Gasket decomposition of 3-colored planar maps

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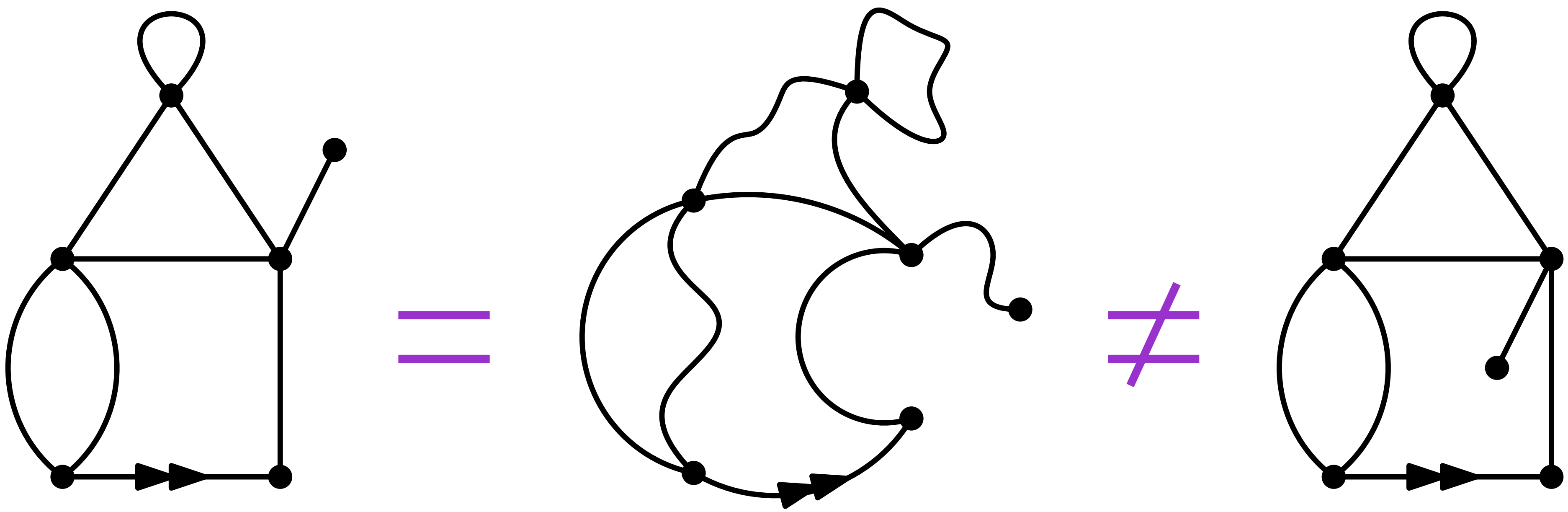
I- Recursive decomposition of colored planar maps

1) Definitions

Planar maps

A **planar map** is the embedding of a connected graph into the sphere, up to orientation preserving homeomorphism.
Multi-edges and loops are allowed.



Planar map = planar graph + cyclic ordering of the edges around each vertex.

All maps are **rooted**, i.e. an oriented edge is marked.

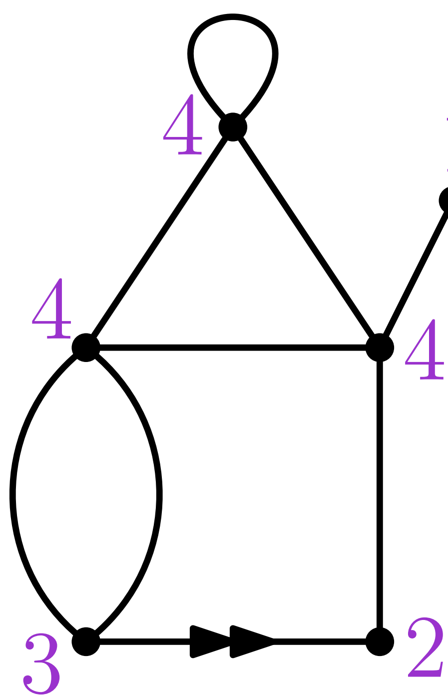
Planar maps

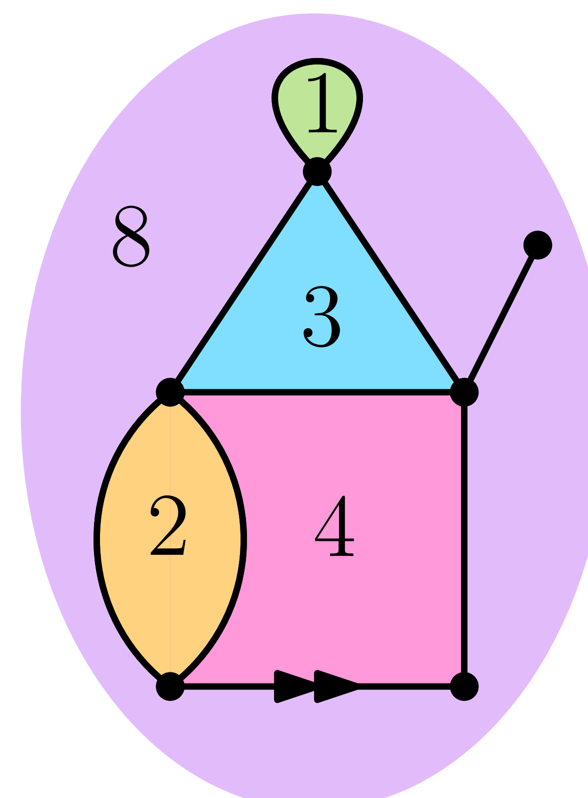
Planar map = planar graph + cyclic ordering of the edges around each vertex.

- **Vertices** and **edges** are inherited from the graph.
- **Faces** are the connected components of the sphere minus the map.

Size: number of edges n.

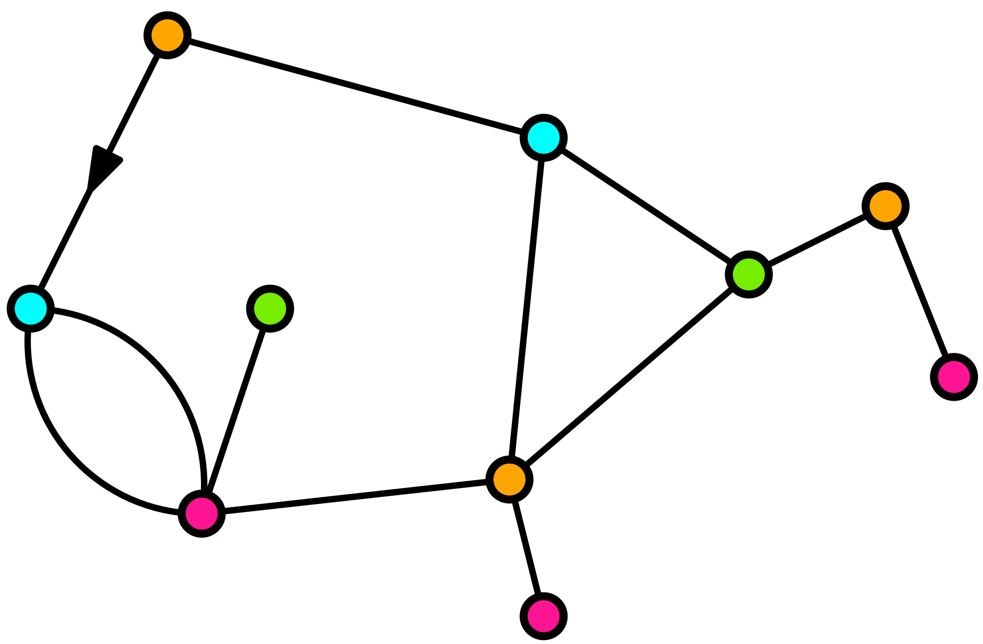
Degree (of a vertex or face) = number of incident half-edges.



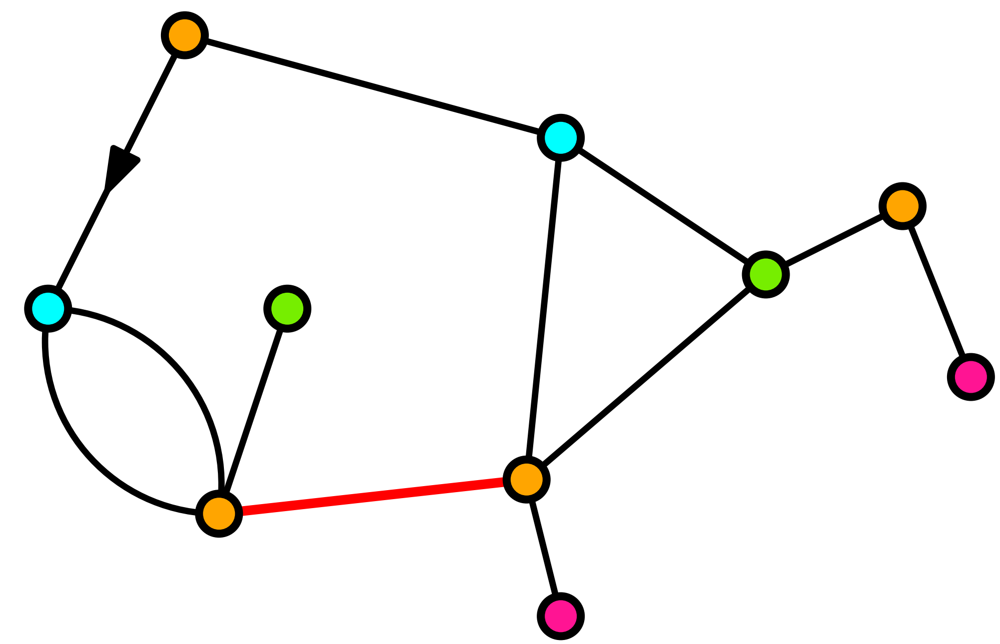


Colored maps

Colored map: assign a color to each vertex so that adjacent vertices receive different colors.



proper coloring



non-proper coloring

I- Recursive decomposition of colored planar maps

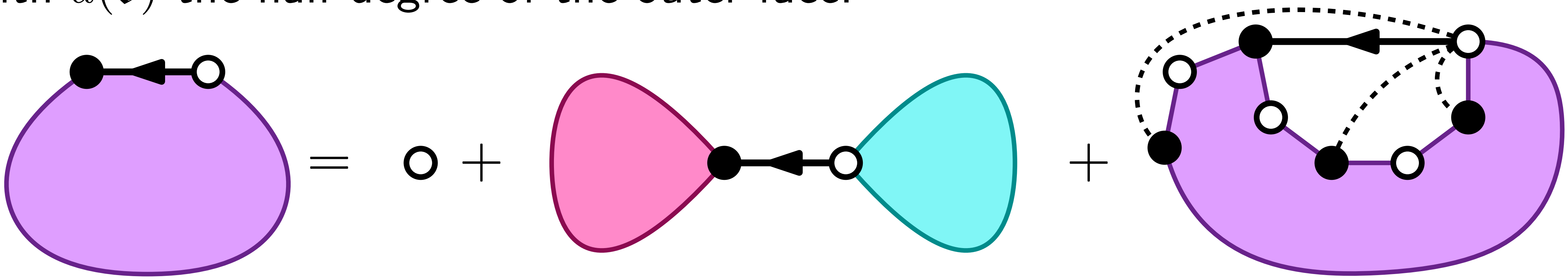
2) Tutte's decompositions

2-colored maps

Generating function of bicolored planar maps:

$$B(t,y) = \sum_{\mathfrak{b}\in\mathcal{B}} t^{e(\mathfrak{b})} y^{d(\mathfrak{b})}$$

with $d(\mathfrak{b})$ the half degree of the outer face.



$$B(t,y) \quad = \quad 1 \quad + \quad tyB(t,y)^2 \quad + ty \sum_{p\geqslant 1} \sum_{k=1}^{p} y^{p-k}[y^p]B(t,y)$$

2-colored maps

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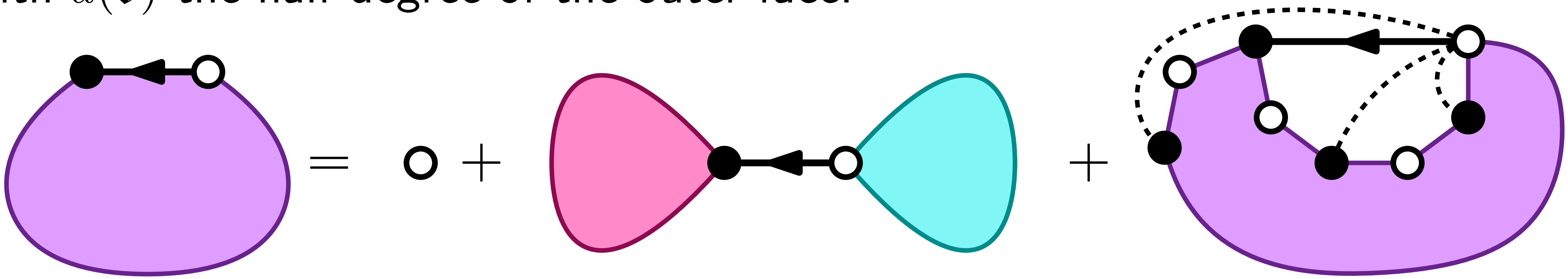

$$B(t,y) \quad = \quad 1 \quad + \quad tyB(t,y)^2 \quad + ty\frac{B(t,y)-B(t,1)}{y-1}$$

$\hookrightarrow$ Can be solved with the quadratic method [Brown '60s].

$+$ This method provides an algebraic equation of degree 4 for $B(t,y)$.

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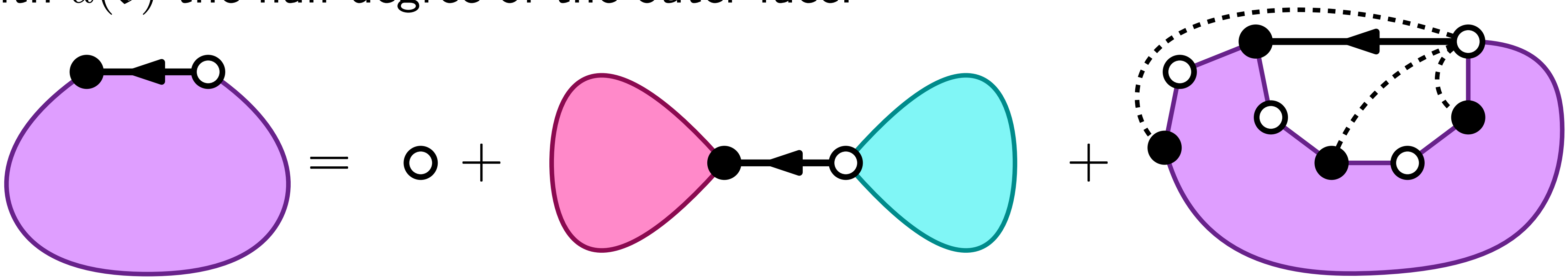

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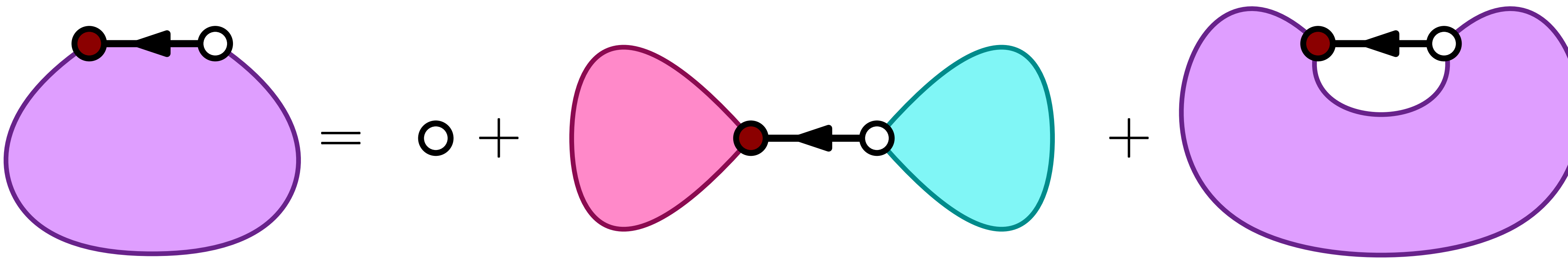
$+$ This method provides an algebraic equation of degree 4 for $B(t,y)$.

$\hookrightarrow$ Explicit bijection with a families of trees [Schaeffer '97].

3-colored maps

$M(t, y)$: generating function of 3-colored planar maps with $t \to$ number of edges and $y \to$ degree of the outer face.



$$M(t, y) \quad = \quad 1 \quad + \quad 2ty^2 M(t, y)^2 \quad + \quad ty???$$

3-colored maps

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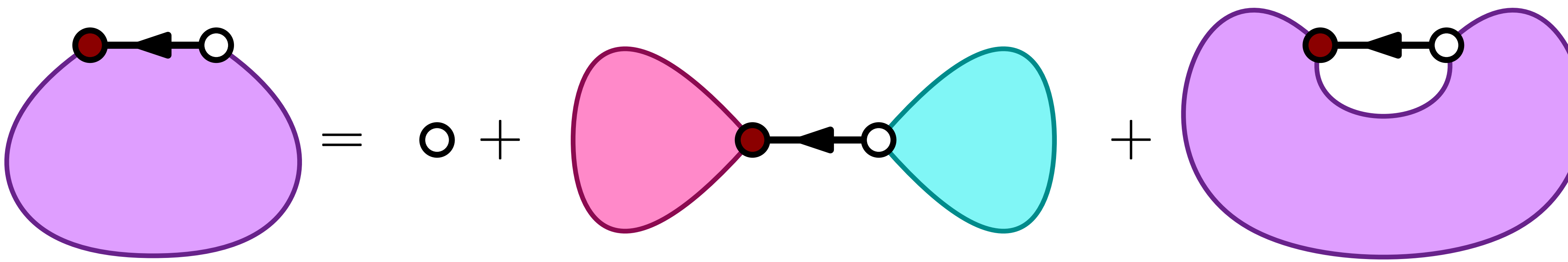

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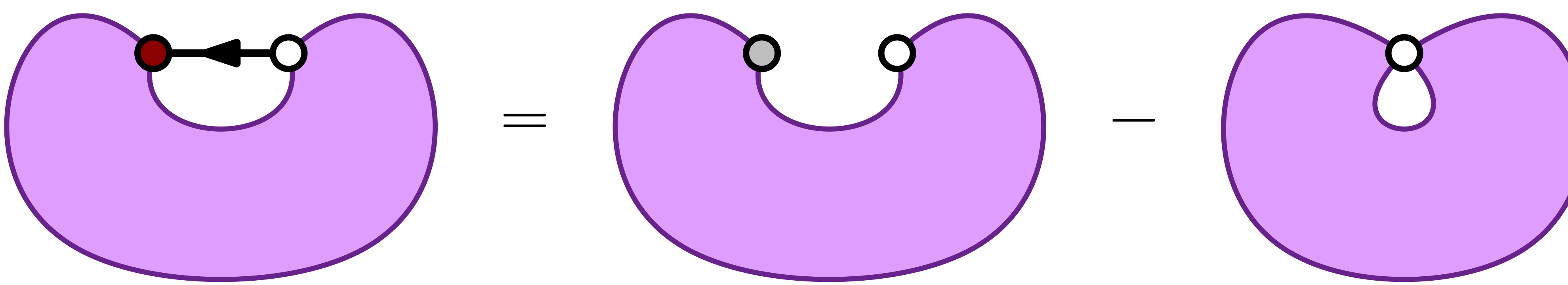
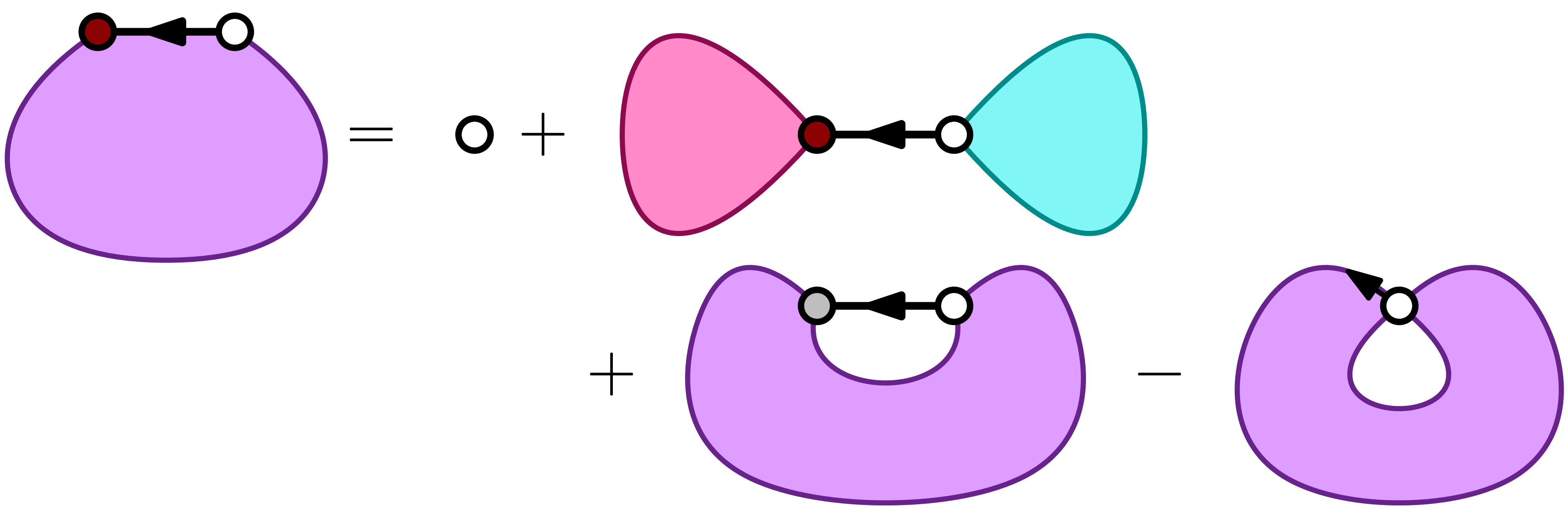
Solution:



$\Longrightarrow$ need a second catalytic variable recording the degree of the root vertex.

3-colored maps

$M(t, x, y)$: generating function of 3-colored planar maps with $t \to$ number of edges, $y \to$ degree of the outer face and $x \to$ degree of the root vertex.



Theorem. [Tutte '68, Bernardi & Bousquet-Mélou '11]

$$M(x,y) = 1 + xyt(2y-1)M(x,y)M(1,y) - xytM(x,y)M(x,1) \\ -xyt\frac{xM(x,y)-M(1,y)}{x-1} + xyt\frac{yM(x,y)-M(x,1)}{y-1}.$$

3-colored maps

Tutte equation for 3-colored planar maps:

$$\begin{aligned} M(x,y) &= 1 + xyt(2y-1))M(x,y)M(1,y) - xytM(x,y)M(x,1) \\ &\quad -xyt\frac{xM(x,y)-M(1,y)}{x-1} + xyt\frac{yM(x,y)-M(x,1)}{y-1}. \end{aligned}$$

Not an algebraic/D-finite/... equation $\Longrightarrow$ no toolbox ?

2 catalytic variables $\Longrightarrow$ no method to get an algebraic equation.

Negative coefficients $\Longrightarrow$ cannot build a random generator.

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2 catalytic variables $\implies$ no method to get an algebraic equation.
Negative coefficients $\implies$ cannot build a random generator.

Theorem. [Bernardi & Bousquet-Mélou '11] The generating function of 3-colored planar map is algebraic and is equal to

$$M(1,1) = \frac{(1+2S)(1-2S^2-4S^3-4S^4)}{(1-2S^3)^2}$$

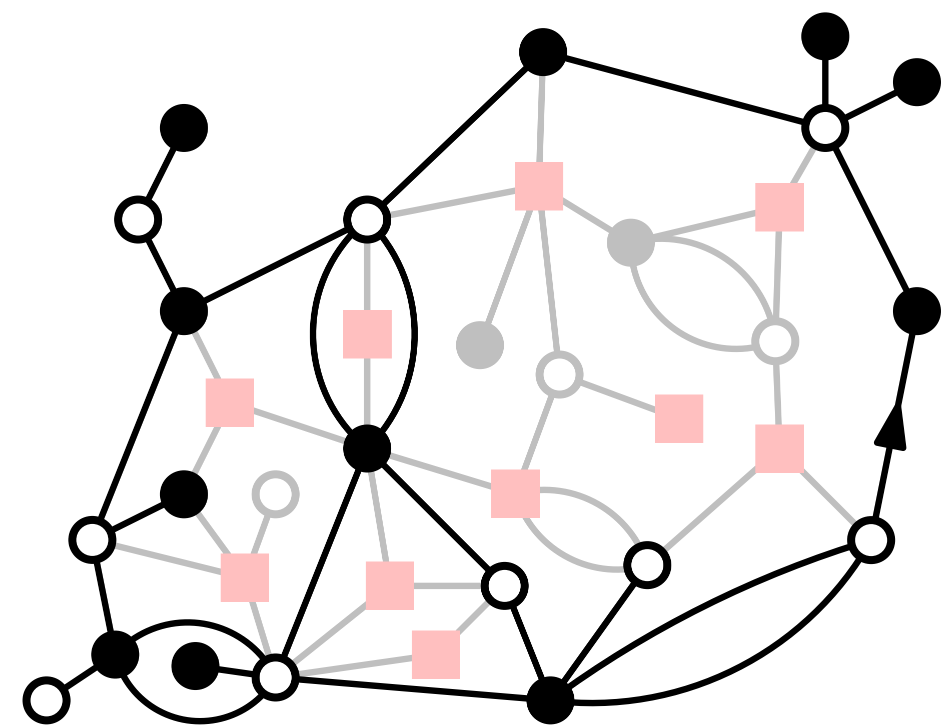
with S the solution of $t = \dfrac{S(1-S^3)}{(1+2S)^3}$ with constant term 0.

$\hookrightarrow$ Obtained via a reduction to 1-catalytic equation.

I- Recursive decomposition of colored planar maps

3) Gasket decompositions

A car gasket.

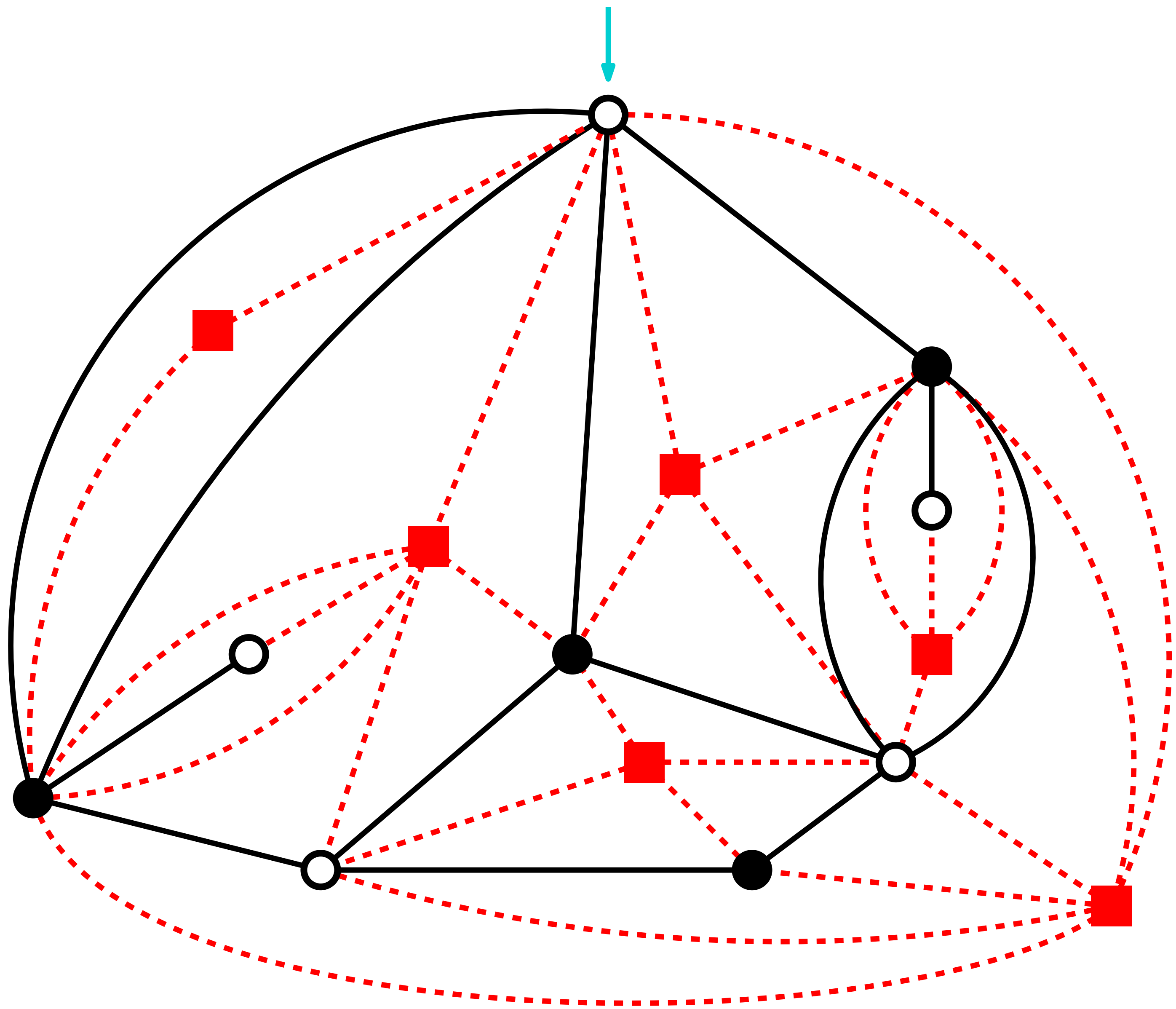


A map gasket.

Term from [Borot, Bouttier, Guitter '12].

Warm-up: Triangulations

Triangulation = all faces have degree 3.



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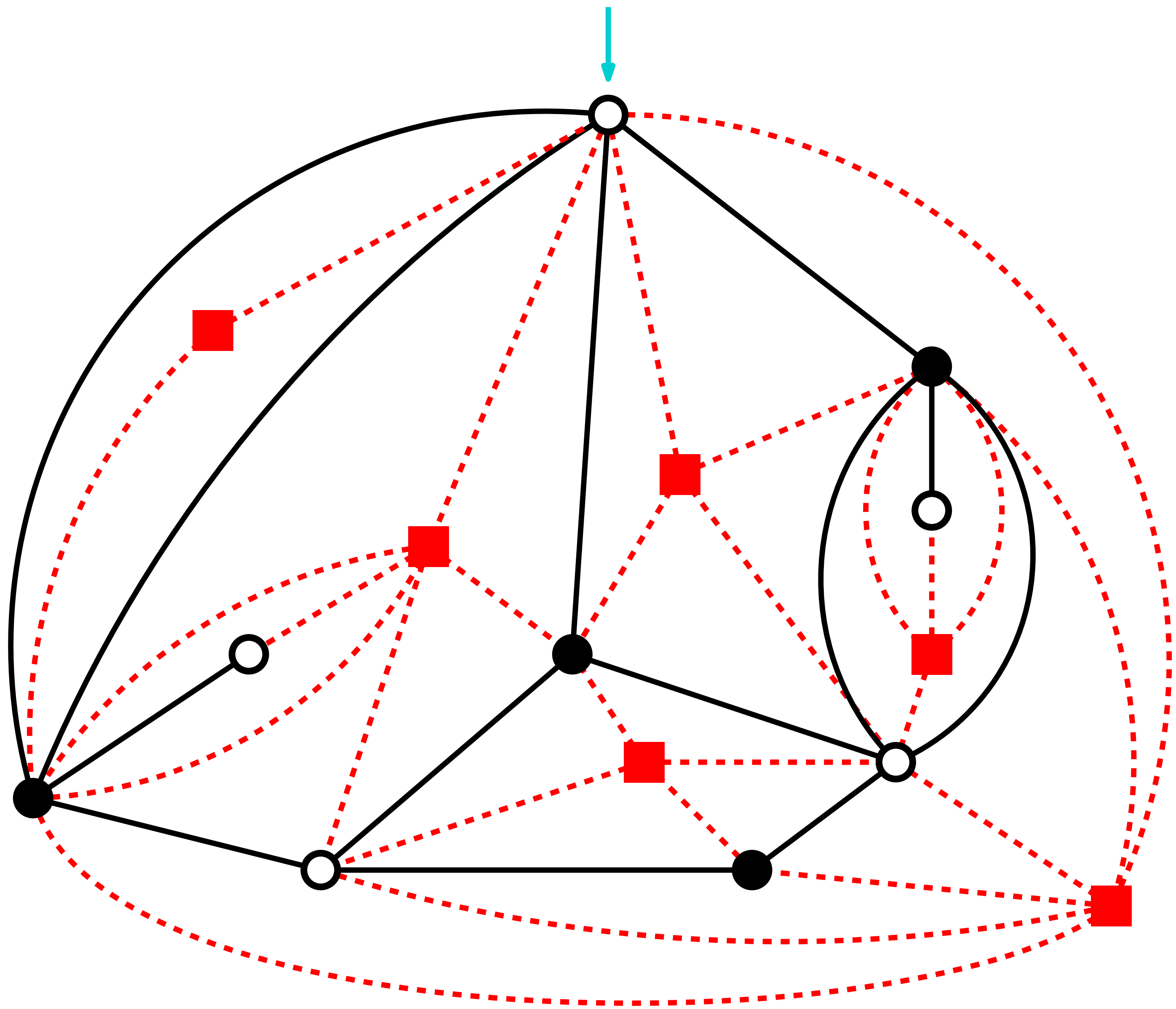
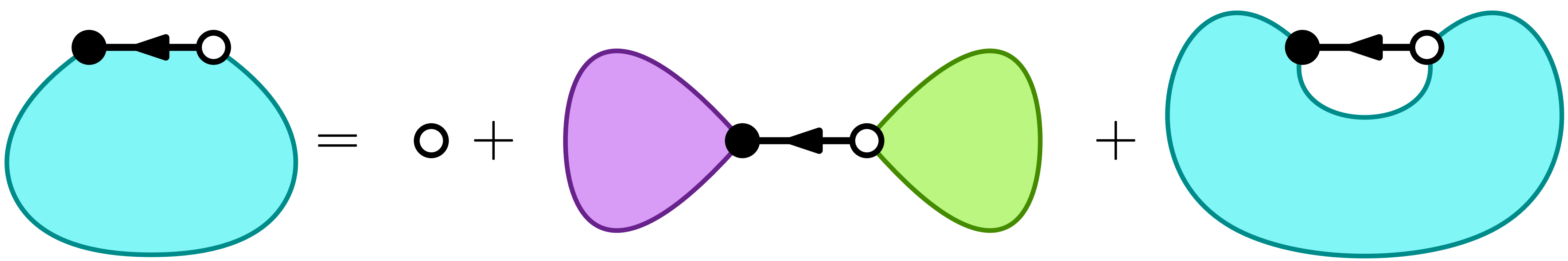


3-colored triangulations $=$ 2-colored general maps.

Gasket decomposition

Idea: Use the structure from the coloring and get back to bipartite maps.

Let $M(t,y) \equiv M(y)$ be the generating function of 3-colored planar maps with a **black and white outer face**, with $t \rightarrow$ edges and $y \rightarrow$ half the degree of the outer face.



$$M(y) \quad = \quad 1 \quad + \quad tyM(y)^2 \quad + \quad ???$$

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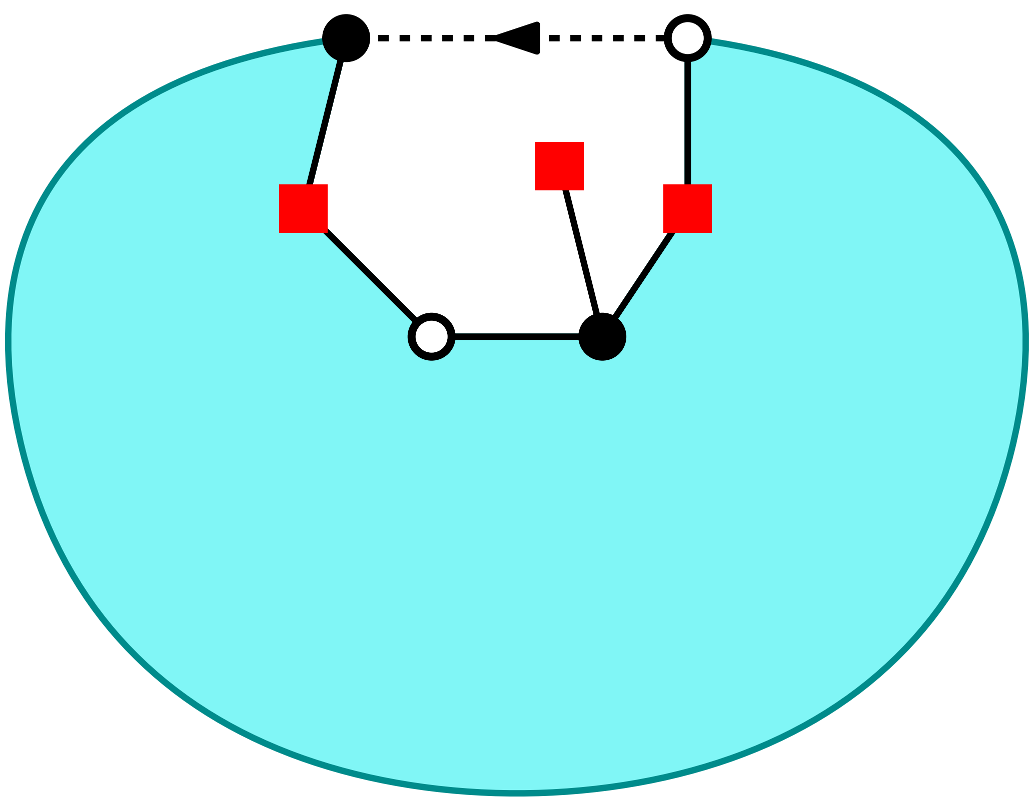

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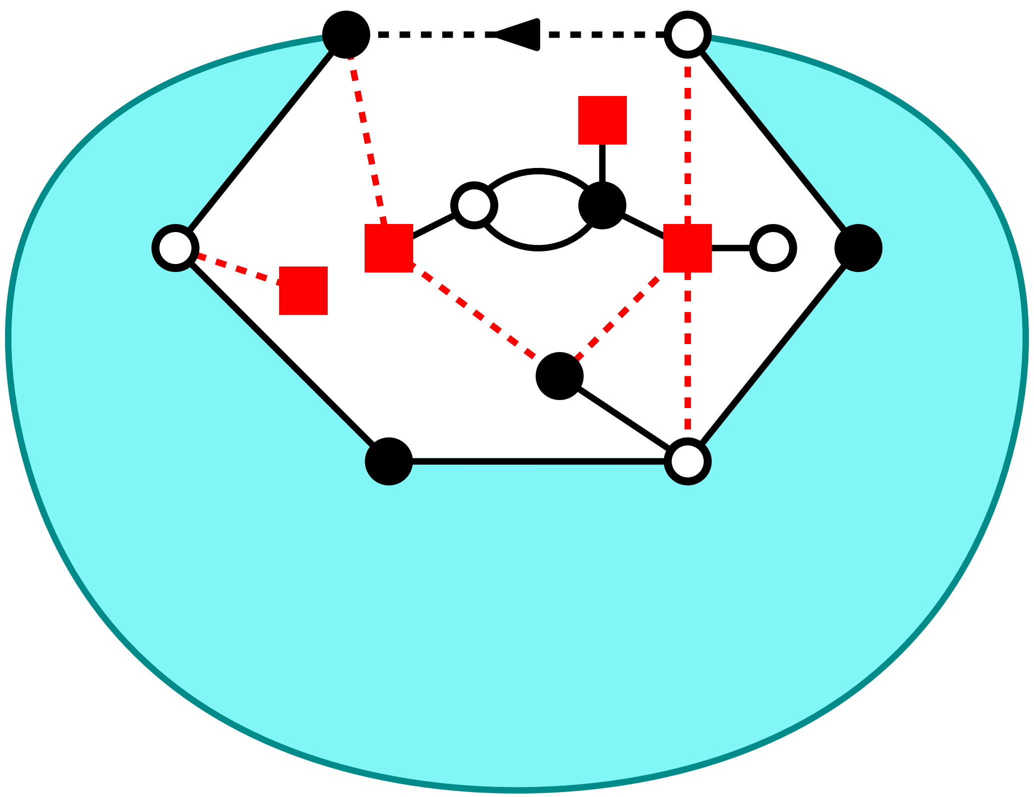

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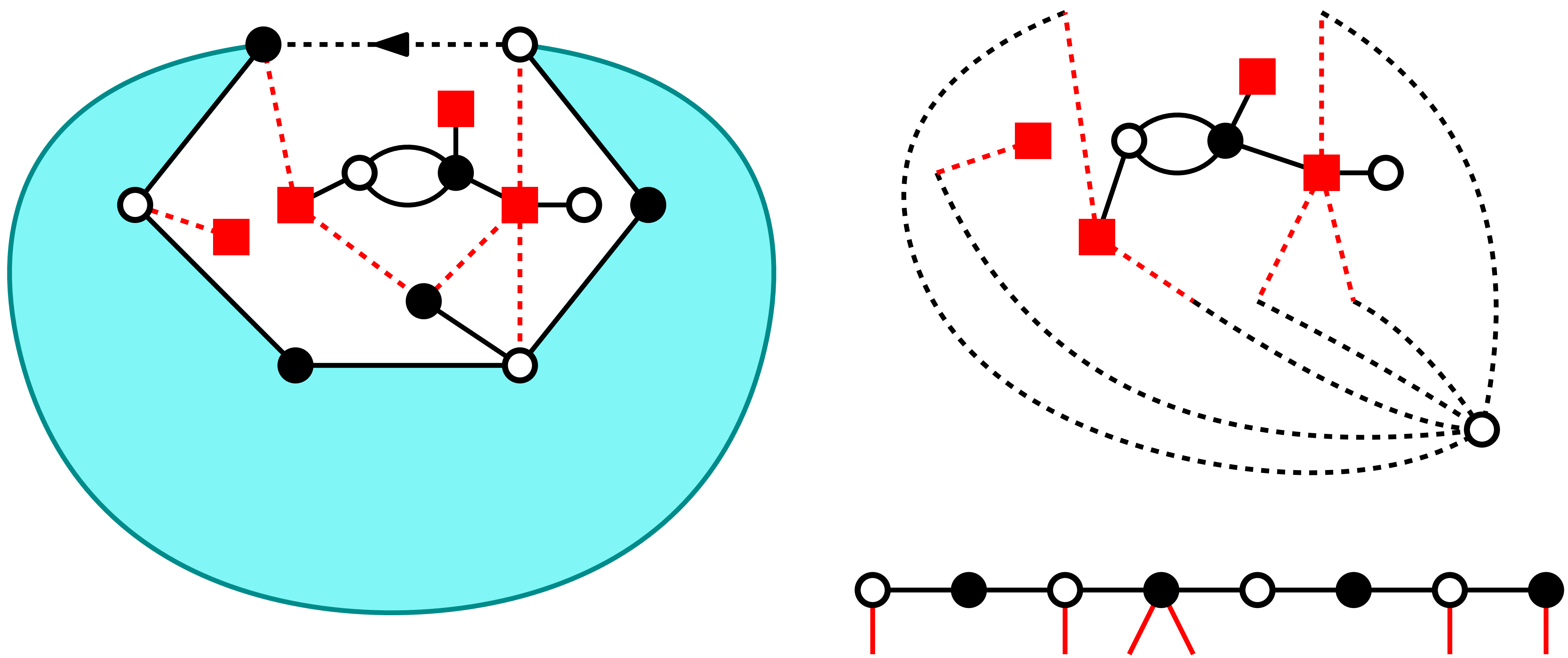

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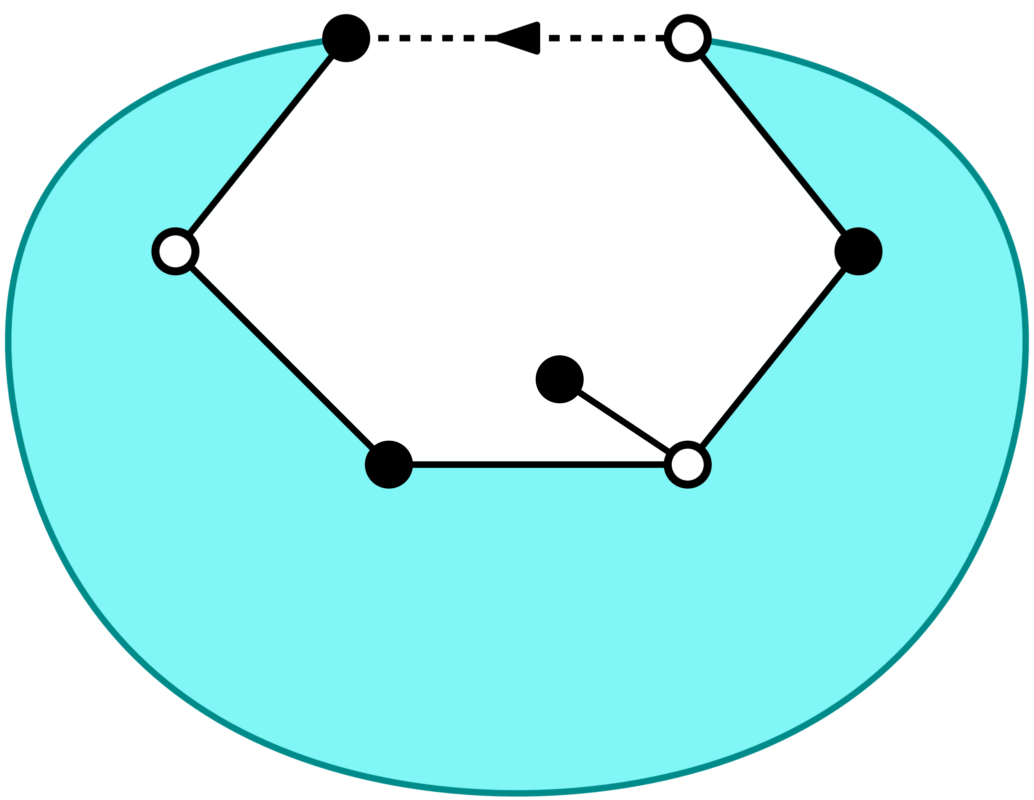
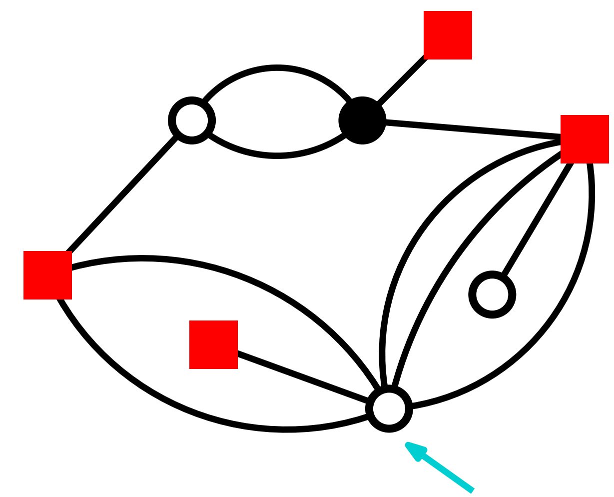
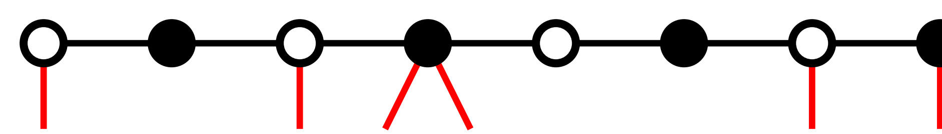




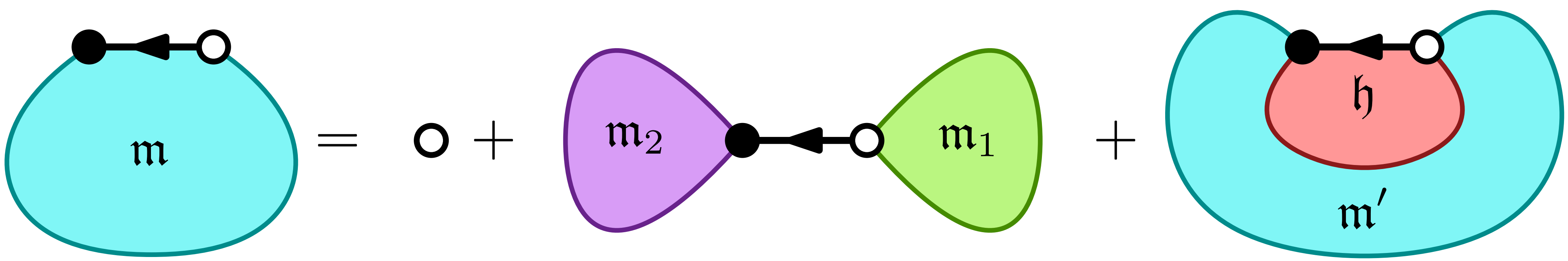


3-colored map where all neighbors of the root vertex are red.
$+$ connection caterpillar.

Gasket decomposition

Let $M(t,y) \equiv M(y)$ be the generating function of 3-colored planar maps with a **black and white outer face**, with $t \to$ edges and $y \to$ half the degree of the outer face.

Let $H(t,x) \equiv H(x)$ be the generating function of 3-colored planar maps where all the **neighbors** of the **root vertex** are **red**, with $t \to$ edges and $x \to$ degree of the root vertex.



$$M(y) = 1 + tyM(y)^2 + ty \sum_{p \geqslant 1} \sum_{k=1}^{p} h_k y^{p-k} [y^p] M(y),$$

with $h_k = \sum_{d \geqslant 0} \binom{2k+d-1}{2k-1} [x^d] H(x)$.

Gasket decomposition

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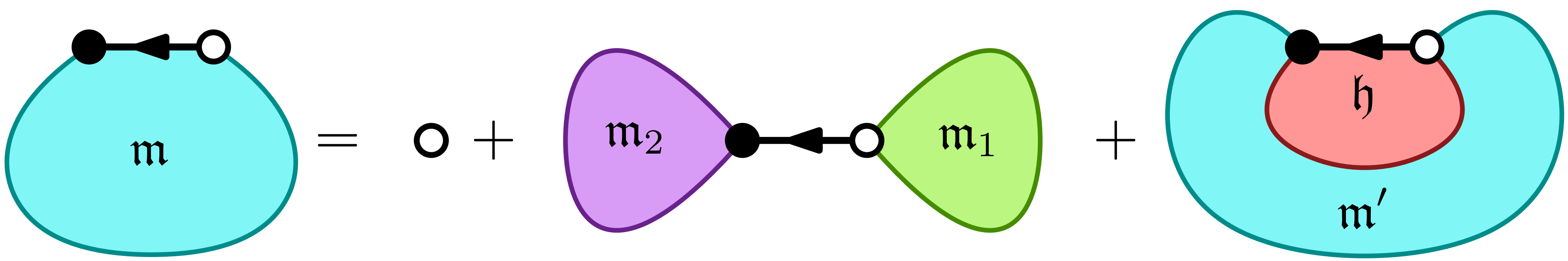

$$M(y) = 1 + tyM(y)^2 + ty[y^{\geqslant 0}](M(y)-1)\widetilde{H}(\overline{y}), \qquad \overline{y} = 1/y$$

$$\widetilde{H}(y) = \sum_{k\geqslant 1} y^k \sum_{d\geqslant 0} \binom{2k+d-1}{2k-1}[x^d]H(x)$$
$$= \frac{1}{2}\left[\frac{\sqrt{y}}{1-\sqrt{y}}H\left(\frac{1}{1-\sqrt{y}}\right) - \frac{\sqrt{y}}{1+\sqrt{y}}H\left(\frac{1}{1+\sqrt{y}}\right)\right].$$

Gasket decomposition

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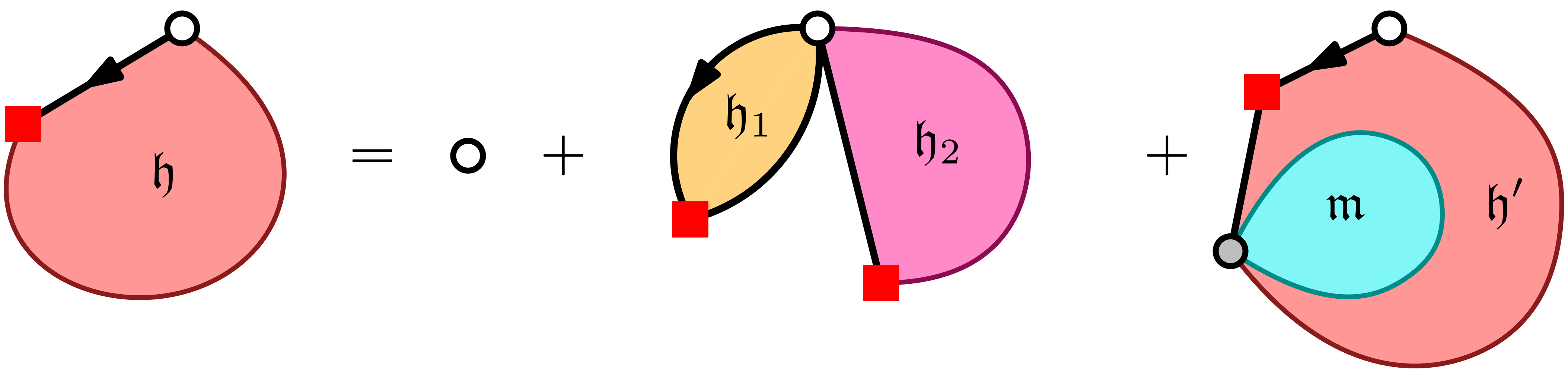
Let $H(t, x) \equiv H(x)$ be the generating function of 3-colored planar maps where all the **neighbors** of the **root vertex** are **red**, with $t \rightarrow$ edges and $x \rightarrow$ degree of the root vertex.



$$H(x) = 1 + txH(x)^2 + 2tx[x^{\geqslant 0}](H(x) - 1)\widetilde{M}(\overline{x}),$$

$$\widetilde{M}(x) = \sum_{i \geqslant 1} x^i \sum_{p \geqslant 0} \binom{2p+i-1}{i-1} [y^p] M(t, y) = \frac{x}{(1-x)^2} M\left(t, \frac{1}{(1-x)^2}\right).$$

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Theorem. [S. '26]

$$\begin{cases} M(t,y) = 1 + tyM(t,y)^2 + ty[y^{\geqslant 0}](M(t,y)-1)\widetilde{H}(t,\overline{y}), \\ H(t,x) = 1 + txH(t,x)^2 + 2tx[x^{\geqslant 0}](H(t,x)-1)\widetilde{M}(t,\overline{x}), \end{cases}$$

with

$$\widetilde{M}(t,x) = \tfrac{x}{(1-x)^2} M\Big(t, \tfrac{1}{(1-x)^2}\Big),$$

$$\widetilde{H}(t,y) = \tfrac{1}{2}\Big[\tfrac{\sqrt{y}}{1-\sqrt{y}} H\Big(t, \tfrac{1}{1-\sqrt{y}}\Big) - \tfrac{\sqrt{y}}{1+\sqrt{y}} H\Big(t, \tfrac{1}{1+\sqrt{y}}\Big)\Big].$$

$$P_M(S,y,M) = 4y^5(y-1)^2S^5(2S^3-1)^6M^6+24y^4(y-1)^2(1+2S)^3(2S^3-1)^5S^4M^5-(1+2S)(2S^3-1)^4((64S^6+96S^5+88S^4+32S^3-2S+1)y^2-(16S^4+456S^3+684S^2+334S+57)(1+2S)^2y+56(1+2S)^5)S^3(y-1)y^3M^4-4(2S^3-1)^3(1+2S)^4((64S^6+96S^5+88S^4+32S^3-2S+1)y^2-(16S^4+136S^3+204S^2+94S+17)(1+2S)^2y+16(1+2S)^5)S^2(y-1)y^2M^3-(1+2S)^2(2S^3-1)^2(2S(64S^6+96S^5+94S^4+50S^3+18S^2+4S+1)(S+1)^3y^4+(1+2S)(6144S^{10}+20928S^9+33600S^8+32064S^7+18380S^6+5244S^5-162S^4-536S^3-87S^2+18S+5)y^3-2(384S^6+1120S^5+1914S^4+1604S^3+706S^2+171S+23)(1+2S)^5y^2+(96S^4+616S^3+924S^2+414S+77)(1+2S)^7y-36(1+2S)^{10})SyM^2-2(2S^3-1)(1+2S)^5(2S(64S^6+96S^5+94S^4+50S^3+18S^2+4S+1)(S+1)^3y^4+(1+2S)(2048S^{10}+6592S^9+9536\text{[illegible]})y^3-2(128S^6+224S^5+266S^4+164S^3+50S^2+11S+3)(1+2S)^5y^2+(32S^4+72S^3+108S^2+38S+9)(1+2S)^7y-4(1+2S)^{10})M-(1+2S)^3(S(4S^2+2S+1)^3(S+1)^6y^4+2(64S^6+96S^5+94S^4+50S^3+18S^2+4S+1)(S+1)^3(1+2S)^5y^3+(1024S^9+3008S^8+3520S^7+1344S^6-1460S^5-2436S^4-1642S^3-616S^2-127S-12)(1+2S)^6y^2-2(64S^5+64S^4+14S^3-36S^2-34S-9)(1+2S)^{10}y+8(2S^3-1)(1+2S)^{12})$$

II- A new algebraicity proof

$$P_H(S,\sqrt{\Delta},x,H) = S^3x^5(2S^3-1)^4(2x-1)H^4-2S^2x^3(2x-1)(x-2)(1+2S)^3(2S^3-1)^3H^3+Sx(1+2S)(2S^3-1)^2(2S(2S^3+4S^2+2S+1)x^4-2(8S^4+8S^3+12S^2+2S+1)(1+2S)^2x^3+(8S^4-24S^3-36S^2-22S-3)(1+2S)^2x^2+10(1+2S)^5x-4(1+2S)^5)H^2-2(1+2S)(2S^3-1)(S((8S^5+28S^4+38S^3+26S^2+10S+2)\sqrt{\Delta}+16S^6+56S^5+76S^4+58S^3+28S^2+8S+1)x^4-2(16S^6+48S^5+86S^4+76S^3+34S^2+9S+1)(1+2S)^3x^3+(20S^4+56S^3+84S^2+32S+7)(1+2S)^5x^2-(8S^4+56S^3+84S^2+38S+7)(1+2S)^5x+2(1+2S)^8)H(1+2S)^3((32S^8+(16S^6+64S^5+104S^4+90S^3+46S^2+14S+2)\sqrt{\Delta}+176S^7+384S^6+452S^5+356S^4+182S^3+64S^2+15S+2)x^3-2(1+2S)(-64S^9-320S^8-656S^7+(8S^5+28S^4+38\sqrt{\Delta}S^3+26S^2+10S+2)\sqrt{\Delta}-560S^6-72S^5+332S^4+354S^3+172S^2+44S+5)x^2-2(8S^7+96S^6+144S^5+60S^4-46S^3-80S^2-38S-7)(1+2S)^3x+4(2S^3-1)(1+2S)^6)$$

The strategy: Guess and Check

$$\begin{cases} M(t,y) = 1 + tyM(t,y)^2 + ty[y^{\geqslant 0}](M(t,y)-1)\widetilde{H}(t,\overline{y}), \\ H(t,x) = 1 + txH(t,x)^2 + 2tx[x^{\geqslant 0}](H(t,x)-1)\widetilde{M}(t,\overline{x}). \end{cases}$$

Idea: The equations allow us to recursively compute all the coefficients of the series M and H

$\implies$ unique solution in $\mathbb{Q}[y][[t]] \times \mathbb{Q}[x][[t]]$.

Guess and Check:

1. Guess polynomial equations satisfied by our series.
2. Prove that the solutions of the polynomial equations satisfy the original equations.

Guess

$$\begin{cases} M(t,y) = 1 + tyM(t,y)^2 + ty[y^{\geqslant 0}](M(t,y)-1)\widetilde{H}(t,\overline{y}), \\ H(t,x) = 1 + txH(t,x)^2 + 2tx[x^{\geqslant 0}](H(t,x)-1)\widetilde{M}(t,\overline{x}). \end{cases}$$

Guess:

1. Use the equations to compute the first (250) terms of the series M and H.
2. Guess polynomial equations with `algeqtoseries` (package `gfun`).
3. Determine which solution of the polynomial equation is the candidate.

Guess

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$\hookrightarrow$ We use the intermediate field $\mathbb{Q}(t,y) \to \mathbb{Q}(S,y) \to \mathbb{Q}(t,y,M(t,y))$ where $t = \frac{S(1-S^3)}{(1+2S)^3}$ to have smaller equations.

$\hookrightarrow$ Equations of degree 6 for $M(t,y)$ over $\mathbb{Q}(S,y)$ and of degree 8 for $H(t,x)$ over $\mathbb{Q}(S,x)$.
Both polynomial equations have only one solution that is a power series in t.
$\hookrightarrow$ We denote them M_c and H_c.

Guess

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3. Determine which solution of the polynomial equation is the candidate.

$\hookrightarrow$ Equations of degree 6 for $M(t,y)$ over $\mathbb{Q}(S,y)$ and of degree 8 for $H(t,x)$ over $\mathbb{Q}(S,x)$.

$$P_M(S,y,M) = 4y^5(y-1)^2S^5(2S^3-1)^6M^6 + 24y^4(y-1)^2(1+2S)^3(2S^3-1)^5S^4M^5 - (1+2S)(2S^3-1)^4((64S^6+96S^5+88S^4+32S^3-2S+1)y^2 - (16S^4+456S^3+684S^2+334S+57)(1+2S)^2y + 56(1+2S)^5)S^3(y-1)y^3M^4 - 4(2S^3-1)^3(1+2S)^4((64S^6+96S^5+88S^4+32S^3-2S+1)y^2 - (16S^4+136S^3+204S^2+94S+17)(1+2S)^2y + 16(1+2S)^5)S^2(y-1)y^2M^3 - (1+2S)^2(2S^3-1)^2(2S(64S^6+96S^5+94S^4+50S^3+18S^2+4S+1)(S+1)^3y^4 + (1+2S)(6144S^{10}+20928S^9+33600S^8+32064S^7+18380S^6+5244S^5-162S^4-536S^3-87S^2+18S+5)y^3 - 2(384S^6+1120S^5+1914S^4+1604S^3+706S^2+171S+23)(1+2S)^5y^2 + (96S^4+616S^3+924S^2+414S+77)(1+2S)^7y - 36(1+2S)^{10})SyM^2 - 2(2S^3-1)(1+2S)^5(2S(64S^6+96S^5+94S^4+50S^3+18S^2+4S+1)(S+1)^3y^4 + (1+2S)(2048S^{10}+6592S^9+9536S^8+7488S^7+2508S^6-900S^5-1346S^4-600S^3-119S^2-6S+1)y^3 - 2(128S^6+224S^5+266S^4+164S^3+50S^2+11S+3)(1+2S)^5y^2 + (32S^4+72S^3+108S^2+38S+9)(1+2S)^7y - 4(1+2S)^{10})M - (1+2S)^3(S(4S^2+2S+1)^3(S+1)^6y^4 + 2(64S^6+96S^5+94S^4+50S^3+18S^2+4S+1)(S+1)^3(1+2S)^5y^3 + (1024S^9+3008S^8+3520S^7+1344S^6-1460S^5-2436S^4-1642S^3-616S^2-127S-12)(1+2S)^6y^2 - 2(64S^5+64S^4+14S^3-36S^2-34S-9)(1+2S)^{10}y + 8(2S^3-1)(1+2S)^{12})$$

Check

Check: Prove that M_c and H_c satisfy

$$\begin{cases} M_c(t,y) = 1 + tyM_c(t,y)^2 + ty[y^{\geqslant 0}](M_c(t,y)-1)\widetilde{H}_c(t,\overline{y}), \\ H_c(t,x) = 1 + txH_c(t,x)^2 + 2tx[x^{\geqslant 0}](H_c(t,x)-1)\widetilde{M}_c(t,\overline{x}), \end{cases}$$

with $\widetilde{M}_c(t,x) = \frac{x}{(1-x)^2} M_c\Big(t, \frac{1}{(1-x)^2}\Big)$

and $\widetilde{H}_c(t,y) = \frac{1}{2}\Big[\frac{\sqrt{y}}{1-\sqrt{y}} H_c\Big(t, \frac{1}{1-\sqrt{y}}\Big) - \frac{\sqrt{y}}{1+\sqrt{y}} H_c\Big(t, \frac{1}{1+\sqrt{y}}\Big)\Big]$.

Check

Check: Prove that M_c and H_c satisfy

$$\begin{cases} M_c(t,y) = 1 + tyM_c(t,y)^2 + ty[y^{\geqslant 0}](M_c(t,y)-1)\widetilde{H}_c(t,\overline{y}), \\ H_c(t,x) = 1 + txH_c(t,x)^2 + 2tx[x^{\geqslant 0}](H_c(t,x)-1)\widetilde{M}_c(t,\overline{x}), \end{cases}$$

with $\widetilde{M}_c(t,x) = \frac{x}{(1-x)^2} M_c\Big(t, \frac{1}{(1-x)^2}\Big)$

and $\widetilde{H}_c(t,y) = \frac{1}{2}\Big[\frac{\sqrt{y}}{1-\sqrt{y}} H_c\Big(t, \frac{1}{1-\sqrt{y}}\Big) - \frac{\sqrt{y}}{1+\sqrt{y}} H_c\Big(t, \frac{1}{1+\sqrt{y}}\Big)\Big]$.

1. Reformulate the equations to some property visible on the polynomials:

$$\begin{cases} [y^{\geqslant 0}]R_M(y) = 0, \\ [x^{\geqslant 0}]R_H(x) = 0, \end{cases}$$

where $R_M(y) = (M_c(y)-1)\widetilde{H}_c(\overline{y}) - \frac{1}{ty}\left(M_c(y) - 1 - tyM_c(y)^2\right) \in \mathbb{Q}((\overline{y}))[[t]]$
and $R_H(x) = (H_c(x)-1)\widetilde{M}_c(\overline{x}) - \frac{1}{2tx}\left(H_c(x) - 1 - txH_c(x)^2\right) \in \mathbb{Q}((\overline{x}))[[t]]$.

Check

Check: Prove that M_c and H_c satisfy

wit

anc

1.

Let $F(t, y) \in \mathbb{Q}((y))[[t]]$ be an algebraic series in t
given by its minimal polynomial and first coefficients.
How to decide whether F has a **pole** at $y = 0$?

wh $[[t]]$

anc $t]]$.

Check

Check: Prove that $[y^{\geqslant 0}]R_M(y) = [x^{\geqslant 0}]R_H(x) = 0$.

2. Compute minimal polynomials for R_M and R_H by eliminating M_c and H_c using resultants.

$\hookrightarrow$ An equation of degree 12 for R_M over $\mathbb{Q}(S, y)$ and one of degree 12 for R_H over $\mathbb{Q}(S, x)$. Both are too big to work with.

$Q_M(S, y, R_M) = S^{16}(y-1)^8(2S^3-1)^{24}R_M^{12} + 24S^{14}y(y-1)^8(1+2S)^3(2S^3-1)^{22}(4S^4 +$
$8S^3y + 12S^2y + 6Sy - 2S + y)R_M^{11} - 2S^{12}y(y-1)^6(1+2S)(2S^3-1)^{20}(-124(1+2S)^{11}y^5 -$
$(992S^4 - 2008S^3 - 3012S^2 - 2002S - 251)(1+2S)^8y^4 - 2(976S^8 - 7968S^7 - 7792S^6 + 5508S^5 +$
$18538S^4 + 16336S^3 + 7112S^2 + 1273S + 65)(1+2S)^5y^3 + (128S^{12} + 31296S^{11} - 17568S^{10} -$
$168528S^9 - 262136S^8 - 162720S^7 + 27492S^6 + 123148S^5 + 96298S^4 + 37652S^3 + 7353S^2 +$
$550S + 3)(1+2S)^2y^2 - 2S(512S^{13} + 32000S^{12} + 77792S^{11} + 74176S^{10} + 1440S^9 - 73060S^8 -$
$76978S^7 - 27504S^6 + 13750S^5 + 20880S^4 + 10619S^3 + 2647S^2 + 279S + 3)y + 2S^2(64S^6 + 96S^5 +$
$88S^4 + 32S^3 - 2S + 1)(2S^3-1)^2)R_M^{10} - 8S^{10}y^2(y-1)^6(1+2S)^2(2S^3-1)^{18}(-180(1+2S)^{16}y^6 -$
$(2160S^4 - 2984S^3 - 4476S^2 - 3318S - 373)(1+2S)^{13}y^5 - 2(4240S^8 - 17648S^7 - 19880S^6 +$
$2208S^5 + 31108S^4 + 29532S^3 + 13766S^2 + 2316S + 103)(1+2S)^{10}y^4 - (10240S^{12} - 137216S^{11} -$
$55808S^{10} + 323552S^9 + 649936S^8 + 447744S^7 - 36204S^6 - 313484S^5 - 256670S^4 - 103496S^3 -$
$19625S^2 - 1366S - 13)(1+2S)^7y^3 + 2S(1280S^{15} + 82560S^{14} - 159808S^{13} - 766080S^{12} -$
$1072032S^{11} - 419680S^{10} + 685888S^9 + 1140832S^8 + 684746S^7 + 2048S^6 - 293620S^5 -$
$226596S^4 - 86349S^3 - 17168S^2 - 1475S - 26)(1+2S)^4y^2 - 2S^2(1+2S)(10240S^{14} + 184320S^{13} +$
$489472S^{12} + 515648S^{11} + 42368S^{10} - 543048S^9 - 666152S^8 - 308280S^7 + 83652S^6 +$

Check

Check: Prove that $[y^{\geqslant 0}]R_M(y) = [x^{\geqslant 0}]R_H(x) = 0$.

2. Compute minimal polynomials for R_M and R_H by eliminating M_c and H_c using resultants.

$\hookrightarrow$ An equation of degree 12 for R_M over $\mathbb{Q}(S, y)$ and one of degree 12 for R_H over $\mathbb{Q}(S, x)$. Both are too big to work with.

3. Find intermediate field extensions.

$$\mathbb{Q}(S, y) \xrightarrow{3} \mathbb{Q}(S, L) \xrightarrow{2} \mathbb{Q}(S, K) \xrightarrow{2} \mathbb{Q}(S, y, R_M)$$

$$L = \frac{1}{y}\frac{(1+2S)^3(1+2S+4S^2-4S^2L)(1-L)}{(1+2S+2S^2+4S^3-4S^3L)^2},$$

$$K = \frac{S^2 L}{(4S^2+2S+1)(1-K)}.$$

Check

Check: Prove that $[y^{\geqslant 0}]R_M(y) = [x^{\geqslant 0}]R_H(x) = 0$.

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$$L = \frac{1}{y} \frac{(1+2S)^3(1+2S+4S^2-4S^2L)(1-L)}{(1+2S+2S^2+4S^3-4S^3L)^2},$$
$$K = \frac{S^2 L}{(4S^2+2S+1)(1-K)}.$$

4. Solve the equation for R_M over $\mathbb{Q}(S, K)$.

$$\hookrightarrow \quad R_M(y) = \frac{B - C\sqrt{D}}{2A} \qquad \text{with } A, B, C, D \in \mathbb{Q}[S, K].$$

Check

Check: Prove that $[y^{\geqslant 0}]R_M(y) = [x^{\geqslant 0}]R_H(x) = 0$.

3. Find intermediate field extensions:

$$\mathbb{Q}(S,y) \xrightarrow{3} \mathbb{Q}(S,L) \xrightarrow{2} \mathbb{Q}(S,K) \xrightarrow{2} \mathbb{Q}(S,y,R_M)$$

$$L = \frac{1}{y}\frac{(1+2S)^3(1+2S+4S^2-4S^2L)(1-L)}{(1+2S+2S^2+4S^3-4S^3L)^2},$$

$$K = \frac{S^2L}{(4S^2+2S+1)(1-K)}.$$

4. Solve the equation for R_M over $\mathbb{Q}(S,K)$.

$$\hookrightarrow \quad R_M(y) = \frac{B - C\sqrt{D}}{2A} \qquad \text{with } A,B,C,D \in \mathbb{Q}[S,K].$$

5. Prove that L then K then R_M only have negative powers of y.

6. Apply the same method to R_H.

Algebraicity

a. $$\begin{cases} M(t,y) = 1 + tyM(t,y)^2 + ty[y^{\geqslant 0}](M(t,y)-1)\widetilde{H}(t,\overline{y}), \\ H(t,x) = 1 + txH(t,x)^2 + 2tx[x^{\geqslant 0}](H(t,x)-1)\widetilde{M}(t,\overline{x}). \end{cases}$$

has a unique solution in $\mathbb{Q}[y][[t]] \times \mathbb{Q}[x][[t]]$.

b. There are polynomials $P_M(S, y, M)$ and $P_H(S, x, H)$ whose only power series solutions satisfy the system.

Theorem. [S. '26] The generating function $M(t,y)$ of 3-colored planar maps with black and white boundary is algebraic over $\mathbb{Q}(t,y)$ of degree 24.

- Complete map of the subfields of $\mathbb{Q}(t,y,M(y))$.

$$\mathbb{Q}(t,y) \xrightarrow{4} \mathbb{Q}(S,y) \xrightarrow{3} \mathbb{Q}(S,L(y)) \xrightarrow{2} \mathbb{Q}(S,y,M(y))$$

- All $[y^p]M(y)$ are rational fractions in S.

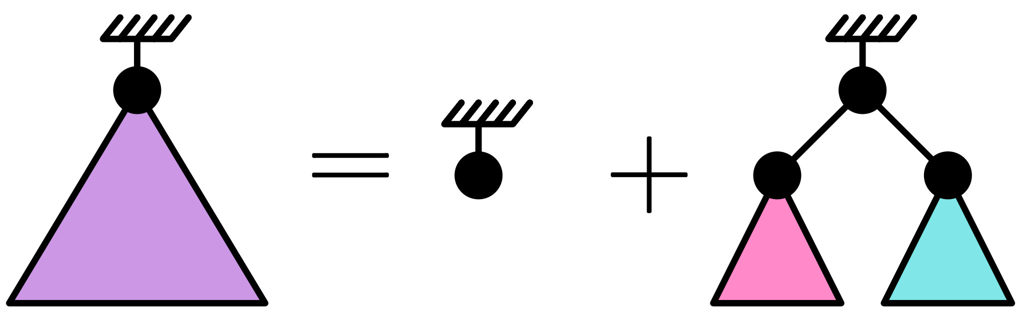
III- Random sampling

Boltzmann samplers

Exact size sampling

Principle: use the recursive decomposition to sample uniformly objects of a given size n.

For binary trees :



$$t_1 = 1$$

$$t_n = \sum_{k=1}^{n-1} t_k t_{n-k-1}$$

BinTreeSampler`(n) =`

  if `n = 1` **then** •

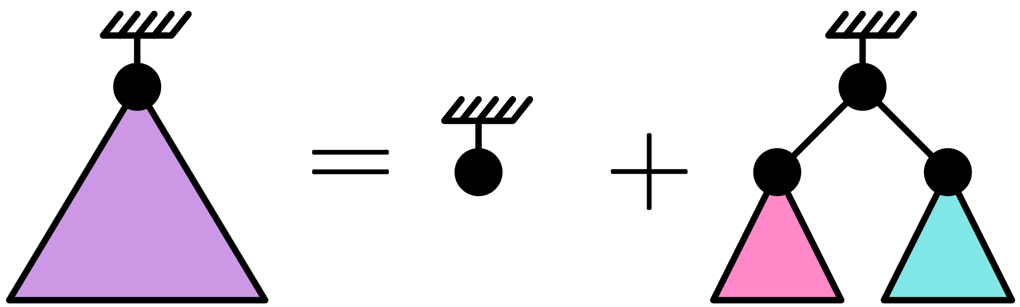
  else `draw k with` $\mathbb{P}(k) = t_k t_{n-k-1} / t_n$

    `(BinTreeSampler(k), BinTreeSampler(n-1-k))`

Exact size sampling

Principle: use the recursive decomposition to sample uniformly objects of a given size n.

For binary trees :



$$t_1 = 1$$

$$t_n = \sum_{k=1}^{n-1} t_k t_{n-k-1}$$

BinTreeSampler(n) =
  if n = 1 **then** •
  else draw k with $\mathbb{P}(k) = t_k t_{n-k-1} / t_n$ ← very expensive
    (BinTreeSampler(k), BinTreeSampler(n-1-k))

- Need to write the explicit recurrence.
- Very slow for large size objects.

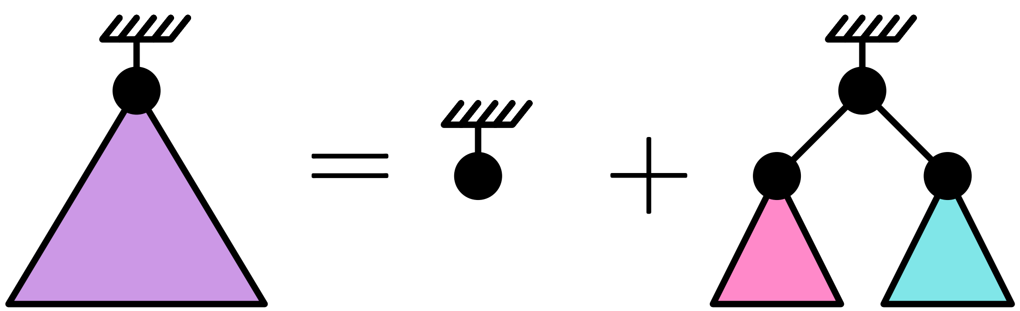
Boltzmann samplers [Duchon, Flajolet, Louchard, Schaeffer '04]

Principle: sample objects $\gamma \in \mathcal{C}$ according to the probability distribution

$$\mathbb{P}(\gamma) = \frac{t^{|\gamma|}}{C(t)}$$

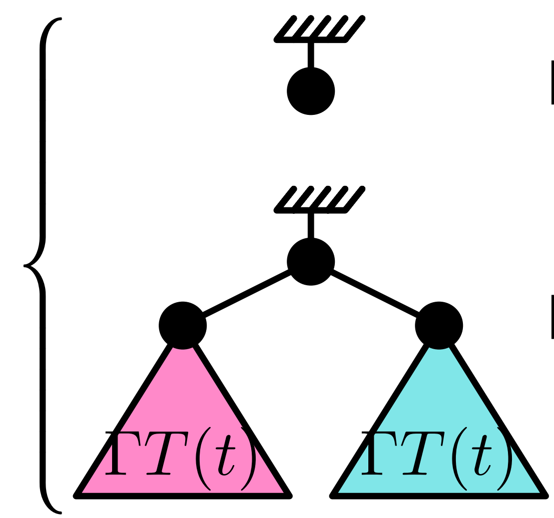
with t a parameter, by following a recursive decomposition of $\mathcal{C}$.

For binary trees :



$$T(t) = t + tT(t)^2$$

$\Gamma T(t) =$



probability $\frac{t}{T(t)}$

probability $tT(t)$

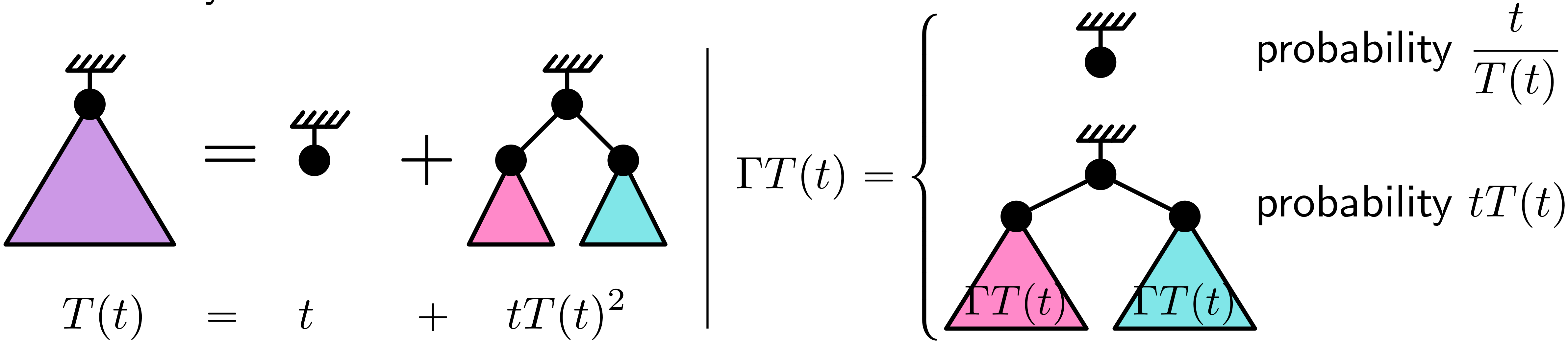
Boltzmann samplers [Duchon, Flajolet, Louchard, Schaeffer '04]

Principle: sample objects $\gamma \in \mathcal{C}$ according to the probability distribution

$$\mathbb{P}(\gamma) = \frac{t^{|\gamma|}}{C(t)}$$

with t a parameter, by following a recursive decomposition of $\mathcal{C}$.

For binary trees :



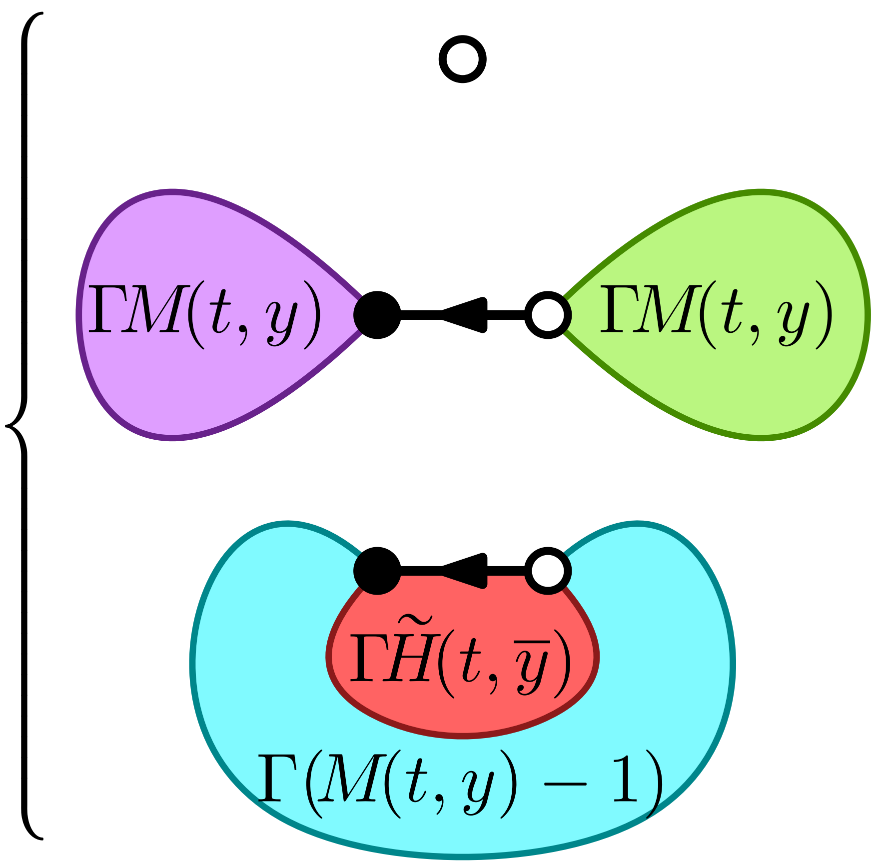
- Unfixed size (adjust the value of t to aim for a given size).
- Uniform distribution when conditioned by the size of the object.
- Can be built as long as we have a positive combinatorial equation.
- Very efficient for large objects.

Boltzmann sampler for 3-colored maps

- The catalytic variable becomes a second parameter: $\mathbb{P}(\mathfrak{m}) = \dfrac{t^{e(\mathfrak{m})} y^{d(\mathfrak{m})}}{M(t,y)}$.
- We use rejection to handle the $[y^{\geqslant 0}]$ and $[x^{\geqslant 0}]$.

$$\begin{cases} M(t,y) = 1 + tyM(t,y)^2 + ty[y^{\geqslant 0}](M(t,y)-1)\widetilde{H}(t,\overline{y}), \\ H(t,x) = 1 + txH(t,x)^2 + 2tx[x^{\geqslant 0}](H(t,x)-1)\widetilde{M}(t,\overline{x}). \end{cases}$$

$\Gamma M(t,y) =$



with probability $\dfrac{1}{M(t,y)}$

with probability $tyM(t,y)$

Otherwise.
Reject while the lengths of the borders of the maps are incompatible.

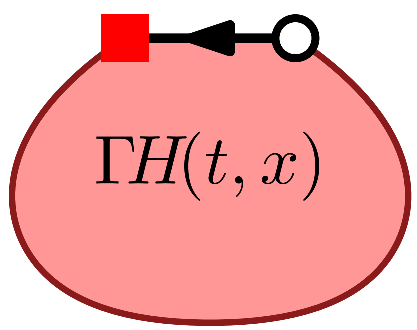
$\Gamma H(t,x)$ similarly.

Boltzmann sampler for 3-colored maps

- The catalytic variable becomes a second parameter: $\mathbb{P}(\mathfrak{m}) = \dfrac{t^{e(\mathfrak{m})} y^{d(\mathfrak{m})}}{M(t,y)}$.
- We use rejection to handle the $[y^{\geqslant 0}]$ and $[x^{\geqslant 0}]$.

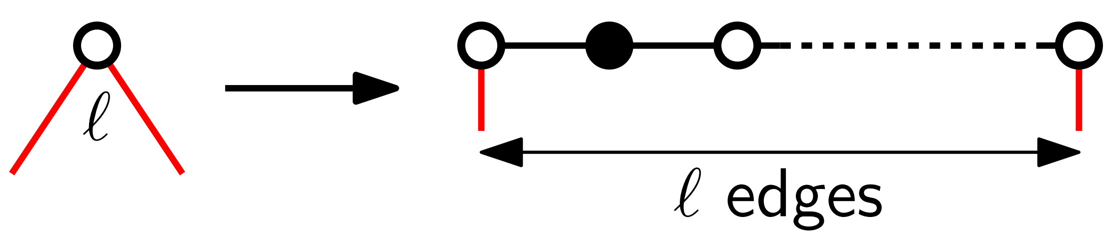
$$\begin{cases} M(t,y) = 1 + tyM(t,y)^2 + ty[y^{\geqslant 0}](M(t,y)-1)\widetilde{H}(t,\overline{y}), \\ H(t,x) = 1 + txH(t,x)^2 + 2tx[x^{\geqslant 0}](H(t,x)-1)\widetilde{M}(t,\overline{x}). \end{cases}$$

$\Gamma\widetilde{H}(t,\overline{y}) =$



$+$ an integer $\ell \sim$ Geom$(\overline{y})$ for each corner

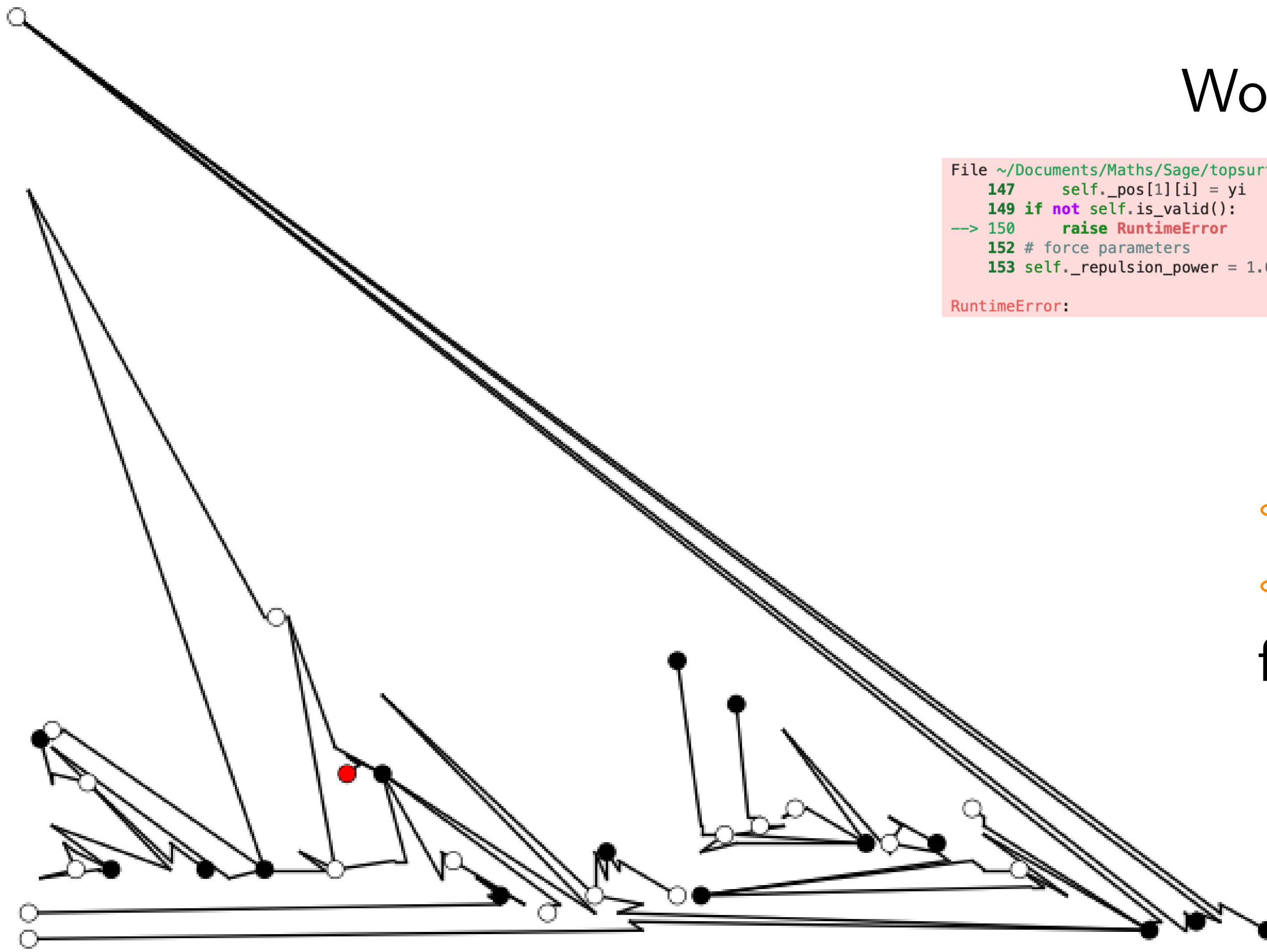
ℓ indicates the number of edges to insert:



$\hookrightarrow$ Reject while the sum of the labels is odd.

Boltzmann sampler for 3-colored maps

- The catalytic variable becomes a second parameter: $\mathbb{P}(\mathfrak{m}) = \dfrac{t^{e(\mathfrak{m})} y^{d(\mathfrak{m})}}{M(t,y)}$.
- We use rejection to handle the $[y^{\geqslant 0}]$ and $[x^{\geqslant 0}]$.



A random map with 33 edges.

Work in progress ...

```
File ~/Documents/Maths/Sage/topsurf/topsurf/planar_layout.py:150, in PlanarLayout.__init__(self, m, root)
    147     self._pos[1][i] = yi
    149 if not self.is_valid():
--> 150     raise RuntimeError
    152 # force parameters
    153 self._repulsion_power = 1.0

RuntimeError:
```

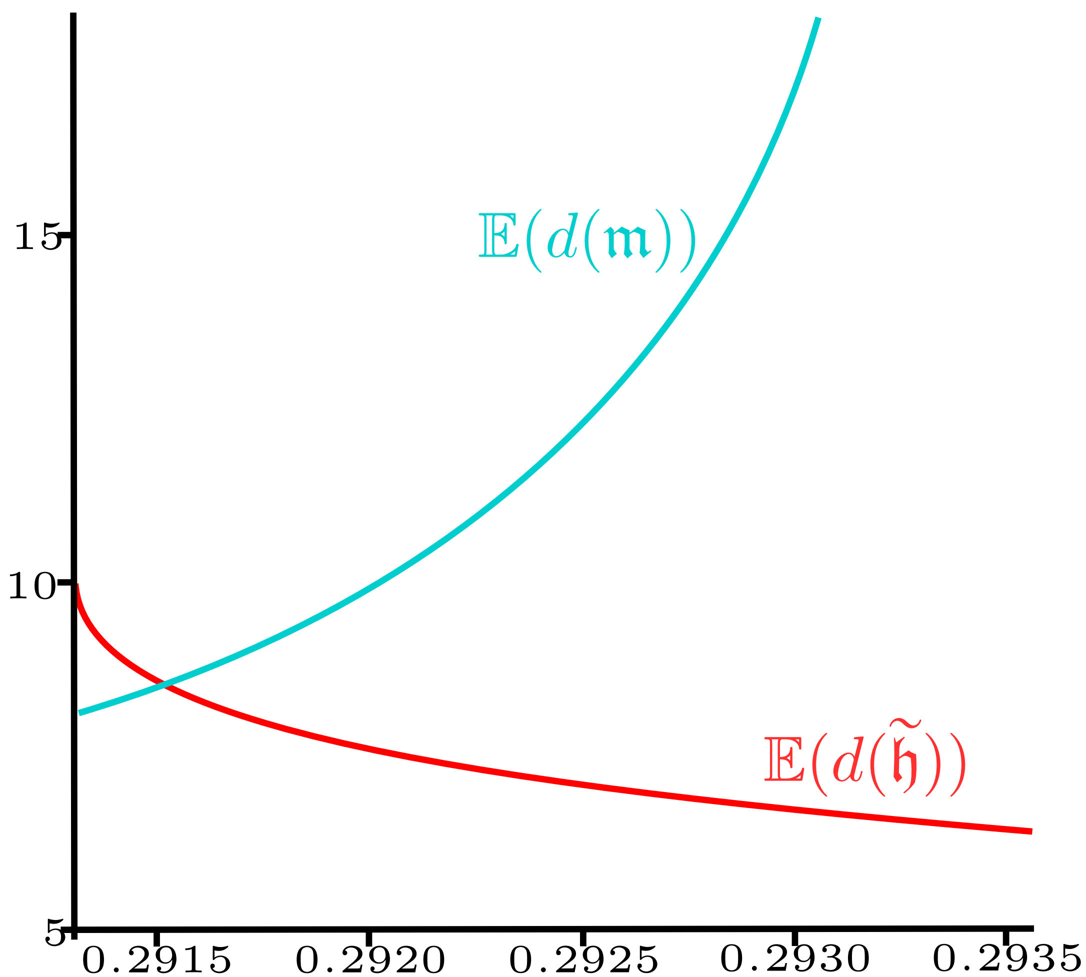
$\hookrightarrow$ Lots of rejection.

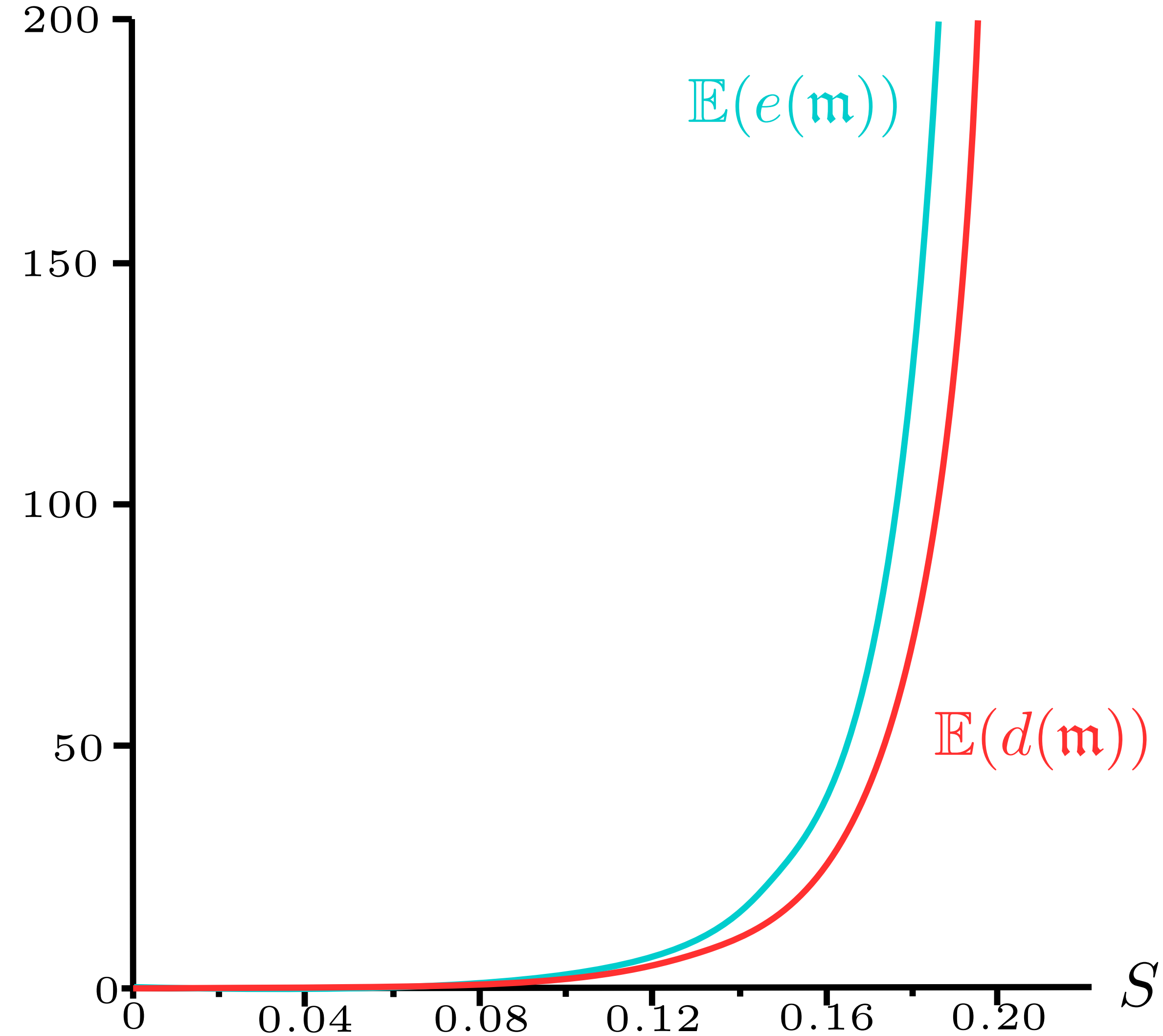
$\hookrightarrow$ Hard to build an oracle for the pointed series.

Boltzmann sampler for 3-colored maps

- The catalytic variable becomes a second parameter: $\mathbb{P}(\mathfrak{m}) = \dfrac{t^{e(\mathfrak{m})} y^{d(\mathfrak{m})}}{M(t,y)}$.
- We use rejection to handle the $[y^{\geqslant 0}]$ and $[x^{\geqslant 0}]$.

$\hookrightarrow$ Lots of rejection.





Boltzmann sampler for 3-colored maps

- Make the catalytic variable recursive.

Equations for $Mp = [y^p]M(y)$, $H_d = [x^d]H(x)$, $\widetilde{M}_k = [x^k]\widetilde{M}(x)$ and $\tilde{H}_k = [y^k]\tilde{H}(y)$:

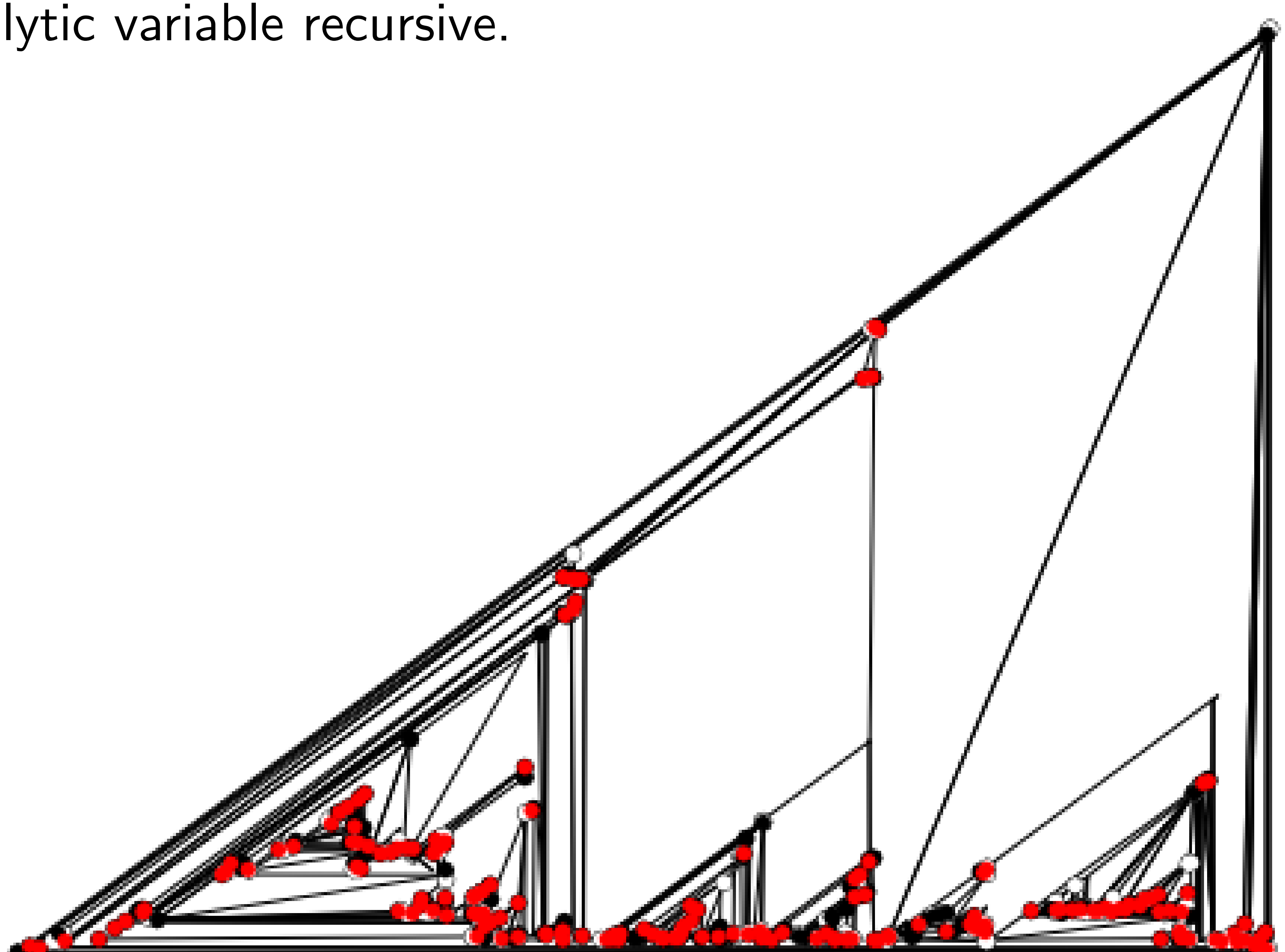
$$\begin{cases} M_0 = H_0 = 1, \\ M_p = t \sum_{k=0}^{p-1} M_k M_{p-1-k} + t \sum_{k \geqslant 1} M_{p+k-1} \tilde{H}_k \quad \text{for } p > 0, \\ H_d = t \sum_{k=0}^{d-1} H_k H_{d-1-k} + t \sum_{k \geqslant 1} H_{d+k-1} \widetilde{M}_k \quad \text{for } d > 0, \\ \widetilde{M}_k = \sum_{p \geqslant 0} \binom{2p+k-1}{k-1} M_p \quad \text{for } k > 0, \\ \tilde{H}_k = \sum_{d \geqslant 0} \binom{2k+d-1}{2k-1} H_d \quad \text{for } k > 0. \end{cases}$$

$\hookrightarrow$ Need to truncate the sums at some K

$\Longrightarrow$ boundary size at most K (recursively).

Boltzmann sampler for 3-colored maps

- Make the catalytic variable recursive.



A random 3-colored map with 616 edges and a horrible layout.

Boltzmann sampler for 3-colored maps

- The catalytic variable becomes a second parameter: $\mathbb{P}(\mathfrak{m}) = \dfrac{t^{e(\mathfrak{m})} y^{d(\mathfrak{m})}}{M(t,y)}$.

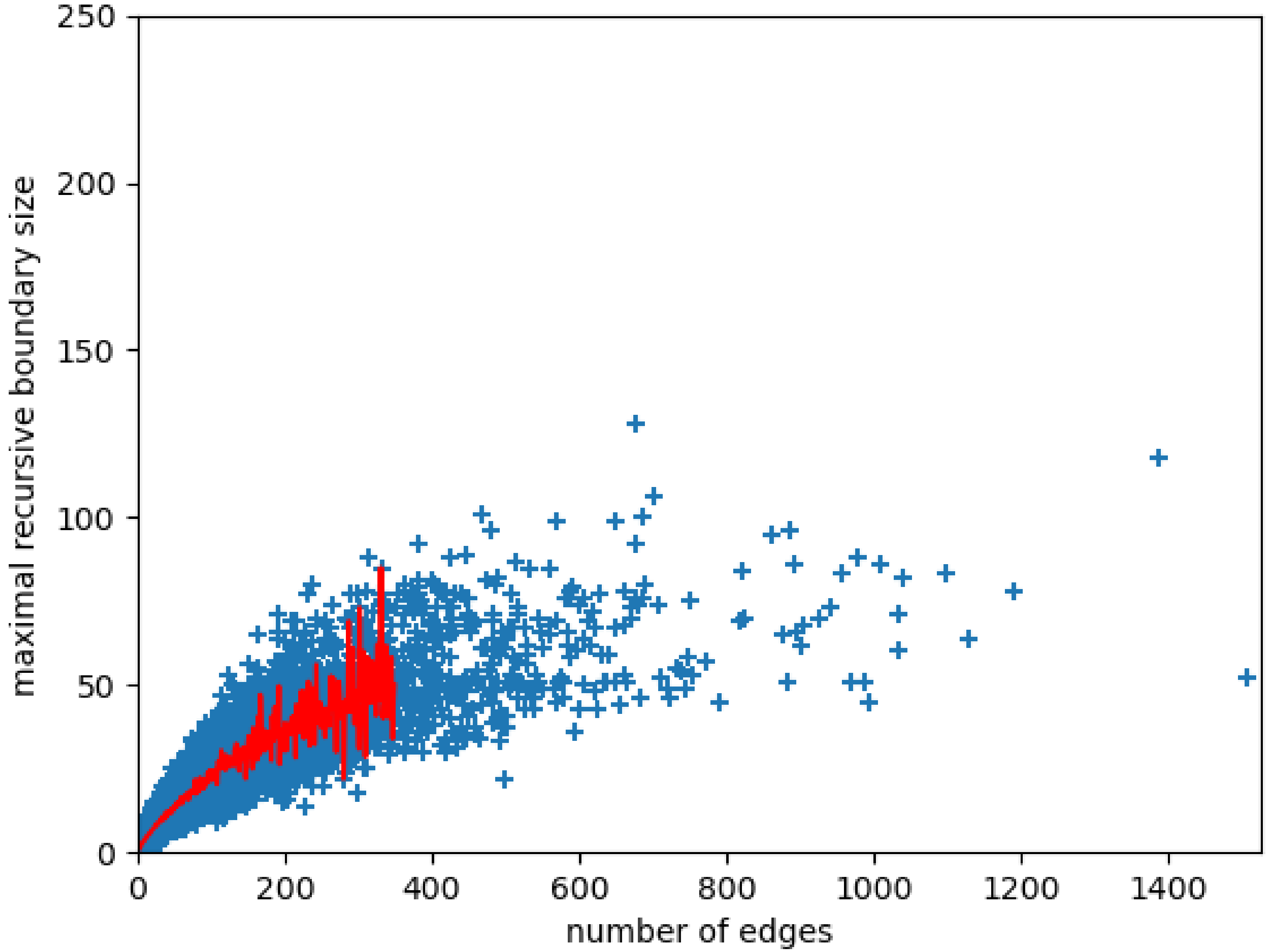
or

- Make the catalytic variable recursive (and bound the maximal outer degree in the decomposition by K) .

	Boltzmann catalytic variable	Recursive catalytic variable
Distribution at fixed n	Uniform	Uniform only if $n \leqslant K$
Precalculus	0	$\mathcal{O}(K^2)$
One step	$\mathcal{O}(1)$	$\mathcal{O}(\log(K))$
Rejection	A lot	None

Boltzmann sampler for 3-colored maps

- Make the catalytic variable recursive.



The size and width of maps generated with $K = 250$.

Perspectives

- Find a nice family of trees in bijection with 3-colored planar maps.
- Others applications of the decomposition?
- Adapt the gasket decomposition to others families of colored maps.
  - $\hookrightarrow$ Colorful quadrangulations [Budd '25], (D-algebraic and solved [Bousquet-Mélou, Elvey Price '20]).
  - $\hookrightarrow$ 4-colored triangulations (D-algebraic [Tutte '82]).
- Analyse the samplers.
- For which series $G \in \mathbb{Q}[[t,z]]$ are the solutions of

$$F(z) = \mathrm{Pol}(t, z, F(z)) + \mathrm{Pol}(t, z)[z^{\geqslant 0}]F(z)G(\overline{z})$$

algebraic?

Perspectives

- Find a nice family of trees in bijection with 3-colored planar maps.
- Others applications of the decomposition?
- Adapt the gasket decomposition to others families of colored maps.

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algebraic?