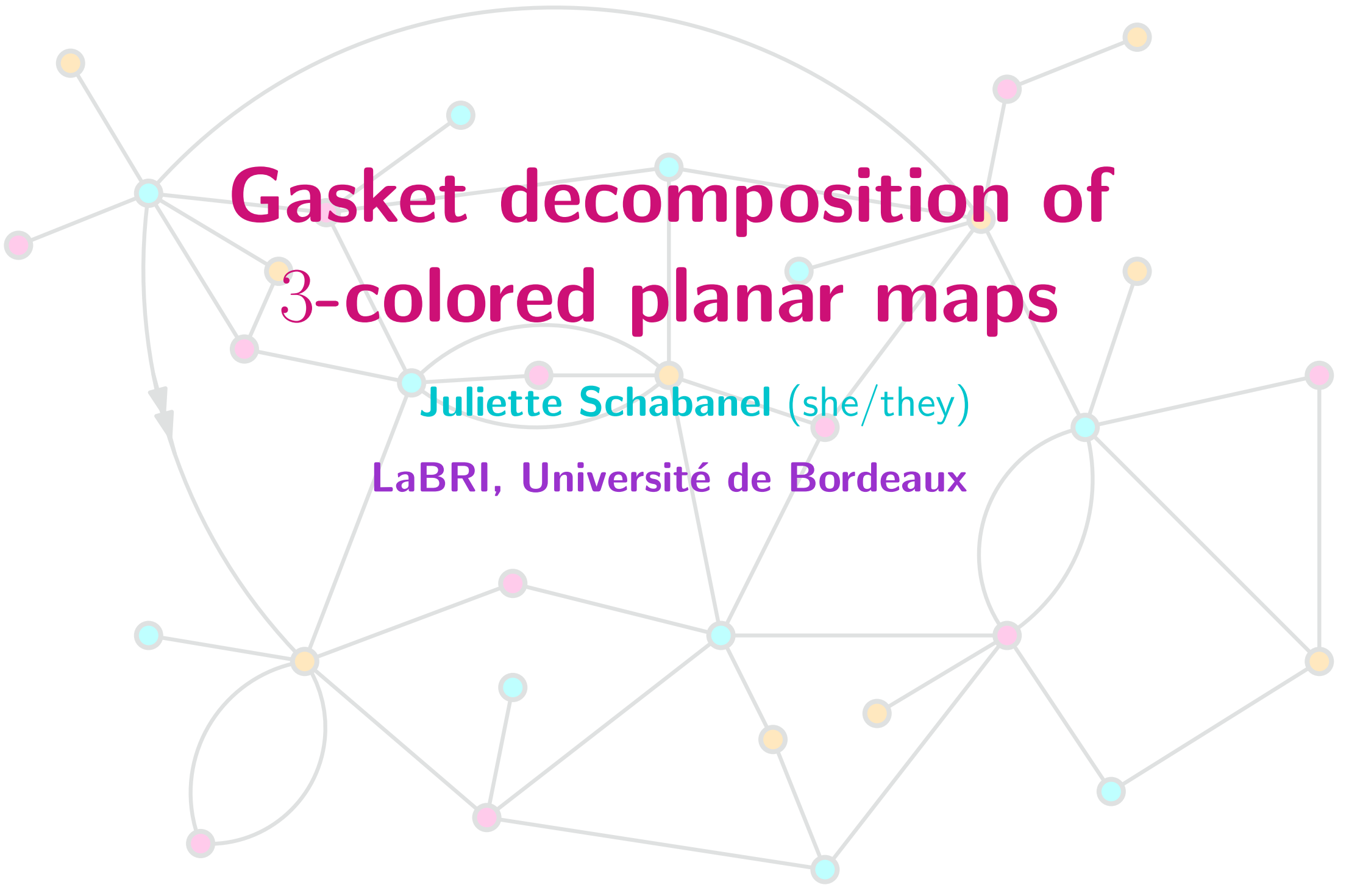
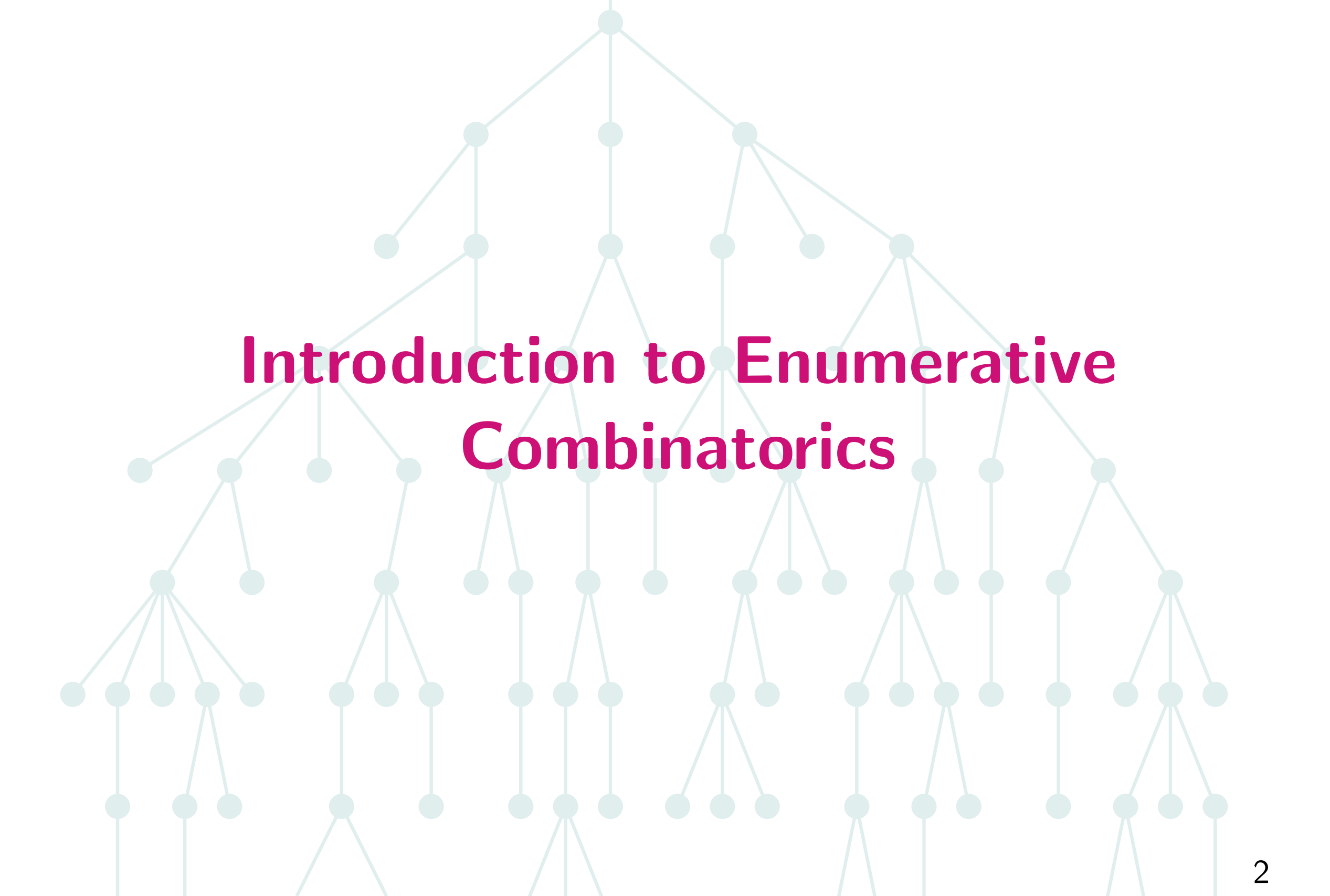




Gasket decomposition of 3-colored planar maps

Juliette Schabanel (she/they)

LaBRI, Université de Bordeaux

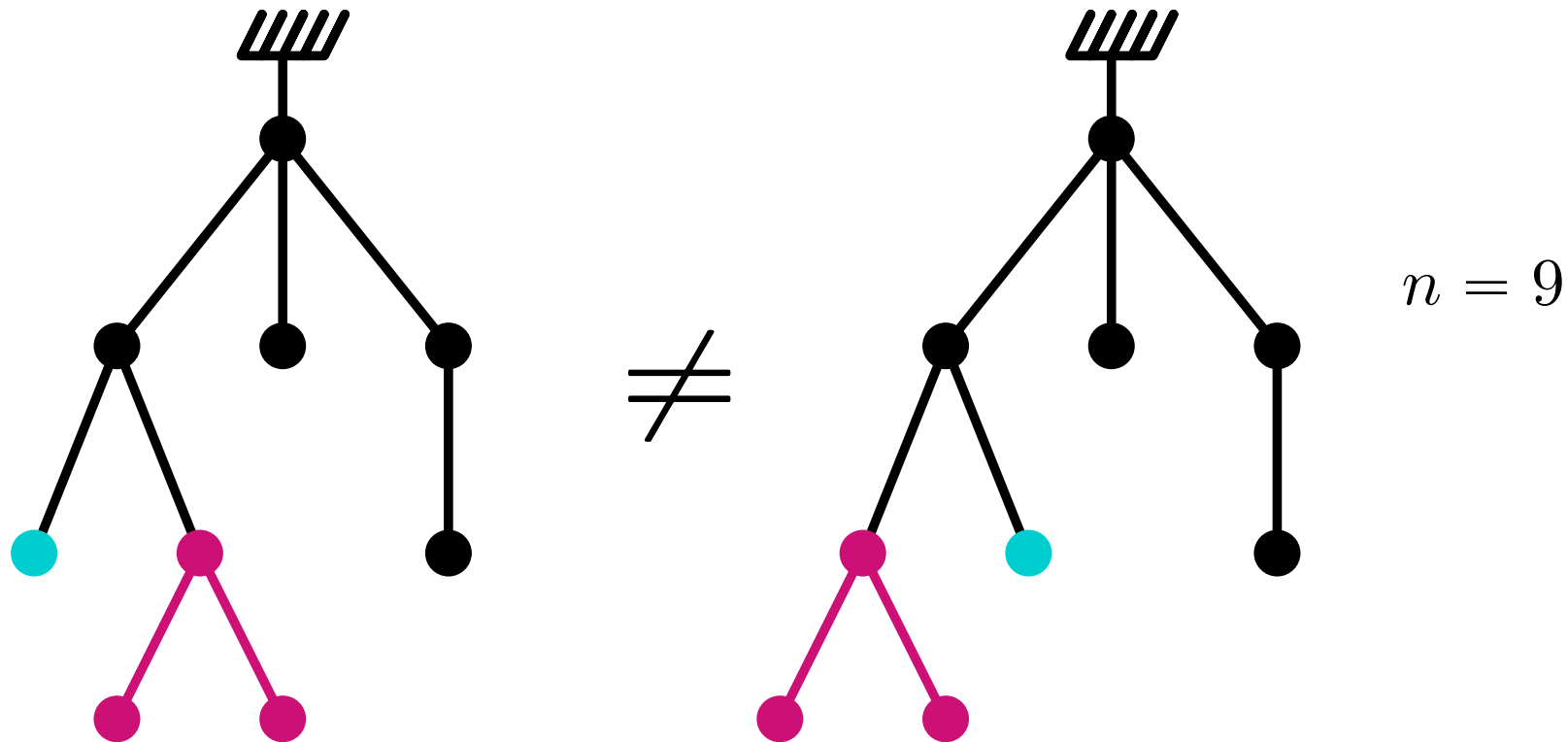


Introduction to Enumerative Combinatorics

First example: Trees

Plane tree = rooted tree with ordered children.

Size n = number of vertices.



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Binary trees: all vertices have 0 or 2 children.

size 1 :



$$t_1 = 1$$

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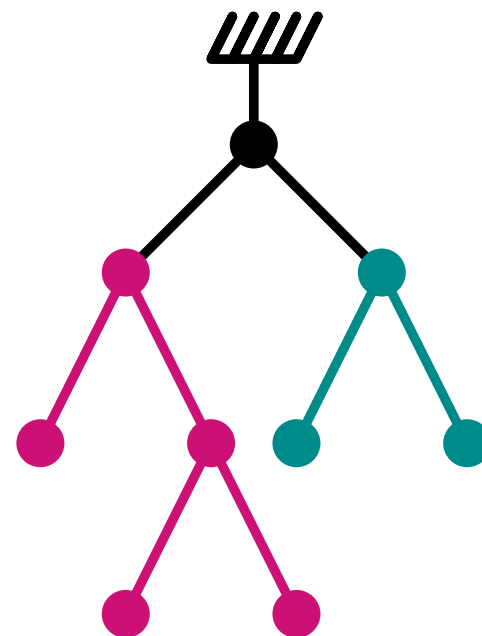
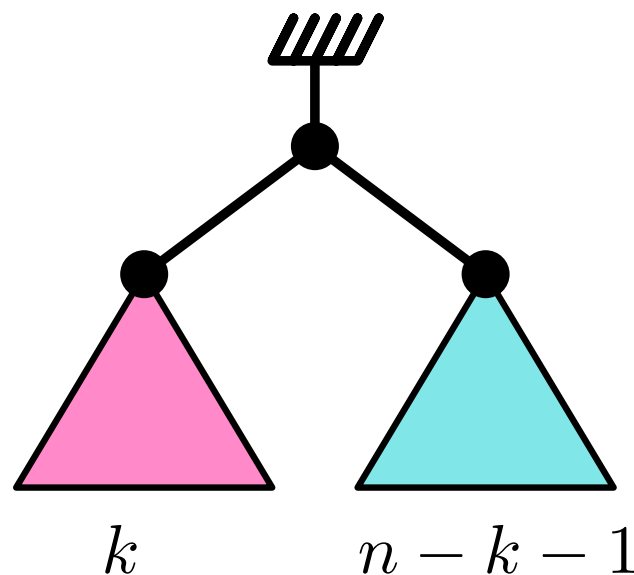
Binary trees: all vertices have 0 or 2 children.

size 1 :



$t_1 = 1$

size $n > 1$:



$n = 9$

$k = 5$

$n - k - 1 = 3$

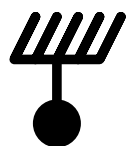
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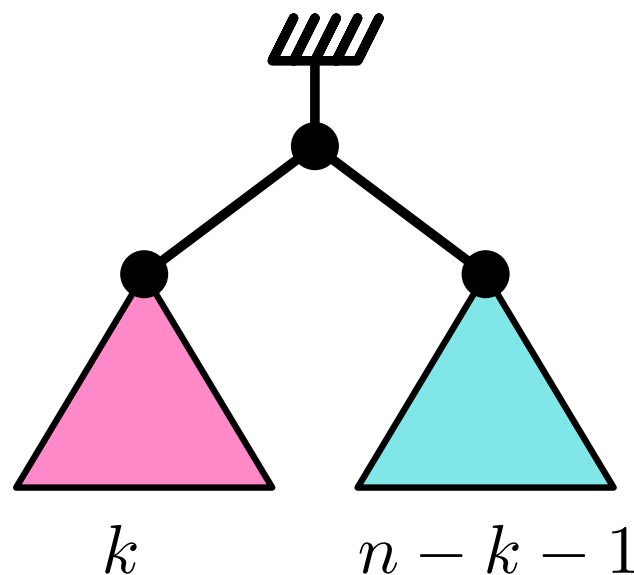
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$$t_n = \sum_{k=1}^{n-1} t_k t_{n-k-1}$$

Solution :
$$t_n = \frac{1}{n+1} \binom{2n}{n}$$

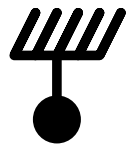
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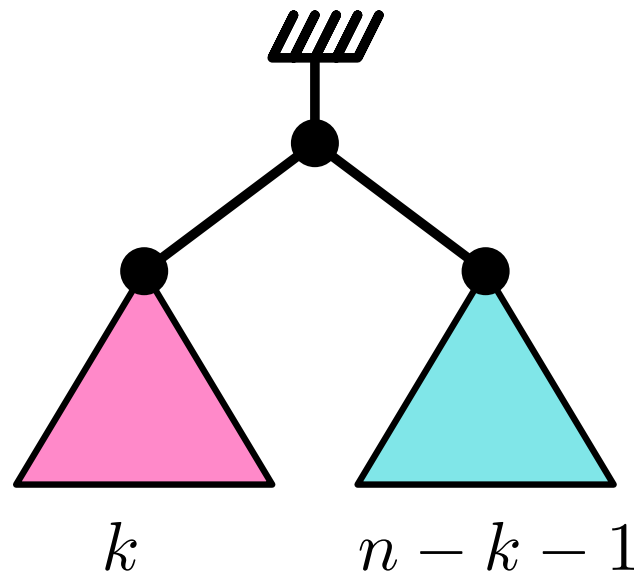
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Trees with nodes



Generating functions

The **generating function** of a combinatorial class $\mathcal{C}$ is the series

$$C(t) = \sum_{\gamma \in \mathcal{C}} t^{|\gamma|} = \sum_{n \geq 0} c_n t^n \in \mathbb{Q}[[t]].$$

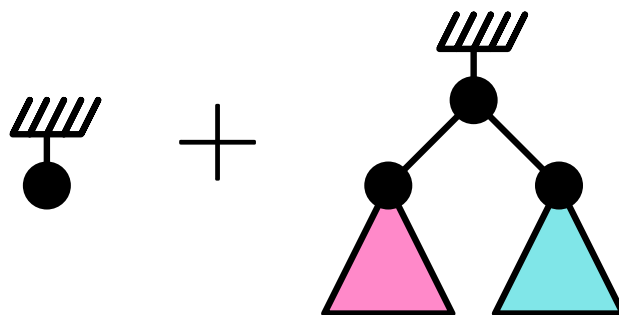
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A recursive decomposition translates into an equation on the generating function.

Binary trees:



$$T(t) = t + tT(t)^2$$

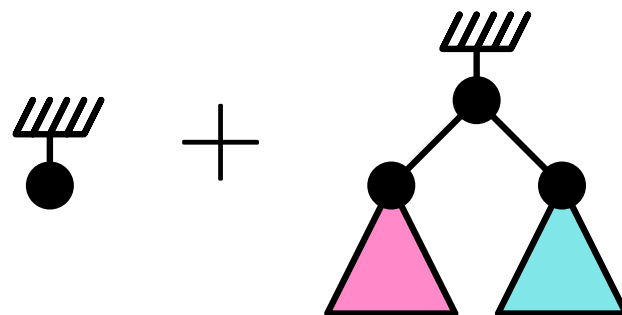
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Trees with nodes



$$T(t) = 2t + 3tT(t)^2 + tT(t)^3 + tT(t)^7$$

Generating functions

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$$C(t) = \sum_{\gamma \in \mathcal{C}} t^{|\gamma|} = \sum_{n \geq 0} c_n t^n \in \mathbb{Q}[[t]].$$

What to do with it ?

- ▶ If we can solve the equation and get a closed form for $C(t)$, we can recover the c_n through $C^{(n)}(0) = n!c_n$.
- ▶ Use the equation to compute asymptotics of c_n .
- ▶ Build random generator (Boltzmann).

Generating functions

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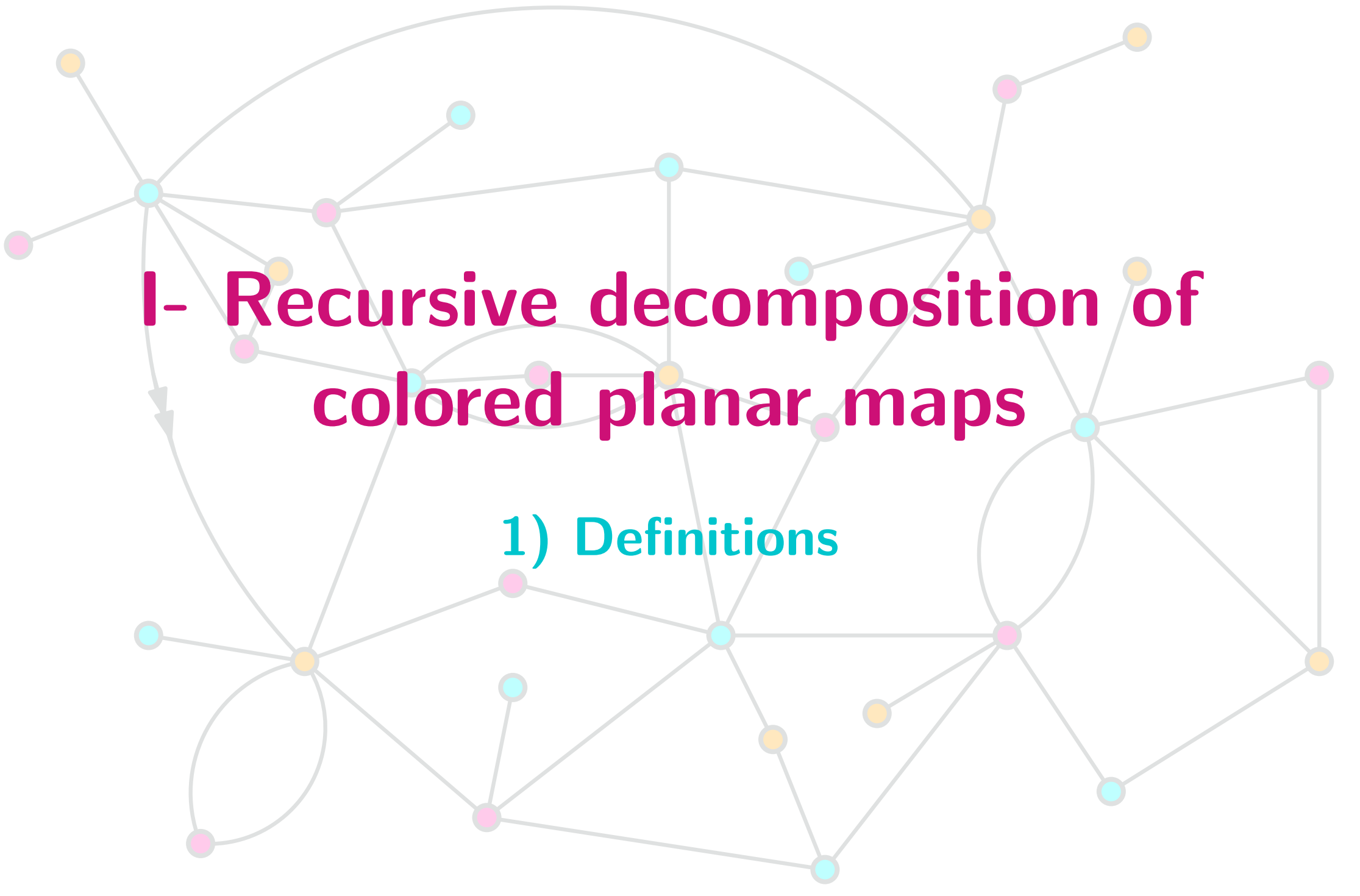
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- ▶ Build random generator (Boltzmann).

The form of the equation gives information on the class:

- ▶ Rational function: $C(t) = P(t) + Q(t)C(t)$, P, Q polynomials.
 $\implies c_n \sim \kappa \lambda^n n^\alpha$, $\alpha \in \mathbb{N}$. $\mathcal{C} \approx$ regular language?
- ▶ Algebraic function: $P(t, C(t)) = 0$, $P(t, C)$ polynomial.
 $\implies c_n \sim \kappa \lambda^n n^\alpha$, $\alpha \in \mathbb{Q}$. $\mathcal{C} \approx$ non-ambiguous grammar (trees)?



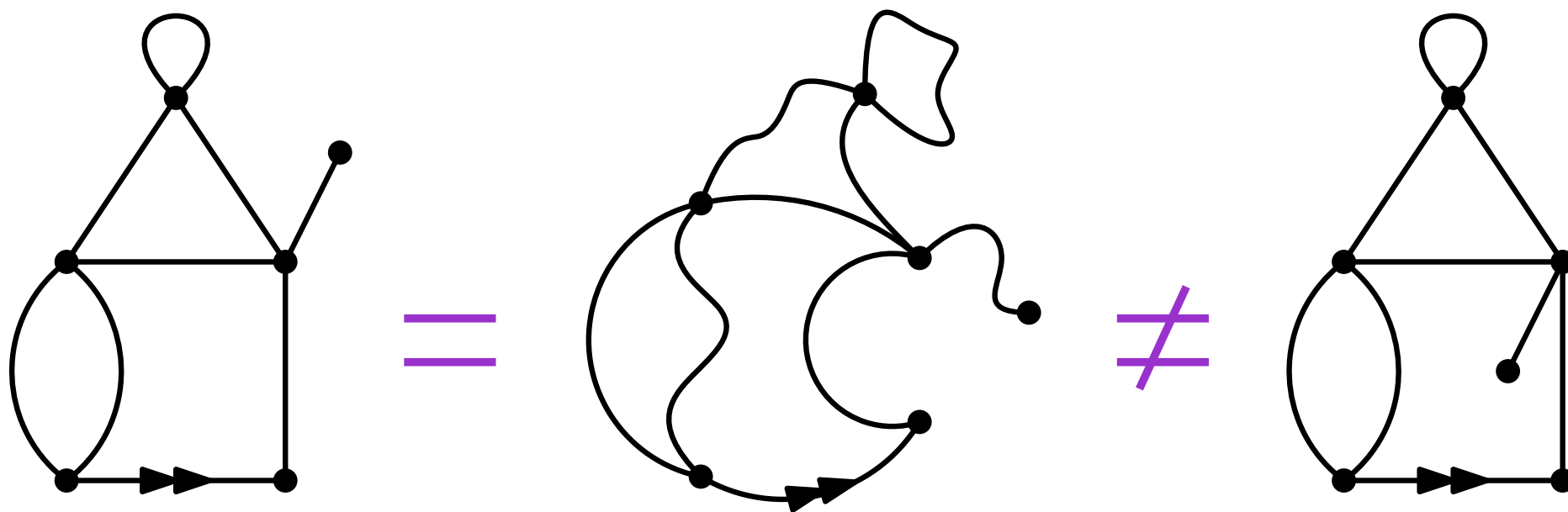
I- Recursive decomposition of colored planar maps

1) Definitions

Planar maps

A **planar map** is the embedding of a connected graph onto the sphere, up to orientation preserving homeomorphism.

Multi-edges and loops are allowed.



Planar map = planar graph + cyclic ordering of the edges around each vertex.

All maps are **rooted**, i.e. an oriented edge is marked.

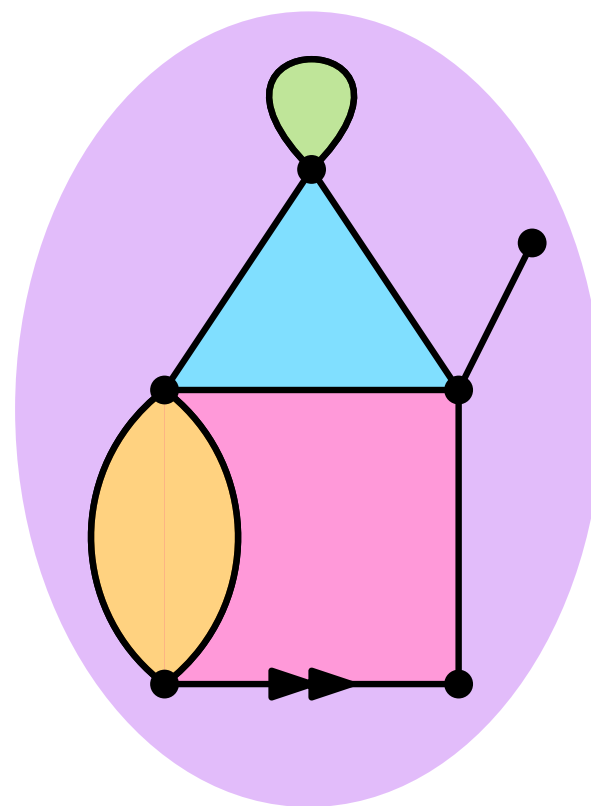
Planar maps

Planar map = planar graph + cyclic ordering of the edges around each vertex.

- **Vertices** and **edges** are inherited from the graph.
- **Faces** are the connected components of the sphere minus the map.

Size: number of edges n .

Here: 6 vertices, 9 edges
et 5 faces.



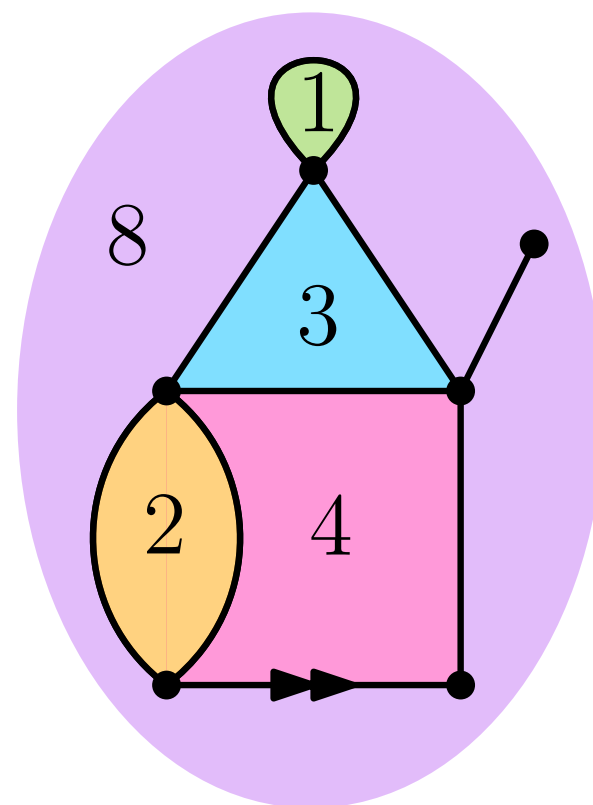
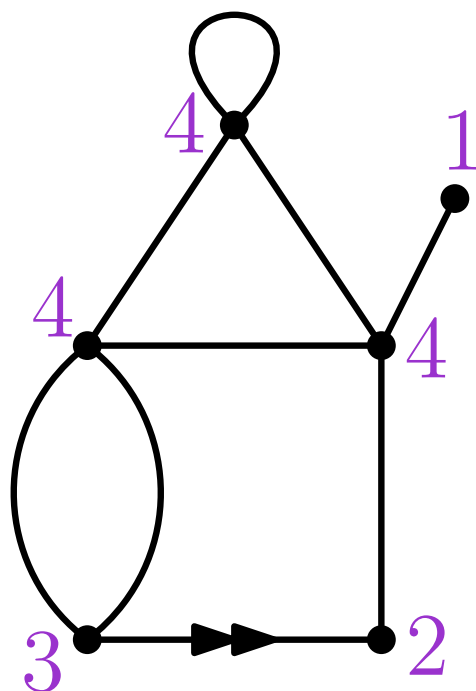
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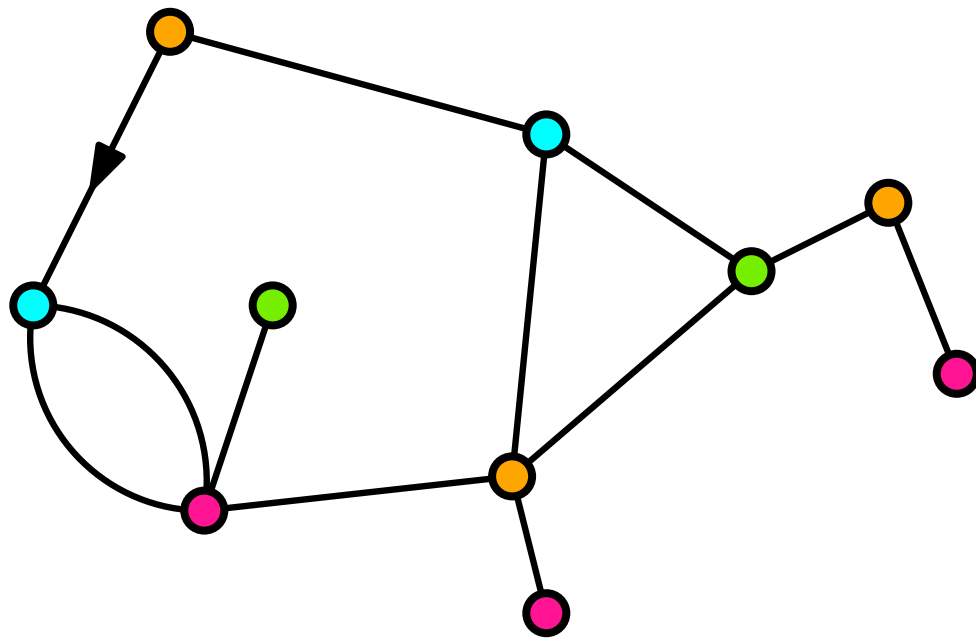
Size: number of edges n .

Degree (of a vertex or face) = number of incident half-edges.

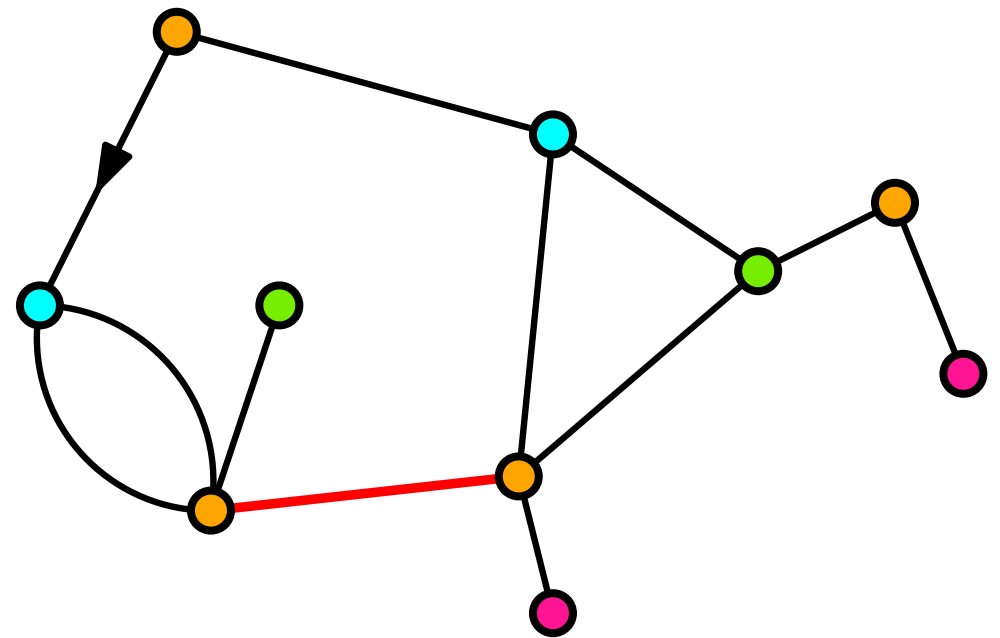


Colored maps

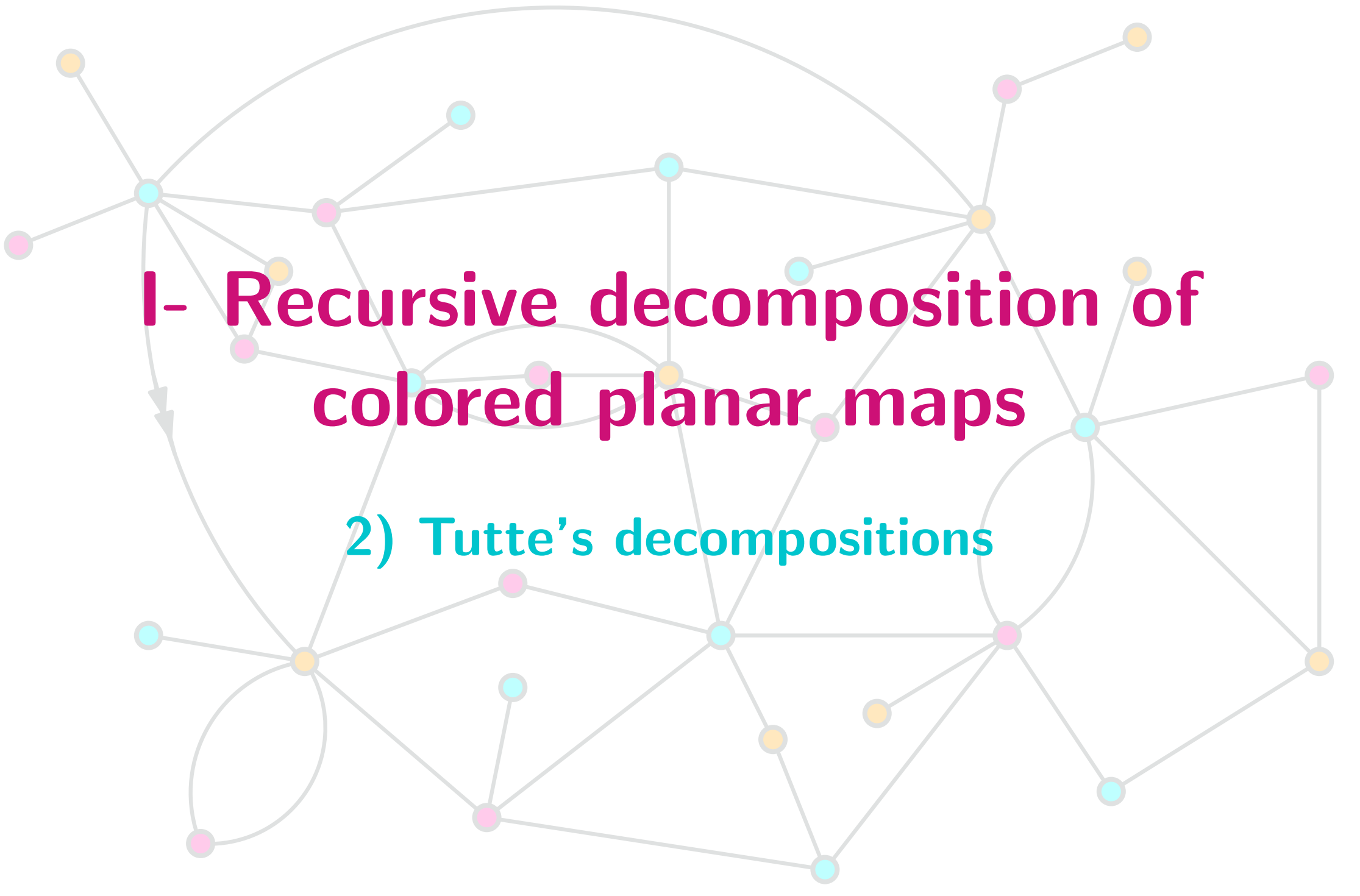
Colored map: assign a color to each vertex so that adjacent vertices receive different colors.



proper coloring



non-proper coloring

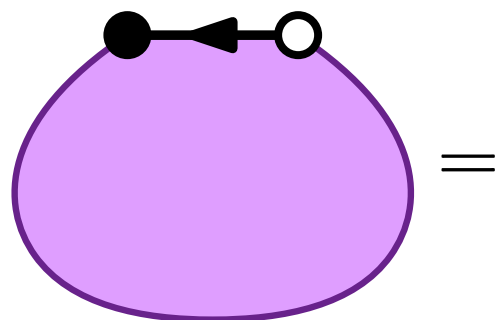


I- Recursive decomposition of colored planar maps

2) Tutte's decompositions

2-colored maps

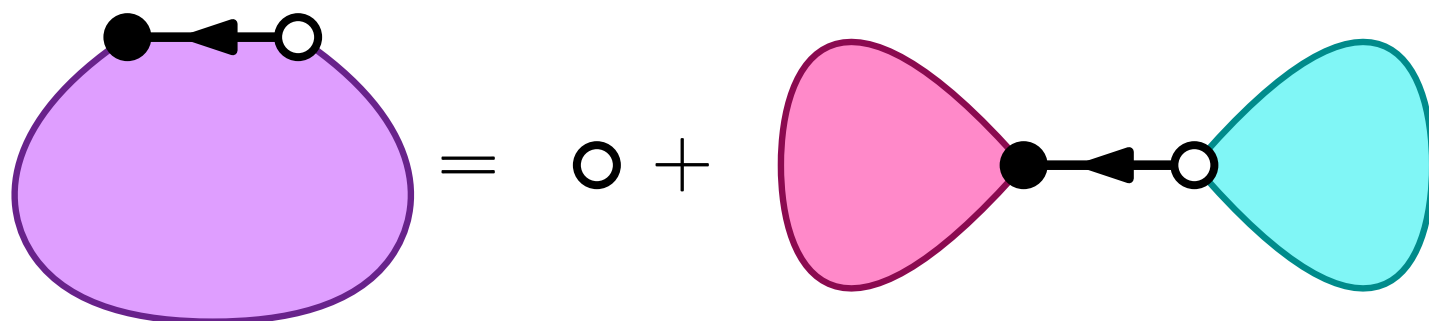
$B(t)$: generating function of bicolored planar maps counted by number of edges.



$$B(t) =$$

2-colored maps

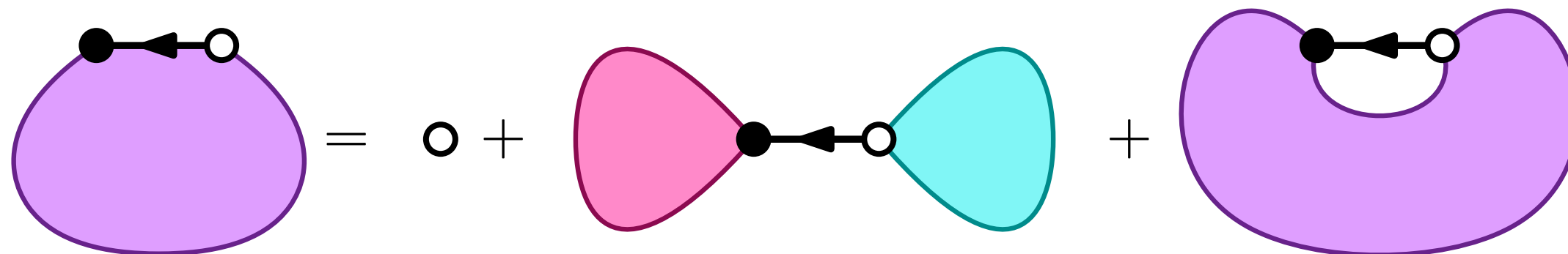
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$$B(t) = 1 + tB(t)^2$$

2-colored maps

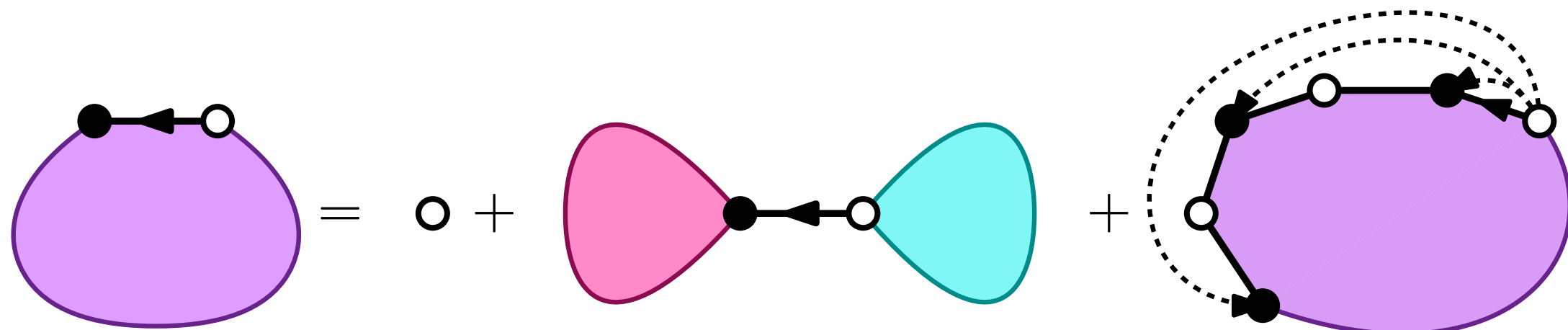
$B(t)$: generating function of bicolored planar maps counted by number of edges.



$$B(t) = 1 + tB(t)^2 + ???$$

2-colored maps

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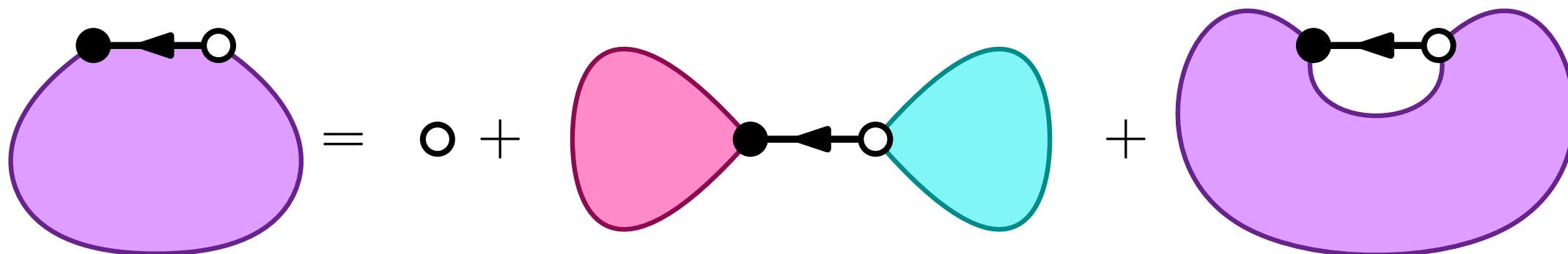
$$B(t) = 1 + tB(t)^2 + \text{???}$$

2-colored maps

$B(t)$: generating function of bicolored planar maps counted by number of edges.

$\Rightarrow$ add a **catalytic variable** recording the half-length of the boundary

$$\hookrightarrow B(t, y) = \sum_{\mathfrak{b} \in \mathcal{B}} t^{e(\mathfrak{b})} y^{b(\mathfrak{b})}.$$



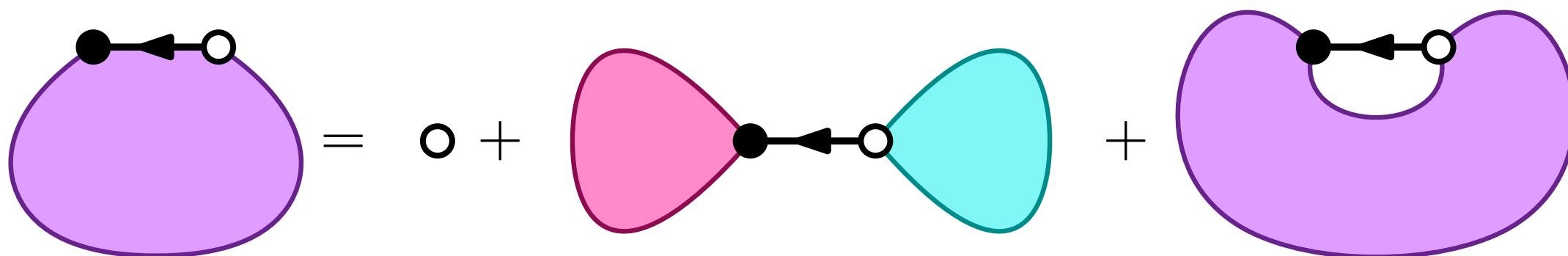
$$B(t, y) = 1 + ty B(t, y)^2 + ty \sum_{p \geq 1} \left(\sum_{k \geq 1}^p y^{p-k} \right) [y^p] B(t, y)$$

2-colored maps

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$$\hookrightarrow B(t, y) = 1 + ty B(t, y)^2 + ty \frac{B(t, y) - B(t, 1)}{y - 1}.$$

2-colored maps

$B(t, y)$: generating function of bicolored planar maps, $t \rightarrow$ edges, $y \rightarrow$ half degree of the outer face.

$$B(t, y) = 1 + tyB(t, y)^2 + ty \frac{B(t, y) - B(t, 1)}{y - 1}$$

→ Can be solved with the quadratic method [Brown '60s].

+ This method provides an algebraic equation of degree 4 for $B(t, y)$.

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- ↪ Can be solved with the quadratic method [Brown '60s].
 - + This method provides an algebraic equation of degree 4 for $B(t, y)$.
- ↪ Explicit bijection with a families of trees [Schaeffer '97].
 - + Additional control on all face degrees.

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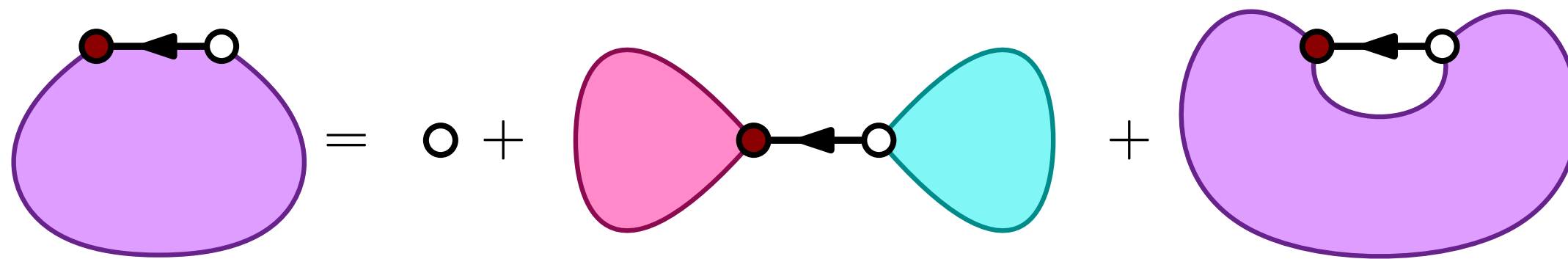
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Take away: 2-colored maps are very well understood.

3-colored maps

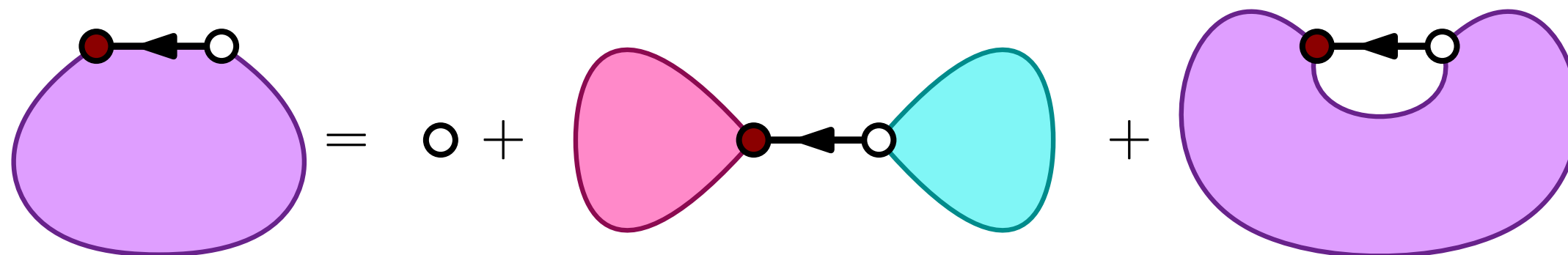
$M(t, y)$: generating function of 3-colored planar maps with $t \rightarrow$ number of edges and $y \rightarrow$ degree of the outer face.



$$M(t, y) = 1 + 2ty^2 M(t, y)^2 + ty^{??}$$

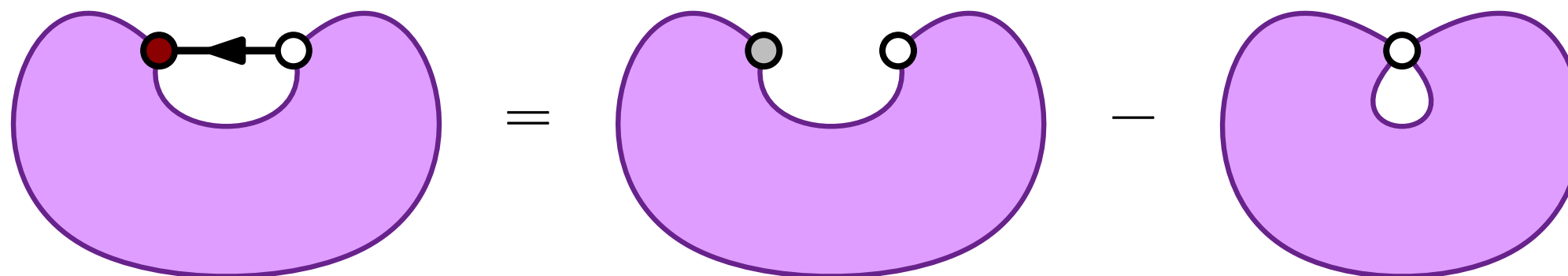
3-colored maps

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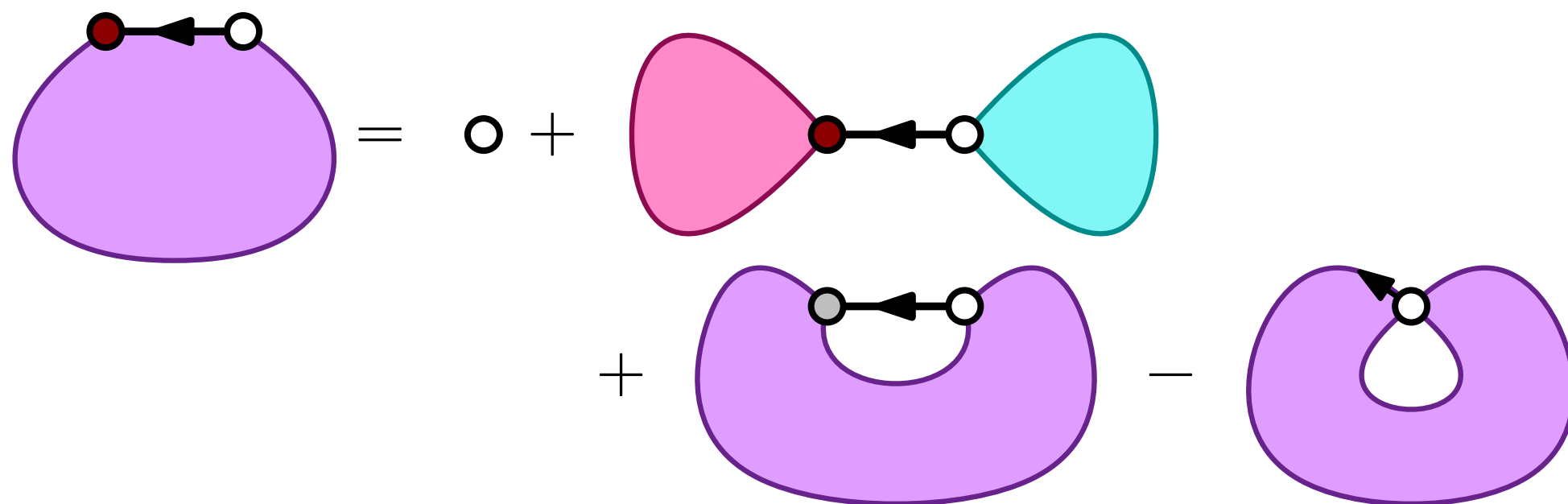
Solution:



$\Rightarrow$ need a second catalytic variable recording the degree of the root vertex.

3-colored maps

$M(t, x, y)$: generating function of 3-colored planar maps with $t \rightarrow$ number of edges, $y \rightarrow$ degree of the outer face and $x \rightarrow$ degree of the root vertex.



Theorem. [Tutte '68, Bernardi & Bousquet-Mélou '11]

$$M(x, y) = 1 + xyt(2y - 1)M(x, y)M(1, y) - xytM(x, y)M(x, 1) - xyt \frac{xM(x, y) - M(1, y)}{x - 1} + xyt \frac{yM(x, y) - M(x, 1)}{y - 1}.$$

3-colored maps

Tutte equation for 3-colored planar maps:

$$M(x, y) = 1 + xyt(2y - 1)M(x, y)M(1, y) - xytM(x, y)M(x, 1) - xyt \frac{xM(x, y) - M(1, y)}{x - 1} + xyt \frac{yM(x, y) - M(x, 1)}{y - 1}.$$

► Not an algebraic/D-finite/... equation $\implies$ no toolbox ?

Theorem. [Bousquet-Mélou, Jehanne '06] If $F(y) \in \mathbb{Q}[y][[t]]$ satisfies an equation

$$F(y) = P(y) + tQ(t, y, F(y), \Delta F(y), \dots, \Delta^k F(y))$$

where P, Q are polynomials and $\Delta F(y) = \frac{F(y) - F(1)}{y - 1}$ then F is algebraic.

$\hookrightarrow$ We have 2 catalytic variables so it does not apply.

► Negative coefficients $\implies$ cannot build a random generator.

3-colored maps

Tutte equation for 3-colored planar maps:

$$M(x, y) = 1 + xyt(2y - 1)M(x, y)M(1, y) - xytM(x, y)M(x, 1) - xyt \frac{xM(x, y) - M(1, y)}{x - 1} + xyt \frac{yM(x, y) - M(x, 1)}{y - 1}.$$

Not an algebraic/D-finite/... equation $\implies$ no toolbox ?

2 catalytic variables $\implies$ no method to get an algebraic equation.

Negative coefficients $\implies$ cannot build a random generator.

Theorem. [Bernardi & Bousquet-Mélou '11] The generating function of 3-colored planar map is algebraic and is equal to

$$M(1, 1) = \frac{(1 + 2S)(1 - 2S^2 - 4S^3 - 4S^4)}{(1 - 2S^3)^2}$$

with S the solution of $t = \frac{S(1 - S^3)}{(1 + 2S)^3}$ with constant term 0.

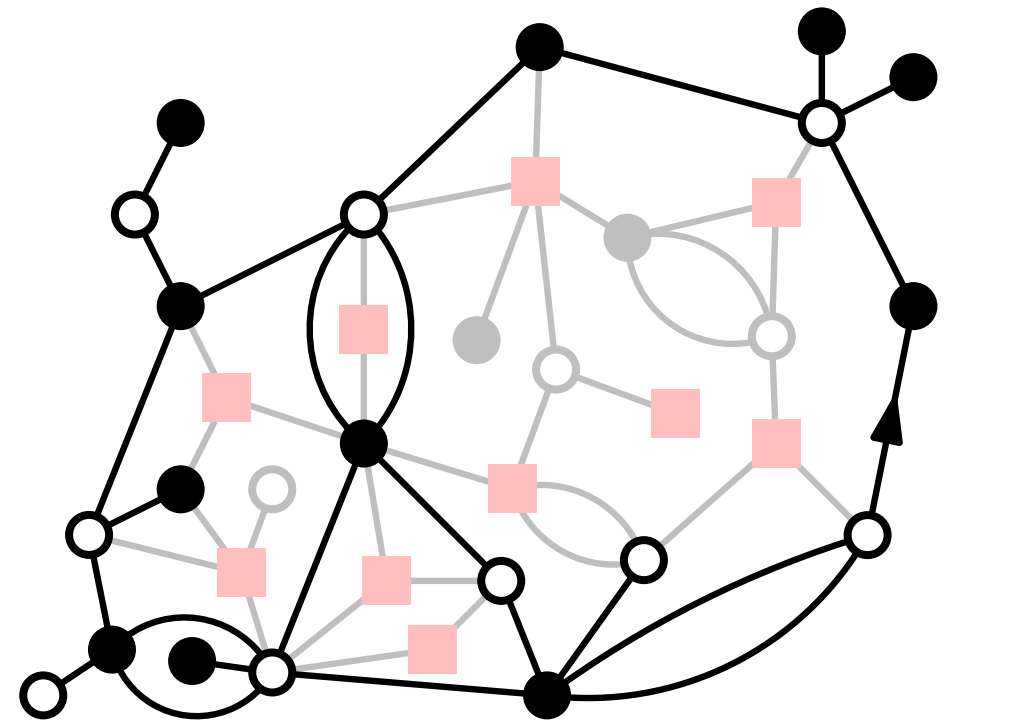
$\hookrightarrow$ Obtained via a reduction to 1-catalytic equation.

I- Recursive decomposition of colored planar maps

3) Gasket decompositions



A car gasket.

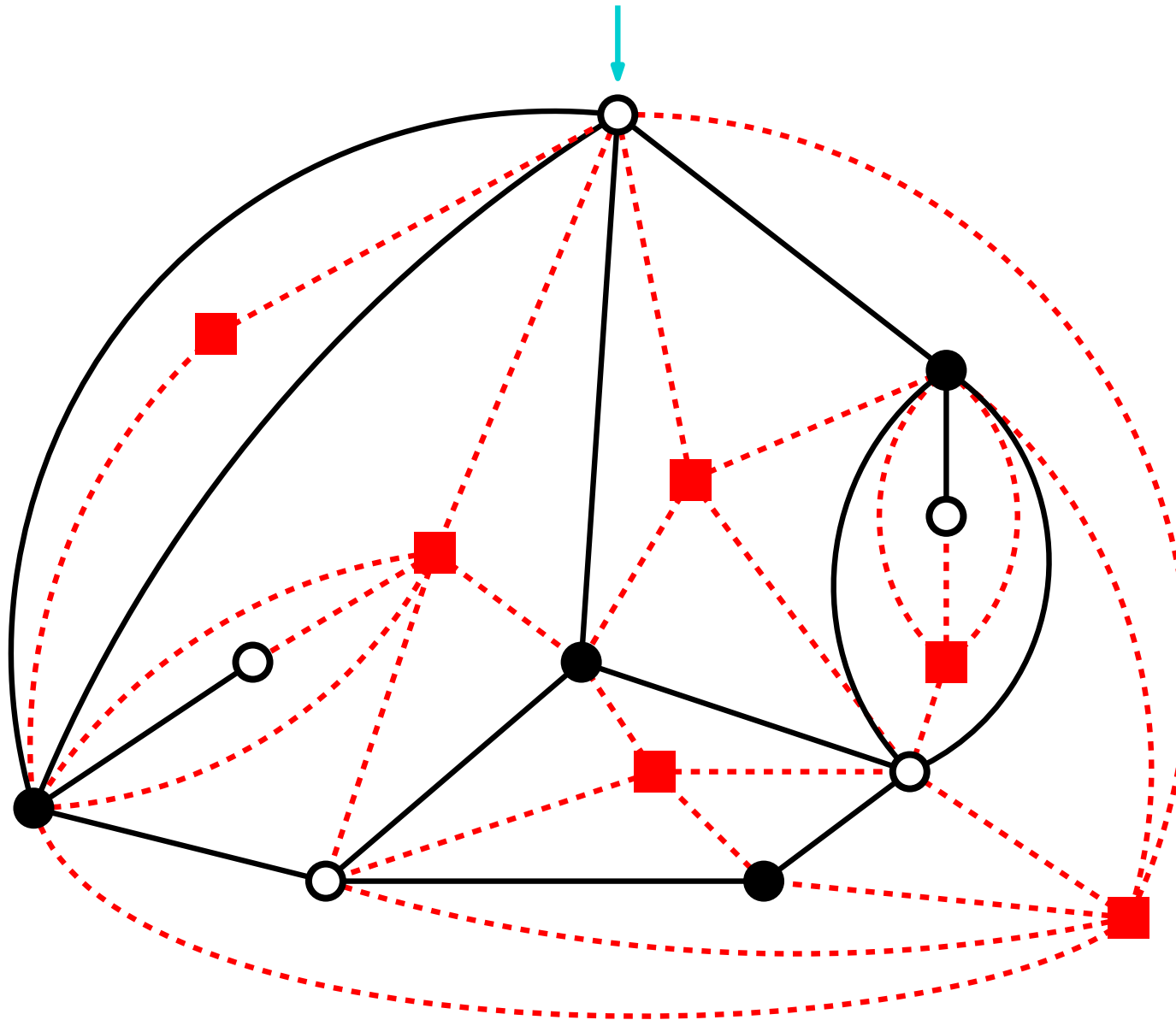


A map gasket.

Term from [Borot, Bouttier, Guitter '12].

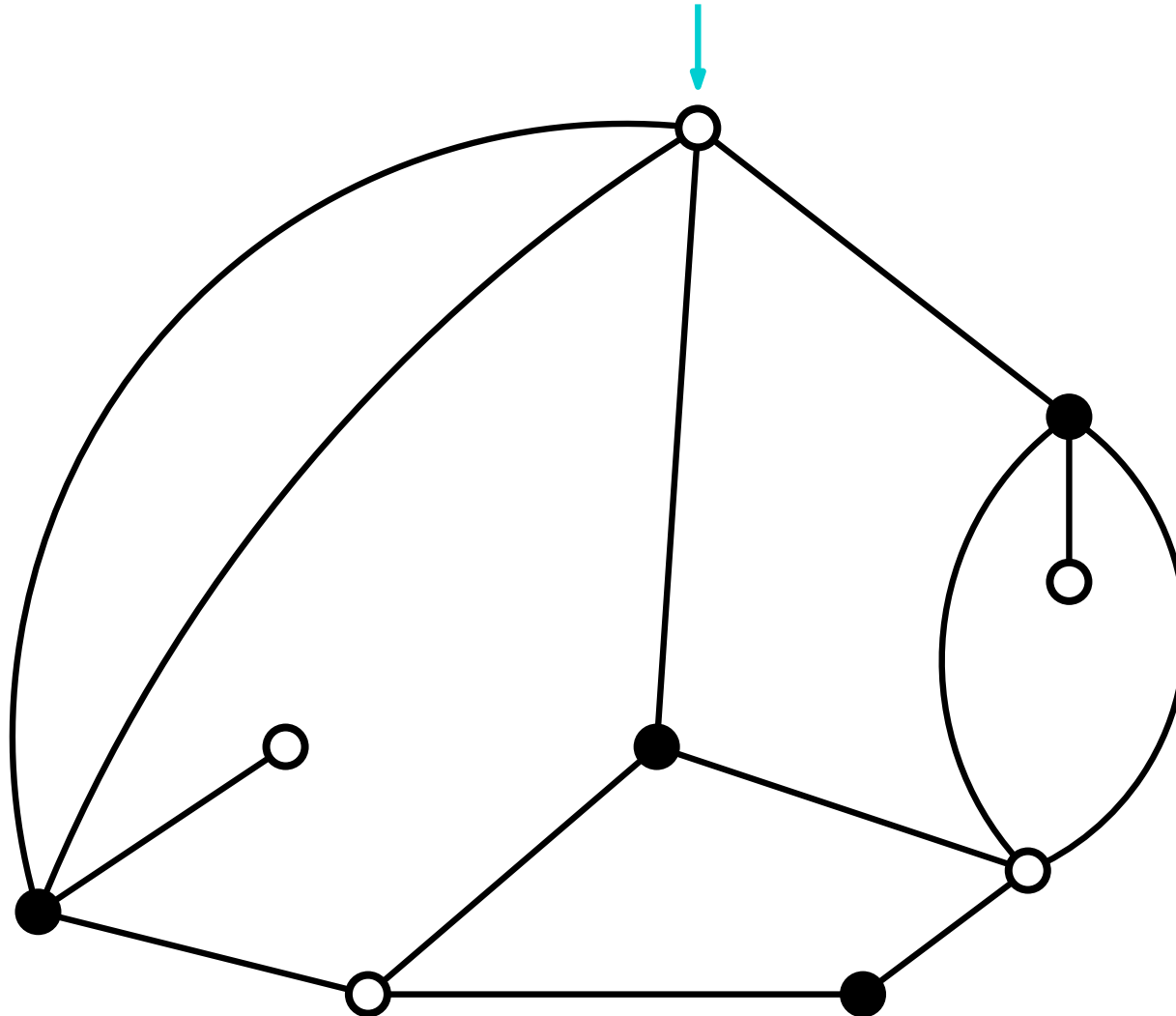
Warm-up: Triangulations

Triangulation = all faces have degree 3.



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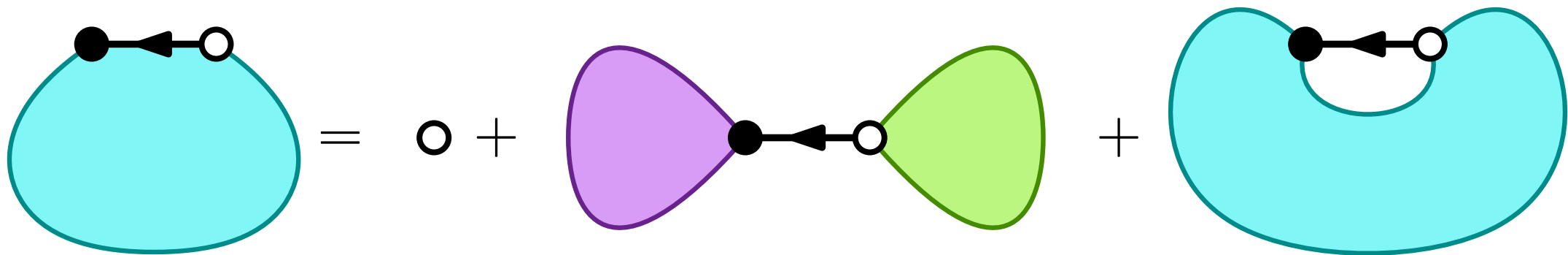


3-colored triangulations = 2-colored general maps.

Gasket decomposition

Idea: Use the structure from the coloring and get back to bipartite maps.

Let $M(t, y) \equiv M(y)$ be the generating function of 3-colored planar maps with a **black and white outer face**, with $t \rightarrow$ edges and $y \rightarrow$ half the degree of the outer face.

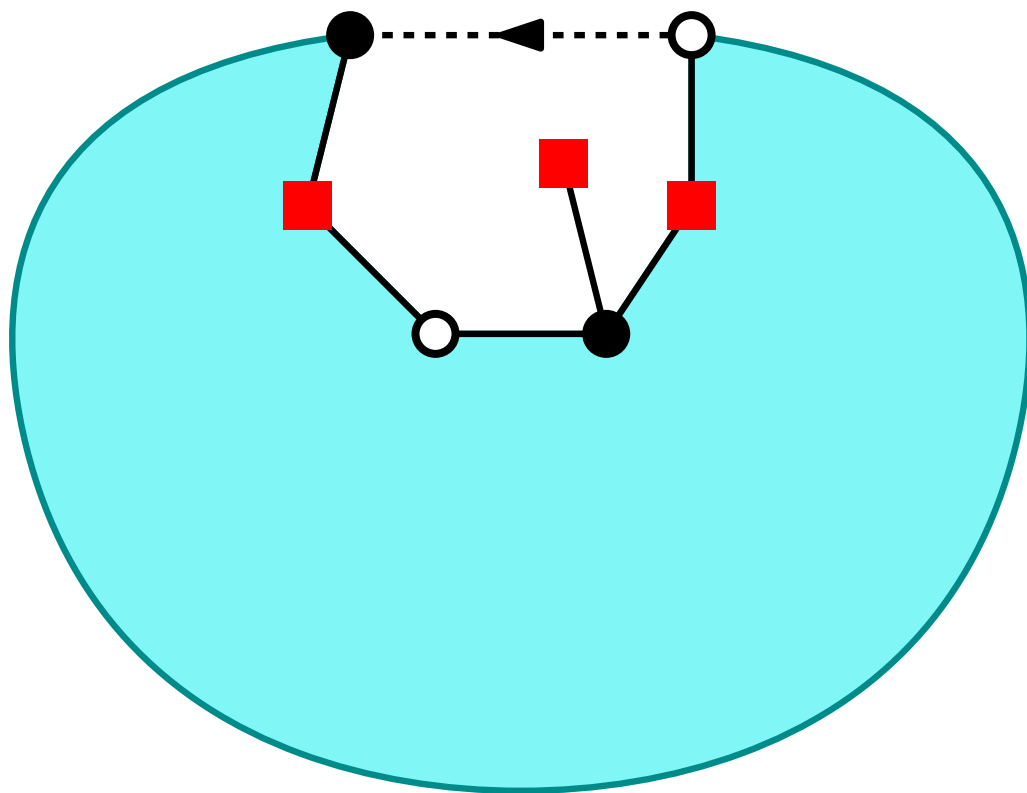


$$M(y) = 1 + tyM(y)^2 + ???$$

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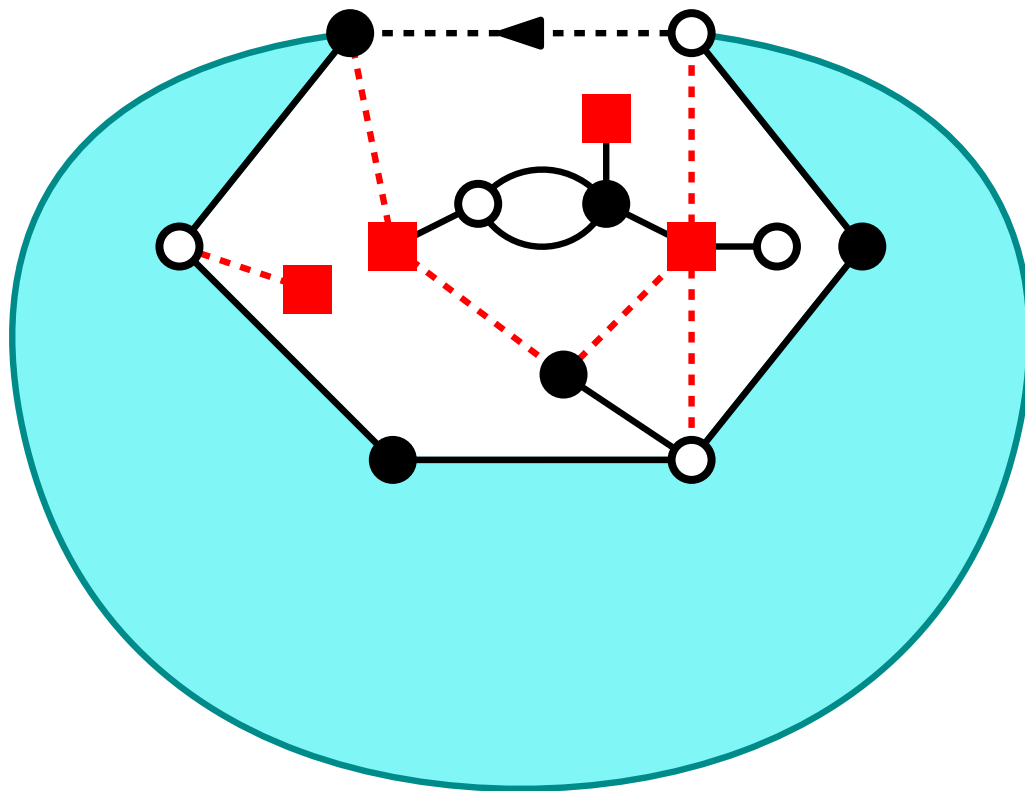
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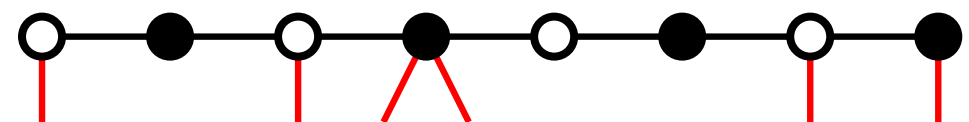
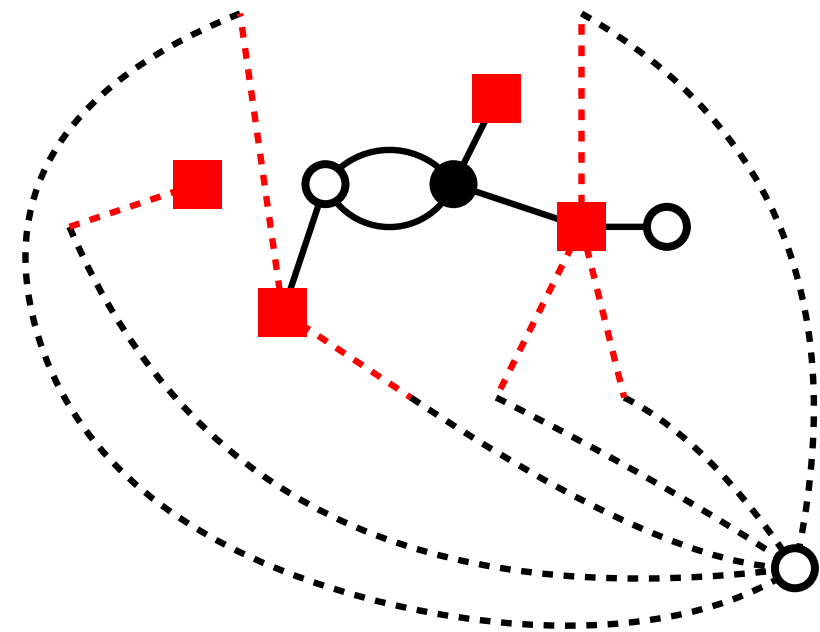
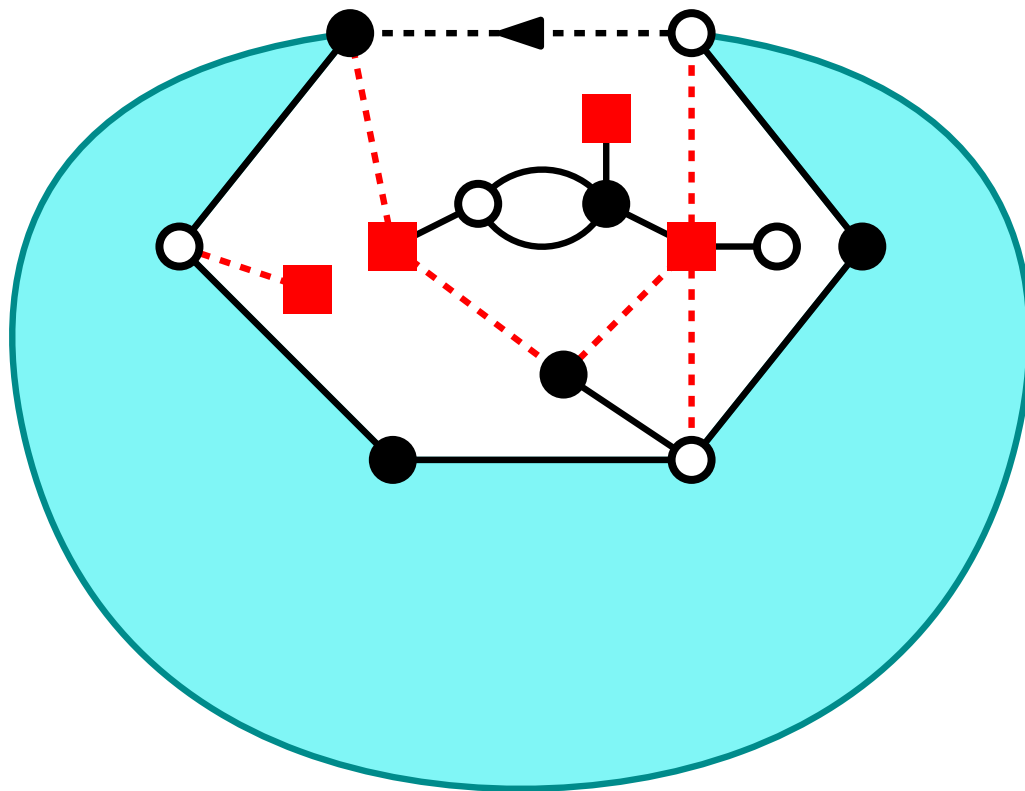
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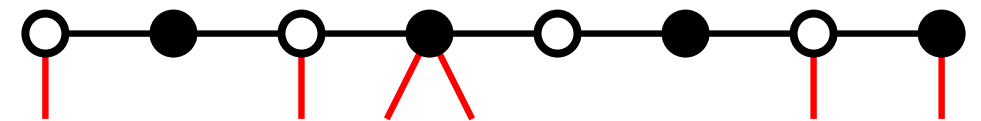
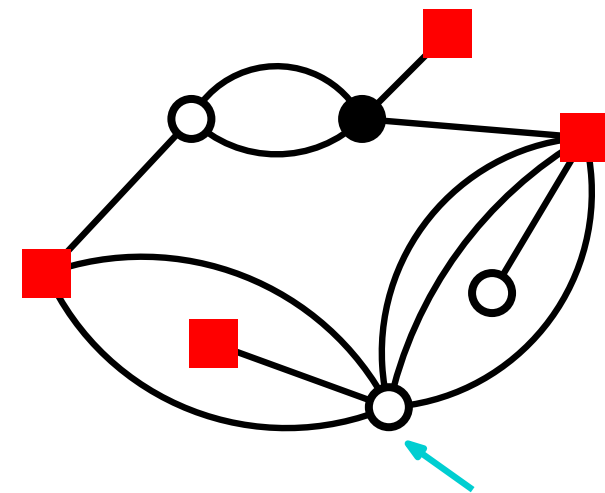
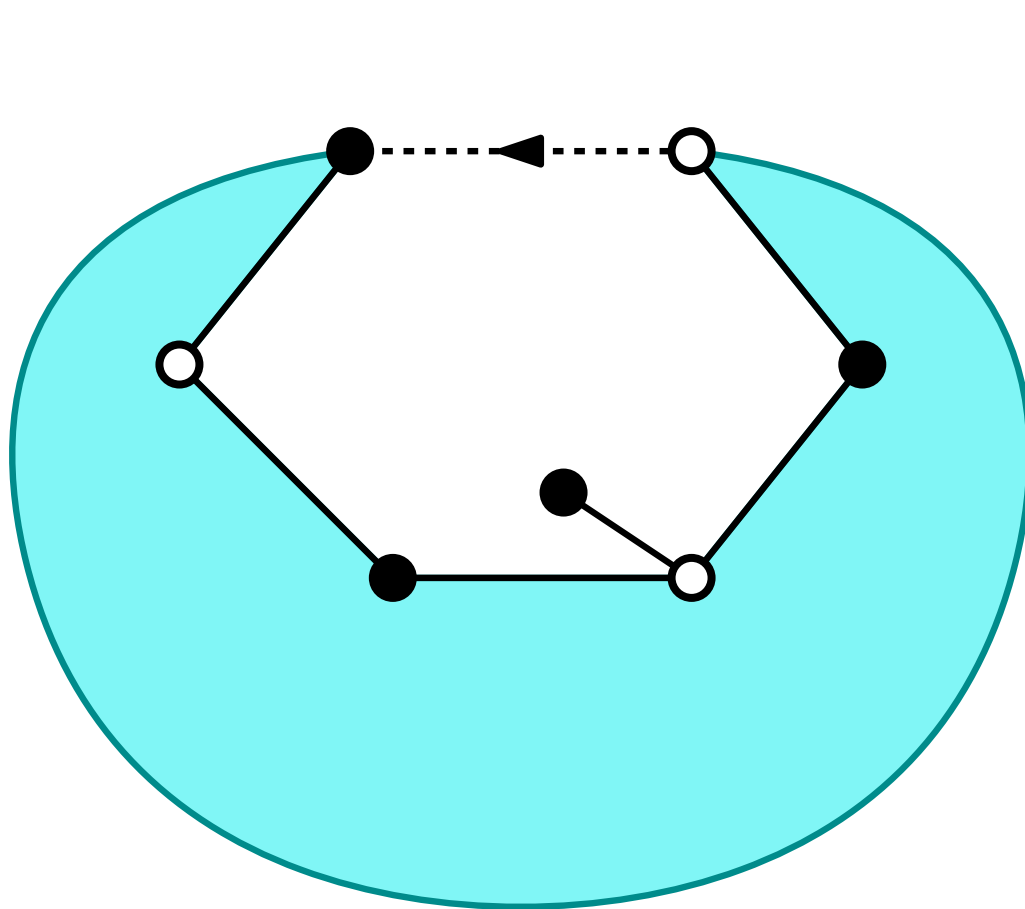
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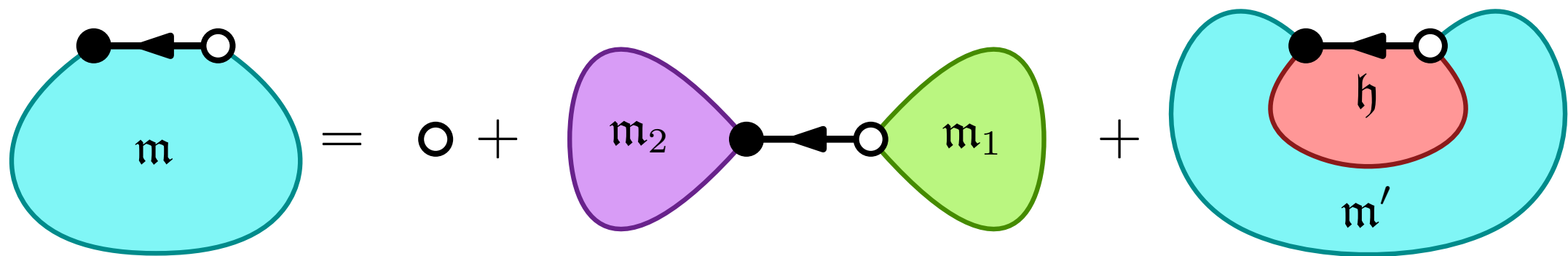


3-colored map where all neighbors of the root vertex are red.
+ connection caterpillar.

Gasket decomposition

Let $M(t, y) \equiv M(y)$ be the generating function of 3-colored planar maps with a **black and white outer face**, with $t \rightarrow$ edges and $y \rightarrow$ half the degree of the outer face.

Let $H(t, x) \equiv H(x)$ be the generating function of 3-colored planar maps where all the **neighbors** of the **root vertex** are **red**, with $t \rightarrow$ edges and $x \rightarrow$ degree of the root vertex.



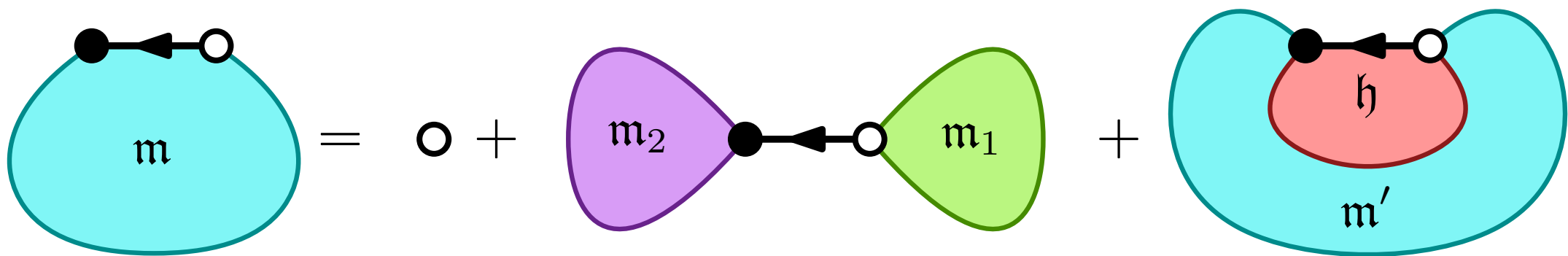
$$M(y) = 1 + tyM(y)^2 + ty \sum_{p \geq 1} \sum_{k=1}^p h_k y^{p-k} [y^p] M(y),$$

with $h_k = \sum_{d \geq 0} \binom{2k+d-1}{2k-1} [x^d] H(x).$

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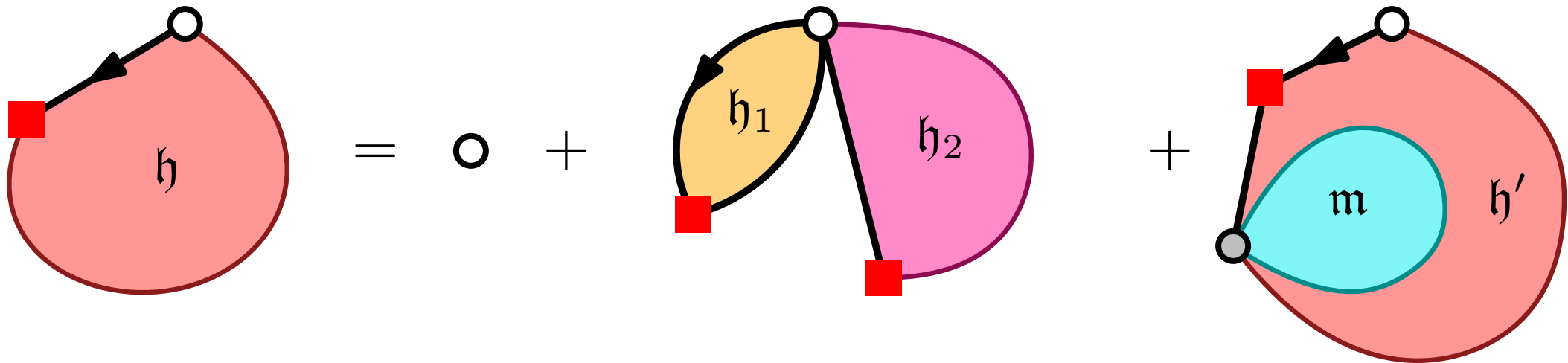
$$M(y) = 1 + tyM(y)^2 + ty[y^{\geq 0}](M(y) - 1)\tilde{H}(\bar{y}), \quad \bar{y} = 1/y$$

$$\begin{aligned} \tilde{H}(y) &= \sum_{k \geq 1} y^k \sum_{d \geq 0} \binom{2k+d-1}{2k-1} [x^d] H(x) \\ &= \frac{1}{2} \left[\frac{\sqrt{y}}{1-\sqrt{y}} H\left(\frac{1}{1-\sqrt{y}}\right) - \frac{\sqrt{y}}{1+\sqrt{y}} H\left(\frac{1}{1+\sqrt{y}}\right) \right]. \end{aligned}$$

Gasket decomposition

Let $M(t, y) \equiv M(y)$ be the generating function of 3-colored planar maps with a **black and white outer face**, with $t \rightarrow$ edges and $y \rightarrow$ half the degree of the outer face.

Let $H(t, x) \equiv H(x)$ be the generating function of 3-colored planar maps where all the **neighbors** of the **root vertex** are **red**, with $t \rightarrow$ edges and $x \rightarrow$ degree of the root vertex.



$$H(x) = 1 + txH(x)^2 + 2tx[x^{\geq 0}](H(x) - 1)\widetilde{M}(\overline{x}),$$

$$\widetilde{M}(x) = \sum_{i \geq 1} x^i \sum_{p \geq 0} \binom{2p+i-1}{i-1} [y^p] M(t, y) = \frac{x}{(1-x)^2} M\left(t, \frac{1}{(1-x)^2}\right).$$

Gasket decomposition

Let $M(t, y) \equiv M(y)$ be the generating function of 3-colored planar maps with a **black and white outer face**, with $t \rightarrow$ edges and $y \rightarrow$ half the degree of the outer face.

Let $H(t, x) \equiv H(x)$ be the generating function of 3-colored planar maps where all the **neighbors** of the **root vertex** are **red**, with $t \rightarrow$ edges and $x \rightarrow$ degree of the root vertex.

Theorem. [S. '26]

$$\begin{cases} M(t, y) = 1 + tyM(t, y)^2 + ty[y^{\geq 0}](M(t, y) - 1)\tilde{H}(t, \bar{y}), \\ H(t, x) = 1 + txH(t, x)^2 + 2tx[x^{\geq 0}](H(t, x) - 1)\widetilde{M}(t, \bar{x}), \end{cases}$$

with

$$\begin{aligned} \widetilde{M}(t, x) &= \frac{x}{(1-x)^2} M\left(t, \frac{1}{(1-x)^2}\right), \\ \tilde{H}(t, y) &= \frac{1}{2} \left[\frac{\sqrt{y}}{1-\sqrt{y}} H\left(t, \frac{1}{1-\sqrt{y}}\right) - \frac{\sqrt{y}}{1+\sqrt{y}} H\left(t, \frac{1}{1+\sqrt{y}}\right) \right]. \end{aligned}$$

$$P_M(S, y, M) = 4y^5(y-1)^2S^5(2S^3-1)^6M^6 + 24y^4(y-1)^2(1+2S)^3(2S^3-1)^5S^4M^5 - (1+2S)(2S^3-1)^4((64S^6+96S^5+88S^4+32S^3-2S+1)y^2 - (16S^4+456S^3+684S^2+334S+57)(1+2S)^2y + 56(1+2S)^5)S^3(y-1)y^3M^4 - 4(2S^3-1)^3(1+2S)^4((64S^6+96S^5+88S^4+32S^3-2S+1)y^2 - (16S^4+136S^3+204S^2+94S+17)(1+2S)^2y + 16(1+2S)^5)S^2(y-1)y^2M^3 - (1+2S)^2(2S^3-1)^2(2S(64S^6+96S^5+94S^4+50S^3+18S^2+4S+1)(S+1)^3y^4 + (1+2S)(6144S^{10}+20928S^9+33600S^8+32064S^7+18380S^6+5244S^5-162S^4-536S^3-87S^2+18S+5)y^3 - 2(384S^6+1120S^5+1914S^4+1604S^3+706S^2+171S+23)(1+2S)^5y^2 + (96S^4+616S^3+924S^2+414S+77)(1+2S)^7y - 36(1+2S)^{10})SyM^2 - 2(2S^3-1)(1+2S)^5(2S(64S^6+96S^5+94S^4+50S^3+18S^2+4S+1)(S+1)^3y^4 + (1+2S)(2048S^{10}+6592S^9+9536S^8+7488S^7+1808S^6-144S^5-609S^4-119S^3-6S^2-4S-1)y^3 - 2(128S^6+224S^5+266S^4+164S^3+50S^2+11S+3)(1+2S)^5y^2 + (32S^4+72S^3+108S^2+38S+9)(1+2S)^7y - 4(1+2S)^{10})M - (1+2S)^3(S(4S^2+2S+1)^3(S+1)^6y^4 + 2(64S^6+96S^5+94S^4+50S^3+18S^2+4S+1)(S+1)^3(1+2S)^5y^3 + (1024S^9+3008S^8+3520S^7+1344S^6-1460S^5-2436S^4-1642S^3-616S^2-127S-12)(1+2S)^6y^2 - 2(64S^5+64S^4+14S^3-36S^2-34S-9)(1+2S)^{10}y + 8(2S^3-1)(1+2S)^{12})$$

II- A new algebraicity proof

$$P_H(S, \sqrt{\Delta}, x, H) = S^3x^5(2S^3-1)^4(2x-1)H^4 - 2S^2x^3(2x-1)(x-2)(1+2S)^3(2S^3-1)^3H^3 + Sx(1+2S)(2S^3-1)^2(2S(2S^3+4S^2+2S+1)x^4 - 2(8S^4+8S^3+12S^2+2S+1)(1+2S)^2x^3 + (8S^4-24S^3-36S^2-22S-3)(1+2S)^2x^2 + 10(1+2S)^5x - 4(1+2S)^5)H^2 - 2(1+2S)(2S^3-1)(S((8S^5+28S^4+38S^3+26S^2+10S+2)\sqrt{\Delta} + 16S^6+56S^5+76S^4+58S^3+28S^2+8S+1)x^4 - 2(16S^6+48S^5+86S^4+76S^3+34S^2+9S+1)(1+2S)^3x^3 + (20S^4+56S^3+84S^2+32S+7)(1+2S)^5x^2 - (8S^4+56S^3+84S^2+38S+7)(1+2S)^5x + 2(1+2S)^8)H(1+2S)^3((32S^8+(16S^6+64S^5+104S^4+90S^3+46S^2+14S+2)\sqrt{\Delta} + 176S^7+384S^6+452S^5+356S^4+182S^3+64S^2+15S+2)x^3 - 2(1+2S)(-64S^9-320S^8-656S^7+(8S^5+28S^4+38\sqrt{\Delta}S^3+26S^2+10S+2)\sqrt{\Delta} - 560S^6-72S^5+332S^4+354S^3+172S^2+44S+5)x^2 - 2(8S^7+96S^6+144S^5+60S^4-46S^3-80S^2-38S-7)(1+2S)^3x + 4(2S^3-1)(1+2S)^6)$$

The strategy: Guess and Check

$$\begin{cases} M(t, y) = 1 + tyM(t, y)^2 + ty[y^{\geq 0}](M(t, y) - 1)\tilde{H}(t, \bar{y}), \\ H(t, x) = 1 + txH(t, x)^2 + 2tx[x^{\geq 0}](H(t, x) - 1)\widetilde{M}(t, \bar{x}). \end{cases}$$

Idea: The equations allow us to recursively compute all the coefficients of the series M and H

$\implies$ unique solution in $\mathbb{Q}[y][[t]] \times \mathbb{Q}[x][[t]]$.

Guess and Check:

1. Guess polynomial equations satisfied by our series.
2. Prove that the solutions of the polynomial equations satisfy the original equations.

Guess

$$\begin{cases} M(t, y) = 1 + tyM(t, y)^2 + ty[y^{\geq 0}](M(t, y) - 1)\tilde{H}(t, \bar{y}), \\ H(t, x) = 1 + txH(t, x)^2 + 2tx[x^{\geq 0}](H(t, x) - 1)\tilde{M}(t, \bar{x}). \end{cases}$$

Guess:

1. Use the equations to compute the first (250) terms of the series M and H .
2. Guess polynomial equations with `seriestoalgeq` (package gfun).
3. Determine which solution of the polynomial equation is the candidate.

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$$\begin{cases} M(t, y) = 1 + tyM(t, y)^2 + ty[y^{\geq 0}](M(t, y) - 1)\tilde{H}(t, \bar{y}), \\ H(t, x) = 1 + txH(t, x)^2 + 2tx[x^{\geq 0}](H(t, x) - 1)\tilde{M}(t, \bar{x}). \end{cases}$$

Guess:

1. Use the equations to compute the first (250) terms of the series M and H .
2. Guess polynomial equations with `seriestoalgeq` (package gfun).
3. Determine which solution of the polynomial equation is the candidate.

→ We use the intermediate field $\mathbb{Q}(t, y) \rightarrow \mathbb{Q}(S, y) \rightarrow \mathbb{Q}(t, y, M(t, y))$
where $t = \frac{S(1-S^3)}{(1+2S)^3}$ to have smaller equations.

→ Equations of degree 6 for $M(t, y)$ over $\mathbb{Q}(S, y)$ and of degree 8 for $H(t, x)$ over $\mathbb{Q}(S, x)$.

Both polynomial equations have only one solution that is a power series in t .

→ We denote them M_c and H_c .

Guess

1. Use the equations to compute the first (250) terms of the series M and H .
2. Guess polynomial equations with **algeqto**series (package gfun).
3. Determine which solution of the polynomial equation is the candidate.

→ Equations of degree 6 for $M(t, y)$ over $\mathbb{Q}(S, y)$ and of degree 8 for $H(t, x)$ over $\mathbb{Q}(S, x)$.

$$\begin{aligned} P_M(S, y, M) = & 4y^5(y-1)^2S^5(2S^3-1)^6M^6 + 24y^4(y-1)^2(1+2S)^3(2S^3-1)^5S^4M^5 - (1+ \\ & 2S)(2S^3-1)^4((64S^6+96S^5+88S^4+32S^3-2S+1)y^2 - (16S^4+456S^3+684S^2+334S+57)(1+ \\ & 2S)^2y + 56(1+2S)^5)S^3(y-1)y^3M^4 - 4(2S^3-1)^3(1+2S)^4((64S^6+96S^5+88S^4+32S^3- \\ & 2S+1)y^2 - (16S^4+136S^3+204S^2+94S+17)(1+2S)^2y + 16(1+2S)^5)S^2(y-1)y^2M^3 - (1+ \\ & 2S)^2(2S^3-1)^2(2S(64S^6+96S^5+94S^4+50S^3+18S^2+4S+1)(S+1)^3y^4 + (1+2S)(6144S^{10}+ \\ & 20928S^9+33600S^8+32064S^7+18380S^6+5244S^5-162S^4-536S^3-87S^2+18S+5)y^3 - \\ & 2(384S^6+1120S^5+1914S^4+1604S^3+706S^2+171S+23)(1+2S)^5y^2 + (96S^4+616S^3+ \\ & 924S^2+414S+77)(1+2S)^7y - 36(1+2S)^{10})SyM^2 - 2(2S^3-1)(1+2S)^5(2S(64S^6+96S^5+ \\ & 94S^4+50S^3+18S^2+4S+1)(S+1)^3y^4 + (1+2S)(2048S^{10}+6592S^9+9536S^8+7488S^7+ \\ & 2508S^6-900S^5-1346S^4-600S^3-119S^2-6S+1)y^3 - 2(128S^6+224S^5+266S^4+164S^3+ \\ & 50S^2+11S+3)(1+2S)^5y^2 + (32S^4+72S^3+108S^2+38S+9)(1+2S)^7y - 4(1+2S)^{10})M - (1+ \\ & 2S)^3(S(4S^2+2S+1)^3(S+1)^6y^4 + 2(64S^6+96S^5+94S^4+50S^3+18S^2+4S+1)(S+1)^3(1+ \\ & 2S)^5y^3 + (1024S^9+3008S^8+3520S^7+1344S^6-1460S^5-2436S^4-1642S^3-616S^2-127S- \\ & 12)(1+2S)^6y^2 - 2(64S^5+64S^4+14S^3-36S^2-34S-9)(1+2S)^{10}y + 8(2S^3-1)(1+2S)^{12}) \end{aligned}$$

Check

Check: Prove that M_c and H_c satisfy

$$\begin{cases} M_c(t, y) = 1 + tyM_c(t, y)^2 + ty[y \geq 0](M_c(t, y) - 1)\tilde{H}_c(t, \bar{y}), \\ H_c(t, x) = 1 + txH_c(t, x)^2 + 2tx[x \geq 0](H_c(t, x) - 1)\widetilde{M}_c(t, \bar{x}), \end{cases}$$

with $\widetilde{M}_c(t, x) = \frac{x}{(1-x)^2} M_c\left(t, \frac{1}{(1-x)^2}\right)$

and $\tilde{H}_c(t, y) = \frac{1}{2} \left[\frac{\sqrt{y}}{1-\sqrt{y}} H_c\left(t, \frac{1}{1-\sqrt{y}}\right) - \frac{\sqrt{y}}{1+\sqrt{y}} H_c\left(t, \frac{1}{1+\sqrt{y}}\right) \right].$

Check

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1. Reformulate the equations to some property visible on the polynomials:

$$\begin{cases} [y^{\geq 0}]R_M(y) = 0, \\ [x^{\geq 0}]R_H(x) = 0, \end{cases}$$

where $R_M(y) = (M_c(y) - 1)\tilde{H}_c(\bar{y}) - \frac{1}{ty} (M_c(y) - 1 - tyM_c(y)^2) \in \mathbb{Q}((\bar{y}))[[t]]$

and $R_H(x) = (H_c(x) - 1)\widetilde{M}_c(\bar{x}) - \frac{1}{2tx} (H_c(x) - 1 - txH_c(x)^2) \in \mathbb{Q}((\bar{x}))[[t]]$.

Check

Check: Prove that M_c and H_c satisfy

Let $F(t, y) \in \mathbb{Q}((y))[[t]]$ be an algebraic series in t given by its minimal polynomial and first coefficients.
1. How to decide whether F has a **pole** at $y = 0$?

Check

Check: Prove that M_c and H_c satisfy

Let $F(t, y) \in \mathbb{Q}((y))[[t]]$ be an algebraic series in t given by its minimal polynomial and first coefficients.

How to decide whether F has a **pole** at $y = 0$?

Sometimes easy to see:

$$F(y) = y \frac{Pol(t, F(y))}{1 + tF(y)}$$

Check

Check: Prove that $[y^{\geq 0}]R_M(y) = [x^{\geq 0}]R_H(x) = 0$.

2. Compute minimal polynomials for R_M and R_H by eliminating M_c and H_c using resultants.

→ An equation of degree 12 for R_M over $\mathbb{Q}(S, y)$ and one of degree 12 for R_H over $\mathbb{Q}(S, x)$. Both are too big to work with.

$$\begin{aligned} Q_M(S, y, R_M) = & S^{16}(y-1)^8(2S^3-1)^{24}R_M^{12} + 24S^{14}y(y-1)^8(1+2S)^3(2S^3-1)^{22}(4S^4 + \\ & 8S^3y + 12S^2y + 6Sy - 2S + y)R_M^{11} - 2S^{12}y(y-1)^6(1+2S)(2S^3-1)^{20}(-124(1+2S)^{11}y^5 - \\ & (992S^4 - 2008S^3 - 3012S^2 - 2002S - 251)(1+2S)^8y^4 - 2(976S^8 - 7968S^7 - 7792S^6 + 5508S^5 + \\ & 18538S^4 + 16336S^3 + 7112S^2 + 1273S + 65)(1+2S)^5y^3 + (128S^{12} + 31296S^{11} - 17568S^{10} - \\ & 168528S^9 - 262136S^8 - 162720S^7 + 27492S^6 + 123148S^5 + 96298S^4 + 37652S^3 + 7353S^2 + \\ & 550S + 3)(1+2S)^2y^2 - 2S(512S^{13} + 32000S^{12} + 77792S^{11} + 74176S^{10} + 1440S^9 - 73060S^8 - \\ & 76978S^7 - 27504S^6 + 13750S^5 + 20880S^4 + 10619S^3 + 2647S^2 + 279S + 3)y + 2S^2(64S^6 + 96S^5 + \\ & 88S^4 + 32S^3 - 2S + 1)(2S^3-1)^2)R_M^{10} - 8S^{10}y^2(y-1)^6(1+2S)^2(2S^3-1)^{18}(-180(1+2S)^{16}y^6 - \\ & (2160S^4 - 2984S^3 - 4476S^2 - 3318S - 373)(1+2S)^{13}y^5 - 2(4240S^8 - 17648S^7 - 19880S^6 + \\ & 2208S^5 + 31108S^4 + 29532S^3 + 13766S^2 + 2316S + 103)(1+2S)^{10}y^4 - (10240S^{12} - 137216S^{11} - \\ & 55808S^{10} + 323552S^9 + 649936S^8 + 447744S^7 - 36204S^6 - 313484S^5 - 256670S^4 - 103496S^3 - \\ & 19625S^2 - 1366S - 13)(1+2S)^7y^3 + 2S(1280S^{15} + 82560S^{14} - 159808S^{13} - 766080S^{12} - \\ & 1072032S^{11} - 419680S^{10} + 685888S^9 + 1140832S^8 + 684746S^7 + 2048S^6 - 293620S^5 - \\ & 226596S^4 - 86349S^3 - 17168S^2 - 1475S - 26)(1+2S)^4y^2 - 2S^2(1+2S)(10240S^{14} + 184320S^{13} + \\ & 489472S^{12} + 515648S^{11} + 42368S^{10} - 543048S^9 - 666152S^8 - 308280S^7 + 83652S^6 + \end{aligned}$$

Check

Check: Prove that $[y^{\geq 0}]R_M(y) = [x^{\geq 0}]R_H(x) = 0$.

2. Compute minimal polynomials for R_M and R_H by eliminating M_c and H_c using resultants.
- An equation of degree 12 for R_M over $\mathbb{Q}(S, y)$ and one of degree 12 for R_H over $\mathbb{Q}(S, x)$. Both are too big to work with.
3. Find intermediate field extensions.

$$\mathbb{Q}(S, y) \xrightarrow{3} \mathbb{Q}(S, L) \xrightarrow{2} \mathbb{Q}(S, K) \xrightarrow{2} \mathbb{Q}(S, y, R_M)$$

$$L = \frac{1}{y} \frac{(1 + 2S)^3 (1 + 2S + 4S^2 - 4S^2 L)(1 - L)}{(1 + 2S + 2S^2 + 4S^3 - 4S^3 L)^2},$$

$$K = \frac{S^2 L}{(1 + 2S + 4S^2)(1 - K)}.$$

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$$K = \frac{S^2 L}{(1 + 2S + 4S^2)(1 - K)}.$$

4. Solve the equation for R_M over $\mathbb{Q}(S, K)$.

$$\hookrightarrow R_M(y) = \frac{B - C\sqrt{D}}{2A} \quad \text{with } A, B, C, D \in \mathbb{Q}[S, K].$$

Check

Check: Prove that $[y^{\geq 0}]R_M(y) = [x^{\geq 0}]R_H(x) = 0$.

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$$\hookrightarrow R_M(y) = \frac{B - C\sqrt{D}}{2A} \quad \text{with } A, B, C, D \in \mathbb{Q}[S, K].$$

5. Prove that L then K then R_M only have negative powers of y .

6. Apply the same method to R_H .

Algebraicity

a.
$$\begin{cases} M(t, y) = 1 + tyM(t, y)^2 + ty[y^{\geq 0}](M(t, y) - 1)\tilde{H}(t, \bar{y}), \\ H(t, x) = 1 + txH(t, x)^2 + 2tx[x^{\geq 0}](H(t, x) - 1)\widetilde{M}(t, \bar{x}). \end{cases}$$

has a unique solution in $\mathbb{Q}[y][[t]] \times \mathbb{Q}[x][[t]]$.

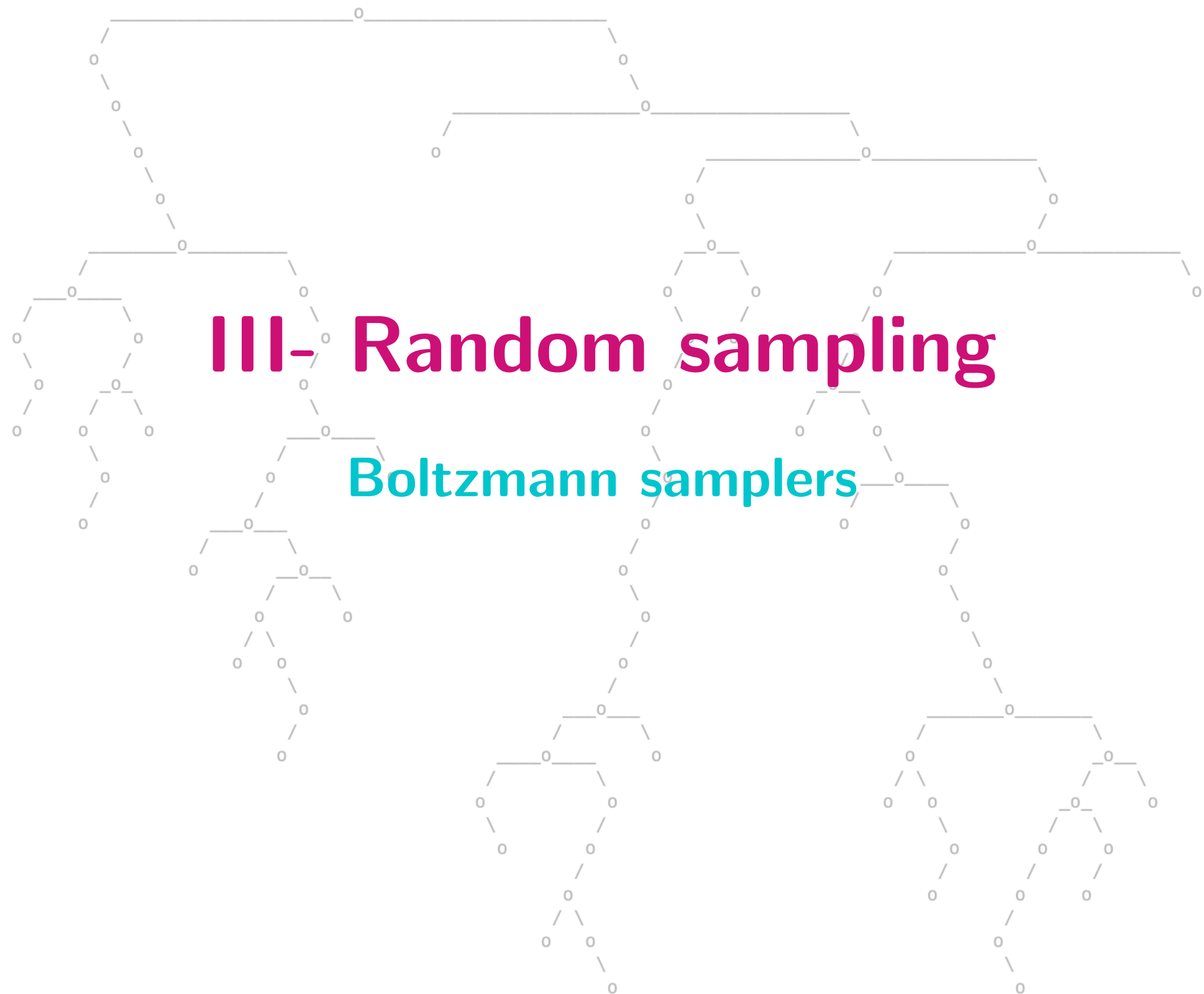
b. There are polynomials $P_M(S, y, M)$ and $P_H(S, x, H)$ whose only power series solutions satisfy the system.

Theorem. [S. '26] The generating function $M(t, y)$ of 3-colored planar maps with black and white boundary is algebraic over $\mathbb{Q}(t, y)$ of degree 24.

► Complete map of the subfields of $\mathbb{Q}(t, y, M(y))$.

$$\mathbb{Q}(t, y) \xrightarrow{4} \mathbb{Q}(S, y) \xrightarrow{3} \mathbb{Q}(S, L(y)) \xrightarrow{2} \mathbb{Q}(S, y, M(y))$$

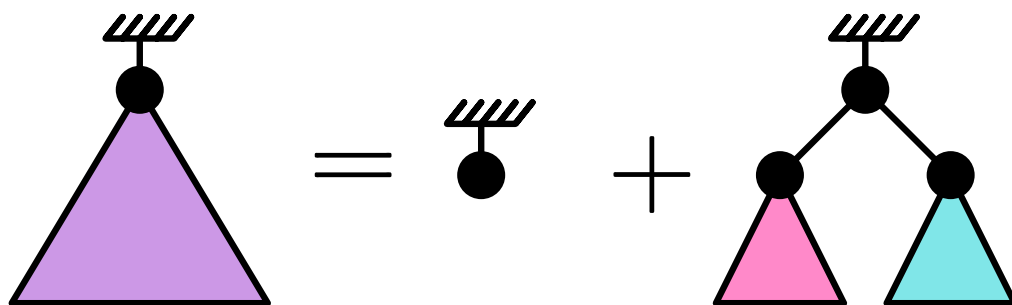
► All $[y^p]M(y)$ are rational fractions in S .



Exact size sampling

Principle: use the recursive decomposition to sample uniformly objects of a given size n .

For binary trees :



$$t_1 = 1$$

$$t_n = \sum_{k=1}^{n-1} t_k t_{n-k-1}$$

BinTreeSampler(n) =

if $n = 1$ **then** •

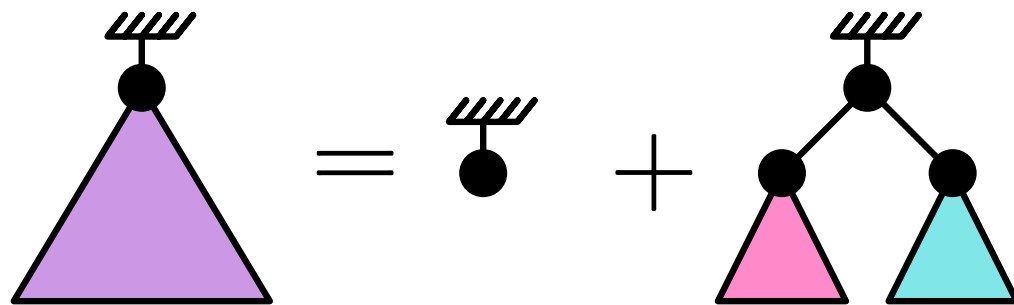
else draw k with $\mathbb{P}(k) = t_k t_{n-k-1} / t_n$

(BinTreeSampler(k), BinTreeSampler($n-1-k$))

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if $n = 1$ then •

else draw k with $\mathbb{P}(k) = t_k t_{n-k-1} / t_n$ ← very expensive

(BinTreeSampler(k), BinTreeSampler($n-1-k$))

- ▶ Need to write the explicit recurrence.
- ▶ Very slow for large size objects.

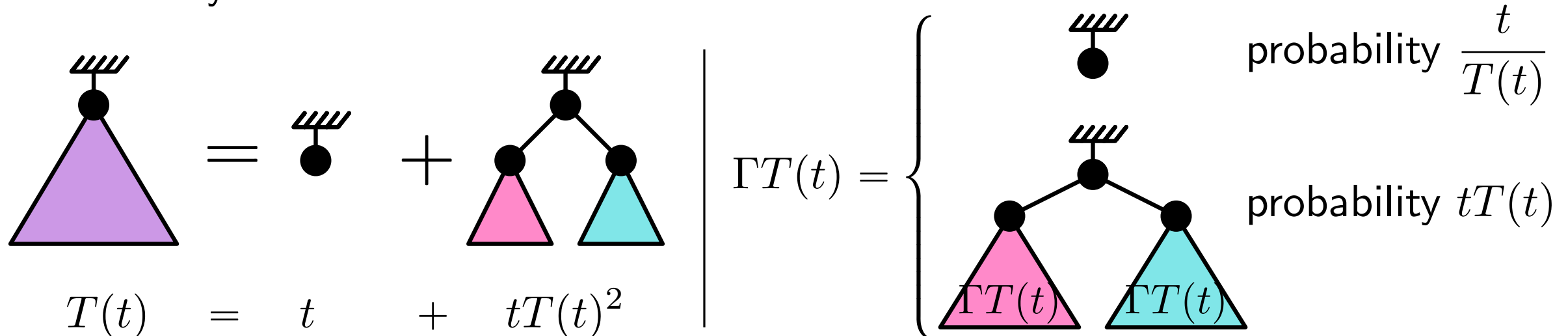
Boltzmann samplers [Duchon, Flajolet, Louchard, Schaeffer '04]

Principle: sample objects $\gamma \in \mathcal{C}$ according to the probability distribution

$$\mathbb{P}(\gamma) = \frac{t^{|\gamma|}}{C(t)}$$

with t a parameter, by following a recursive decomposition of $\mathcal{C}$.

For binary trees :



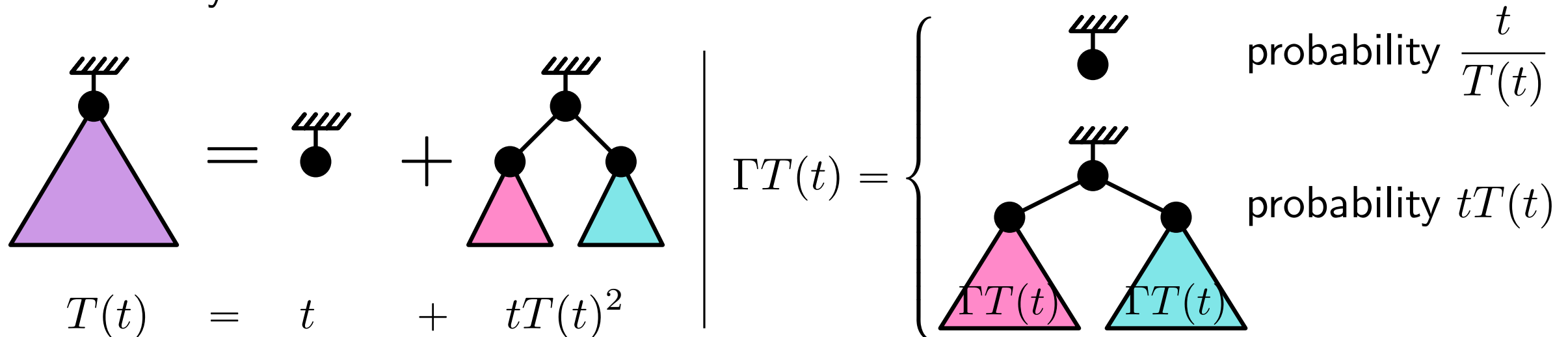
Boltzmann samplers [Duchon, Flajolet, Louchard, Schaeffer '04]

Principle: sample objects $\gamma \in \mathcal{C}$ according to the probability distribution

$$\mathbb{P}(\gamma) = \frac{t^{|\gamma|}}{C(t)}$$

with t a parameter, by following a recursive decomposition of $\mathcal{C}$.

For binary trees :

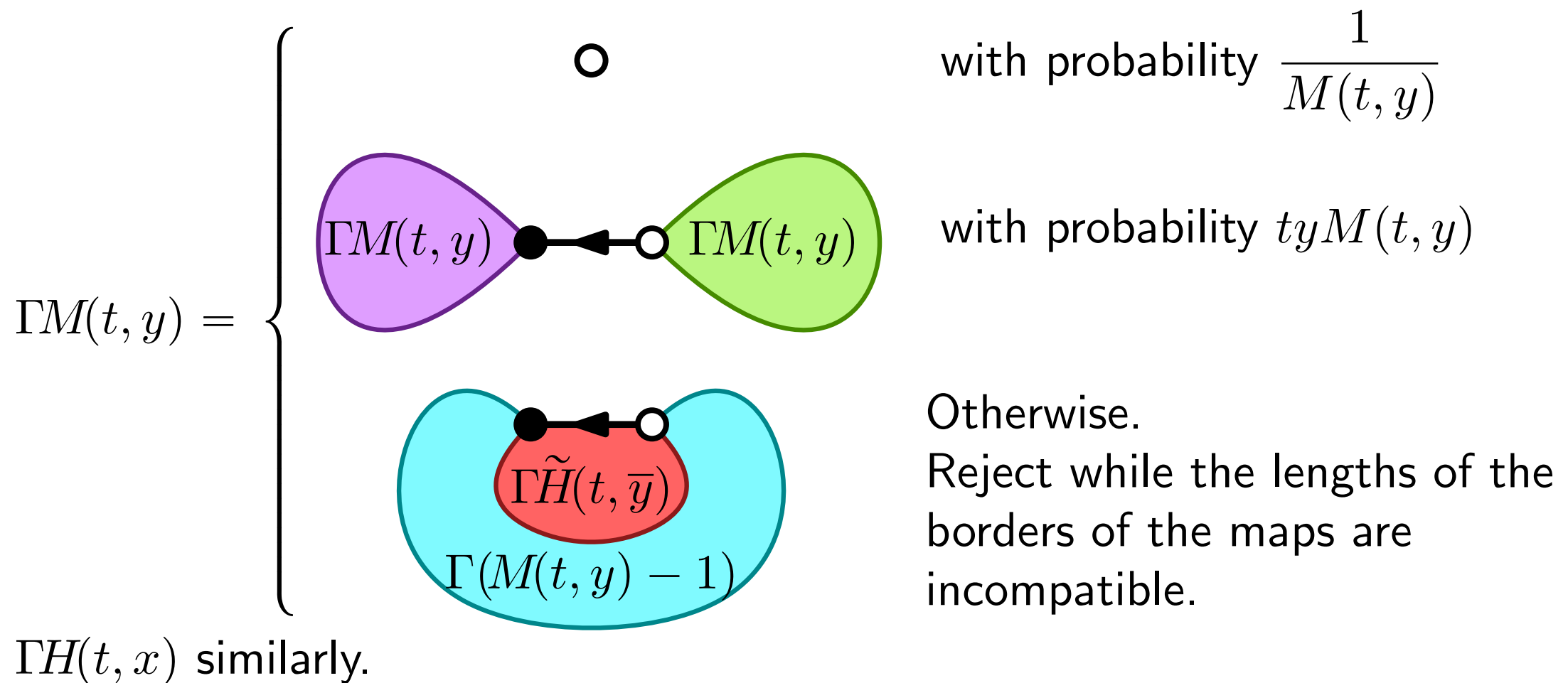


- ▶ Unfixed size (adjust the value of t to aim for a given size).
- ▶ Uniform distribution when conditioned by the size of the object.
- ▶ Can be built as long as we have a positive combinatorial equation.
- ▶ Very efficient for large objects.

Boltzmann sampler for 3-colored maps

- ▶ The catalytic variable becomes a second parameter: $\mathbb{P}(\mathfrak{m}) = \frac{t^{e(\mathfrak{m})} y^{d(\mathfrak{m})}}{M(t, y)}$.
- ▶ We use rejection to handle the $[y^{\geq 0}]$ and $[x^{\geq 0}]$.

$$\begin{cases} M(t, y) = 1 + tyM(t, y)^2 + ty[y^{\geq 0}](M(t, y) - 1)\tilde{H}(t, \bar{y}), \\ H(t, x) = 1 + txH(t, x)^2 + 2tx[x^{\geq 0}](H(t, x) - 1)\tilde{M}(t, \bar{x}). \end{cases}$$



Boltzmann sampler for 3-colored maps

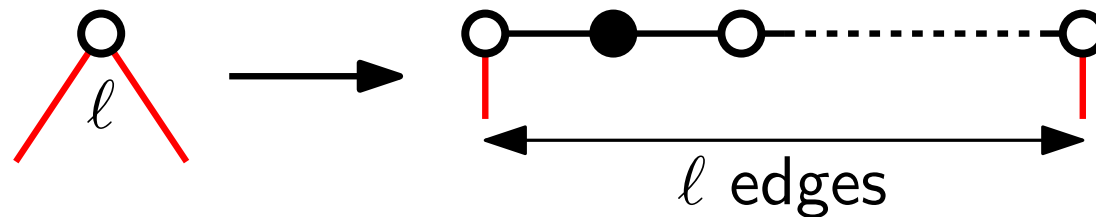
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$$\Gamma\tilde{H}(t, \bar{y}) = \text{Diagram} \quad + \quad \text{an integer } \ell \sim \text{Geom}(\bar{y}) \text{ for each corner}$$

The diagram shows a red oval labeled $\Gamma H(t, x)$. At the top, there is a red square and a white circle connected by a horizontal line with an arrow pointing from the circle to the square.

ℓ indicates the number of edges to insert:



→ Reject while the sum of the labels is odd.

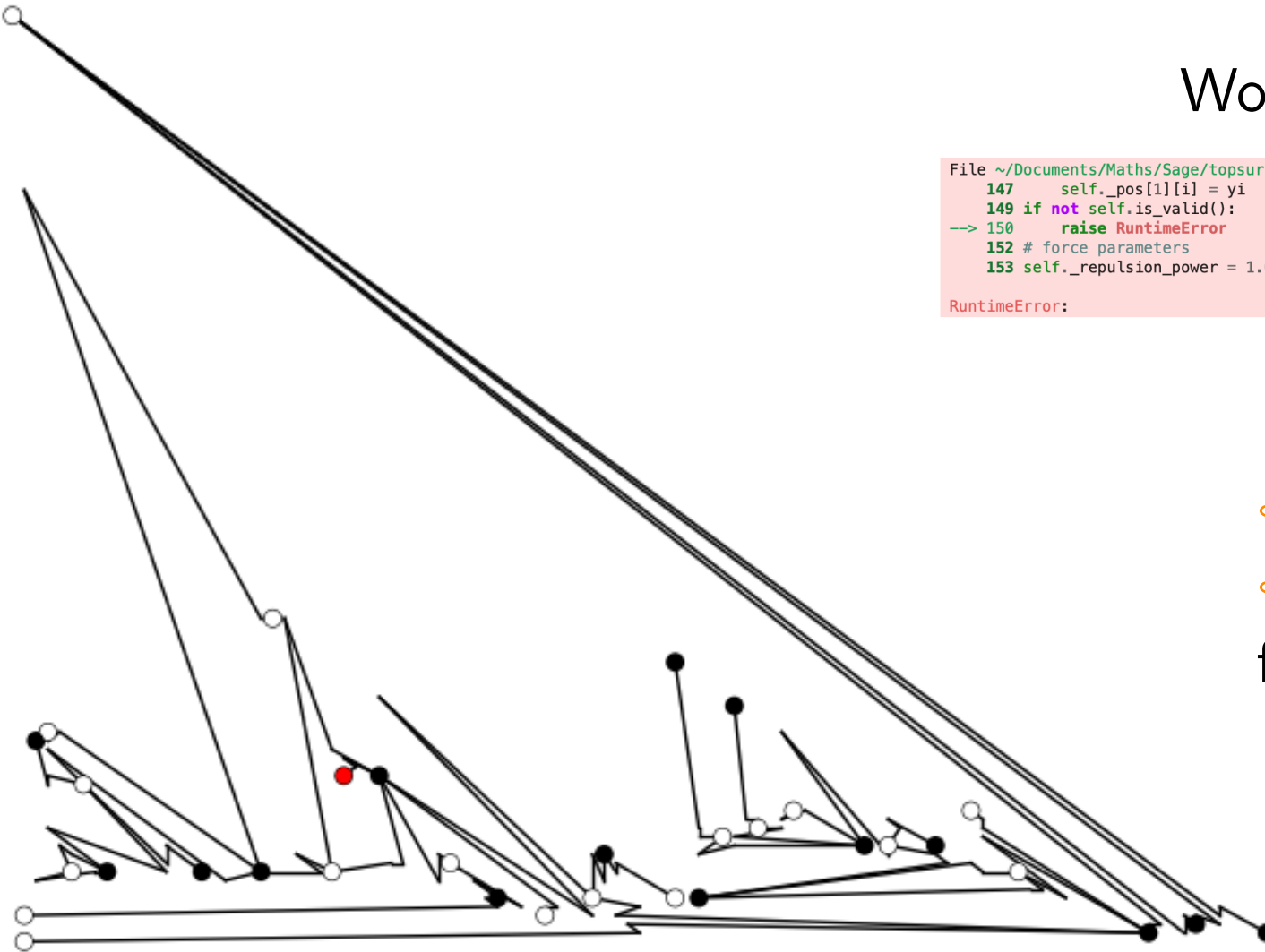
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Work in progress ...

```
File ~/Documents/Maths/Sage/topsurf/topsurf/planar_layout.py:150, in PlanarLayout.__init__(self, m, root)
    147     self._pos[1][i] = yi
    149 if not self.is_valid():
-> 150     raise RuntimeError
    152 # force parameters
    153 self._repulsion_power = 1.0

RuntimeError:
```



A random map with 33 edges.

- Lots of rejection.
- Hard to build an oracle for the pointed series.

Boltzmann sampler for 3-colored maps

- Make the catalytic variable recursive.

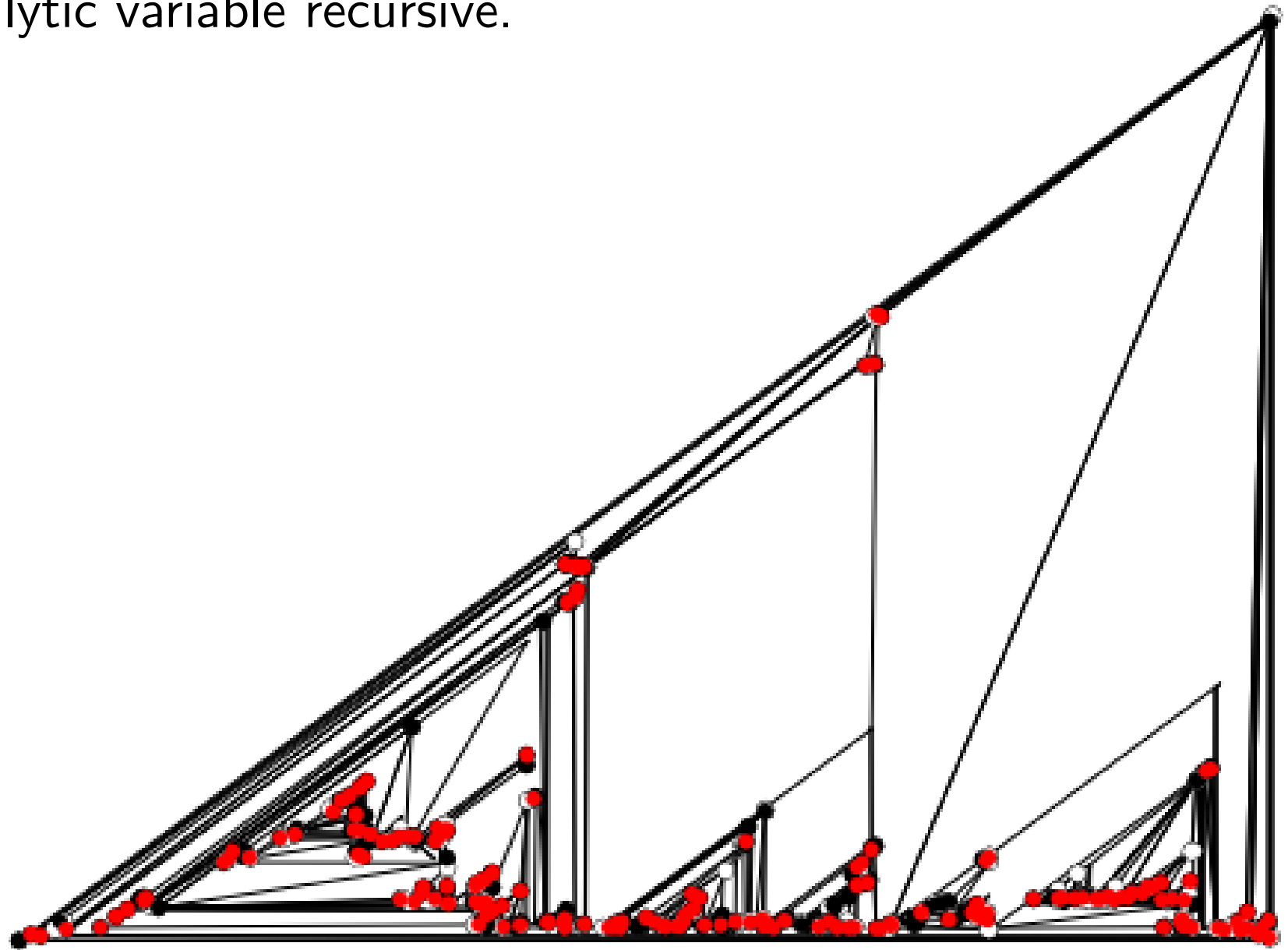
Equations for $M_p = [y^p]M(y)$, $H_d = [x^d]H(x)$, $\widetilde{M}_k = [x^k]\widetilde{M}(x)$ and $\widetilde{H}_k = [y^k]\widetilde{H}(y)$:

$$\left\{ \begin{array}{l} M_0 = H_0 = 1, \\ M_p = t \sum_{k=0}^{p-1} M_k M_{p-1-k} + t \sum_{k \geq 1} M_{p+k-1} \widetilde{H}_k \quad \text{for } p > 0, \\ H_d = t \sum_{k=0}^{d-1} H_k H_{d-1-k} + t \sum_{k \geq 1} H_{d+k-1} \widetilde{M}_k \quad \text{for } d > 0, \\ \widetilde{M}_k = \sum_{p \geq 0} \binom{2p+k-1}{k-1} M_p \quad \text{for } k > 0, \\ \widetilde{H}_k = \sum_{d \geq 0} \binom{2k+d-1}{2k-1} H_d \quad \text{for } k > 0. \end{array} \right.$$

→ Only one infinite sum per equation $\implies$ can handle them by computing the terms on the fly.

Boltzmann sampler for 3-colored maps

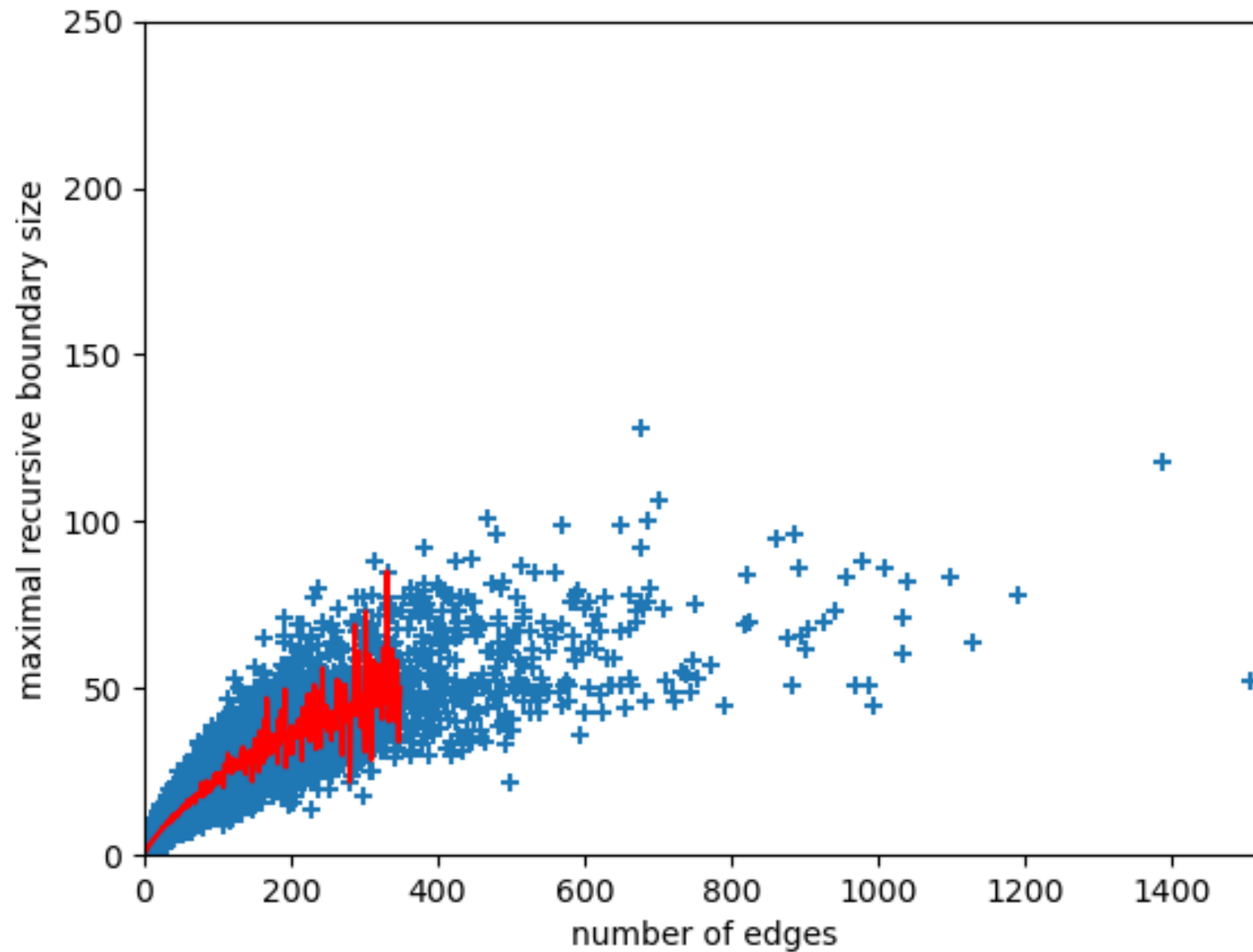
- Make the catalytic variable recursive.



A random 3-colored map with 616 edges and a horrible layout.

Boltzmann sampler for 3-colored maps

- Make the catalytic variable recursive.



The size and maximal recursive border size of random maps.

Perspectives

- ▶ Find a nice family of trees in bijection with 3-colored planar maps.
- ▶ Others applications of the decomposition?
- ▶ Adapt the gasket decomposition to others families of colored maps.
 - ↪ Colorful quadrangulations [Budd '25], (D-algebraic and solved [Bousquet-Mélou, Elvey Price '20]).
 - ↪ 4-colored triangulations (D-algebraic [Tutte '82]).
- ▶ Analyse the samplers.
- ▶ For which series $G \in \mathbb{Q}[[t, z]]$ are the solutions of
$$F(z) = \text{Pol}(t, z, F(z)) + \text{Pol}(t, z)[z^{\geq 0}]F(z)G(\bar{z})$$
algebraic?

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