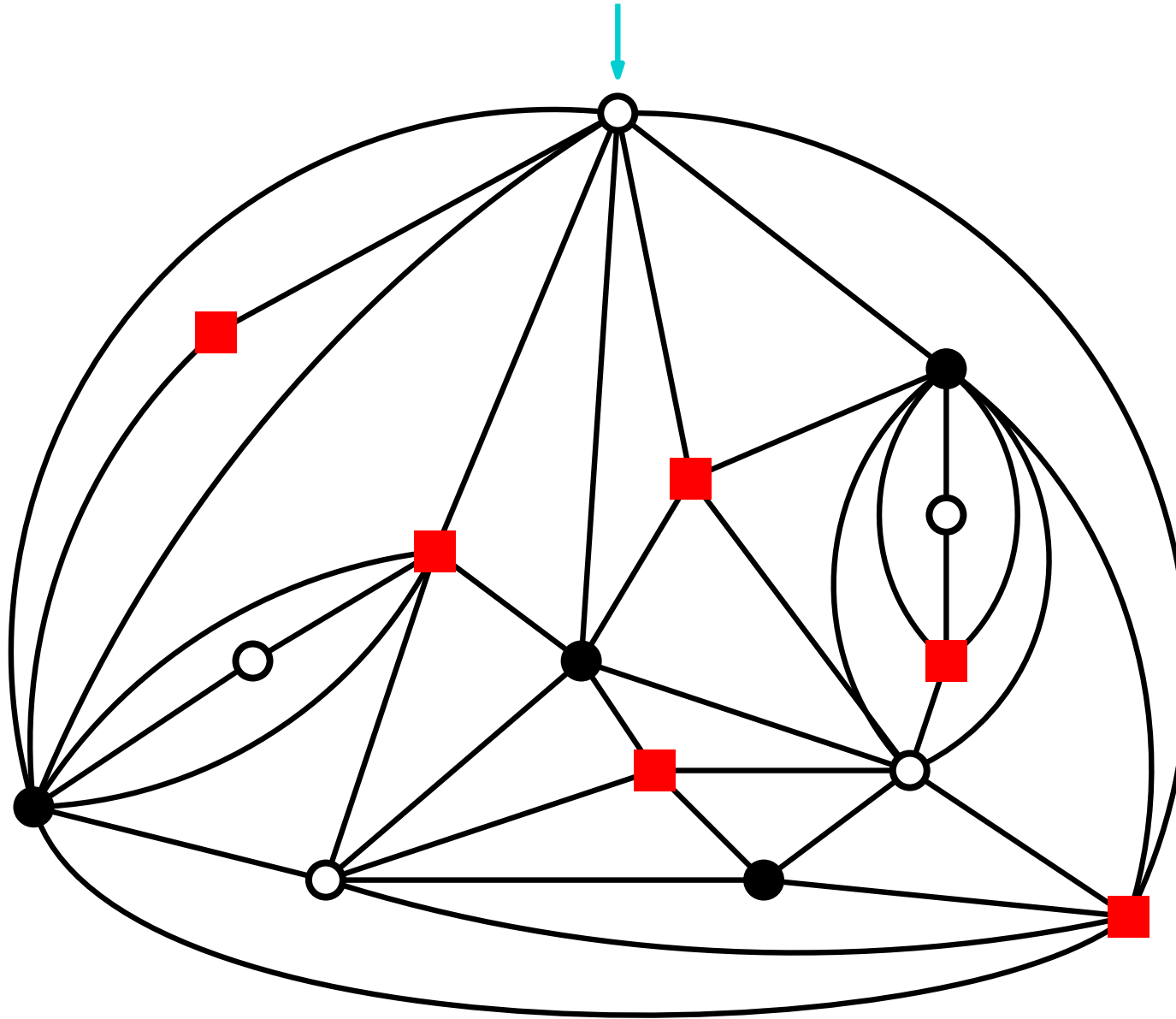


A map gasket.

Term from [Borot Bouttier Guitter '12].

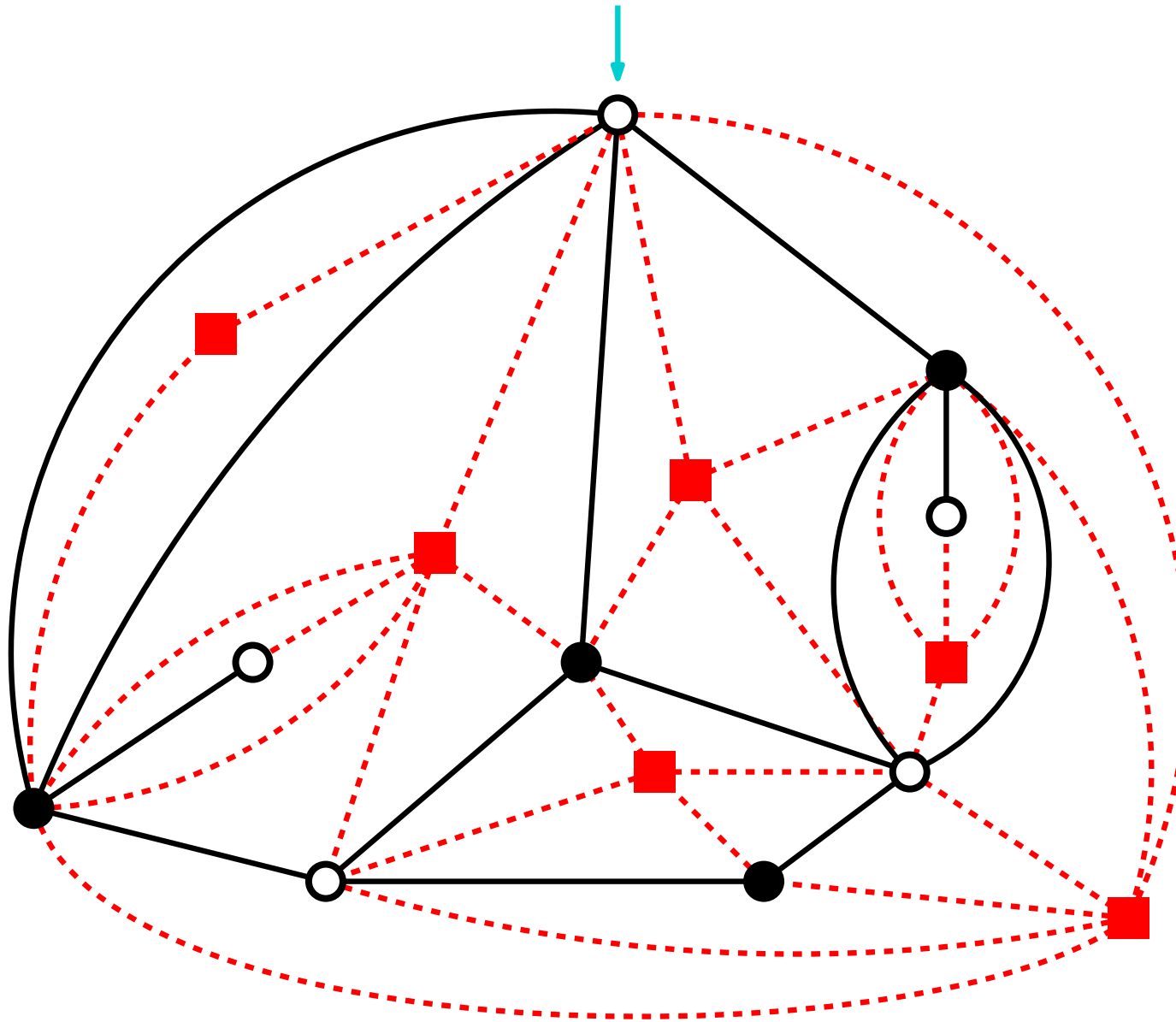
Warm-up: Triangulations

Triangulation = all faces have degree 3.



Warm-up: Triangulations

Triangulation = all faces have degree 3.

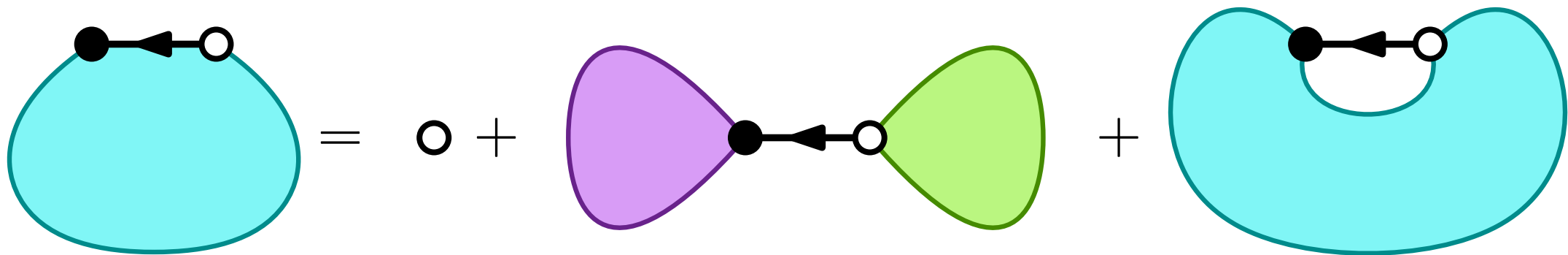


3-colored triangulations = 2-colored general maps.

Gasket decomposition

Idea: Use the structure from the coloring and get back to bipartite maps.

Let $M(t, y) \equiv M(y)$ be the generating function of 3-colored planar maps with a **black and white outer face**, with $t \rightarrow$ edges and $y \rightarrow$ half the degree of the outer face.

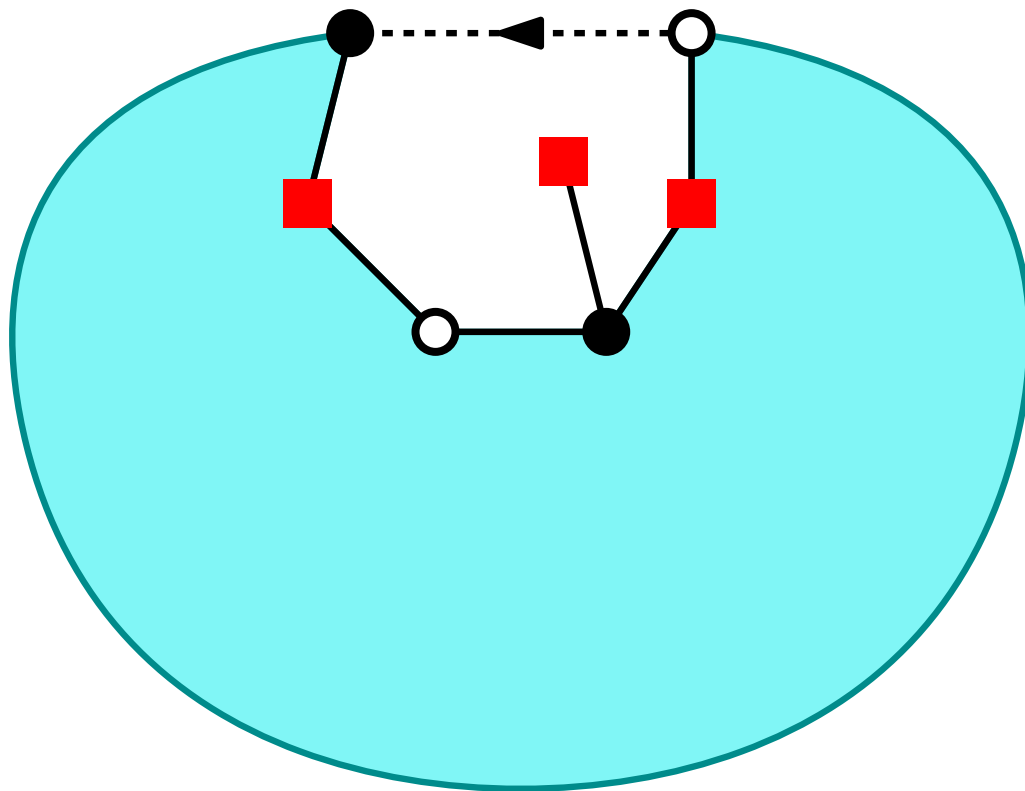


$$M(y) = 1 + tyM(y)^2 + ???$$

Gasket decomposition

Idea: Use the structure from the coloring and get back to bipartite maps.

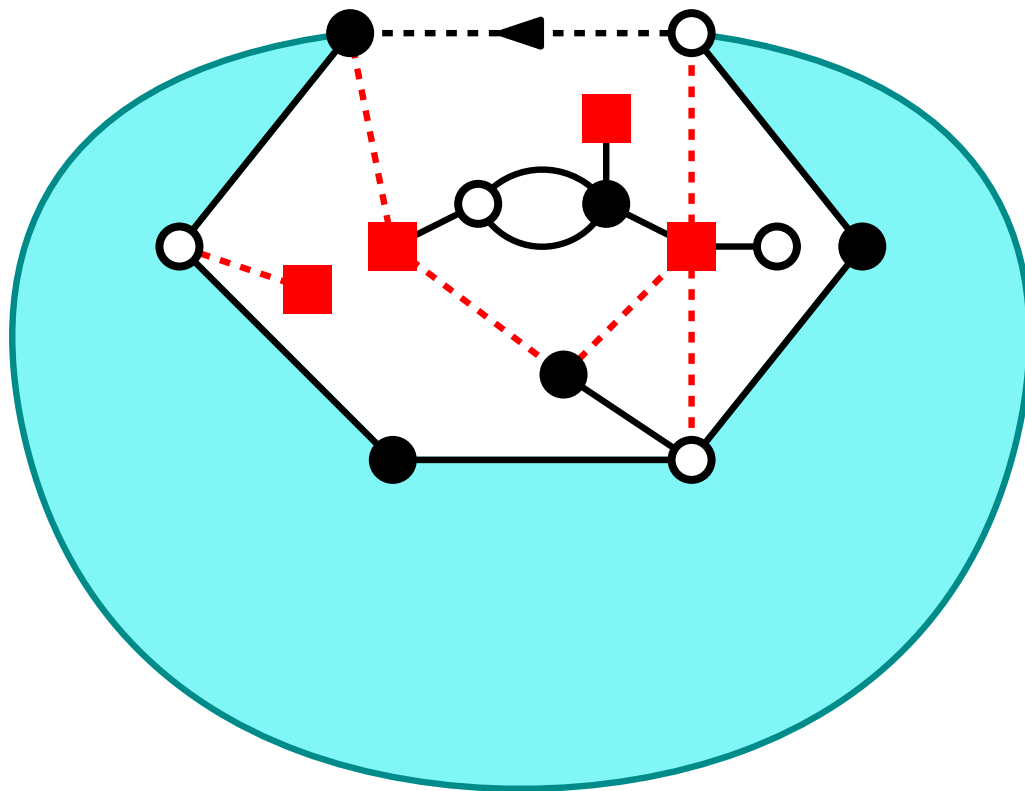
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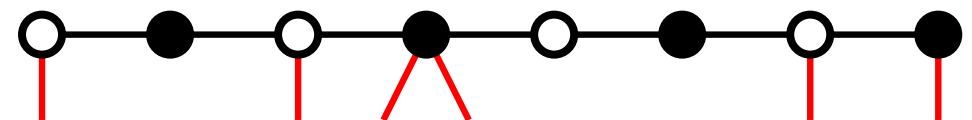
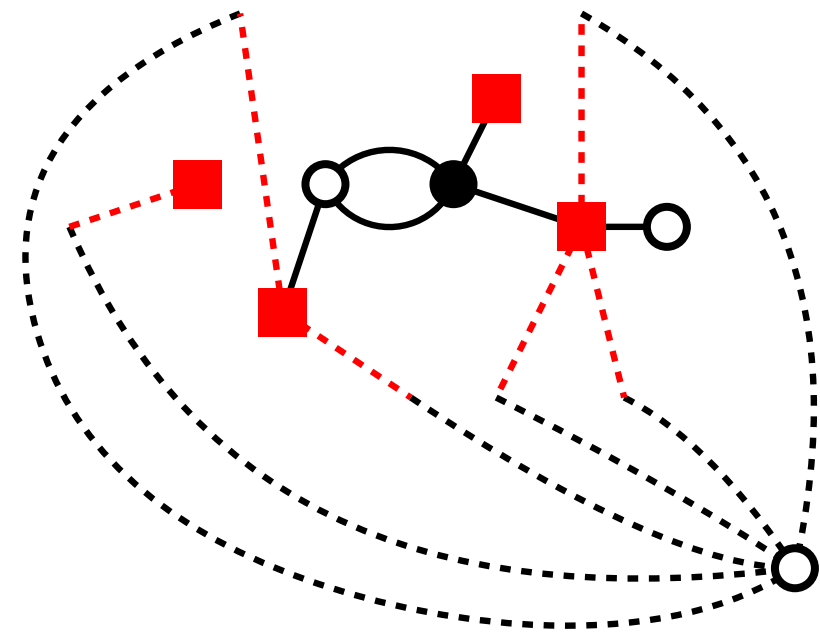
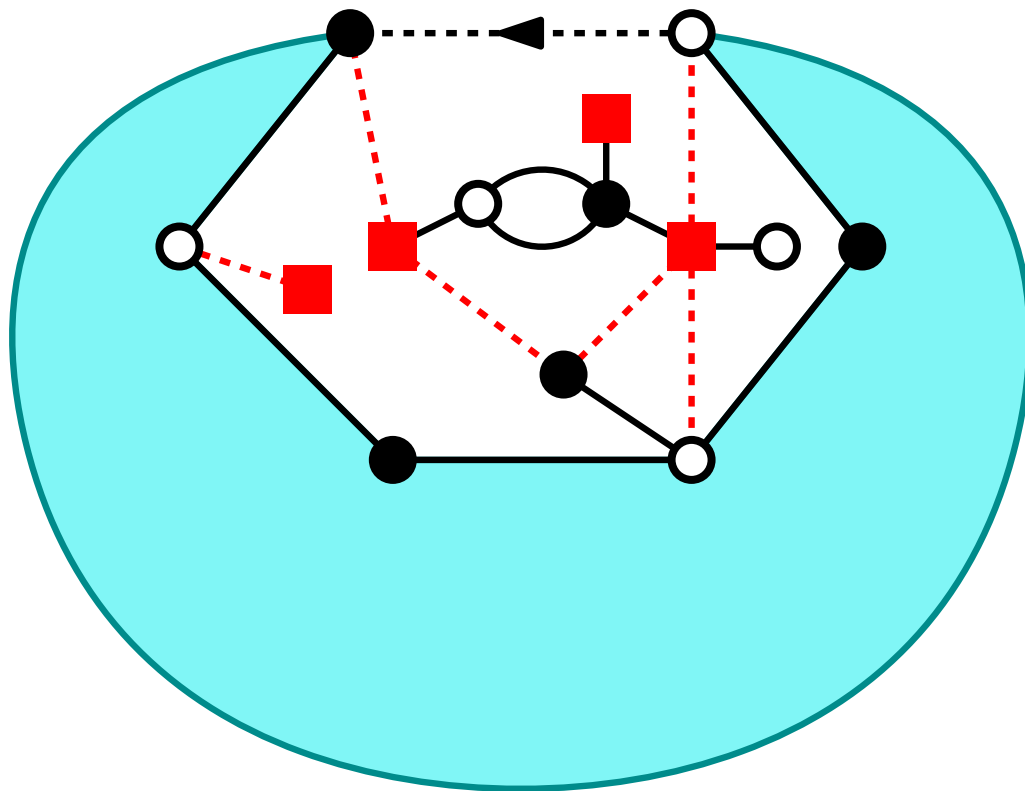
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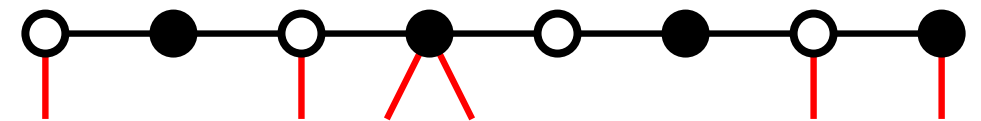
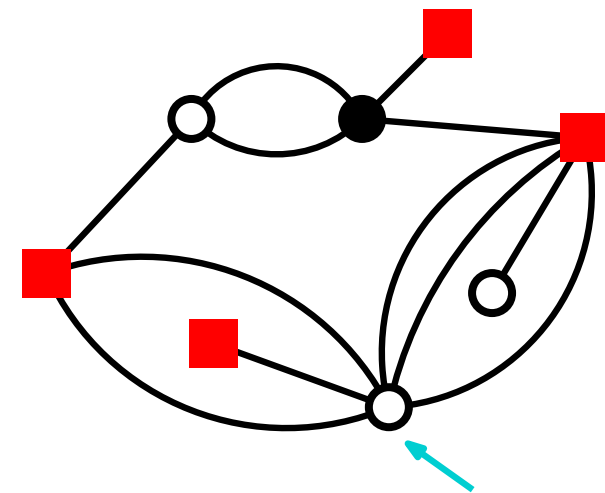
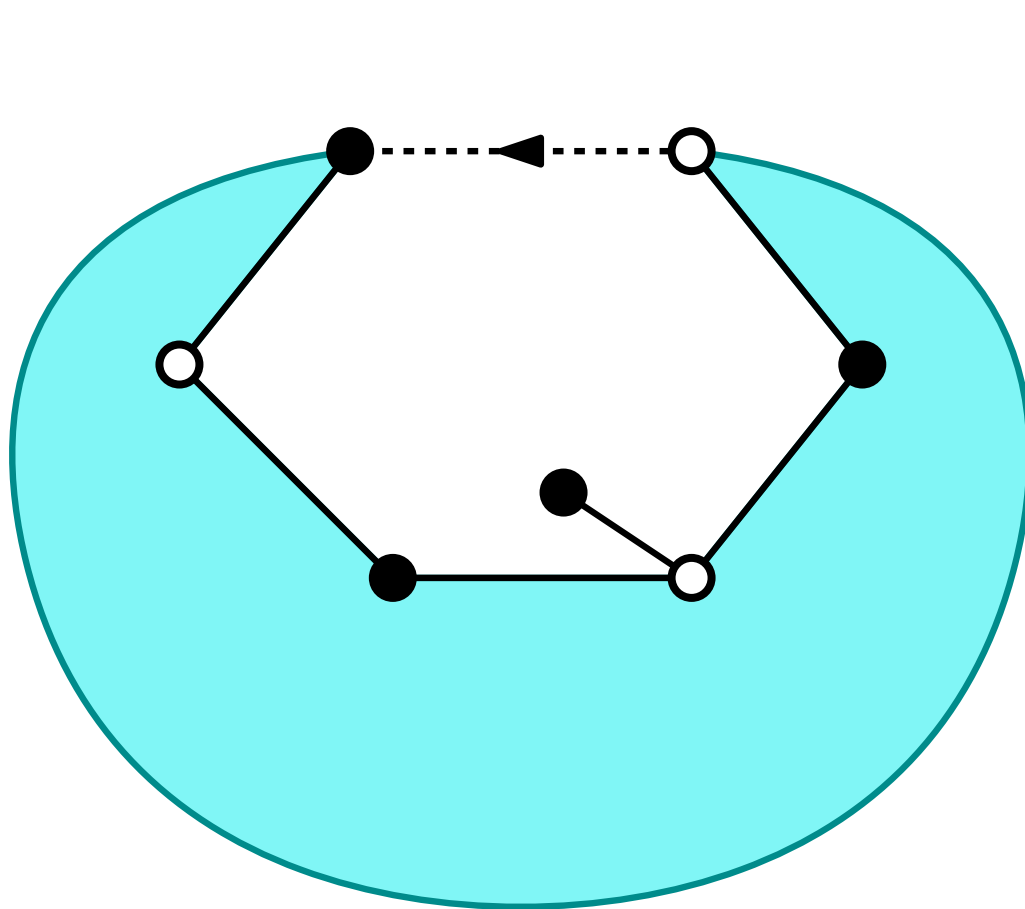
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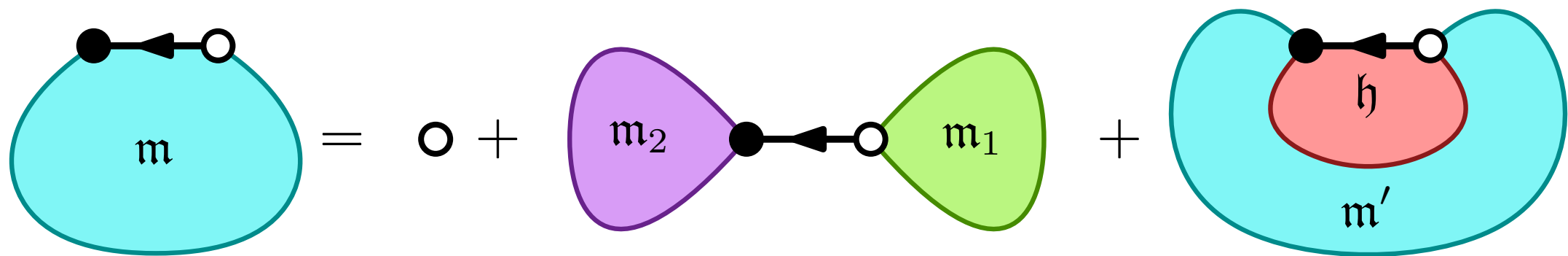


3-colored map where all neighbors of the root vertex are red.
+ connection caterpillar.

Gasket decomposition

Let $M(t, y) \equiv M(y)$ be the generating function of 3-colored planar maps with a **black and white outer face**, with $t \rightarrow$ edges and $y \rightarrow$ half the degree of the outer face.

Let $H(t, x) \equiv H(x)$ be the generating function of 3-colored planar maps where all the **neighbors** of the **root vertex** are **red**, with $t \rightarrow$ edges and $x \rightarrow$ degree of the root vertex.



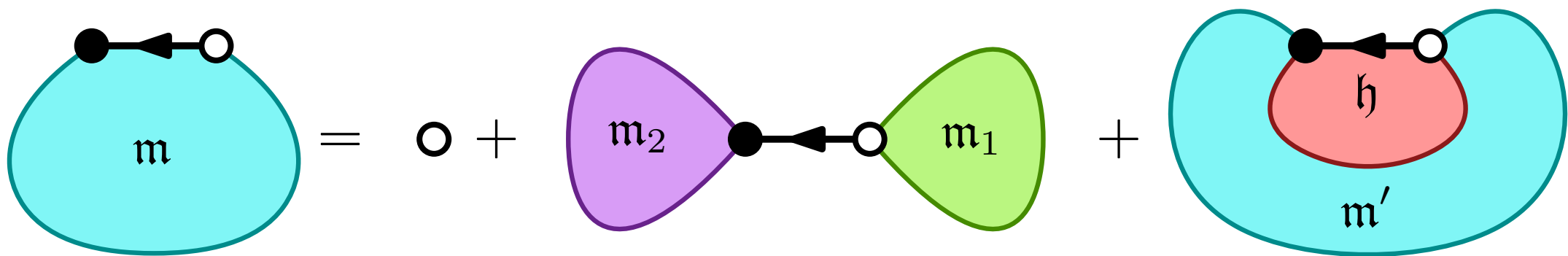
$$M(y) = 1 + tyM(y)^2 + ty \sum_{p \geq 1} \sum_{k=1}^p h_k y^{p-k} [y^p] M(y),$$

with $h_k = \sum_{d \geq 0} \binom{2k+d-1}{2k-1} [x^d] H(x).$

Gasket decomposition

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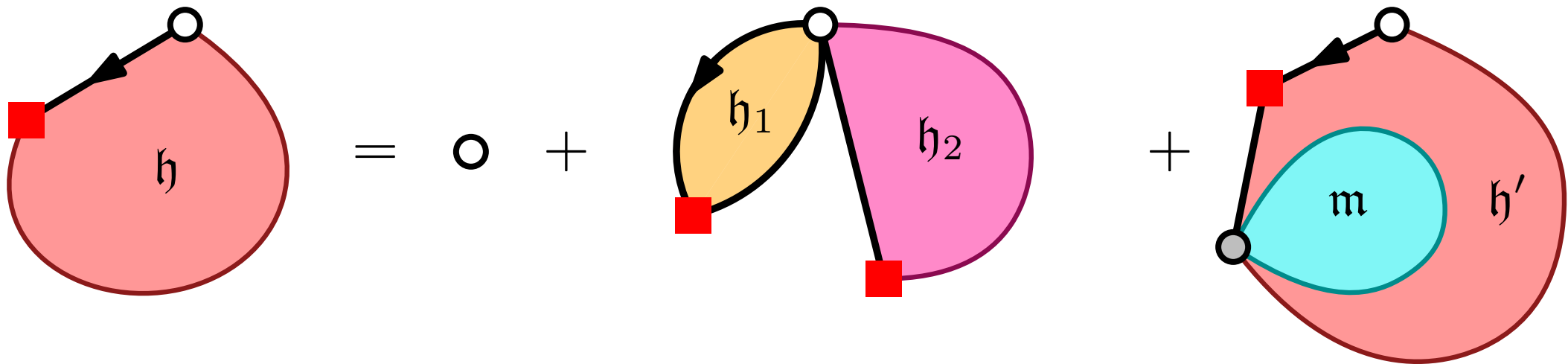
$$M(y) = 1 + tyM(y)^2 + ty[y^{\geq 0}](M(y) - 1)\tilde{H}(\bar{y}), \quad \bar{y} = 1/y$$

$$\begin{aligned} \tilde{H}(y) &= \sum_{k \geq 1} y^k \sum_{d \geq 0} \binom{2k+d-1}{2k-1} [x^d] H(x) \\ &= \frac{1}{2} \left[\frac{\sqrt{y}}{1-\sqrt{y}} H\left(\frac{1}{1-\sqrt{y}}\right) - \frac{\sqrt{y}}{1+\sqrt{y}} H\left(\frac{1}{1+\sqrt{y}}\right) \right]. \end{aligned}$$

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$$H(x) = 1 + txH(x)^2 + 2tx[x^{\geq 0}](H(x) - 1)\widetilde{M}(\overline{x}),$$

$$\widetilde{M}(x) = \sum_{i \geq 1} x^i \sum_{p \geq 0} \binom{2p+i-1}{i-1} [y^p] M(t, y) = \frac{x}{(1-x)^2} M\left(t, \frac{1}{(1-x)^2}\right).$$

Gasket decomposition

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Let $H(t, x) \equiv H(x)$ be the generating function of 3-colored planar maps where all the **neighbors** of the **root vertex** are **red**, with $t \rightarrow$ edges and $x \rightarrow$ degree of the root vertex.

Theorem. [S. '26]

$$\begin{cases} M(t, y) = 1 + tyM(t, y)^2 + ty[y^{\geq 0}](M(t, y) - 1)\tilde{H}(t, \bar{y}), \\ H(t, x) = 1 + txH(t, x)^2 + 2tx[x^{\geq 0}](H(t, x) - 1)\widetilde{M}(t, \bar{x}), \end{cases}$$

with

$$\begin{aligned} \widetilde{M}(t, x) &= \frac{x}{(1-x)^2} M\left(t, \frac{1}{(1-x)^2}\right), \\ \tilde{H}(t, y) &= \frac{1}{2} \left[\frac{\sqrt{y}}{1-\sqrt{y}} H\left(t, \frac{1}{1-\sqrt{y}}\right) - \frac{\sqrt{y}}{1+\sqrt{y}} H\left(t, \frac{1}{1+\sqrt{y}}\right) \right]. \end{aligned}$$

$$P_M(S, y, M) = 4y^5(y-1)^2S^5(2S^3-1)^6M^6 + 24y^4(y-1)^2(1+2S)^3(2S^3-1)^5S^4M^5 - (1+2S)(2S^3-1)^4((64S^6+96S^5+88S^4+32S^3-2S+1)y^2 - (16S^4+456S^3+684S^2+334S+57)(1+2S)^2y + 56(1+2S)^5)S^3(y-1)y^3M^4 - 4(2S^3-1)^3(1+2S)^4((64S^6+96S^5+88S^4+32S^3-2S+1)y^2 - (16S^4+136S^3+204S^2+94S+17)(1+2S)^2y + 16(1+2S)^5)S^2(y-1)y^2M^3 - (1+2S)^2(2S^3-1)^2(2S(64S^6+96S^5+94S^4+50S^3+18S^2+4S+1)(S+1)^3y^4 + (1+2S)(6144S^{10}+20928S^9+33600S^8+32064S^7+18380S^6+5244S^5-162S^4-536S^3-87S^2+18S+5)y^3 - 2(384S^6+1120S^5+1914S^4+1604S^3+706S^2+171S+23)(1+2S)^5y^2 + (96S^4+616S^3+924S^2+414S+77)(1+2S)^7y - 36(1+2S)^{10})SyM^2 - 2(2S^3-1)(1+2S)^5(2S(64S^6+96S^5+88S^4+32S^3-2S+1)(S+1)^3y^4 + (1+2S)(2048S^{10}+6592S^9+9536S^8+7488S^7+4508S^6-1003S^5-1546S^4-551S^3-119S^2-6S+1)y^3 - 2(128S^6+224S^5+266S^4+164S^3+50S^2+11S+3)(1+2S)^5y^2 + (32S^4+72S^3+108S^2+38S+9)(1+2S)^7y - 4(1+2S)^{10})M - (1+2S)^3(S(4S^2+2S+1)^3(S+1)^6y^4 + 2(64S^6+96S^5+94S^4+50S^3+18S^2+4S+1)(S+1)^3(1+2S)^5y^3 + (1024S^9+3008S^8+3520S^7+1344S^6-1460S^5-2436S^4-1642S^3-616S^2-127S-12)(1+2S)^6y^2 - 2(64S^5+345S^4+414S^3-36S^2-34S-9)(1+2S)^{10}y + 8(2S^3-1)(1+2S)^{12})$$

II- Applications

1) Algebraicity proof

$$P_H(S, \sqrt{\Delta}, x, H) = S^3x^5(2S^3-1)^4(2x-1)H^4 - 2S^2x^3(2x-1)(x-2)(1+2S)^3(2S^3-1)^3H^3 + Sx(1+2S)(2S^3-1)^2(2S(2S^3+4S^2+2S+1)x^4 - 2(8S^4+8S^3+12S^2+2S+1)(1+2S)^2x^3 + (8S^4-24S^3-36S^2-22S-3)(1+2S)^2x^2 + 10(1+2S)^5x - 4(1+2S)^5)H^2 - 2(1+2S)(2S^3-1)(S((8S^5+28S^4+38S^3+26S^2+10S+2)\sqrt{\Delta} + 16S^6+56S^5+76S^4+58S^3+28S^2+8S+1)x^4 - 2(16S^6+48S^5+86S^4+76S^3+34S^2+9S+1)(1+2S)^3x^3 + (20S^4+56S^3+84S^2+32S+7)(1+2S)^5x^2 - (8S^4+56S^3+84S^2+38S+7)(1+2S)^5x + 2(1+2S)^8)H(1+2S)^3((32S^8+(16S^6+64S^5+104S^4+90S^3+46S^2+14S+2)\sqrt{\Delta} + 176S^7+384S^6+452S^5+356S^4+182S^3+64S^2+15S+2)x^3 - 2(1+2S)(-64S^9-320S^8-656S^7+(8S^5+28S^4+38\sqrt{\Delta}S^3+26S^2+10S+2)\sqrt{\Delta} - 560S^6-72S^5+332S^4+354S^3+172S^2+44S+5)x^2 - 2(8S^7+96S^6+144S^5+60S^4-46S^3-80S^2-38S-7)(1+2S)^3x + 4(2S^3-1)(1+2S)^6)$$

The strategy: Guess and Check

$$\begin{cases} M(t, y) = 1 + tyM(t, y)^2 + ty[y^{\geq 0}](M(t, y) - 1)\tilde{H}(t, \bar{y}), \\ H(t, x) = 1 + txH(t, x)^2 + 2tx[x^{\geq 0}](H(t, x) - 1)\widetilde{M}(t, \bar{x}). \end{cases}$$

Idea: The equations allow us to recursively compute all the coefficients of the series M and H

$\implies$ unique solution in $\mathbb{Q}[y][[t]] \times \mathbb{Q}[x][[t]]$.

Guess and Check:

1. Guess polynomial equations satisfied by our series.
2. Prove that the solutions of the polynomial equations satisfy the original equations.

Guess

$$\begin{cases} M(t, y) = 1 + tyM(t, y)^2 + ty[y^{\geq 0}](M(t, y) - 1)\tilde{H}(t, \bar{y}), \\ H(t, x) = 1 + txH(t, x)^2 + 2tx[x^{\geq 0}](H(t, x) - 1)\tilde{M}(t, \bar{x}). \end{cases}$$

Guess:

1. Use the equations to compute the first (250) terms of the series M and H .
2. Guess polynomial equations with `seriestoalgeq` (package gfun).
3. Determine which solution of the polynomial equation is the candidate.

Guess

$$\begin{cases} M(t, y) = 1 + tyM(t, y)^2 + ty[y^{\geq 0}](M(t, y) - 1)\tilde{H}(t, \bar{y}), \\ H(t, x) = 1 + txH(t, x)^2 + 2tx[x^{\geq 0}](H(t, x) - 1)\tilde{M}(t, \bar{x}). \end{cases}$$

Guess:

1. Use the equations to compute the first (250) terms of the series M and H .
2. Guess polynomial equations with `seriestoalgeq` (package gfun).
3. Determine which solution of the polynomial equation is the candidate.

→ We use the intermediate field $\mathbb{Q}(t, y) \rightarrow \mathbb{Q}(S, y) \rightarrow \mathbb{Q}(t, y, M(t, y))$
where $t = \frac{S(1-S^3)}{(1+2S)^3}$ to have smaller equations.

→ Equations of degree 6 for $M(t, y)$ over $\mathbb{Q}(S, y)$ and of degree 8 for $H(t, x)$ over $\mathbb{Q}(S, x)$.

Both polynomial equations have only one solution that is a power series in t .

→ We denote them M_c and H_c .

Guess

1. Use the equations to compute the first (250) terms of the series M and H .
2. Guess polynomial equations with **algeqto**series (package gfun).
3. Determine which solution of the polynomial equation is the candidate.

→ Equations of degree 6 for $M(t, y)$ over $\mathbb{Q}(S, y)$ and of degree 8 for $H(t, x)$ over $\mathbb{Q}(S, x)$.

$$\begin{aligned} P_M(S, y, M) = & 4y^5(y-1)^2S^5(2S^3-1)^6M^6 + 24y^4(y-1)^2(1+2S)^3(2S^3-1)^5S^4M^5 - (1+ \\ & 2S)(2S^3-1)^4((64S^6+96S^5+88S^4+32S^3-2S+1)y^2 - (16S^4+456S^3+684S^2+334S+57)(1+ \\ & 2S)^2y + 56(1+2S)^5)S^3(y-1)y^3M^4 - 4(2S^3-1)^3(1+2S)^4((64S^6+96S^5+88S^4+32S^3- \\ & 2S+1)y^2 - (16S^4+136S^3+204S^2+94S+17)(1+2S)^2y + 16(1+2S)^5)S^2(y-1)y^2M^3 - (1+ \\ & 2S)^2(2S^3-1)^2(2S(64S^6+96S^5+94S^4+50S^3+18S^2+4S+1)(S+1)^3y^4 + (1+2S)(6144S^{10}+ \\ & 20928S^9+33600S^8+32064S^7+18380S^6+5244S^5-162S^4-536S^3-87S^2+18S+5)y^3 - \\ & 2(384S^6+1120S^5+1914S^4+1604S^3+706S^2+171S+23)(1+2S)^5y^2 + (96S^4+616S^3+ \\ & 924S^2+414S+77)(1+2S)^7y - 36(1+2S)^{10})SyM^2 - 2(2S^3-1)(1+2S)^5(2S(64S^6+96S^5+ \\ & 94S^4+50S^3+18S^2+4S+1)(S+1)^3y^4 + (1+2S)(2048S^{10}+6592S^9+9536S^8+7488S^7+ \\ & 2508S^6-900S^5-1346S^4-600S^3-119S^2-6S+1)y^3 - 2(128S^6+224S^5+266S^4+164S^3+ \\ & 50S^2+11S+3)(1+2S)^5y^2 + (32S^4+72S^3+108S^2+38S+9)(1+2S)^7y - 4(1+2S)^{10})M - (1+ \\ & 2S)^3(S(4S^2+2S+1)^3(S+1)^6y^4 + 2(64S^6+96S^5+94S^4+50S^3+18S^2+4S+1)(S+1)^3(1+ \\ & 2S)^5y^3 + (1024S^9+3008S^8+3520S^7+1344S^6-1460S^5-2436S^4-1642S^3-616S^2-127S- \\ & 12)(1+2S)^6y^2 - 2(64S^5+64S^4+14S^3-36S^2-34S-9)(1+2S)^{10}y + 8(2S^3-1)(1+2S)^{12}) \end{aligned}$$

Check

Check: Prove that M_c and H_c satisfy

$$\begin{cases} M_c(t, y) = 1 + tyM_c(t, y)^2 + ty[y \geq 0](M_c(t, y) - 1)\tilde{H}_c(t, \bar{y}), \\ H_c(t, x) = 1 + txH_c(t, x)^2 + 2tx[x \geq 0](H_c(t, x) - 1)\widetilde{M}_c(t, \bar{x}), \end{cases}$$

with $\widetilde{M}_c(t, x) = \frac{x}{(1-x)^2} M_c\left(t, \frac{1}{(1-x)^2}\right)$

and $\tilde{H}_c(t, y) = \frac{1}{2} \left[\frac{\sqrt{y}}{1-\sqrt{y}} H_c\left(t, \frac{1}{1-\sqrt{y}}\right) - \frac{\sqrt{y}}{1+\sqrt{y}} H_c\left(t, \frac{1}{1+\sqrt{y}}\right) \right].$

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1. Reformulate the equations:

$$\begin{cases} [y^{\geq 0}]R_M(y) = 0, \text{ with } R_M(y) \in \mathbb{Q}((\bar{y}))[[t]] \\ [x^{\geq 0}]R_H(x) = 0, \text{ with } R_H(x) \in \mathbb{Q}((\bar{x}))[[t]] \end{cases}$$

↪ Let $F(t, y) \in \mathbb{Q}((\bar{y}))[[t]]$ be an algebraic series in t given by its minimal polynomial and first coefficients. How to decide whether some coefficient of F has a **pole** at $\bar{y} = 0$?

↪ Sometimes easy to see: $F(y) = y \frac{Pol(t, F(y))}{1 + tF(y)}.$

Check

Check: Prove that M_c and H_c satisfy

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2. Compute minimal polynomials for R_M and R_H by eliminating M_c and H_c using resultants.
3. Find intermediate field extensions to lower the degrees.
4. Prove negativity of the powers of y and x .

Algebraicity

a.
$$\begin{cases} M(t, y) = 1 + tyM(t, y)^2 + ty[y^{\geq 0}](M(t, y) - 1)\tilde{H}(t, \bar{y}), \\ H(t, x) = 1 + txH(t, x)^2 + 2tx[x^{\geq 0}](H(t, x) - 1)\widetilde{M}(t, \bar{x}). \end{cases}$$

has a unique solution in $\mathbb{Q}[y][[t]] \times \mathbb{Q}[x][[t]]$.

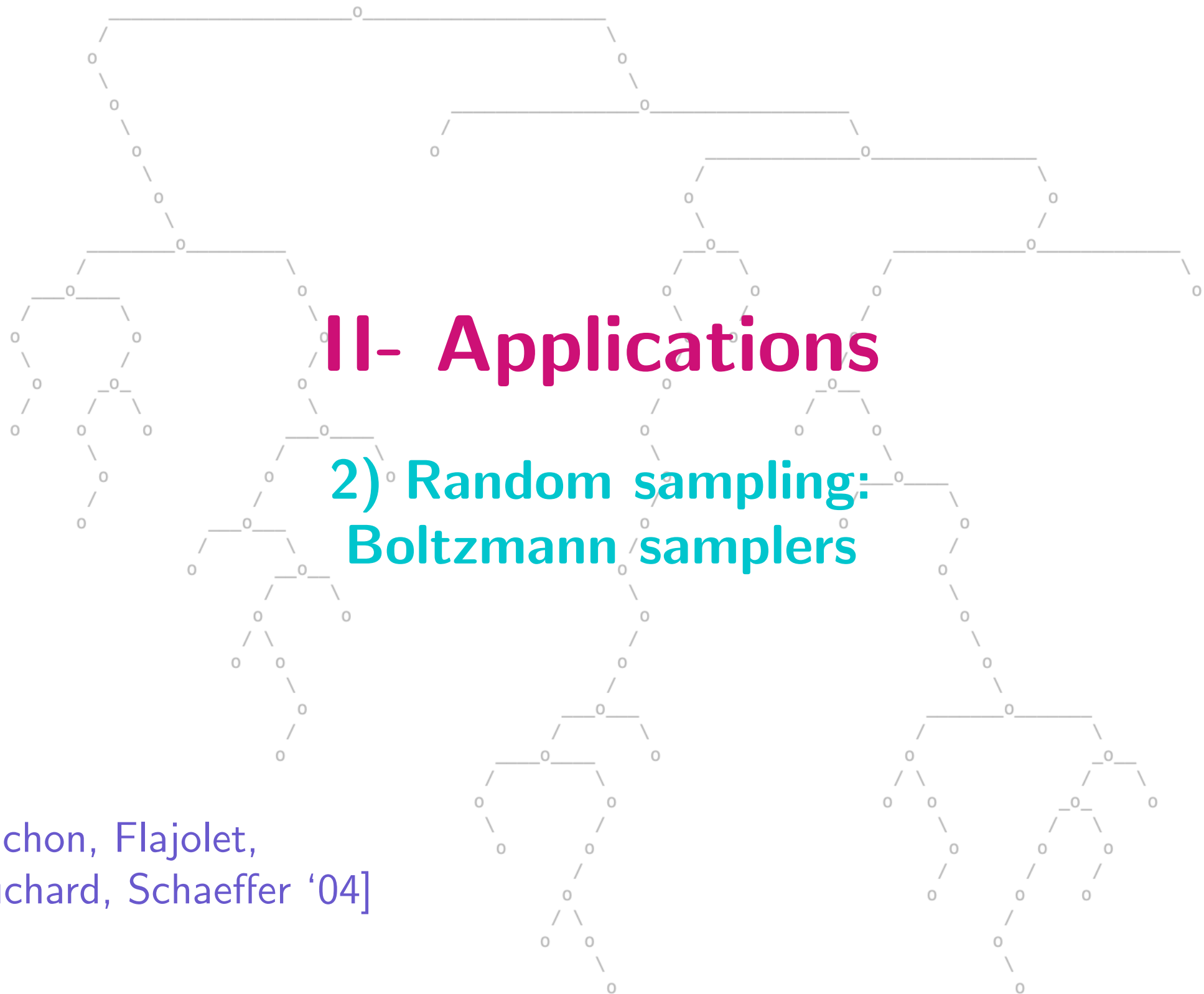
b. There are polynomials $P_M(S, y, M)$ and $P_H(S, x, H)$ whose only power series solutions satisfy the system.

Theorem. [S. '26] The generating function $M(t, y)$ of 3-colored planar maps with black and white boundary is algebraic over $\mathbb{Q}(t, y)$ of degree 24.

► Complete map of the subfields of $\mathbb{Q}(t, y, M(y))$.

$$\mathbb{Q}(t, y) \xrightarrow{4} \mathbb{Q}(S, y) \xrightarrow{3} \mathbb{Q}(S, L(y)) \xrightarrow{2} \mathbb{Q}(S, y, M(y))$$

► All $[y^p]M(y)$ are rational fractions in S and they satisfy a recurrence.



[Duchon, Flajolet,
Louchard, Schaeffer '04]

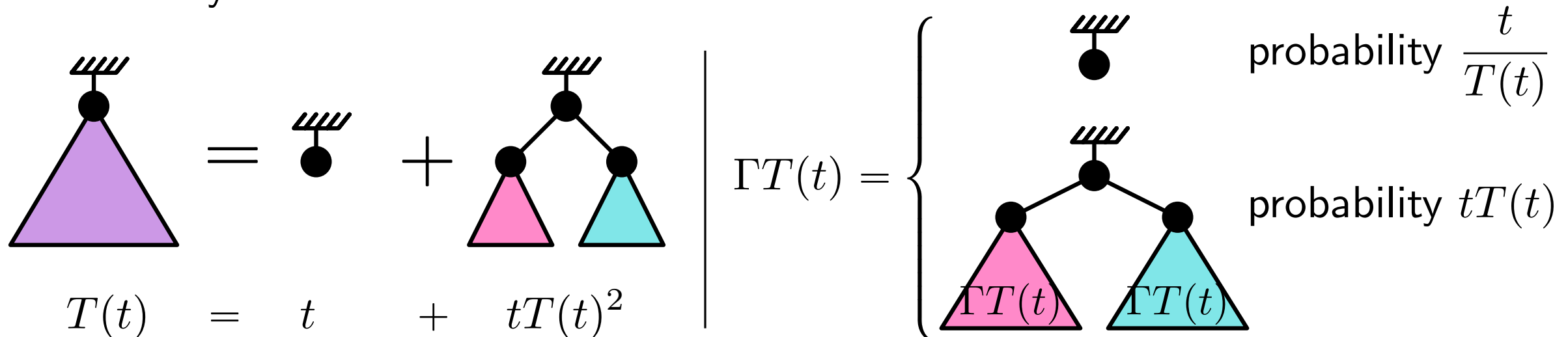
Boltzmann samplers

Principle: sample objects $\gamma \in \mathcal{C}$ according to the probability distribution

$$\mathbb{P}(\gamma) = \frac{t^{|\gamma|}}{C(t)}$$

with t a parameter, by following a recursive decomposition of $\mathcal{C}$.

For binary trees :



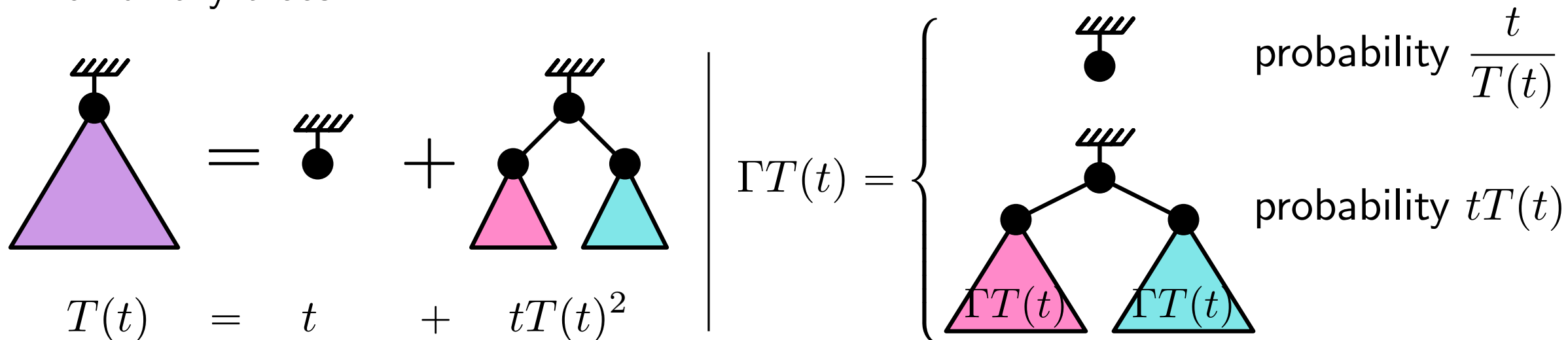
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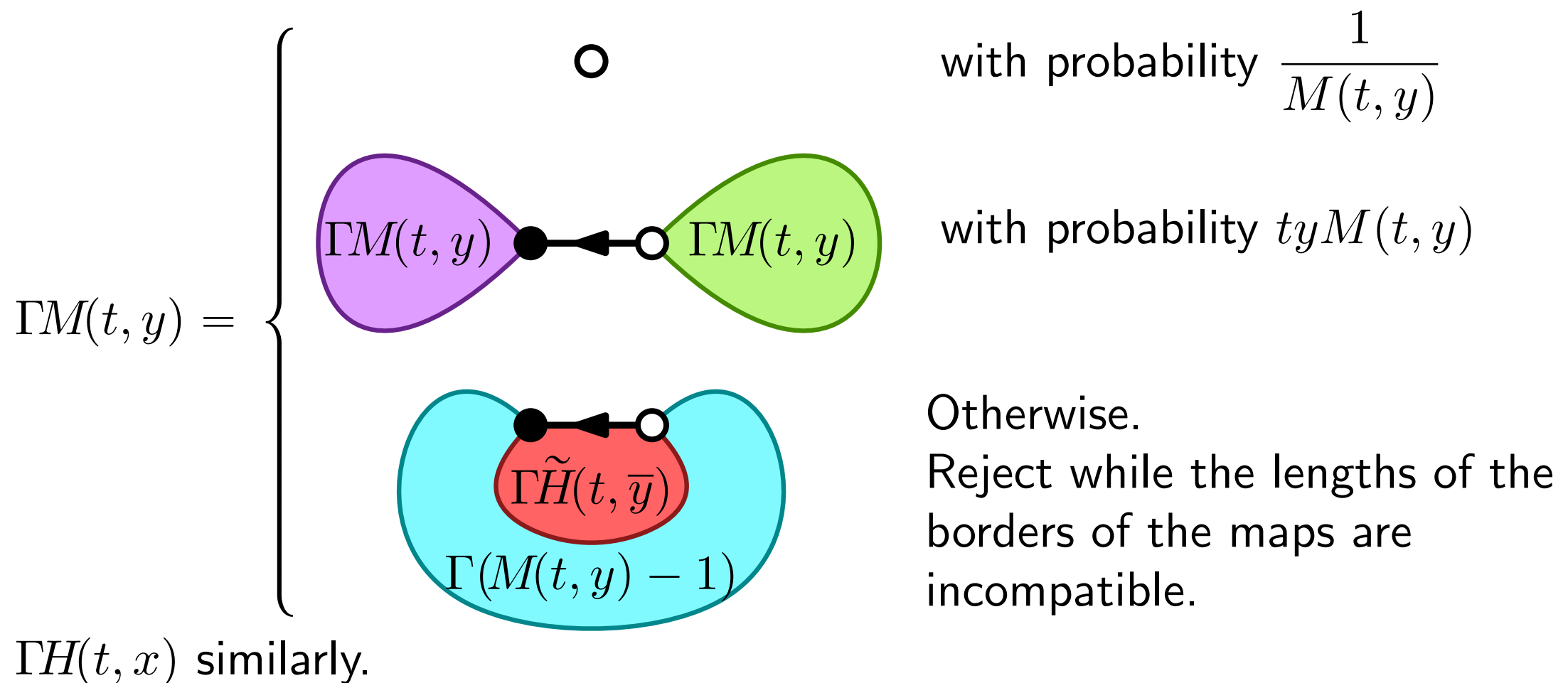


- Unfixed size (adjust the value of t to aim a given size).
- Uniform distribution when conditioned by the size of the object.
- Can be built as long as we have a positive combinatorial equation.
- Very efficient for large objects.

Boltzmann sampler for 3-colored maps

- ▶ The catalytic variable becomes a second parameter: $\mathbb{P}(\mathfrak{m}) = \frac{t^{e(\mathfrak{m})} y^{d(\mathfrak{m})}}{M(t, y)}$.
- ▶ We use rejection to handle the $[y^{\geq 0}]$ and $[x^{\geq 0}]$.

$$\begin{cases} M(t, y) = 1 + tyM(t, y)^2 + ty[y^{\geq 0}](M(t, y) - 1)\tilde{H}(t, \bar{y}), \\ H(t, x) = 1 + txH(t, x)^2 + 2tx[x^{\geq 0}](H(t, x) - 1)\tilde{M}(t, \bar{x}). \end{cases}$$



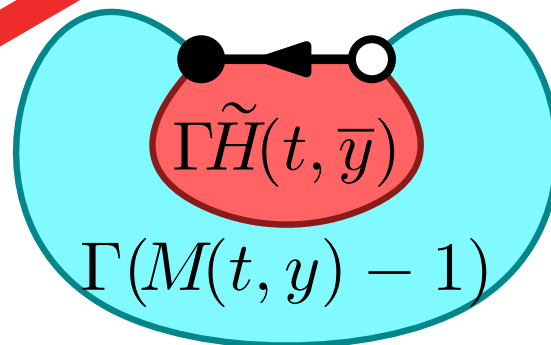
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Too much rejection :(

$\Gamma M(t, y)$



$\Gamma H(t, x)$ similarly.

Otherwise.

Reject while the lengths of the borders of the maps are incompatible.

Boltzmann sampler for 3-colored maps

- ▶ Recursive catalytic variable = samplers for map with a given boundary size.

Equations for $M_p = [y^p]M(y)$, $H_d = [x^d]H(x)$, $\widetilde{M}_k = [x^k]\widetilde{M}(x)$ and $\widetilde{H}_k = [y^k]\widetilde{H}(y)$:

$$\left\{ \begin{array}{l} M_0 = H_0 = 1, \\ M_p = t \sum_{k=0}^{p-1} M_k M_{p-1-k} + t \sum_{k \geq 1} M_{p+k-1} \widetilde{H}_k \quad \text{for } p > 0, \\ H_d = t \sum_{k=0}^{d-1} H_k H_{d-1-k} + t \sum_{k \geq 1} H_{d+k-1} \widetilde{M}_k \quad \text{for } d > 0, \\ \widetilde{M}_k = \sum_{p \geq 0} \binom{2p+k-1}{k-1} M_p \quad \text{for } k > 0, \\ \widetilde{H}_k = \sum_{d \geq 0} \binom{2k+d-1}{2k-1} H_d \quad \text{for } k > 0. \end{array} \right.$$

→ Only one infinite sum per equation so we can compute the terms on the fly.

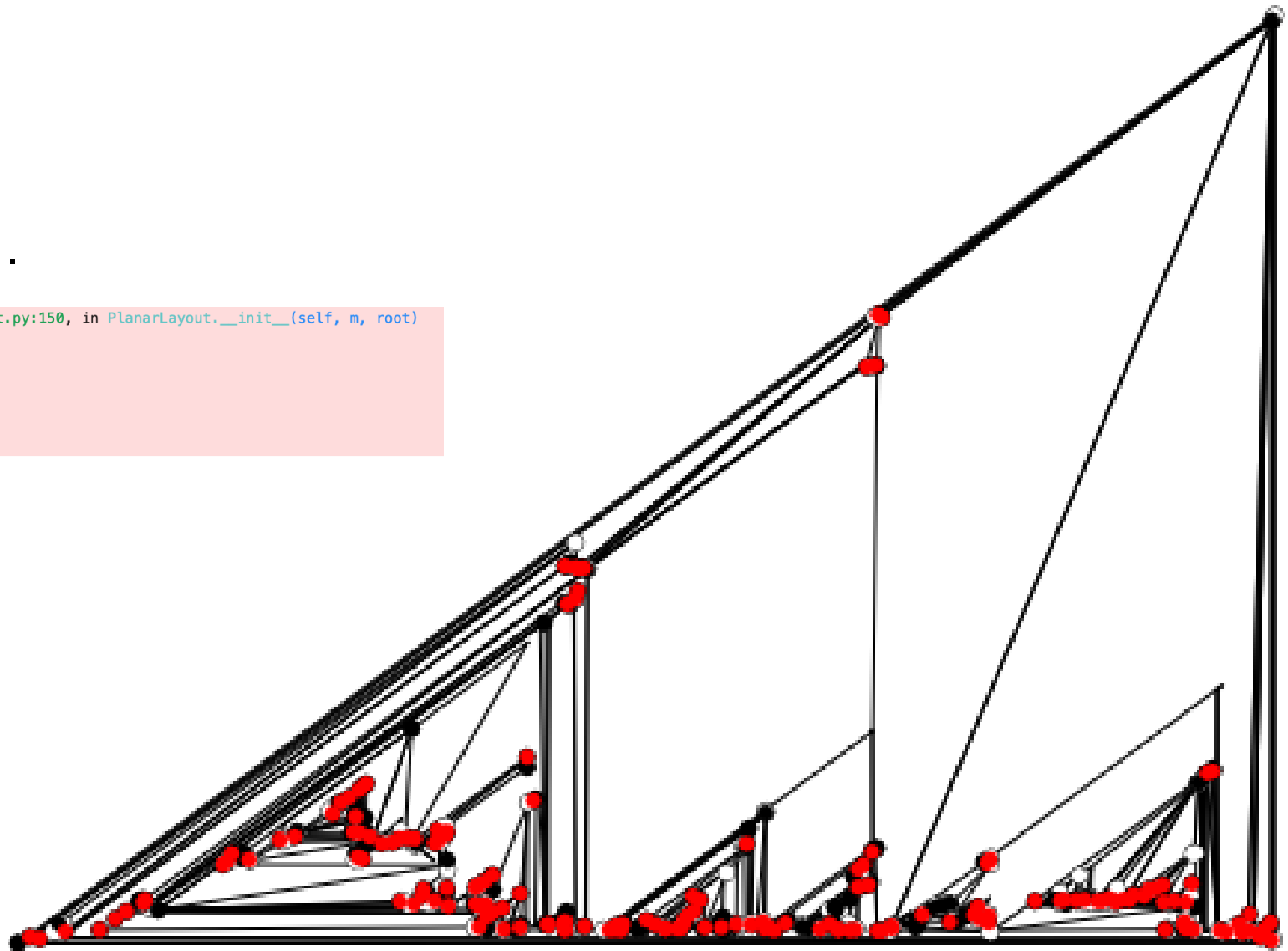
Boltzmann sampler for 3-colored maps

- Recursive catalytic variable = samplers for map with a given boundary size.

Work in progress ...

```
File ~/Documents/Maths/Sage/topsurf/topsurf/planar_layout.py:150, in PlanarLayout.__init__(self, m, root)
    147     self._pos[1][i] = yi
    149 if not self.is_valid():
--> 150     raise RuntimeError
    152 # force parameters
    153 self._repulsion_power = 1.0

RuntimeError:
```



A random 3-colored map with 616 edges and a horrible layout.

Perspectives

- ▶ Find a nice family of trees in bijection with 3-colored planar maps.
- ▶ Others applications of the decomposition?
- ▶ Adapt the gasket decomposition to others families of colored maps.
 - ↪ Colorful quadrangulations [Budd '25], (D-algebraic and solved [Bousquet-Mélou, Elvey Price '20]).
 - ↪ 4-colored triangulations (D-algebraic [Tutte '82]).
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