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Rapid Decay of Bicrossed Products and Representation Theory of Some Semidirect Products

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Titre : Décroissance rapide des biproduits croisés et théorie des représentations de certains produits semidirects

Résumé : On étude la propriété de décroissance rapide (propriété (RD)) et croissance polynomiale pour les duaux des groupes quantiques compacts venant de la construction de biproduits croisés des paires assorties des groupes classiques, et la théorie des représentations des produits semidirects d'un groupe quantique compact avec un groupe fini. On utilise ces théories pour donner des nouveaux exemples des groupes quantiques discrets ayant la propriété (RD) sans la croissance polynomiale.

Mots clefs : Propriété (RD), théories des représentations, règles de fusion, groupes quantiques

Title : Rapid decay of bicrossed products and representation theory of some semidirect products

Abstract : We study the rapid decay property (property (RD)) and polynomial growth of the duals of bicrossed products of matched pairs of classical groups, and the representation theory of semi-direct products of a compact quantum group with a finite group. We use these theories to obtain new examples of discrete quantum groups that has property (RD) but not polynomial growth.

Keywords : Property (RD), representation theory, fusion rules, quantum groups

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Introduction and some background

The central theme of this thesis is the study of a certain approximation property, namely the rapid decay property, a.k.a. property (RD), of the dual of bicrossed products of matched pair of classical groups.

To put the mathematical objects of this thesis into perspective, we now briefly outline some background on property (RD). In the breakthrough paper (Haagerup, 1978/79), Haagerup showed that for the free group $\mathbb{F}_N$ with N generators, the norm of the reduced C^*-algebra $C_r^*(\mathbb{F}_N)$ can be controlled by the more manageable Sobolev ℓ^2-norm associated to the word length function on $\mathbb{F}_N$. This striking phenomenon is later shown to be quite ubiquitous, and is later recognized and systematically studied as the rapid decay property (property (RD)) by Jolissaint (Jolissaint, 1990). Among many of its applications nowadays, let us mention the remarkable connection with K−theory. Property (RD) allowed Jolissaint (Jolissaint, 1989) to show that the K-theory of $C_r^*(\Gamma)$ equals the K-theory of subalgebras of rapidly decreasing functions of Γ (Jolissaint did attribute this result to Connes). This work was later used by V. Lafforgue in his approach to the Baum-Connes conjecture via Banach KK-theory (Lafforgue, 2002; 2000).

We are now witnessing the rapid development of the theory of topological quantum groups in the sense of Woronowicz in the compact case (Woronowicz, 1998; 1987) (and its dual which is the discrete case), and in the sense of Kustermans and Vaes (Kustermans and Vaes, 2003; 2000) in the more general locally compact case. It is natural to develop various approximation properties in this new quantum setting. The bicrossed product construction, which was already present in the framework of Kac algebras (see (Kac, 1968)), and later developed in full generality in the framework of locally quantum groups in (Vaes and Vainerman, 2003), is a powerful process of producing highly nontrivial (both non-commutative and non-cocommuative) quantum groups starting from the so-called matched pair of (quantum) groups. In the paper (Fima et al., 2017), various approximation property for (the dual of) compact bicrossed products of classical matched pair are studied. However, due to the lack of correct understanding the representation theory of these bicrossed products, the study of property (RD) for the dual of these bicrossed products was out of reach, and this is the starting point of the work in this thesis.

The main content of this thesis is divided into three chapters. Chapter I studies the permanence of property (RD), and the closely related property of polynomial growth, under the above mentioned bicrossed product construction. This is achieved by a careful study of the representation theory of these bicrossed products, and the theory of matched pair of length functions designed to reflect this representation theory. Chapter II studies the representation theory of semidirect products of a compact quantum group with a finite group, and can be viewed as a quantum analogue of the classical Mackey's analysis, with the new results on the calculation of the fusion rules of these semidirect products. While the author is led to study the representation of these semidirect products motivated by constructing concrete examples of bicrossed products whose dual has property (*RD*), it is the author's opinion that this theory is of interest of its own, as it satisfactorily describes the representation theory of many semidirect products, which might have wider applications. Chapter III constructs concrete examples of bicrossed products whose dual has (*RD*) but not polynomial growth, hence provides new examples of interesting quantum groups. Here the more theoretical work of both Chapter I and Chapter II are used in an essential way. The author hopes these examples provide more "flesh" to the abstract theory of the first two chapters of this thesis.

Contents

Chapter I

Rapid decay and polynomial growth of bicrossed products

Introduction

This chapter of the thesis is a rewrite of the author's collaborative work with P. Fima (Fima and Wang, 2018). The central theme here is the study of the permanence of property (*RD*) and the closely related property of polynomial growth of the dual of bicrossed products. We refer the reader to the introduction to this thesis and to the article (Fima and Wang, 2018) for some background on property (*RD*) and why they are interesting objects.

We now compare the treatment here with that of (Fima and Wang, 2018). The similarities are obvious: the central tools are always representation theory of the bicrossed product and the theory of matched pair of length functions. However, there are more differences to justify this rewrite. Firstly, (Fima and Wang, 2018) is a research article targeted towards experts in this field, it is written with brevity in mind in order to convey our research efficiently; by contrast, this rewrite takes a more pedagogically friendly approach and is more detailed. Secondly, in the treatment of the representation theory of bicrossed product, a whole section, namely § I.3, is dedicated to motivate the classification of irreducible representations of bicrossed products. Despite its logical independence, the author hopes the treatment there is more natural and easier to understand, and satisfies people who wonders why the classification of irreducible representations of the bicrossed product in (Fima and Wang, 2018) looks like what they do, which could seem to be quite artificial and miraculous without the considerations in § I.3. The key idea in (Fima and Wang, 2018) in the study of irreducible representations of the bicrossed product is by twisting the induced representations of some suitable isotropy subgroups. This idea of course remains important in this thesis if one ignores the motivational § I.3 (and logically speaking, one can safely to do so). On the other hand, our notations in this rewrite is quite different from the notations in (Fima and Wang, 2018). In the latter, the more succinct notation introduces one noteworthy obscurity—one relies on the choices of some orbital sections, and many more closely related mappings that depends on these choices, which makes the calculations there a little bit difficult to track. Here in this thesis, these choices are eliminated wherever possible, and we replace them with the more systematic (yet somewhat equivalent) notion of the so-called $\mathscr{O}$-representations. The latter treatment on matched pair of length functions also uses this approach. This makes our main results, namely the results presented in § I.7 and § I.8, a little more precise and the proof of these results more transparent. As an illustration of the advantage of this more systematic approach, we point out the fusion rules for bicrossed products in (Fima and Wang, 2018) (Theorem I.4.15, statement (c)) is easily simplified (Theorem I.4.19). We also point out that unlike the paper (Fima et al., 2017) that (Fima and Wang, 2018) is based upon, the construction of the bicrossed product here in this thesis is given by a purely algebraic approach as an algebraic compact quantum group, which gives a more clear picture of what is going on. This purely algebraic picture can be easily translated to the now more standard operator-algebraic construction via the GNS construction with respect to the Haar integral, and at the same time has the important advantage of being more suitable of a systematic study of the representation theory of the underlying quantum group, as is manifested in § I.3. Finally, we point out that more background information on generalities of length functions, property (RD) and polynomial growth is given here (§ I.6) to make this thesis more self-contained.

I.1 Matched pair of groups

We begin with some rudimentary observations on locally compact groups. Let H be a locally compact group. Suppose there exists a compact subgroup G of H, and a discrete group Γ of H, such that $H = \Gamma G$ (so $H = G\Gamma$ too) and $\Gamma \cap G = \{e_H\}$, where e_H is the identity of H. Then every element of H can be written uniquely as a product of an element of G and an element of Γ, in either order. In particular, there are mappings $\alpha\colon \Gamma \times G \to G$, $\beta\colon \Gamma \times G \to \Gamma$ such that

$$\forall g \in G, \gamma \in \Gamma, \quad \gamma g = \alpha_\gamma(g)\beta_g(\gamma), \tag{I.1.1}$$

where $\alpha_\gamma = \alpha(\gamma, \cdot)$ and $\beta_g = \beta(\cdot, g)$. Based on this property, we have

$$\forall r, s \in \Gamma, g \in G, \quad rsg = \alpha_{rs}(g)\beta_g(rs) = r\alpha_s(g)\beta_g(s) = \alpha_r\big(\alpha_s(g)\big)\beta_{\alpha_s(g)}(r)\beta_g(s),$$

Then the uniqueness of the corresponding decompositions forces that

$$\forall r, s \in \Gamma, g \in G, \quad \alpha_{rs}(g) = (\alpha_r \circ \alpha_s)(g), \quad \beta_g(rs) = \beta_{\alpha_s(g)}(r)\beta_g(s).$$

Similarly,

$$\forall \gamma \in \Gamma, g, h \in G, \quad \alpha_\gamma(gh)\beta_{gh}(\gamma) = \gamma gh = \alpha_\gamma(g)\beta_g(\gamma)h = \alpha_\gamma(g)\alpha_{\beta_g(\gamma)}(h)\beta_h\big(\beta_g(\gamma)\big),$$

Then this forces that

$$\forall \gamma \in \Gamma, g, h \in G, \quad \alpha_\gamma(gh) = \alpha_\gamma(g)\alpha_{\beta_g(\gamma)}(h), \quad \beta_{gh}(\gamma) = (\beta_h \circ \beta_g)(\gamma).$$

Obviously, we have

$$\forall \gamma \in \Gamma, g \in G, \quad \alpha_e(g) = g, \quad \beta_e(\gamma) = \gamma, \quad \alpha_\gamma(e) = e, \quad \beta_g(e) = e.$$

To recapitulate, we have

(a) α is a left action of the discrete group Γ on G viewed as a compact space;

(b) β is a right action of the compact group G on the discrete space Γ;

(c) the two actions satisfy the following compatibility conditions:

$$\forall \gamma \in \Gamma, g, h \in G, \quad \alpha_\gamma(gh) = \alpha_\gamma(g)\alpha_{\beta_\gamma(g)}(h) \quad \text{and} \quad \alpha_\gamma(e_G) = e_G, \tag{I.1.2}$$

and

$$\forall g \in G, r, s \in \Gamma, \quad \beta_g(rs) = \beta_{\alpha_s(g)}(r)\beta_g(s) \quad \text{and} \quad \beta_g(e_\Gamma) = e_\Gamma. \tag{I.1.3}$$

Lemma I.1.1. *Both $\alpha, \beta \colon \Gamma \times G \to G$ are continuous.*

Proof. Since Γ is a discrete subgroup of H, it is in particular closed. As a continuous bijection from a compact space onto a Hausdorff space, the mapping $\varphi \colon G \to H/\Gamma$, $g \mapsto g\Gamma$ is in fact a homeomorphism. By (I.1.1), we have $\gamma g\Gamma = \alpha_\gamma(g)\Gamma$, thus $\alpha(\gamma, g) = \alpha_\gamma(g) = \varphi^{-1}(\gamma g\Gamma)$, i.e. α is the composite of the multiplication $\Gamma \times G \to H$, the canonical projection $H \to H/\Gamma$, and the inverse $\varphi^{-1} \colon H/\Gamma \to G$ of the homeomorphism φ, all of which are continuous. Hence α itself is continuous. The continuity of β follows from that of α and (I.1.1). □

Corollary I.1.2. *Using the above notations, every β-orbit is finite.*

Proof. For each $\gamma \in \Gamma$, the β-orbit $\gamma \cdot G$ is the rang of G under the continuous mapping $\beta(\gamma, \cdot) \colon G \to \Gamma$. Hence $\gamma \cdot G$ is compact in the discrete space Γ, thus must be finite. □

Conversely, one can easily check that given a pair of groups (Γ, G), where G is compact and Γ is discrete, suppose that there exists a continuous left action α of the group Γ on G, and a continuous right action β of the group G on Γ, such that the compatibility conditions (I.1.2) and (I.1.3) hold, then

$$(g, r)(h, s) := \big(g\alpha_r(h), \beta_h(r)s\big) \tag{I.1.4}$$

defines a group law on $G \times \Gamma$, which makes $G \times \Gamma$ a locally compact topological group. When one identifies $G \times \Gamma$ with H via multiplication, one recovers the multiplication on H from (I.1.4)(see (I.1.1)).

We formalize these observations in the following definition.

Definition I.1.3. A matched pair of groups consists of the following data:

- a pair (Γ, G) of topological groups, where Γ is discrete and G is compact,
- a continuous left action $\alpha \colon \Gamma \curvearrowright G$,
- a continuous right action $\beta \colon \Gamma \curvearrowleft G$,

such that for all $r, s \in \Gamma$, $g, h \in G$, the following compatibility conditions (often referred as the **matched pair relations** in the sequel) are satisfied:

$$\begin{gathered} \alpha_r(gh) = \alpha_r(g)\alpha_{\beta_g(r)}(h), \qquad \beta_g(rs) = \beta_{\alpha_s(g)}(r)\beta_g(s), \\ \alpha_\gamma(e_G) = e_G, \qquad \text{and} \qquad \beta_g(e_\Gamma) = e_\Gamma. \end{gathered} \tag{I.1.5}$$

Such a matched pair of groups is often denoted simply as (Γ, G), suppressing the actions α and β when they are implicitly understood from context.

Remark I.1.4. There is a more general notion of matched pair of locally compact groups. A pair of locally compact groups (G_1, G_2) is called matched if there exists a locally compact group G, such that G_1, G_2 are identified with closed topological subgroup of G that intersect trivially and the complement of G_1G_2 is a null set with respect to the Haar measure of G. In our case, $G_1 = \Gamma$ is discrete (hence closed) , $G_2 = G$ is compact, so G_1G_2 is closed, and its complement is open and hence is of null Haar measure if and only if $G_1G_2 = G$. Thus our notion of matched pair is the particular case of matched pair of locally compact groups where the first group is compact and the second group discrete.

Remark I.1.5. There is an even more general notion of a matched pair of locally compact quantum groups ((Majid, 1990b; 1991),(Takeuchi, 1981),(Vaes and Vainerman, 2003)). There are two important constructions associated to such a matched pair, both yielding new locally compact quantum groups. One is called the double crossed product, which is the quantum analogue of recovering the global group G from the matched pair (G_1, G_2) in the classical case of locally compact groups. We will not treat this construction in this thesis and refer the reader to (Baaj and Vaes, 2005). The other one is called the bicrossed product construction, the most general case in the setting of locally compact quantum groups is treated in (Vaes and Vainerman, 2003). We also refer the reader to the introduction of Chapter II of this thesis for further references on these constructions. Among other important results, it is shown in (Vaes, 2005, Proposition 2.17) that the bicrossed product of a pair of locally compact quantum groups $(\mathbb{G}_1, \mathbb{G}_2)$ is compact if and only if $\mathbb{G}_1$ is discrete and $\mathbb{G}_2$ is compact. The construction of bicrossed product in this generality is rather technically involved and requires the theory of locally compact quantum groups as developed in (Kustermans and Vaes, 2000) and (Kustermans and Vaes, 2003). Since we only consider the classical bicrossed product where the matched pair is given by Definition I.1.3, in this case a simpler construction (albeit still very nontrivial) of bicrossed product is developed in (Fima et al., 2017). The construction of bicrossed products in this thesis uses a purely algebraic one, which the author believes to be pedagogically more suitable for treating the representation theory of such objects, and is long known among experts working on Hopf algebras.

We finish our treatment of matched pair with some technical lemmas.

Lemma I.1.6. *If (Γ, G) is a matched pair of groups with left action α, then for every $\gamma \in \Gamma$, the homeomorphism $\alpha_\gamma \colon G \to G$ preserves the Haar measure of G.*

Proof. Let $\beta \colon \Gamma \times G \to \Gamma$ be the corresponding right action of the matched pair (Γ, G). By Lemma I.1.1, the β-orbit $\gamma \cdot G$ of γ is finite (compact in a discrete space Γ). Suppose $\gamma \cdot G = \{s_1, \dots, s_n\}$, and put $A_k := \{g \in G \ : \ \beta_g(\gamma) = s_k\}$ for $k = 1, \dots, n$. By the continuity of β (Lemma I.1.1), each A_k is clopen in G. Fix an arbitrary $g \in G$, and denote the Haar probability measure on G by μ. Consider the measure $\nu \colon = \alpha_\gamma^* \mu$, we want to show that $\nu = \mu$. Since $\nu(G) = \mu\big(\alpha_\gamma(G)\big) = \mu(G) = 1$, by the uniqueness of the Haar probability measure, it suffices to show that for every Borel set X of G, one has $\nu(Xg) = \nu(X)$. Let $X_k = A_k \cap X$, then each X_k remains a Borel set and X is the disjoint union of $X_1, \dots, X_n$. Moreover, by the right invariance of μ, the definition of X_k and (I.1.2), one has

$$\nu(X_k g) = \mu\big(\alpha_\gamma(X_k g)\big) = \mu\big(\alpha_\gamma(X_k)\alpha_{s_k}(g)\big) = \mu\big(\alpha_\gamma(X_k)\big) = \nu(X_k).$$

Summing up the above equality over $k = 1, \dots, n$ yields $\nu(Xg) = \nu(X)$, which finishes the proof. $\square$

Notations I.1.7. *In the following, the right action β is often simply denoted by a dot, while the left action is always indicated explicitly by α.*

Lemma I.1.8. *For every $\gamma \in \Gamma$, let G_γ be the isotropy subgroup of G fixing γ with respect to the action β, then G_γ is clopen, and α_γ restricts to a topological isomorphism from G_γ onto $G_{\gamma^{-1}}$.*

Proof. That G_γ is clopen follows from the continuity of β and the discreteness of Γ. For every $g \in G$, we have

$$e_\Gamma = \beta_g(\gamma^{-1}\gamma) = \beta_{\alpha_\gamma(g)}(\gamma^{-1})\beta_g(\gamma),$$

hence $(\gamma \cdot g)^{-1} = \gamma^{-1} \cdot \alpha_\gamma(g)$, and $g \in G_\gamma \iff \alpha_\gamma(g) \in G_{\gamma^{-1}}$, which, together with the matched pair relations (which imply that $\alpha_\gamma|_{G_\gamma}$ is multiplicative), proves the second assertion. $\square$

Lemma I.1.9. *Let $\gamma, r \in \Gamma$, then for every $g \in G$, $r = \gamma \cdot g$ if and only if $\gamma^{-1} \cdot \alpha_\gamma(g) = r^{-1}$. In particular, $(\gamma \cdot G)^{-1} = \gamma^{-1} \cdot G$, and $r \mapsto r^{-1}$ is a bijection from $\gamma \cdot G$ onto $\gamma^{-1} \cdot G$.*

Proof. By the matched pair relations, we have

$$e_\Gamma = \beta_g(\gamma^{-1}\gamma) = \beta_{\alpha_\gamma(g)}(\gamma^{-1})\beta_g(\gamma) = \big(\gamma^{-1} \cdot \alpha_\gamma(g)\big)(\gamma \cdot g) \qquad \square$$

Lemma I.1.10. *If $\mathscr{O}_1$, $\mathscr{O}_2$ are β-orbits, then the set*

$$\mathscr{O}_1\mathscr{O}_2 := \{rs \ : \ r \in \mathscr{O}_1, \ s \in \mathscr{O}_2\} \subseteq \Gamma$$

is a disjoint union of β-orbits.

Proof. If $\gamma \in \mathscr{O}_1\mathscr{O}_2$, then $\gamma = rs$ for some $r \in \mathscr{O}_1$ and $s \in \mathscr{O}_2$, and for every $g \in G$, we have

$$\gamma \cdot g = \beta_g(rs) = \beta_{\alpha_s(g)}(r)\beta_g(s) = (r \cdot \alpha_s(g))\,(s \cdot g) \in \mathscr{O}_1\mathscr{O}_2. \qquad \square$$

I.2 Bicrossed product as an algebraic compact quantum group

From now on in this chapter, we fix a matched pair (Γ, G) (see Definition I.1.3) together with the associated actions α and β.

Let $\mathrm{Pol}(G)$ be the subalgebra of matrix coefficients of representations of G, then $\mathrm{Pol}(G)$ is a dense $*$-subalgebra of the abelian C^*-algebra $C(G)$. Moreover, $\mathrm{Pol}(G)$ also possesses a canonical algebraic compact quantum group structure with comultiplication inherited from G viewed as a commutative compact quantum group $(C(G), \Delta)$. Since $\alpha_r \colon G \to G$ is a homeomorphism for every $r \in \Gamma$, $\alpha_r\big(\mathrm{Pol}(G)\big)$ is a dense subspace of $C(G)$, which is also stable under involution (conjugation). Let A_0 be the subalgebra of $C(G)$ generated by $\cup_{\gamma\in\Gamma}\alpha_\gamma^*\big(\mathrm{Pol}(G)\big)$ and $\Big\{\alpha_\gamma^*(v_{r,s}) \ : \ r, s, \gamma \in \Gamma\Big\}$, where $v_{r,s}$ is the character function of the clopen set

$$G_{r,s} = \Big\{g \in G \ : \ \beta_g(r) = s\Big\} \subseteq G.$$

The clopen set $G_{\gamma,\gamma}$ is in fact an open subgroup of G, which will often be denoted be G_γ. We check immediately that for each β-orbit $\gamma \cdot G$, where $\gamma \in \Gamma$, the matrix

$(v_{r,s})_{r,s\in\gamma\cdot G}$ over $C(G)$ is a magic unitary. Our first goal is to establish that $A_0 = \mathrm{Pol}(G)$ (Proposition I.2.4).

With these notations introduced, we first establish some elementary properties of $\mathrm{Pol}(G)$ associated with the matched-pair actions α and β.

Lemma I.2.1. *Suppose H is a topological group, (V,ρ) a continuous finite dimensional representation of H. For any continuous mapping $\varphi\colon H \to \mathbb{C}$, let $\check{\varphi}\colon H \to \mathbb{C}$ be the mapping $h \mapsto \varphi(h^{-1})$, and (V^*,ρ^*) the contragredient representation of (V,ρ), then φ is a matrix coefficient of ρ if and only if $\check{\varphi}$ is a matrix coefficient of ρ^*.*

Proof. This follows directly from the definition of the contragredient representation, namely $\rho^*(h)$ is the transpose of $\rho(h^{-1})$. ◻

Lemma I.2.2. *Suppose H is a topological group, $\varphi\colon H \to \mathbb{C}$ is a continuous function on H, then the following are equivalent:*

(a) *the subspace*
$$\mathscr{L}_H(\varphi) = \mathrm{Vect}\{\varphi(h\,\cdot\,) \ : \ h \in H\}$$
of $C(H)$ is finite-dimensional;

(b) *the subspace*
$$\mathscr{R}_H(\varphi) = \mathrm{Vect}\{\varphi(\,\cdot\, h) \ : \ h \in H\}$$
of $C(H)$ is finite-dimensional;

(c) *the subspace*
$$\mathscr{T}_H(\varphi) = \mathrm{Vect}\{\varphi(h\ \cdot\ k) \ : \ h,k \in H\}$$
of $C(H)$ is finite dimensional;

(d) *there is a continuous representation (V,ρ) of H on some finite dimensional complex vector space V, such that φ is a matrix coefficient of ρ.*

Proof. That (d) implies (c) follows from a routine verification, and obviously (c) implies both (a) and (b). If (b) implies (d), then by Lemma I.2.1, (a) also implies (d). Thus to finish the proof, it suffices to show that (b) implies (d).

Suppose (b) holds. Put $V = \mathscr{R}_H(\varphi)$, then V is a finite dimensional subspace of the function space $C(G)$. Define $\rho\colon H \to \mathrm{GL}(V)$, $h \mapsto R_h^*$, where $R_h\colon H \to H$ is the multiplication on the right by h, and R_h^* is the pull-back along R_h, i.e. $R_h^*\colon \psi \mapsto \psi\circ R_h$. Then (V,ρ) is a finite dimensional representation of H. Note that $\varphi \in V$, and the evaluation at the identity element e, denoted by $\widehat{e}$, is a linear functional on V. Hence the mapping
$$h \mapsto \langle \widehat{e},\ \rho(h)\varphi\rangle = \langle \widehat{e},\ \varphi\circ R_h\rangle = [\varphi\circ R_h](e) = \varphi(h)$$
is a matrix coefficient of ρ. The proof will be complete once we show that the representation (V,ρ) is continuous, which is equivalent to every matrix coefficients of ρ is continuous. For all $h_0 \in H$, let $\widehat{h_0} \in V^*$ be the linear functional of evaluating at h_0, then obviously $\cap_{h_0\in H} \ker \widehat{h_0} = 0$, hence V^* is the linear span of $\left\{\widehat{h_0} \ : \ h_0 \in H\right\}$. Thus to prove that every matrix coefficient of ρ is continuous on H, it suffices to show that mappings from H to $\mathbb{C}$ of the form
$$h \mapsto \left\langle \widehat{h_0}\,,\ \rho(h)\psi\right\rangle = \left\langle \widehat{h_0}\,,\ \psi\circ R_h\right\rangle = \psi(hh_0)$$

are always continuous for all $h_0 \in H$, and $\psi \in V$. But the above mapping is just $\psi \circ R_{h_0}$, whose continuity follows from the continuity of both ψ and R_{h_0}. □

Lemma I.2.3. *Let K be an open subgroup of the compact group G, then* $\mathrm{Pol}(G)|_K = \mathrm{Pol}(K)$, *where*

$$\mathrm{Pol}(G)|_K = \{\varphi|_K \ : \ \varphi \in \mathrm{Pol}(G)\}.$$

Proof. If $\varphi \in \mathrm{Pol}(G)$ is a matrix coefficient of some (continuous) finite dimensional representation (V, ρ), then $\varphi|_K$ is a matrix coefficient of the restricted representation $\rho|_K$. Thus $\mathrm{Pol}(G)|_K \subseteq \mathrm{Pol}(K)$. Since $\mathrm{Pol}(G)$ is dense in $C(G)$, the subspace $\mathrm{Pol}(G)|_K$ is dense in $C(K)$ (any function g in $C(K)$ can be extended to a function f in $C(G)$ by the Tietze extension theorem, and if f is the uniform limit of a sequence (f_n) in $\mathrm{Pol}(G)$, which exists by the density of $\mathrm{Pol}(G)$ in $C(G)$, then in particular, g is the uniform limit of (g_n), where $g_n = f_n|_K$). Hence by the orthogonality relations, we must have $\mathrm{Pol}(G)|_K = \mathrm{Pol}(K)$. □

Proposition I.2.4. *Using the above notations, the following hold:*

(a) $v_{r,s} \in \mathrm{Pol}(G)$ *for all $r, s \in \Gamma$ in the same β-orbit,*

(b) *for every $\gamma \in \Gamma$, and all $\varphi \in C(G)$ with* $\operatorname{supp}\varphi \subseteq G_\gamma$, *we have $\varphi \in \mathrm{Pol}(G)$ if and only if $\varphi|_{G_\gamma} \in \mathrm{Pol}(G_\gamma)$;*

(c) $\mathrm{Pol}(G)$ *is stable under the action of α, i.e. one has $\alpha_\gamma^*\varphi = \varphi \circ \alpha_\gamma \in \mathrm{Pol}(G)$, for every $\varphi \in \mathrm{Pol}(G)$ and every $\gamma \in \Gamma$.*

Proof. (a). Take any β-orbit $\gamma \cdot G$, where γ is some element in Γ. The right permutation representation of G on the finite dimensional Hilbert space $\ell^2(\gamma \cdot G)$ (equipped with the canonical Hilbert space structure on it), when written in the operator form, is exactly $\sum_{r,s \in \gamma \cdot G} e_{r,s} \otimes v_{r,s} \in \mathcal{B}(\ell^2(\gamma \cdot G)) \otimes C(G)$, where $\{e_{r,s} \ : \ r, s \in \gamma \cdot G\}$ is the matrix unit associated with the canonical Hilbert basis $\{\delta_r \ : \ r \in \gamma \cdot G\}$. Thus $v_{r,s}$, where $r, s \in \gamma \cdot G$, are all matrix coefficients of this representation.

(b). It is clear that $\varphi \in \mathrm{Pol}(G)$ implies $\varphi|_{G_\gamma} \in \mathrm{Pol}(G_\gamma)$. Conversely, suppose $\varphi|_{G_\gamma} \in \mathrm{Pol}(G_\gamma)$. Then by Lemma I.2.3, there exists some $\varphi' \in \mathrm{Pol}(G)$ with $\varphi'|_{G_\gamma} = \varphi|_{G_\gamma}$. Since $v_{\gamma,\gamma} \in \mathrm{Pol}(G)$ by (a), $\operatorname{supp} v_{\gamma,\gamma} = G_\gamma$ and $\operatorname{supp}\varphi \subseteq G_\gamma$, we have $\varphi = v_{\gamma,\gamma}\varphi' \in \mathrm{Pol}(G)$.

(c). We first treat the special case in which $\operatorname{supp}\varphi \subseteq G_{\gamma^{-1}}$. Denote $\varphi|_{G_{\gamma^{-1}}}$ by ψ. By Lemma I.2.3, there exists a finite dimensional unitary representation $(\rho, \mathscr{H})$ of $G_{\gamma^{-1}}$, such that ψ is a matrix coefficient of ρ. Hence by Lemma I.1.8, $\alpha_\gamma^*\psi = \psi \circ \alpha_\gamma$ is a matrix coefficient of the unitary representation $(\rho \circ \alpha_\gamma, \mathscr{H})$ of G_γ. Using Lemma I.1.8 again, we see that $\operatorname{supp}(\alpha_\gamma^*\varphi) \subseteq G_\gamma$. Hence $(\alpha_\gamma^*\varphi)|_{G_\gamma} = \alpha_\gamma^*\psi \in \mathrm{Pol}(G_\gamma)$, and $\alpha_\gamma^*\varphi \in \mathrm{Pol}(G)$ by (b).

Take an arbitrary $r \in \gamma \cdot G$, and suppose $\operatorname{supp}\varphi \subseteq G_{\gamma^{-1},r^{-1}}$. Take $g_0 \in G$ such that $r = \gamma \cdot g_0$, then by Lemma I.1.9, $r^{-1} = \gamma^{-1} \cdot \alpha_\gamma(g_0)$. Using Lemma I.1.9 again, we also have $\operatorname{supp}(\alpha_\gamma^*\varphi) \subseteq G_{\gamma,r}$. Let $R_g \colon G \to G$ be the multiplication on the right by $g \in G$. Put $\varphi' = \varphi \circ R_{\alpha_\gamma(g_0)}$ and $\varphi_1 = (\alpha_\gamma^*\varphi) \circ R_{g_0}$, then $\varphi' \in \mathrm{Pol}(G)$ by Lemma I.2.2, and $\operatorname{supp}\varphi' \subseteq G_{\gamma^{-1}}$ so $\operatorname{supp}(\alpha_\gamma^*\varphi') \subseteq G_\gamma$ as we've seen above, while $\operatorname{supp}\varphi_1 \subseteq G_\gamma$. Now for every $x \in G_\gamma$, we have

$$\varphi_1(x) = (\alpha_\gamma^*\varphi)(xg_0) = \varphi\big(\alpha_\gamma(xg_0)\big) = \varphi\big(\alpha_\gamma(x)\alpha_\gamma(g_0)\big) = \varphi'\big(\alpha_\gamma(x)\big) = (\alpha_\gamma^*\varphi')(x).$$

Thus $\varphi_1 = \alpha_\gamma^*\varphi' \in \mathrm{Pol}(G)$ by the previous case, and consequently $\alpha_\gamma^*\varphi \in \mathrm{Pol}(G)$ by Lemma I.2.2.

The general case now follows easily. Indeed, by Lemma I.1.9, $r \mapsto r^{-1}$ is a bijection from $\gamma \cdot G$ onto $\gamma^{-1} \cdot G$. Hence, $\varphi = \sum_{r \in \gamma \cdot G} v_{\gamma^{-1}, r^{-1}} \varphi$. Since the support of each $v_{\gamma^{-1}, r^{-1}} \varphi \in \mathrm{Pol}(G)$ (see (a)) is in $G_{\gamma^{-1}, r^{-1}}$, we have $\alpha_\gamma^*(v_{\gamma^{-1}, r^{-1}} \varphi) \in \mathrm{Pol}(G)$ by our previous argument. Hence

$$\alpha_\gamma^* \varphi = \sum_{r \in \gamma \cdot G} \alpha_\gamma^*(v_{\gamma^{-1}, r^{-1}} \varphi) \in \mathrm{Pol}(G). \qquad \square$$

Remark I.2.5. If β is trivial, i.e. α is an action by continuous group automorphisms, then (c) is almost self-evident. It is remarkable that $\mathrm{Pol}(G)$ remains stable under the action of α even α fails to be an action by group automorphisms (β nontrivial).

With these preparations, we can now construct the bicrossed product of the matched pair (Γ, G) as an algebraic compact quantum group. Consider

$$\begin{aligned} \mathscr{A} &= C_c\big(\Gamma, \mathrm{Pol}(G)\big) \\ &= \big\{\Phi\colon \Gamma \to \mathrm{Pol}(G) \;:\; \Phi(\gamma) = 0 \text{ except for a finite number of } \gamma \in \Gamma\big\}. \end{aligned} \tag{I.2.1}$$

The goal is to construct a multiplication $\widetilde{m} : \mathscr{A} \otimes \mathscr{A} \to \mathscr{A}$ with a (unique) unit $\widetilde{\eta} : \mathbb{C} \to \mathscr{A}$, a comultiplication $\widetilde{\Delta} : \mathscr{A} \to \mathscr{A} \otimes \mathscr{A}$ with a (unique) counit $\widetilde{\epsilon} : \mathscr{A} \to \mathbb{C}$, a (uniquely determined) antipode $\widetilde{S} : \mathscr{A} \to \mathscr{A}$, an involution $*\colon \mathscr{A} \to \mathscr{A}$ and a positive invariant integral (which will be normalized as the Haar state) $\widetilde{\tau} : \mathscr{A} \to \mathbb{C}$, such that equipped with these structures, $(\mathscr{A}, \widetilde{\Delta})$ is an algebraic compact quantum group.

Let $\widetilde{\alpha} : \Gamma \to \mathrm{Aut}\big(C(G)\big)$ be the group morphism $\gamma \mapsto \alpha^*_{\gamma^{-1}}$, then $\big(\Gamma, C(G), \widetilde{\alpha}\big)$ is a C^*-dynamical system. It is well-known that when defining the crossed product of Γ and $C(G)$ with respect to the action $\widetilde{\alpha}$, one starts with the convolution as multiplication on the space $C_c\big(\Gamma, C(G)\big)$ of (automatically continuous) mappings from Γ into $C(G)$ with compact (equivalently, finite) support, which makes $C_c\big(\Gamma, C(G)\big)$ an involutive algebra. For convenience of the reader, we recall briefly here this construction in a slightly more general setting where we replace $C(G)$ with an arbitrary *unital* involutive algebra, and $\widetilde{\alpha} : \Gamma \to \mathrm{Aut}(A)$ is still a group morphism. The idea is that one wants to incorporate the multiplication structures of both the algebra A and the group Γ in a universal way. To achieve this, one considers the vector space $C_c(\Gamma, A)$ of compactly supported (which is the same as finitely supported as the group Γ is discrete), A-valued functions. Since A is a unital algebra, one has the analogue of A-valued Dirac measure, i.e. for each $\gamma \in \Gamma$, one can associate a element $u_\gamma \in C_c(\Gamma, A)$, such that

$$\forall \mu \in \Gamma, \qquad u_\gamma(\mu) = \begin{cases} 0 & \text{if } \mu \neq \gamma, \\ 1_A & \text{if } \mu = \gamma \end{cases}$$

where of course 1_A is the multiplicative unit of the algebra A. In this way, one has a copy of Γ as a set in $C_c(\Gamma, A)$ via the bijective correspondence $\gamma \leftrightarrow u_\gamma$. There is also a distinct element in Γ, namely the multiplicative identity e_Γ, which gives us a copy of A in $C_c(\Gamma, A)$ via the embedding $A \hookrightarrow C_c(\Gamma, A)$, $a \mapsto u_\gamma a$, where the notation $u_\gamma a$ denotes the A-valued function

$$\forall \mu \in \Gamma, \qquad u_\gamma a = \begin{cases} 0 & \text{if } \mu \neq \gamma, \\ a & \text{if } \mu = \gamma. \end{cases}$$

Identifying A and Γ with their respective copies in $C_c(\Gamma, A)$ as explained above, the key idea to define the multiplicative and involutive structure on $C_c(\Gamma, A)$ is that one wants

(a) A is a unital $*$-subalgebra of $C_c(\Gamma, A)$;

(b) Γ is a unitary subgroup of the multiplicative group of $C_c(\Gamma, A)$;

(c) the action $\widetilde{\alpha}_\gamma$ behaves exactly as conjugation by u_γ for every $\gamma \in \Gamma$, i.e. for all $a \in A$, we have $\widetilde{\alpha}_\gamma(a) = u_\gamma a u_\gamma^{-1} = u_\gamma a u_{\gamma^{-1}}$.

With these requirements in mind and using the above notations, and noting that a generic element of $C_c(\Gamma, A)$ is of the form of finite sum $\sum_{\gamma\in\Gamma} u_\gamma a_\gamma$, where all but finitely many $a_\gamma \in A$ are nonzero, the multiplication on $C_c(\Gamma, A)$ is defined by

$$\left(\sum_{\gamma\in\Gamma} u_\gamma a_\gamma\right)\left(\sum_{\mu\in\mu} u_\mu b_\mu\right) := \sum_{\gamma,\mu\in\Gamma} u_{\gamma\mu}\widetilde{\alpha}_{\mu^{-1}}(a_\gamma)b_\mu = \sum_{\gamma\in\Gamma} u_\gamma \sum_{r,s\in\Gamma,\ rs=\gamma} \widetilde{\alpha}_{s^{-1}}(a_r)b_s, \quad \text{(I.2.2)}$$

as by our requirements, we have

$$\forall \mu \in \Gamma, \quad a \in A, \qquad a_\gamma u_\mu = u_\mu u_\mu^{-1} a_\gamma u_\mu = u_\mu(u_{\mu^{-1}} a_\gamma u_\mu) = u_\mu \widetilde{\alpha}_{\mu^{-1}}(a).$$

One checks easily that the multiplication as defined in (I.2.2) makes $C_c(\Gamma, A)$ a unital associative algebra, with $u_{e_\Gamma} 1_A$ being the multiplicative unit. Similarly, the involution on $C_c(\Gamma, A)$ is defined by

$$\left(\sum_{\gamma\in\Gamma} u_\gamma a_\gamma\right) := \sum_{\gamma\in\Gamma} a_\gamma^* u_{\gamma^{-1}} = \sum_{\gamma\in\Gamma} u_{\gamma^{-1}}\widetilde{\alpha}_\gamma(a_\gamma^*) = \sum_{\gamma\in\Gamma} u_\gamma \widetilde{\alpha}_{\gamma^{-1}}(a_{\gamma^{-1}}^*), \qquad \text{(I.2.3)}$$

as one needs $u_\gamma^* = u_\gamma^{-1} = u_{\gamma^{-1}}$, i.e. u_γ to be unitary, in our requirements. Again, one checks easily that (I.2.6) defines an involution on the unital associative algebra $C_c(\Gamma, A)$, making the latter a unital involutive algebra and completes this construction. We call the unital involutive algebra $C_c(\Gamma, A)$ the (algebraic) crossed product of Γ and A with respect to the action $\widetilde{\alpha}$, and also denote it by $A \rtimes_{\widetilde{\alpha}} \Gamma$, or simply by $A \rtimes \Gamma$ when the action $\widetilde{\alpha}$ is clear. Of course, we have the following useful universal property, whose proof is merely a routine verification using the construction above.

Proposition I.2.6. *Using the above notations, the formula*

$$\forall \gamma \in \Gamma, a \in A, \qquad \rho(u_\gamma a) = \rho_\Gamma(u_\gamma)\rho_A(a) \qquad \text{(I.2.4)}$$

determines a bijection between the class of non-degenerate representations ρ of the involutive algebra $A \rtimes \Gamma$ on some Hilbert space H and the class of pairs (ρ_Γ, ρ_A), where ρ_Γ is a unitary representation of the group Γ on H and ρ_A is a non-degenerate representation of the involutive algebra A on H, such that ρ_Γ and ρ_A are covariant in the sense that

$$\forall \gamma \in \Gamma, a \in A, \qquad \rho_A\big(\widetilde{\alpha}_\gamma(a)\big) = \rho_\Gamma(u_\gamma)\rho_A(a)\rho_\Gamma(u_{\gamma^{-1}}). \qquad \text{(I.2.5)}$$

Before we return to our discussion of the bicrossed products, we point out that per our construction, $C_c(\Gamma, A)$ is a free A-module (both left and right) with $\{u_\gamma \ : \ \gamma \in \Gamma\}$ as a base.

We now apply the above procedure to the bicrossed product construction. Since $\mathrm{Pol}(G)$ is a unital $*$-subalgebra of $C(G)$ that is invariant under the action $\widetilde{\alpha}$ (Proposition I.2.4), we can canonically identifies $\mathscr{A} = C_c\big(\Gamma, \mathrm{Pol}(G)\big)$ as a unital $*$-subalgebra of the $*$-algebra $C_c\big(\Gamma, C(G)\big)$. This gives us the multiplication $\widetilde{m}$, the unit $\widetilde{\eta} : \mathbb{C} \longrightarrow \mathscr{A}$ for the multiplication $\widetilde{m}$, and the involution $*$ on $\mathscr{A}$. For $\varphi \in \mathrm{Pol}(G)$, $\gamma \in \Gamma$, let

$u_\gamma\varphi\colon \Gamma \to \mathrm{Pol}(G)$ denote the mapping sending $r \in \Gamma$ to $\delta_{\gamma,r}\varphi \in \mathrm{Pol}(G)$. We write $u_{e_\Gamma}\varphi$ simply as φ, and $u_\gamma 1_G$ simply as u_γ, where 1_G is of course the constant function on G with value 1, which is the common unit of $\mathrm{Pol}(G)$ and $C(G)$. Thus $u_{e_\Gamma} = 1_G$ is the multiplicative identity of $\mathscr{A}$, and is often denoted simply as 1. This allows us to identify Γ with the subgroup $\{u_\gamma : \gamma \in G\}$ of the multiplicative group $\mathscr{A}^\times$, and identify $\mathrm{Pol}(G)$ with the unital $*$-subalgebra $\{u_{e_\Gamma}\varphi : \varphi \in \mathrm{Pol}(G)\}$ of $\mathscr{A}$. These identification will be freely used below without further explanation. As a vector space, $\mathscr{A}$ is spanned by $\{u_\gamma\varphi : \gamma \in \Gamma, \varphi \in \mathrm{Pol}(G)\}$. Thus the multiplication $\widetilde{m}$ on $\mathscr{A}$ is completely determined by the following relations (note that $\widetilde{m}$ is associative)

$$\begin{aligned} &\forall r, s \in \Gamma,\ \varphi, \psi \in \mathrm{Pol}(G), \qquad \widetilde{m}(u_r \otimes u_s) = u_r u_s = u_{rs}, \qquad \widetilde{m}(\varphi \otimes \psi) = \varphi\psi, \\ &\text{and} \qquad \widetilde{m}(\varphi \otimes u_\gamma) = \varphi u_\gamma = u_\gamma\big(\widetilde{\alpha}_{\gamma^{-1}}(\varphi)\big) = u_\gamma(\alpha_\gamma^*\varphi) = \widetilde{m}(u_\gamma \otimes \alpha_\gamma^*\varphi); \end{aligned}$$

where as the involution $*$ on $\mathscr{A}$ is completely determined by

$$\forall \gamma \in \Gamma,\ \varphi, \qquad (u_\gamma\varphi)^* = \varphi^* u_\gamma^* = \overline{\varphi} u_{\gamma^{-1}} = u_\gamma\left(\alpha^*_{\gamma^{-1}}\overline{\varphi}\right). \tag{I.2.6}$$

This completes our description of the multiplicative and involutive structures of $\mathscr{A}$.

To describe the comultiplication $\widetilde{\Delta}$ requires further work. Define

$$\rho_\Gamma : \Gamma \to \mathscr{A} \otimes \mathscr{A}, \qquad \gamma \mapsto \sum_{r \in \gamma \cdot G} u_\gamma v_{\gamma,r} \otimes u_r.$$

Lemma I.2.7. *The mapping ρ_Γ is a unitary representation of the discrete group Γ, i.e. $\rho_\Gamma(e_\Gamma) = 1 \otimes 1$, $\rho_\Gamma(\gamma)$ is unitary, and $\rho_\Gamma(\gamma\mu) = \rho_\Gamma(\gamma)\rho_\Gamma(\mu)$ for all $\gamma, \mu \in \Gamma$.*

Proof. That $\rho_\Gamma(e_\Gamma) = 1 \otimes 1$ follows from the definition of ρ_Γ and the fact that $e_\Gamma \cdot G = \{e_\Gamma\}$. We now show that that ρ_Γ is multiplicative. By definition,

$$\rho_\Gamma(\gamma)\rho_\Gamma(\mu) = \sum_{\substack{r \in \gamma\cdot G,\\ s \in \mu\cdot G}} u_\gamma v_{\gamma,r} u_\mu v_{\mu,s} \otimes u_r u_s = \sum_{\substack{r \in \gamma\cdot G,\\ s \in \mu\cdot G}} u_{\gamma\mu}\left(\alpha_\mu^* v_{\gamma,r}\right) v_{\mu,s} \otimes u_{rs}. \tag{I.2.7}$$

Note that $\left(\alpha_\mu^* v_{\gamma,r}\right) v_{\mu,s} \in \mathrm{Pol}(G)$ (Proposition I.2.4) is the characteristic function of the clopen set $X_{r,s} := \alpha_\mu^{-1}(G_{\gamma,r}) \cap G_{\mu,s}$. By the matched pair relations, we have

$$\forall g \in G, \qquad (\gamma\mu)\cdot g = \beta_g(\gamma\mu) = \beta_{\alpha_\mu(g)}(\gamma)\beta_g(\mu) = \big(\gamma\cdot\alpha_\mu(g)\big)(\mu\cdot g),$$

which implies that for every $t \in (\gamma\mu)\cdot G$, the clopen set $G_{\gamma\mu,t}$ is the disjoint union of clopen sets of the form $X_{r,s}$ where $r \in \gamma \cdot G$, $s \in \mu \cdot G$ and $rs = t$. Hence

$$v_{\gamma\mu,t} = \sum_{\substack{r \in \gamma\cdot G,\\ s \in \mu\cdot G,\\ rs = t}} (\alpha_\mu^* v_{\gamma,r}) v_{\mu,s}. \tag{I.2.8}$$

Combining (I.2.7) and (I.2.8) yields

$$\rho_\Gamma(\gamma)\rho_\Gamma(\mu) = \sum_{t \in (\gamma\mu)\cdot G} u_{\gamma\mu} v_{\gamma\mu,t} \otimes u_t = \rho_\Gamma(\gamma\mu).$$

It remains to show that $\rho_\Gamma(\gamma)$ is unitary, or equivalently (in view of the multiplicativity of ρ_Γ and $\rho_\Gamma(e_\Gamma) = 1 \otimes 1$), that $\rho_\Gamma(\gamma^{-1}) = [\rho_\Gamma(\gamma)]^*$. But this follows immediately from Lemma I.1.9, the definition of ρ_Γ and formula (I.2.6). ◻

The unital $*$-morphism $\Delta : \mathrm{Pol}(G) \longrightarrow \mathrm{Pol}(G) \otimes \mathrm{Pol}(G) \subseteq \mathscr{A} \otimes \mathscr{A}$ can be seen as a representation of the unital $*$-algebra $\mathrm{Pol}(G)$.

Lemma I.2.8. *The representations ρ_Γ and Δ are covariant, i.e. for all $\gamma \in \Gamma$ and $\varphi \in \mathrm{Pol}(G)$, we have*

$$\Delta(\varphi)\rho_\Gamma(\gamma) = \rho_\Gamma(\gamma)\Delta\left(\alpha_\gamma^*\varphi\right). \tag{I.2.9}$$

Proof. By definition, we have

$$\begin{aligned} \Delta(\varphi)\rho_\Gamma(\gamma) &= \sum_{r\in\gamma\cdot G} \Delta(\varphi)(u_\gamma v_{\gamma,r} \otimes u_r) \\ &= \sum_{r\in\gamma\cdot G} (u_\gamma \otimes u_r)\left[\left(\alpha_\gamma^* \otimes \alpha_r^*\right)(\Delta(\varphi))\right](v_{\gamma,r} \otimes 1), \\ \rho_\Gamma(\gamma)\Delta\left(\alpha_\gamma^*\varphi\right) &= \sum_{r\in\gamma\cdot G} (u_\gamma \otimes u_r)(v_{\gamma,r} \otimes 1)\Delta\left(\alpha_\gamma^*\varphi\right). \end{aligned}$$

Thus it suffices to show that

$$(v_{\gamma,r} \otimes 1)\Delta\left(\alpha_\gamma^*\varphi\right) = \left[\left(\alpha_\gamma^* \otimes \alpha_r^*\right)(\Delta(\varphi))\right](v_{\gamma,r} \otimes 1). \tag{I.2.10}$$

As continuous mappings from $G \times G$ into $\mathbb{C}$, both sides of (I.2.10) are supported in $G_{\gamma,r} \times G$. Moreover, for every $(g,h) \in G_{\gamma,r} \times G$, we have

$$\left[(v_{\gamma,r} \otimes 1)\Delta\left(\alpha_\gamma^*\varphi\right)\right](g,h) = \left[\Delta\left(\alpha_\gamma^*\varphi\right)\right](g,h) = \varphi\left(\alpha_\gamma(gh)\right), \tag{I.2.11}$$

and

$$\begin{aligned} &\left\{\left[\left(\alpha_\gamma^* \otimes \alpha_r^*\right)(\Delta(\varphi))\right](v_{\gamma,r} \otimes 1)\right\}(g,h) \\ &= \left[\left(\alpha_\gamma^* \otimes \alpha_r^*\right)(\Delta(\varphi))\right](g,h) = \varphi\left(\alpha_\gamma(g)\alpha_r(h)\right). \end{aligned} \tag{I.2.12}$$

Since $g \in G_{\gamma,r}$, we have $\beta_g(\gamma) = r$, and the matched pair relations yield

$$\alpha_\gamma(gh) = \alpha_\gamma(g)\alpha_r(h). \tag{I.2.13}$$

Combining (I.2.11), (I.2.12) and (I.2.13) establishes (I.2.10), hence proves (I.2.9). ◻

By Lemma I.2.7, Lemma I.2.8 and Proposition I.2.6, the linear mapping $\widetilde{\Delta} : \mathscr{A} \longrightarrow \mathscr{A} \otimes \mathscr{A}$ determined by $u_\gamma\varphi \mapsto \rho_\Gamma(\gamma)\Delta(\varphi)$ is a well-defined (recall that $\mathscr{A}$ is free $\mathrm{Pol}(G)$-module with $\{u_\gamma \ : \ \gamma \in \Gamma\}$ as a base) unital $*$-morphism.

Lemma I.2.9. *The comultiplication $\widetilde{\Delta} : \mathscr{A} \longrightarrow \mathscr{A} \otimes \mathscr{A}$ is coassociative.*

Proof. Since the comultiplication Δ of the Hopf $*$-algebraic structure on $\mathrm{Pol}(G)$ is coassociative and $\mathscr{A}$ is generated by Γ and $\mathrm{Pol}(G)$ as an algebra, it suffices to show that for every $\gamma \in \Gamma$, we have

$$(\mathrm{id} \otimes \widetilde{\Delta})\widetilde{\Delta}(u_\gamma) = (\widetilde{\Delta} \otimes \mathrm{id})\widetilde{\Delta}(u_\gamma). \tag{I.2.14}$$

On the one hand, we have

$$(\mathrm{id} \otimes \widetilde{\Delta})\widetilde{\Delta}(u_\gamma) = \sum_{r\in\gamma\cdot G} u_\gamma v_{\gamma,r} \otimes \widetilde{\Delta}(u_r) = \sum_{r,s\in\gamma\cdot G} u_\gamma v_{\gamma,r} \otimes u_r v_{r,s} \otimes u_s. \tag{I.2.15}$$

On the other hand, we have

$$
\begin{aligned}
(\widetilde{\Delta}\otimes \mathrm{id})\widetilde{\Delta}(u_\gamma) &= \sum_{s\in\gamma\cdot G}\widetilde{\Delta}(u_\gamma v_{\gamma,s})\otimes u_s = \sum_{s\in\gamma\cdot G}\rho_\Gamma(u_\gamma)\Delta(v_{\gamma,s})\otimes u_s \\
&= \sum_{\substack{r\in\gamma\cdot G,\\ s\in\gamma\cdot G}}(u_\gamma v_{\gamma,r}\otimes u_r\otimes u_s)\big(\Delta(v_{\gamma,s})\otimes 1\big).
\end{aligned} \tag{I.2.16}
$$

By comparing (I.2.15) and (I.2.16), it suffices to show that for all $r, s \in \gamma\cdot G$, we have

$$v_{\gamma,r}\otimes v_{r,s} = (v_{\gamma,r}\otimes 1)\Delta(v_{\gamma,s}) \tag{I.2.17}$$

in order to establish (I.2.14). As mappings from $G\times G$ into $\mathbb{C}$, for all $g, h\in G$, we have

$$[(v_{\gamma,r}\otimes 1)\Delta(v_{\gamma,s})](g,h) = v_{\gamma,r}(g)v_{\gamma,s}(gh), \tag{I.2.18}$$

which is 1 if $\gamma\cdot g = r$ and $\gamma\cdot(gh) = s$, and is 0 otherwise. Hence $(v_{\gamma,r}\otimes 1)\Delta(v_{\gamma,s})$ is the characteristic function of the clopen subset $G_{\gamma,r}\times G_{r,s}$ of $G\times G$, i.e. (I.2.18) holds. ◻

Let ϵ be the counit (evaluating at e_G) of the Hopf $*$-algebra $(\mathrm{Pol}(G),\Delta)$. Define the linear mapping $\widetilde{\epsilon}:\mathscr{A}\to\mathbb{C}$ to be the one uniquely determined by

$$\widetilde{\epsilon}:\mathscr{A}\to\mathbb{C},\qquad u_\gamma\varphi\mapsto \delta_{\gamma,e_\Gamma}\epsilon(\varphi) = \delta_{\gamma,e_\Gamma}\varphi(e_G). \tag{I.2.19}$$

Lemma I.2.10. *The linear mapping $\widetilde{\epsilon}$ is a counit for the comultiplication $\widetilde{\Delta}$.*

Proof. For every $\gamma\in\Gamma$, and every $\varphi\in\mathrm{Pol}(G)$, note that $e_\Gamma\cdot G$ is the singleton consisting of only e_Γ, we have

$$
\begin{aligned}
(\mathrm{id}\otimes\widetilde{\epsilon})\widetilde{\Delta}(u_\gamma\varphi) &= (\mathrm{id}\otimes\widetilde{\epsilon})\left(\sum_{r\in\gamma\cdot G}(u_\gamma v_{\gamma,r}\otimes u_r)\Delta(\varphi)\right) \\
&= \sum_{r\in\gamma\cdot G}\delta_{r,e_\Gamma}(u_\gamma v_{\gamma,r})[(\mathrm{id}\otimes\epsilon)\Delta(\varphi)] = \sum_{r\in\gamma\cdot G}\delta_{\gamma,e_\Gamma}(u_\gamma v_{\gamma,r})[(\mathrm{id}\otimes\epsilon)\Delta(\varphi)] \\
&= \delta_{\gamma,e_\Gamma}u_\gamma[(\mathrm{id}\otimes\epsilon)\Delta(\varphi)] = \delta_{\gamma,u_\gamma}u_\gamma\big(\varphi(e_G)1\big) = \widetilde{\epsilon}(u_\gamma\varphi)1.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
(\widetilde{\epsilon}\otimes\mathrm{id})\widetilde{\Delta}(u_\gamma\varphi) &= (\widetilde{\epsilon}\otimes\mathrm{id})\left(\sum_{r\in\gamma\cdot G}(u_\gamma v_{\gamma,r}\otimes u_r)\Delta(\varphi)\right) \\
&= \delta_{\gamma,e_\Gamma}(\widetilde{\epsilon}\otimes\mathrm{id})\left(\sum_{r\in\gamma\cdot G}(u_\gamma v_{\gamma,r}\otimes u_r)\Delta(\varphi)\right) \\
&= \delta_{\gamma,e_\Gamma}(\widetilde{\epsilon}\otimes\mathrm{id})\left(\sum_{r\in e_\Gamma\cdot G}(u_{e_\Gamma}v_{\gamma,r}\otimes u_r)\Delta(\varphi)\right) \\
&= \delta_{\gamma,e_\Gamma}(\widetilde{\epsilon}\otimes\mathrm{id})\left[(u_{e_\Gamma}\otimes u_{e_\Gamma})\Delta(\varphi)\right] \\
&= \delta_{\gamma,e_\Gamma}(\epsilon\otimes\mathrm{id})\Delta(\varphi) = \delta_{\gamma,e_\Gamma}\varphi(e_G)1 = \widetilde{\epsilon}(u_\gamma\varphi)1.
\end{aligned}
$$

This finishes the proof. ◻

Now we've shown that $(\mathscr{A},\widetilde{\Delta})$ is a unital and counital $*$-bialgebra with $\widetilde{\epsilon}$ as its counit, it remains to describe the antipode $\widetilde{S}:\mathscr{A}\to\mathscr{A}$ and the Haar integral $\widetilde{\tau}:\mathscr{A}\to\mathbb{C}$.

Theorem I.2.11. *The pair $\mathbb{G} = (\mathscr{A}, \widetilde{\Delta})$ is an algebraic compact quantum group of Kac type. More precisely, the following hold.*

(a) *The linear mapping*[1]

$$\widetilde{\epsilon} : \mathscr{A} \to \mathbb{C}, \qquad u_\gamma \varphi \mapsto \varphi(e_G) = \epsilon(\varphi) \tag{I.2.20}$$

is a unital $*$-morphism of $*$-algebras, where $\varphi \in \mathrm{Pol}(G)$, and ϵ is the counit for $\mathrm{Pol}(G)$.

Moreover, $\widetilde{\epsilon}$ is the counit for $\widetilde{\Delta}$.

(b) *The linear mapping*

$$\widetilde{S} : \mathscr{A} \to \mathbb{C}, \qquad u_\gamma \varphi \mapsto \sum_{r \in \gamma \cdot G} (S\varphi) v_{r,\gamma} u_{r^{-1}} \tag{I.2.21}$$

is a unital $*$-antihomomorphism of $*$-algebras, where $\varphi \in \mathrm{Pol}(G)$ and S is the antipode for $\mathrm{Pol}(G)$.

Moreover, $\widetilde{S}$ is the antipode for the $*$-bialgebra $(\mathscr{A}, \widetilde{\Delta})$.

(c) *The linear functional*

$$\widetilde{\tau} : \mathscr{A} \to \mathbb{C}, \qquad u_\gamma \varphi \mapsto \delta_{\gamma, e_\Gamma} \tau(\varphi) \tag{I.2.22}$$

is the Haar state for the Hopf $*$-algebra $(\mathscr{A}, \widetilde{\Delta})$, where $\tau : \mathrm{Pol}(G) \to \mathbb{C}$ is the Haar state on $(\mathrm{Pol}(G), \Delta)$.

Moreover, $\widetilde{\tau}$ is tracial.

Proof. (a). Since for all $\gamma, \mu \in \Gamma$, $\varphi_\gamma, \varphi_\mu, \varphi \in \mathrm{Pol}(G)$, we have

$$\begin{aligned} \epsilon(u_\gamma \varphi_\gamma u_\mu \phi_\mu) &= \epsilon\big(u_{\gamma\mu}(\alpha_\mu^* \varphi_\gamma)\varphi_\mu\big) = \varphi_\gamma\big(\alpha_\mu(e_G)\big)\varphi_\mu(e_G) \\ &= \varphi_\gamma(e_G)\varphi_\mu(e_G) = \epsilon(\varphi_\gamma)\epsilon(\phi_\mu) \end{aligned}$$

and

$$\epsilon(u_{\gamma^{-1}}) = u_\gamma, \qquad \epsilon(\overline{\varphi}) = \epsilon(\varphi) = \varphi(e_G),$$

the linear mapping $\widetilde{\epsilon}$ is indeed a unital $*$-morphism.

Since Γ and $\mathrm{Pol}(G)$ generates $\mathscr{A}$ as an algebra, to show that $\widetilde{\epsilon}$ is the counit for $\widetilde{\Delta}$, it suffices to check that the $*$-morphisms $(\mathrm{id} \otimes \widetilde{\epsilon})\widetilde{\Delta}$ and $(\widetilde{\epsilon} \otimes \mathrm{id})\widetilde{\Delta}$ both act as identity on all u_γ, $\gamma \in \Gamma$ and $\varphi \in \mathrm{Pol}(G)$. By definition, $\widetilde{\Delta}|_{\mathrm{Pol}(G)} = \Delta$ and $\widetilde{\epsilon} = \epsilon$, hence this condition on $\varphi \in \mathrm{Pol}(G)$ is automatic. As for u_γ, we calculate

$$\begin{aligned} (\mathrm{id} \otimes \widetilde{\epsilon})\widetilde{\Delta}(u_\gamma) &= \sum_{r \in \gamma \cdot G} \widetilde{\epsilon}(u_r) u_\gamma v_{\gamma,r} = u_\gamma \sum_{r \in \gamma \cdot G} v_{\gamma,r} = u_\gamma \\ &= \sum_{r \in \gamma \cdot G} \delta_{\gamma,r} u_r = \sum_{r \in \gamma \cdot G} \widetilde{\epsilon}(u_\gamma v_{\gamma,r}) u_r = (\widetilde{\epsilon} \otimes \mathrm{id})\widetilde{\Delta}(u_\gamma). \end{aligned}$$

This finishes the proof of (a).

[1] By this, we mean the unique linear mapping sending $u_\gamma \varphi$ to $\varphi(e_G)$. Note that $\mathscr{A}$ is the linear span of $\{u_\gamma \varphi \ : \ \gamma \in \Gamma,\ \varphi \in \mathrm{Pol}(G)\}$, so this makes sense. The same applies to (b) and (c) below without further remark.

(b). Since $\widetilde{S}|_{\mathrm{Pol}(G)} = S$ is a $*$-antihomomorphism, to establish that $\widetilde{S}$ is also a $*$-antihomomorphism, it suffices, by the general theory, to show that $\widetilde{S}$ is the antipode for the $*$-bialgebra $(\mathscr{A}, \widetilde{\Delta})$ and is also $*$-preserving. For all $\gamma \in \Gamma$, $\varphi \in \mathrm{Pol}(G)$, using the Sweedler notations and the Hopf $*$-algebra structure on $\mathrm{Pol}(G)$, we have that

$$\begin{aligned}
\widetilde{m}(\mathrm{id}\otimes\widetilde{S})\widetilde{\Delta}(u_\gamma\varphi) &= \widetilde{m}(\mathrm{id}\otimes\widetilde{S})\left(\sum_{r\in\gamma\cdot G}\sum u_\gamma v_{\gamma,r}\varphi_{(1)}\otimes u_r\varphi_{(2)}\right)\\
&= \sum_{r,s\in\gamma\cdot G}\sum u_\gamma v_{\gamma,r}\varphi_{(1)}S(\varphi_{(2)})v_{s,r}u_{s^{-1}}\\
&= \sum_{r,s\in\gamma\cdot G} u_\gamma v_{\gamma,r}v_{s,r}\left(\sum\varphi_{(1)}S(\varphi_{(2)})\right)u_{s^{-1}}\\
&= \sum_{r,s\in\gamma\cdot G}\delta_{s,\gamma}\epsilon(\varphi)v_{\gamma,r}u_\gamma u_{s^{-1}}\\
&= \sum_{r\in\gamma\cdot G}\epsilon(\varphi)v_{\gamma,r}u_\gamma u_{\gamma^{-1}} = \epsilon(\varphi)1 = \widetilde{\epsilon}(u_\gamma\varphi)1
\end{aligned}$$

and noticing $S(v_{\gamma,r}) = v_{r,\gamma}$, that

$$\begin{aligned}
\widetilde{m}(\widetilde{S}\otimes\mathrm{id})\widetilde{\Delta}(u_\gamma\varphi) &= \widetilde{m}(\widetilde{S}\otimes\mathrm{id})\left(\sum_{r\in\gamma\cdot G}\sum u_\gamma v_{\gamma,r}\varphi_{(1)}\otimes u_r\varphi_{(2)}\right)\\
&= \sum_{r,s\in\gamma\cdot G}\sum v_{r,\gamma}S(\varphi_{(1)})v_{s,\gamma}u_{s^{-1}}u_r\varphi_{(2)}\\
&= \sum_{r,s\in\gamma\cdot G}\delta_{r,s}v_{r,\gamma}S(\varphi_{(1)})u_{s^{-1}r}\varphi_{(2)}\\
&= \sum_{r\in\gamma\cdot G}\sum v_{r,\gamma}S(\varphi_{(1)})\varphi_{(2)} = \sum_{r\in\gamma\cdot G}\epsilon(\varphi)v_{r,\gamma}\\
&= \epsilon(\varphi)1 = \widetilde{\epsilon}(u_\gamma\varphi)1.
\end{aligned}$$

Hence $\widetilde{S}$ is indeed the antipode for $(\mathscr{A}, \widetilde{\Delta})$, and is in particular a unital antihomomorphism of algebras. That $\widetilde{S}$ is $*$-preserving follows from the fact that S is $*$-preserving on $\mathrm{Pol}(G)$, and that

$$\begin{aligned}
\left[\widetilde{S}(u_\gamma)\right]^* &= \left\{\sum_{r\in\gamma\cdot G} v_{r,\gamma}u_{r^{-1}}\right\}^* = \sum_{r\in\gamma\cdot G} u_r v_{r,\gamma} = \sum_{r\in\gamma\cdot G}(\alpha^*_{r^{-1}}v_{r,\gamma})u_r\\
&\overset{(\spadesuit)}{=} \sum_{r\in\gamma\cdot G} v_{r^{-1},\gamma^{-1}}u_r = \sum_{s\in\gamma^{-1}\cdot G} v_{s,\gamma^{-1}}u_{s^{-1}} \qquad \text{(see Lemma I.1.9 for } (\spadesuit))\\
&= \widetilde{S}(u_{\gamma^{-1}}) = \widetilde{S}(u_\gamma^*).
\end{aligned}$$

(c). By definition, $\widetilde{\tau}(1) = \tau(1_G) = 1$. To show that $\widetilde{\tau}$ is a state, it suffices to check that it is positive. Take an arbitrary element of $\mathscr{A}$, this element is a finite sum $\sum_{\gamma\in\Gamma} u_\gamma\varphi_\gamma$, where all but finitely many $\varphi_\gamma \in \mathrm{Pol}(G)$ are nonzero. Recall that as a set $\mathscr{A} = C_c(\Gamma, \mathrm{Pol}(G))$ and note that for a map $F \in C_c(\Gamma, \mathrm{Pol}(G))$, by definition, $\widetilde{\tau}(F) = \tau\big(F(e_G)\big)$. Motivating by these observations, we calculate

$$\left\{\left(\sum_{\gamma\in\Gamma}u_\gamma\varphi_\gamma\right)^*\left(\sum_{\gamma\in\Gamma}u_\gamma\varphi_\gamma\right)\right\}(e_G) = \left\{\sum_{\gamma,\mu\in\Gamma}\overline{\varphi_\gamma}u_{\gamma^{-1}\mu}\varphi_\mu\right\}(e_G) = \sum_{\gamma\in\Gamma}\overline{\varphi_\gamma}\varphi_\gamma \geq 0,$$

which implies $\widetilde{\tau}$ is positive since τ is positive.

We now show that $\widetilde{\tau}$ is an integral, i.e. it is invariant. Take any $\gamma \in \Gamma$, $\varphi \in \mathrm{Pol}(G)$, using the Sweedler's notation for $(\mathrm{Pol}(G), \Delta)$, we have

$$\begin{aligned}(\mathrm{id} \otimes \widetilde{\tau})\widetilde{\Delta}(u_\gamma \varphi) &= \sum_{r \in \gamma \cdot G} \sum \widetilde{\tau}(u_r \varphi_{(2)}) u_\gamma v_{\gamma,r} \varphi_{(1)} \\ &= \sum_{r \in \gamma \cdot G} u_\gamma v_{\gamma,r} \sum \delta_{r,e_\Gamma} \tau(\varphi_{(2)}) \varphi_{(1)} \\ &= \delta_{\gamma,e_\Gamma} \sum_{r \in \gamma \cdot G} u_\gamma v_{\gamma,r} \sum \tau(\varphi_{(2)}) \varphi_{(1)} \\ &= \delta_{\gamma,e_\Gamma} \sum_{r \in \gamma \cdot G} \tau(\varphi) v_{\gamma,r} 1 = \delta_{\gamma,e_\Gamma} \tau(\varphi) 1 = \widetilde{\tau}(u_\gamma \varphi) 1,\end{aligned}$$

and

$$\begin{aligned}(\widetilde{\tau} \otimes \mathrm{id})\widetilde{\Delta}(u_\gamma \varphi) &= \sum_{r \in \gamma \cdot G} \sum \widetilde{\tau}(u_\gamma v_{\gamma,r} \varphi_{(1)}) u_r \varphi_{(2)} \\ &= \delta_{\gamma,e_G} \sum_{r \in \gamma \cdot G} \sum \tau(v_{\gamma,r} \varphi_{(1)}) u_r \varphi_{(2)} \\ &= \delta_{\gamma,e_G} \sum \tau(v_{e_\Gamma,e_\Gamma} \varphi_{(1)}) u_{e_\Gamma} \varphi_{(2)} \\ &= \delta_{\gamma,e_G} \sum \tau(\varphi_{(1)}) \varphi_{(2)} = \delta_{\gamma,e_G} \tau(\varphi) 1 = \widetilde{\tau}(u_\gamma \varphi) 1.\end{aligned}$$

Therefore, $\widetilde{\tau}$ is indeed invariant.

Now that $(\mathscr{A}, \widetilde{\Delta})$ is an algebraic compact quantum group of Kac type (since $\widetilde{S}$ is $*$-preserving, see (b)), the Haar state $\widetilde{\tau}$ is tracial by the general theory. Alternatively, one can check directly that $\widetilde{\tau}$ is tracial by a routine calculation and Lemma I.1.6. □

Definition I.2.12. We call the algebraic compact quantum group $\mathbb{G}$ in Theorem I.2.11 the bicrossed product of the matched pair (Γ, G) of groups[2].

Using the GNS construction with respect to the Haar state $\widetilde{\tau}$, we obtain immediately the C^*-algebraic and von Neumann algebraic version of $\mathbb{G}$.

Corollary I.2.13. *(Fima, Mukherjee & Patri (Fima et al., 2017)) Using the above notations, and recall that $\widetilde{\alpha} : \Gamma \curvearrowright C(G)$ is the left action $\gamma \mapsto \alpha^*_{\gamma^{-1}}$, the reduced (resp. full) C^*-version of $\mathbb{G}$ is given by the reduced (resp. full) crossed product $\Gamma \ltimes_{\widetilde{\alpha},\mathrm{red}} C(G)$ (resp. $\Gamma \ltimes_{\widetilde{\alpha},\mathrm{full}} C(G)$) of C^*-algebras, and the von Neumann algebraic version of $\mathbb{G}$ is given by the von Neumann algebraic cross product $\Gamma \ltimes_{\widetilde{\alpha}} L(G)$, with the comultiplications in each case being the unique extension of $\widetilde{\Delta}$ by continuity (weak continuity in the von Neumann case, id. for the Haar state), and the Haar states in each case being the unique extension of $\widetilde{\tau}$ by continuity.* □

I.3 Decomposition of $\mathscr{A}$ as a comodule

We begin by recalling some generalities about algebraic compact quantum groups, their corepresentations and comodules. Let $\mathbb{H} = (\mathscr{H}, \Delta)$ be an algebraic compact quantum group with Haar state h. Then as a vector space, $\mathscr{H}$ is equipped with the inner product $\langle \cdot, \cdot \rangle_h$ induced by h, i.e. $\langle x, y \rangle_h = h(x^* y)$ for all $x, y \in \mathscr{H}$. A right comodule over the coalgebra $\mathscr{H}$ is a vector space V equipped with a linear map (called the structure map) $\delta : V \to V \otimes \mathscr{H}$ such that $(\delta \otimes \mathrm{id})\delta = (\mathrm{id} \otimes \Delta)\delta$, and

[2] Of course, $\mathbb{G}$ depends on the actions, see Definition I.1.3 and the remark on our notations after it

$(\mathrm{id} \otimes \epsilon)\delta = \mathrm{id}$, where $\epsilon : \mathscr{H} \to \mathbb{C}$ is the counit. If in addition V is also an inner-product space, then the comodule (V, δ) is called unitary when

$$\forall \xi, \eta \in V, \qquad \langle \delta(\xi) , \delta(\eta) \rangle = \langle \xi , \eta \rangle 1_{\mathscr{H}} \tag{I.3.1}$$

where the "inner product" on the left side of (I.3.1) is on the (algebraic) tensor product $V \otimes \mathscr{H}$ with values in $\mathscr{H}$ defined by $\langle a \otimes x , b \otimes y \rangle = \langle a , b \rangle x^* y$, and $1_{\mathscr{H}}$ is the multiplicative identity of the algebra $\mathscr{H}$. Given a finite dimensional right module (V, δ) of $\mathscr{H}$, there exists a unique representation $U \in \mathcal{B}(V) \otimes \mathscr{H}$ of $\mathbb{H}$ on V such that $U(\xi \otimes 1_{\mathscr{H}}) = \delta(\xi)$ for all $\xi \in V$. Conversely, if $U \in \mathcal{B}(V) \otimes \mathscr{H}$ is a representation of $\mathbb{H}$ on V, then V is a right comodule of $\mathscr{H}$ whose structure map δ is defined by the same formula. By the general theory of algebraic compact quantum groups, one has $\mathscr{H}$ is a unitary right comodule over $\mathscr{H}$ itself, where the structure map is the comultiplication Δ, and $\mathscr{H}$ admits a unique decomposition $\mathscr{H} = \oplus_{i \in I} \mathscr{H}_i$ as an orthogonal algebraic direct sum of comodules, such that each component $\mathscr{H}_i$ corresponds to a pure representation which is a direct sum of finitely many copies of some irreducible unitary representation U_i of $\mathbb{H}$, and the unitary representations corresponding to different $\mathscr{H}_i$'s are pairwise orthogonal. Moreover, each $\mathscr{H}_i$ is spanned by the matrix coefficients of U_i, hence is in fact a sub-coalgebra of $\mathscr{H}$ and the multiplicity of U_i in the representation corresponding to $\mathscr{H}_i$ is exactly $\dim U_i$. If we fix a Hilbert basis for U_i, and write the matrix coefficients of U_i as $u_{p,q}^{(i)}$, then since $\Delta(u_{p,q}^{(i)}) = \sum_r u_{p,r}^{(i)} \otimes u_{r,q}^{(j)}$, one has $\mathscr{H}_i = \oplus_p \mathscr{H}_p^{(i)}$, where each $\mathscr{H}_p^{(i)}$ is spanned by elements of the form $u_{p,q}^{(i)}$ with p fixed and q arbitrary. All of the irreducible sub-comodules $\mathscr{H}_p^{(i)}$ are equivalent, and corresponds to a copy of U_i. It follows from the orthogonality relations that every irreducible unitary representation of $\mathbb{H}$ is a copy of a unique U_i.

With the above picture of representations (which are corepresentations of the underlying Hopf algebras) and comodules of algebraic compact quantum groups in mind, together with the construction described in I.2, one can now describe all irreducible unitary representations, up to equivalence, of the bicrossed product $\mathbb{G} = (\mathscr{A}, \widetilde{\Delta})$—one simply study the (irreducible) sub-comodules of $\mathscr{A}$ generated by a single suitable element, since every irreducible unitary representation of $\mathbb{G}$ is equivalent to the representation corresponding to a simple sub-comodule of $\mathscr{A}$, and all such simple sub-comodules are generated by any of its nonzero element.

By the definition of $\widetilde{\Delta}$ and $\widetilde{\tau}$, it is obvious that one has the following orthogonal decomposition of $\mathscr{A}$ as comodules over $\mathscr{A}$:

$$\mathscr{A} = \bigoplus_{\gamma \in \Gamma} \mathscr{A}_\gamma = \bigoplus_{\gamma \in \Gamma} u_\gamma \operatorname{Pol}(G), \tag{I.3.2}$$

where $\mathscr{A}_\gamma := u_\gamma \operatorname{Pol}(G)$ is the sub-comodule $\{u_\gamma \varphi \ : \ \varphi \in \operatorname{Pol}(G)\}$. As we've pointed out, all irreducible sub-comodules are generated by any of its nonzero elements, by the co-semisimplicity of $\mathscr{A}$, one has only to describe the structure of sub-comodules of each $\mathscr{A}_\gamma$ that is generated by a single element, i.e., an element of the form $u_\gamma \varphi$, where $\varphi \in \operatorname{Pol}(G)$.

Before we proceed, we recall the notion of rank of an algebraic tensor. Let V, W be two vector spaces, recall the rank $\operatorname{rank}(t)$ of a tensor $t \in V \otimes W$ is the smallest integer $n \in \mathbb{N}$, such that there exists $v_1, \ldots, v_n \in V$ and $w_1, \ldots, w_n \in W$ satisfying

$$t = \sum_{i=1}^n v_i \otimes w_i. \tag{I.3.3}$$

It is easy to see that if one has a decomposition (I.3.3), then $\mathrm{rank}(t) = n$ if and only if both $(v_1, \dots, v_n)$ and $(w_1, \dots, w_n)$ are linearly independent.

We also recall the notations in Lemma I.2.2, namely for $\varphi \in \mathrm{Pol}(G)$, the subspace $\mathscr{L}_G(\varphi)$ (resp. $\mathscr{R}_G(\varphi)$) is the finite dimensional subspace generated by left (right) translations of φ.

Finally, we return to the general case discussed at the beginning of this section, the right sub-comodule generated by an arbitrary element $x \in \mathscr{H}$ is exactly $\mathscr{H}' \cdot x = \{(\mathrm{id} \otimes l)\Delta(x) \ : \ l \in \mathscr{H}'\}$, where $\mathscr{H}'$ is the algebraic dual of $\mathscr{H}$. In particular, if $\mathbb{H}$ is some classical compact group H viewed as an algebraic compact quantum group, i.e. $\mathscr{H} = \mathrm{Pol}(H)$, then the right sub-comodule generated by $\varphi \in \mathrm{Pol}(H)$ is exactly $\mathscr{R}_H(\varphi)$, since all linear functionals on $\mathrm{Pol}(H)$ are linear combinations of evaluations on some point of G, and $[\Delta(\varphi)](x, y) = \varphi(xy)$ for all $x, y \in H$.

Lemma I.3.1. *For all $\gamma \in \Gamma$, and all nonzero $\varphi \in \mathrm{Pol}(G)$, the sub-comodule $C_\gamma(\varphi)$ of $\mathscr{A}_\gamma$ generated by $u_\gamma\varphi$ is exactly*

$$u_\gamma \left(\bigoplus_{\mu \in \gamma \cdot G} v_{\gamma,\mu} \mathscr{R}_G(\varphi) \right) = \mathrm{Vect}\big\{ u_\gamma v_{\gamma,\mu} \varphi(\,\cdot\, g) \ : \ g \in G,\ \mu \in \gamma \cdot G \big\},$$

where the direct sum decomposition is orthogonal.

Proof. By definition, we have

$$\widetilde{\Delta}(u_\gamma \varphi) = \sum_{\mu \in \gamma \cdot G} (u_\gamma \otimes u_\mu)[(v_{\gamma,\mu} \otimes 1)\Delta(\varphi)] = \sum_{\mu \in \gamma \cdot G} \sum_{i=1}^{n} (u_\gamma v_{\gamma,\mu} \varphi_{1,i}) \otimes (u_\mu \varphi_{2,i}), \quad \text{(I.3.4)}$$

where $n = \mathrm{rank}\big(\Delta(\varphi)\big)$, and

$$\Delta(\varphi) = \sum_{i=1}^{n} \varphi_{1,i} \otimes \varphi_{2,i} \in \mathrm{Pol}(G) \otimes \mathrm{Pol}(G). \quad \text{(I.3.5)}$$

Since $\varphi_{2,i}$, $i = 1, \dots, n$ are linearly independent, so are $u_\mu \varphi_{2,i}$, $\mu \in \gamma \cdot G$, $i = 1, \dots, n$. Thus the sub-comodule generated by $u_\gamma \varphi$ is exactly the subspace spanned by elements of the form $u_\gamma v_{\gamma,\mu} \varphi_{1,i}$ with $\mu \in \gamma \cdot G$ and $i = 1, \dots, n$. On the other hand, by the discussion above the lemma, we also have

$$\mathrm{Vect}\big\{ \varphi_{1,i} \ : \ i = 1, \dots, n \big\} = \mathscr{R}_G(\varphi).$$

Hence the sub-comodule generated by $u_\gamma \varphi$ is indeed spanned by $u_\gamma \sum_{\mu \in \gamma \cdot G} v_{\gamma,\mu} \mathscr{R}_G(\varphi)$. The fact that this is an orthogonal direct sum decomposition follows by a direct calculation using the definition of the faithful Haar state $\widetilde{\tau}$ on $\mathscr{A}$. ◻

Lemma I.3.2. *Let $\gamma \in \Gamma$, $\varphi \in \mathrm{Pol}(G)$, and take any $\mu \in \gamma \cdot G$ and any $g \in G_{\gamma,\mu}$, the following hold:*

(a) *the linear mapping $R_g : \psi \mapsto \psi(\,\cdot\, g)$ is an isomorphism from $v_{\gamma,\mu} \mathscr{R}_G(\varphi)$ onto $v_{\gamma,\gamma} \mathscr{R}_G(\varphi)$;*

(b) $\dim v_{\gamma,\mu} \mathscr{R}_G(\varphi) = \dim v_{\gamma,\gamma} \mathscr{R}_G(\varphi) = \mathrm{rank}\big((v_{\gamma,\mu} \otimes 1)\Delta(\varphi)\big) = \mathrm{rank}\big((v_{\gamma,\gamma} \otimes 1)\Delta(\varphi)\big)$*;*

Proof. Since $\mathscr{R}_G(\varphi)$ is right invariant, the linear mapping R_g is well-defined (note that $g \in G_{\gamma,\mu}$ implies that $R_g(v_{\gamma,\mu}) = v_{\gamma,\gamma}$). Similarly, since $g^{-1} \in G_{\mu,\gamma}$, the linear mapping $R_{g^{-1}} : v_{\gamma,\gamma}\mathscr{R}_G(\varphi) \to v_{\gamma,\mu}\mathscr{R}_G(\varphi)$ is well-defined, and it is obviously the inverse of $R_g : v_{\gamma,\mu}\mathscr{R}_G(\varphi) \to v_{\gamma,\gamma}\mathscr{R}_G(\varphi)$. This proves (a).

We now prove (b). By $[\Delta(\varphi)](x,y) = \varphi(xy)$ for all $x, y \in G$, it is clear that $\Delta(\varphi) \in \mathscr{R}_G(\varphi)\otimes\mathscr{L}_G(\varphi)$. Since $\mathscr{L}_G(\varphi)$ is left-invariant, the mapping $L_{g^{-1}} : \mathscr{L}_G(\varphi) \to \mathscr{L}_G(\varphi)$, $\psi \mapsto \psi(g^{-1}\,\cdot\,)$ is a well-defined linear isomorphism. Hence by (a),

$$R_g \otimes L_{g^{-1}} : \big(v_{\gamma,\mu}\mathscr{R}_G(\varphi)\big) \otimes \mathscr{L}_G(\varphi) \to \big(v_{\gamma,\gamma}\mathscr{R}_G(\varphi)\big) \otimes \mathscr{L}_G(\varphi) \tag{I.3.6}$$

is a well-defined linear isomorphism. On the other hand, for all $g_1, g_2 \in G$, we have

$$\begin{aligned}&\big\{(R_g \otimes L_{g^{-1}})[(v_{\gamma,\mu} \otimes 1)\Delta(\varphi)]\big\}(g_1,g_2) = [(v_{\gamma,\mu} \otimes 1)\Delta(\varphi)](g_1g, g^{-1}g_2)\\ &= v_{\gamma,\mu}(g_1g)\varphi(g_1g_2) = v_{\gamma,\gamma}(g_1)\varphi(g_1g_2) = [(v_{\gamma,\gamma} \otimes 1)\Delta(\varphi)](g_1,g_2).\end{aligned}$$

Thus $(R_g \otimes L_{g^{-1}})[(v_{\gamma,\mu} \otimes 1)\Delta(\varphi)] = (v_{\gamma,\gamma} \otimes 1)\Delta(\varphi)$, and

$$\operatorname{rank}\big((v_{\gamma,\mu} \otimes 1)\Delta(\varphi)\big) = \operatorname{rank}\big((v_{\gamma,\gamma} \otimes 1)\Delta(\varphi)\big),$$

as it is clear that the isomorphism (I.3.6) preserves the rank of all tensors. On the other hand, it is clear from (a) that

$$\dim v_{\gamma,\mu}\mathscr{R}_G(\varphi) = \dim v_{\gamma,\gamma}\mathscr{R}_G(\varphi).$$

Put $d = \dim v_{\gamma,\gamma}\mathscr{R}_G(\varphi)$ and $d' = \operatorname{rank}\big((v_{\gamma,\gamma} \otimes 1)\Delta(\varphi)\big)$, it remains to show that $d = d'$. Let $\mathscr{C}_\gamma(\varphi)$ be the sub-comodule of $\mathscr{A}$ generated $u_\gamma\varphi$. Then by Lemma I.3.1, we have $\dim\mathscr{C}_\gamma(\varphi) = d \cdot |\gamma \cdot G|$. On the other hand, for all $r \in \gamma \cdot G$, there is a decomposition

$$(v_{\gamma,r} \otimes 1)\Delta(\varphi) = \sum_{i=1}^{d'} \varphi_{r,i} \otimes \psi_{r,i},$$

where $\varphi_{r,i} \in v_{\gamma,r}\mathscr{R}_G(\varphi)$ and $\psi_{r,i} \in \mathscr{L}_G(\varphi)$, since $\operatorname{rank}\big((v_{\gamma,r} \otimes 1)\Delta(\varphi)\big) = d'$ and $\Delta(\varphi) \in \mathscr{R}_G(\varphi) \otimes \mathscr{L}_G(\varphi)$. Hence

$$\widetilde{\Delta}(u_\gamma\varphi) = \sum_{r\in\gamma\cdot G} (u_\gamma \otimes u_r)[(v_{\gamma,r} \otimes 1)\Delta(\varphi)] = \sum_{r\in\gamma\cdot G}\sum_{i=1}^{d'} u_\gamma\varphi_{r,i} \otimes u_r\psi_{r,i}. \tag{I.3.7}$$

Since $d' = \operatorname{rank}\big((v_{\gamma,r} \otimes 1)\Delta(\varphi)\big)$, the families $(\varphi_{r,i} \ : \ i = 1,\dots,d')$ and $(\psi_{r,i} \ : \ i = 1,\dots,d')$ are both linearly independent. Note that $\varphi_{r,i}$ is supported in $G_{\gamma,r}$ and $G_{\gamma,r}$, $r \in \gamma \cdot G$ are a partition of G, it follows that the families

$$\big(u_\gamma\varphi_{r,i} \ : \ r \in \gamma \cdot G,\ i = 1,\dots,d'\big)$$

and

$$\big(u_r\psi_{r,i} \ : \ r \in \gamma \cdot G,\ i = 1,\dots,d'\big)$$

are also both linearly independent. Together with (I.3.7), this implies that

$$\big(u_\gamma\varphi_{r,i} \ : \ r \in \gamma \cdot G,\ i = 1,\dots,d'\big)$$

is a basis for $\mathscr{C}_\gamma(\varphi)$, and $d' \cdot |\gamma \cdot G| = \dim\mathscr{C}_\gamma(\varphi) = d \cdot |\gamma \cdot G|$. Hence $d' = d$. □

Theorem I.3.3. *Let $\gamma \in \Gamma$, $0 \neq \varphi \in \operatorname{Pol}(G)$. Suppose $\mathscr{C}$ is the sub-comodule of $\mathscr{A}$ generated by $u_\gamma\varphi$, and put $d = \dim v_{\gamma,\gamma}\mathscr{R}_G(\varphi)$, then the following hold:*

(a) *$\mathscr{C}$ is a finite dimensional Hilbert space as a subspace of the inner product space $\mathscr{A}$;*

(b) *there is an orthonormal basis* $\mathbf{B} := \left(u_\gamma \varphi_\mu^{(i)} \ : \ \mu \in \gamma \cdot G,\ i = 1, \ldots, d\right)$ *for the Hilbert space $\mathscr{C}$, such that for all $1 \leq i \leq d$, $\mu \in \gamma \cdot G$, we have* $\operatorname{supp} \varphi_\mu^{(i)} \subseteq G_{\gamma,\mu}$*;*

(c) $\dim_{\mathbb{C}} \mathscr{C} = d \cdot |\gamma \cdot G|$*;*

(d) *if $\left(A_{r,s}^{i,j}\right)$ is the matrix coefficients of $\mathscr{C}$ with respect to the basis $\mathbf{B}$, i.e.*

$$\widetilde{\Delta}\left(u_\gamma \varphi_s^{(j)}\right) = \sum_{i=1}^{d} \sum_{r \in \gamma \cdot G} \left(u_\gamma \varphi_r^{(i)}\right) \otimes A_{r,s}^{i,j} \tag{I.3.8}$$

for all $1 \leq j \leq d$, $s \in \gamma \cdot G$, then all $A_{r,s}^{i,j} \in \mathscr{A}$ are of the form $u_r a_{r,s}^{i,j}$ for a unique $a_{r,s}^{i,j} \in v_{r,s} \operatorname{Pol}(G)$. Moreover, define $u_{r,s} := \sum_{i,j=1}^{d} e_{i,j} \otimes a_{r,s}^{i,j} \in \mathcal{B}(\mathbb{C}^d) \otimes \operatorname{Pol}(G)$, where $(e_{i,j})$ is the matrix unit corresponding to the canonical basis of $\mathbb{C}^d$, and $r, s \in \gamma \cdot G$, then

$$u_{r,s}^* u_{r,s} = u_{r,s} u_{r,s}^* = \mathrm{id}_{\mathbb{C}^d} \otimes v_{r,s}, \tag{I.3.9}$$

and

$$(\mathrm{id} \otimes \Delta)(u_{r,s}) = \sum_{t \in \gamma \cdot G} \left(u_{r,t}\right)_{12} \left(u_{t,s}\right)_{13}. \tag{I.3.10}$$

Proof. (a) follows from Lemma I.3.1 since $\mathscr{R}_G(\varphi)$ is finite dimensional. (c) follows directly from (b), while (b) is a direct consequence of the orthogonal decomposition in Lemma I.3.1 and (b) of Lemma I.3.2.

It remains only to show (d). By Lemma I.3.2, we see that the subspace

$$\bigoplus_{\mu \in \gamma \cdot G} v_{\gamma,\mu} \mathscr{R}_G(\varphi)$$

of $\operatorname{Pol}(G)$ is right invariant, hence it is a sub-comodule of $\operatorname{Pol}(G)$ over $\operatorname{Pol}(G)$. It is clear from (b) and Lemma I.3.1 that $\mathbf{B}_\mu := \left(\varphi_\mu^{(i)} \ : \ i = 1, \ldots, d\right)$ is an orthonormal basis for $v_{\gamma,\mu} \mathscr{R}_G(\varphi)$ and the disjoint union $\cup_{\mu \in \gamma \cdot G} \mathbf{B}_\mu$ is a basis for the comodule $\oplus_{\mu \in \gamma \cdot G} v_{\gamma,\mu} \mathscr{R}_G(\mu)$. Note that the comodule $\oplus_{\mu \in \gamma \cdot G} v_{\gamma,\mu} \mathscr{R}_G(\varphi)$ is stable under multiplication by $v_{\gamma,r}$ for all $r \in \gamma \cdot G$. Since $\varphi_\mu^{(i)} \in v_{\gamma,\mu} \mathscr{R}_G(\varphi)$, we have

$$(v_{\gamma,r} \otimes 1)\Delta\left(\varphi_\mu^{(i)}\right) = (v_{\gamma,r} \otimes v_{r,\mu})\Delta\left(\varphi_\mu^{(i)}\right)$$

for all $r \in \gamma \cdot G$. Hence there exists $a_{r,\mu}^{j,i} \in v_{r,\mu} \operatorname{Pol}(G)$, $j = 1, \ldots, d$, such that

$$(v_{\gamma,r} \otimes 1)\Delta\left(\varphi_\mu^{(i)}\right) = \sum_{j=1}^{d} \varphi_r^{(j)} \otimes a_{r,\mu}^{j,i}. \tag{I.3.11}$$

By (I.3.11), we have

$$\begin{aligned}
\widetilde{\Delta}\left(u_\gamma \varphi_s^{(j)}\right) &= \sum_{r \in \gamma \cdot G} (u_\gamma \otimes u_r) \left[(v_{\gamma,r} \otimes 1)\Delta\left(\varphi_s^{(j)}\right)\right] \\
&= \sum_{r \in \gamma \cdot G} (u_\gamma \otimes u_r) \sum_{i=1}^{d} \varphi_r^{(i)} \otimes a_{r,s}^{i,j} \\
&= \sum_{r \in \gamma \cdot G} \sum_{i=1}^{d} \left(u_\gamma \varphi_r^{(i)}\right) \otimes \left(u_r a_{r,s}^{i,j}\right).
\end{aligned} \tag{I.3.12}$$

Hence $A^{i,j}_{r,s} = u_r a^{i,j}_{r,s}$. Since $A^{i,j}_{r,s}$, $r, s \in \gamma \cdot G$, $i, j = 1, \ldots, d$ are matrix coefficients of $\mathscr{C}$ with respect to the orthonormal basis $\mathbf{B}$, for all $r, s \in \gamma \cdot G$, $i, j = 1, \ldots, d$, we have

$$\sum_{t\in\gamma\cdot G} \sum_{k=1}^{d} A^{i,k}_{r,t}\left(A^{j,k}_{s,t}\right)^* = \delta_{i,j}\delta_{r,s}1 = \sum_{t\in\gamma\cdot G} \sum_{k=1}^{d} \left(A^{k,i}_{t,r}\right)^* A^{k,j}_{t,s} \tag{I.3.13}$$

and

$$\widetilde{\Delta}\left(A^{i,j}_{r,s}\right) = \sum_{t\in\gamma\cdot G} \sum_{k=1}^{d} A^{i,k}_{r,t} \otimes A^{k,j}_{t,s}. \tag{I.3.14}$$

Using $A^{i,j}_{r,s} = u_r a^{i,j}_{r,s}$, (I.3.13) becomes

$$\sum_{t\in\gamma\cdot G} \sum_{k=1}^{d} a^{i,k}_{r,t}\left(a^{j,k}_{s,t}\right)^* = \delta_{i,j}\delta_{r,s}1 = \sum_{t\in\gamma\cdot G} \sum_{k=1}^{d} \left(a^{k,i}_{t,r}\right)^* a^{k,j}_{t,s}. \tag{I.3.15}$$

Since $v_{r,t}v_{s,t} = \delta_{r,s}1$, $\sum_{t\in\gamma\dot{G}} v_{r,t} = 1$ and $a^{i,j}_{r,s}$ is supported in $G_{r,s}$ (i.e. $a^{i,j}_{r,s} = v_{r,s}a^{i,j}_{r,s}$), (I.3.15) is equivalent to

$$\forall r, s \in \gamma \cdot G,\ \forall i, j = 1, \ldots, d, \quad \sum_{k=1}^{d} a^{i,k}_{r,s}\left(a^{j,k}_{r,s}\right)^* = \delta_{i,j}v_{r,s} = \sum_{k=1}^{d} \left(a^{k,i}_{r,s}\right)^* a^{k,j}_{r,s}. \tag{I.3.16}$$

A simple calculation shows that (I.3.16) is equivalent to (I.3.9), thus the latter is established.

On the other hand,

$$\widetilde{\Delta}\left(A^{i,j}_{r,s}\right) = \widetilde{\Delta}\left(u_r a^{i,j}_{r,s}\right) = \sum_{t\in\gamma\cdot G} (u_r \otimes u_t)\left[(v_{r,t}\otimes 1)\Delta\left(a^{i,j}_{r,s}\right)\right],$$

hence (I.3.14) is equivalent to

$$\left[(v_{r,t}\otimes 1)\Delta\left(a^{i,j}_{r,s}\right)\right] = \sum_{k=1}^{d} a^{i,k}_{r,t} \otimes a^{k,j}_{t,s}. \tag{I.3.17}$$

A simple calculation shows that (I.3.10) is equivalent to

$$\Delta\left(a^{i,j}_{r,s}\right) = \sum_{t\in\gamma\cdot G} \sum_{k=1}^{d} a^{i,k}_{r,t} \otimes a^{k,j}_{t,s}. \tag{I.3.18}$$

Since $\sum_{t\in\gamma\cdot G} v_{r,t} = 1$, summing (I.3.17) over $t \in \gamma \cdot G$ yields (I.3.18). This finishes the proof of the theorem. □

Remark I.3.4. Let $\left(e_{r,s} : r, s \in \gamma \cdot G\right)$ be the matrix units of $\mathcal{B}\left(\ell^2(\gamma\cdot G)\right)$ that corresponds to the canonical orthonormal basis $(\delta_r : r \in \gamma \cdot G)$. Part (d) of Theorem I.3.3 implies that the operator

$$U := \sum_{r,s\in\gamma\cdot G} e_{r,s} \otimes u_{r,s} \in \mathcal{B}\left(\ell^2(\gamma\cdot G)\right) \otimes \mathcal{B}(\mathbb{C}^d) \otimes \mathrm{Pol}(G)$$

is a unitary representation of G on the tensor product $\ell^2(\gamma \cdot G) \otimes \mathbb{C}^d$. This leads naturally to the notion of $\mathscr{O}$-representations of G to be introduced in § I.4 (Definition I.4.1), which will play a central role in our description of the representation theory of the quantum group $\mathbb{G}$.

Theorem I.3.3, together with the correspondence between comodules over $\mathscr{A}$ as a Hopf-algebra and representations of $\mathbb{G}$, provides a useful clue of how to describe the representation theory of $\mathbb{G}$ in terms of representation theories of some clopen subgroups of G and the dynamics manifested by the actions α and β. We pursue this description in § I.4.

I.4 Representation theory of the bicrossed product

This section aims to describe the representation theory of $\mathbb{G}$ using some more basic data—representation theory of various isotropy (with respect to the action β) subgroups of G and the dynamics of the bicrossed product actions α and β. We point out here that even though § I.3 is not logically necessary to the treatment here, it does provide the motivation to study $\mathscr{O}$-representations (see Definition I.4.1 and Remark I.3.4) of G, which are crucial for our description of the representation theory of the bicrossed product $\mathbb{G}$. In fact, the proof of many results presented in this section can be greatly simplified using Theorem I.3.3, but we prefer to give an independent treatment here, so that readers in a hurry could ignore the materials in § I.3 which serves only as motivation.

First, we recall some notations. Let $\mathscr{O}$ be a β-orbit. There is a preferred Hilbert basis for the finite dimensional Hilbert space $\ell^2(\mathscr{O})$, namely the set $\{\delta_\gamma : \gamma \in \mathscr{O}\}$ of Dirac measures on $\mathscr{O}$. Let $(e_{r,s} : r, s \in \mathscr{O})$ be the matrix units of $\mathcal{B}(\ell^2(\mathscr{O}))$ with respect to this basis, i.e. $e_{r,s}(\delta_t) = \delta_{s,t}\delta_r$ for all $t \in \mathscr{O}$.

Definition I.4.1. Let $\mathscr{O}$ be a β-orbit. An $\mathscr{O}$-representation (of G) is a finite dimensional *unitary* representation U of the compact group G on the tensor product $\ell^2(\mathscr{O}) \otimes \mathscr{H}$, where $\mathscr{H}$ is a finite dimensional Hilbert space, such that if we write U uniquely as[3]

$$U = \sum_{r,s\in\mathscr{O}} e_{r,s} \otimes u_{r,s} \in \mathcal{B}(\ell^2(\mathscr{O})) \otimes \mathcal{B}(\mathscr{H}) \otimes \mathrm{Pol}(G)$$

then

$$\forall r, s \in \mathscr{O}, \qquad u_{r,s}u_{r,s}^* = u_{r,s}^*u_{r,s} = \mathrm{id}_{\mathscr{H}} \otimes v_{r,s}.$$

Using the canonical identification of $\mathcal{B}(\mathscr{H}) \otimes C(G)$ with $C(G, \mathcal{B}(\mathscr{H}))$, we can view each $u_{r,s}$ as a mapping from G to $\mathcal{B}(\mathscr{H})$ whose support is exactly $G_{r,s}$. Here are some elementary properties of $\mathscr{O}$-representations.

Proposition I.4.2. *Let $\mathscr{O}$ be a β-orbit, $\mathscr{H}$ a finite dimensional Hilbert space, and $U = \sum_{r,s\in\mathscr{O}} e_{r,s} \otimes u_{r,s}$ an $\mathscr{O}$-representation on $\ell^2(\mathscr{O}) \otimes \mathscr{H}$. The following hold:*

(a) *For all $r, s, t \in \mathscr{O}$ and $g \in G_{r,s}$, $h \in G_{s,t}$, we have $u_{r,s}(g)u_{s,t}(h) = u_{r,t}(gh)$; or equivalently,*

$$(\mathrm{id} \otimes \Delta)(u_{r,t}) = \sum_{\gamma\in\mathscr{O}} (u_{r,\gamma})_{12}(u_{\gamma,t})_{13} \in \mathcal{B}(\mathscr{H}) \otimes \mathrm{Pol}(G) \otimes \mathrm{Pol}(G). \qquad \text{(I.4.1)}$$

(b) *For every $\gamma \in \mathscr{O}$, the restriction of $u_{\gamma,\gamma}$ onto G_γ is a unitary representation of G_γ on $\mathscr{H}$.*

(c) *For all $r, s \in \mathscr{O}$ and $g \in G_{r,s}$, we have $u_{s,r}(g^{-1}) = [u_{r,s}(g)]^*$.*

[3] Here, we use the canonical identification of $\mathcal{B}(\ell^2(\mathscr{O}) \otimes \mathscr{H})$ with $\mathcal{B}(\ell^2(\mathscr{O}) \otimes \mathcal{B}(H))$.

(d) *For all $r,s\in\mathscr{O}$, if $g\in G_{r,s}$, then $\psi_g^{r,s}:G_r\to G_s$, $x\mapsto gxg^{-1}$ is an isomorphism and a homeomorphism, and the representation $u_{r,r}|_{G_r}$ is equivalent to $\left(u_{s,s}|_{G_s}\right)\circ\psi_g^{r,s}$, with $u_{s,r}(g)$ as a unitary intertwiner from $u_{r,r}|_{G_r}$ to $\left(u_{s,s}|_{G_s}\right)\circ\psi_g^{r,s}$.*

(e) *For all $r\in\mathscr{O}$, the representation U is equivalent to $\operatorname{Ind}_{G_r}^{G}\left(u_{r,r}|_{G_r}\right)$; or equivalently,*

$$\chi_U=\sum_{\gamma\in\mathscr{O}}\iota_\gamma\left(\chi_{u_{\gamma,\gamma}|_{G_\gamma}}\right)\in\operatorname{Pol}(G),\tag{I.4.2}$$

where χ_U (resp. $\chi_{u_{\gamma,\gamma}|_{G_\gamma}}$) is the character of the representation U (resp. $u_{\gamma,\gamma}|_{G_\gamma}$), and $\iota_\gamma:C(G_\gamma)\to C(G)$ is the extension by taking the value 0 outside G_γ.

Proof. (a). Canonically identifying $\mathcal{B}(\mathscr{H})\otimes C(G)\otimes C(G)$ with $C\big(G\times G,\mathcal{B}(\mathscr{H})\big)$, we have

$$[(\mathrm{id}\otimes\Delta)(u_{r,t})](g,h)=u_{r,t}(gh)\tag{I.4.3}$$

and

$$\left[\left(u_{r,\gamma}\right)_{12}\left(u_{\gamma,t}\right)_{13}\right](g,h)=u_{r,\gamma}(g)u_{\gamma,t}(h).\tag{I.4.4}$$

Since the support of $u_{r,\gamma}$ is exactly $G_{r,\gamma}$, the equivalence in (a) follows directly from (I.4.3) and (I.4.4). On the other hand, (I.4.1) holds because U is a representation of G.

(b). This follows directly from (a) by putting $r=s=t=\gamma$ and the condition that (U is an $\mathscr{O}$-representation)

$$u_{\gamma,\gamma}u_{\gamma,\gamma}^*=u_{\gamma,\gamma}^*u_{\gamma,\gamma}=\mathrm{id}_H\otimes v_{\gamma,\gamma}.$$

(c). Since $g\in G_{r,s}\iff r\cdot g=s$, we have $s\cdot g^{-1}=r$ and $g^{-1}\in G_{s,r}$. Thus by (a), we have

$$u_{r,s}(g)u_{s,r}(g^{-1})=u_{r,r}(gg^{-1})=u_{r,r}(e_G)=\mathrm{id}_{\mathscr{H}}=u_{s,s}(e_G)=u_{s,r}(g^{-1})u_{r,s}(g).$$

Hence

$$u_{s,r}(g^{-1})=\left[u_{r,s}(g)\right]^{-1}=\left[u_{r,s}(g)\right]^*.$$

(d). It is clear that $s\cdot g^{-1}=r$, and $\psi_{g^{-1}}^{s,r}:G_s\to G_r$ is the inverse of $\psi_g^{r,s}$. Since both $\psi_{g^{-1}}^{s,r}$ and $\psi_g^{r,s}$ are continuous group morphisms, $\psi_g^{r,s}$ is an isomorphism and a homeomorphism. Moreover, by (a) and (c), we have

$$\begin{aligned}\forall x\in G_r,\qquad \left[\left(u_{s,s}|_{G_s}\right)\circ\psi_g^{r,s}\right](x)&=u_{s,s}(gxg^{-1})\\&=u_{s,r}(g)u_{r,r}(x)u_{r,s}(g^{-1})\\&=u_{s,r}(g)u_{r,r}(x)\left[u_{s,r}(g)\right]^*.\end{aligned}$$

It follows that the unitary operator $u_{s,r}(g)$ is an isomorphism from the representation $u_{r,r}|_{G_r}$ to $u_{s,s}|_{G_s}\circ\psi_g^{r,s}$.

(e). It follows from the definition of U that

$$\chi_U=\sum_{\gamma\in\mathscr{O}}(\mathrm{Tr}_{\mathcal{B}(\mathscr{H})}\otimes\mathrm{id})(u_{\gamma,\gamma})=\sum_{\gamma\in\mathscr{O}}\iota_\gamma\left(\chi_{u_{\gamma,\gamma}|_{G_\gamma}}\right).$$

This proves (I.4.2). By the general theory of induced representations and (d), the character of the induced representation $\operatorname{Ind}_{G_r}^{G}\left(u_{r,r}|_{G_r}\right)$ is exactly

$$\sum_{\gamma\in\mathscr{O}}\iota_\gamma\left(\chi_{u_{\gamma,\gamma}|_{G_\gamma}}\right)=\chi_U.$$

Hence U is equivalent to $\operatorname{Ind}_{G_r}^{G}\left(u_{r,r}|_{G_r}\right)$. □

Part (e) of Proposition I.4.2 says that $\mathscr{O}$-representations are exactly copies of induced representations of finite dimensional unitary representation of the isotropy subgroup G_γ of any point $\gamma \in \mathscr{O}$ with respect to the action β. Conversely, taking any $\gamma \in \mathscr{O}$ and given a finite dimensional unitary representation u of G_γ, one can construct an $\mathscr{O}$-representation $U = \sum_{r,s\in\mathscr{O}} e_{r,s} \otimes u_{r,s}$ from u with $u = u_{\gamma,\gamma}|_{G_\gamma}$. This is shown in the following proposition.

Proposition I.4.3. *Let $\mathscr{O}$ be a β-orbit, $\gamma \in \mathscr{O}$, and $u : G_\gamma \to \mathcal{B}(\mathscr{H})$ a finite dimensional unitary representation of G_γ. Take $\sigma_\mu \in G_{\gamma,\mu}$ for each $\mu \in \mathscr{O}$ with $\sigma_\gamma = e_G$ and define*

$$u_{r,s}(g) = \begin{cases} u\left(\sigma_r g \sigma_s^{-1}\right), & \text{if } g \in G_{r,s};\\ 0, & \text{if } g \notin G_{r,s}.\end{cases}$$

Then the operator $U = \sum_{r,s\in\mathscr{O}} e_{r,s} \otimes u_{r,s}$ is an $\mathscr{O}$-representation on $\ell^2(\mathscr{O}) \otimes \mathscr{H}$.

Proof. From our construction, one checks immediately that

$$u_{r,s}^* u_{r,s} = u_{r,s} u_{r,s}^* = \mathrm{id}_{\mathscr{H}} \otimes v_{r,s} \in \mathcal{B}(\mathscr{H}) \otimes \mathrm{Pol}(G),$$

and the support of $u_{r,s}$ is exactly $G_{r,s}$ for all $r, s \in \mathscr{O}$. Using this, we have the following calculations:

$$\begin{aligned}
UU^* &= \left(\sum_{r,s\in\mathscr{O}} e_{r,s} \otimes u_{r,s}\right)\left(\sum_{r',s'\in\mathscr{O}} e_{s',r'} \otimes u_{r',s'}^*\right) = \sum_{r,r'\in\mathscr{O}} e_{r,r'} \otimes \left(\sum_{s\in\mathscr{O}} u_{r,s} u_{s,r'}^*\right)\\
&= \sum_{r,r'\in\mathscr{O}} \delta_{r,r'} e_{r,r'} \otimes \left(\sum_{s\in\mathscr{O}} u_{r,s} u_{r',s}^*\right) \qquad (\text{since } G_{r,s} \cap G_{r',s} \neq \emptyset \iff r = r')\\
&= \sum_{r\in\mathscr{O}} e_{r,r} \otimes \left(\sum_{s\in\mathscr{O}} u_{r,s} u_{r,s}^*\right) = \sum_{r\in\mathscr{O}} e_{r,r} \otimes \left(\sum_{s\in\mathscr{O}} \mathrm{id}_{\mathscr{H}} \otimes v_{r,s}\right)\\
&= \sum_{r\in\mathscr{O}} e_{r,r} \otimes \mathrm{id}_{\mathscr{H}} \otimes 1 = \mathrm{id}_{\ell^2(\mathscr{O})} \otimes \mathrm{id}_{\mathscr{H}} \otimes 1
\end{aligned}$$

and

$$\begin{aligned}
U^*U &= \left(\sum_{r,s\in\mathscr{O}} e_{s,r} \otimes u_{r,s}^*\right)\left(\sum_{r',s'\in\mathscr{O}} e_{r',s'} \otimes u_{r',s'}\right) = \sum_{s,s'\in\mathscr{O}} e_{s,s'} \otimes \left(\sum_{r\in\mathscr{O}} u_{r,s}^* u_{r,s'}\right)\\
&= \sum_{s,s'\in\mathscr{O}} \delta_{s,s'} e_{s,s'} \otimes \left(\sum_{r\in\mathscr{O}} u_{r,s}^* u_{r,s'}\right) \qquad (\text{since } G_{r,s} \cap G_{r,s'} \neq \emptyset \iff s = s')\\
&= \sum_{s\in\mathscr{O}} e_{s,s} \otimes \left(\sum_{r\in\mathscr{O}} u_{r,s}^* u_{r,s}\right) = \sum_{s\in\mathscr{O}} e_{s,s} \otimes \left(\sum_{r\in\mathscr{O}} \mathrm{id}_{\mathscr{H}} \otimes v_{r,s}\right)\\
&= \sum_{s\in\mathscr{O}} e_{s,s} \otimes \mathrm{id}_{\mathscr{H}} \otimes 1 = \mathrm{id}_{\ell^2(\mathscr{O})} \otimes \mathrm{id}_{\mathscr{H}} \otimes 1.
\end{aligned}$$

Hence U is unitary.

By construction again, for all $r, s, t \in \mathscr{O}$, $g \in G_{r,s}$ and $h \in G_{s,t}$, we have

$$\sigma_r g \sigma_s^{-1}, \sigma_s g \sigma_t^{-1} \in G_\gamma$$

and

$$\begin{aligned}
&\forall g \in G_{r,s}, h \in G_{s,t}, \\
&\qquad u_{r,t}(gh) = u(\sigma_r g h \sigma_t^{-1}) = u\big((\sigma_r g \sigma_s^{-1})(\sigma_s h \sigma_t^{-1})\big) \\
&\qquad\qquad = u(\sigma_r g \sigma_s^{-1}) u(\sigma_s h \sigma_t^{-1}) = u_{r,s}(g) u_{s,t}(h).
\end{aligned} \tag{I.4.5}$$

Using the proof of Proposition I.4.2(a), (I.4.5) implies that (in fact, is equivalent to)

$$\forall r, s \in \mathscr{O}, \qquad (\mathrm{id} \otimes \Delta)(u_{r,s}) = \sum_{\gamma \in \mathscr{O}} (u_{r,\gamma})_{12} (u_{\gamma,s})_{13}.$$

Hence U is indeed a representation of G on $\ell^2(\mathscr{O}) \otimes \mathscr{H}$. □

Since $(\mathrm{Pol}(G), \Delta)$ is a Hopf $*$-subalgebra of $(\mathscr{A}, \widetilde{\Delta})$, any representation $U \in \mathcal{B}(\mathscr{H}) \otimes \mathrm{Pol}(G)$ of G is automatically a representation of the bicrossed product $\mathbb{G}$ via the natural embedding

$$\mathcal{B}(\mathscr{H}) \otimes \mathrm{Pol}(G) \hookrightarrow \mathcal{B}(\mathscr{H}) \otimes \mathscr{A}.$$

But by the general theory, we have $\mathrm{Pol}(\mathbb{G}) = \mathscr{A}$. In order to find enough representations of $\mathbb{G}$, we need to construct representations of the bicrossed product $\mathbb{G}$ whose matrix coefficients contain $u_\gamma \in \mathscr{A}$ for all $\gamma \in \Gamma$. This can be achieved via the following lemma.

Lemma I.4.4. *Let $\mathscr{O}$ be a β-orbit, $\mathscr{H}$ a finite dimensional Hilbert space. Suppose*

$$U = \sum_{r,s \in \mathscr{O}} e_{r,s} \otimes u_{r,s}$$

is an $\mathscr{O}$-representation on $\ell^2(\mathscr{O}) \otimes \mathscr{H}$, then the operator

$$\begin{aligned}
\mathfrak{R}_{\mathscr{O}}(U) :&= \left(\sum_{\gamma \in \mathscr{O}} e_{\gamma,\gamma} \otimes \mathrm{id}_{\mathscr{H}} \otimes u_\gamma \right) U \\
&= \sum_{r,s \in \mathscr{O}} e_{r,s} \otimes [(\mathrm{id}_{\mathscr{H}} \otimes u_r) u_{r,s}] \in \mathcal{B}\big(\ell^2(\mathscr{O})\big) \otimes \mathcal{B}(\mathscr{H}) \otimes \mathscr{A}
\end{aligned} \tag{I.4.6}$$

is a unitary representation of the bicrossed product $\mathbb{G}$ on $\ell^2(\mathscr{O}) \otimes \mathscr{H}$, and the character $\chi(\mathfrak{R}_{\mathscr{O}}(U))$ of $\mathfrak{R}_{\mathscr{O}}(U)$ is given by

$$\chi(\mathfrak{R}_{\mathscr{O}}(U)) = \sum_{\gamma \in \mathscr{O}} u_\gamma \iota_\gamma \Big(\chi\big(u_{\gamma,\gamma}|_{G_\gamma}\big) \Big). \tag{I.4.7}$$

Here for a fixed $\gamma \in \mathscr{O}$, we let $\iota_\gamma : \mathrm{Pol}(G_\gamma) \to \mathrm{Pol}(G)$ denote the unique extension of functions by assigning 0 outside G_γ. Define

$$C_\gamma(U) := \left\{ \iota_\gamma(\varphi) \;:\; \begin{array}{r} \varphi \in \mathrm{Pol}(G_\gamma) \textit{ is a matrix coefficient for} \\ \textit{the representation } u_{\gamma,\gamma}|_{G_\gamma} \end{array} \right\}, \tag{I.4.8}$$

and for any $r, s \in \mathscr{O}$, take any $g \in G_{r,\gamma}$ and $h \in G_{\gamma,s}$, define

$$C_{r,s}(U) := \big\{ \psi(g \cdot h) \;:\; \psi \in C_\gamma(U) \big\}. \tag{I.4.9}$$

Then the following hold:

(a) *$C_{r,s}(U)$ does not depend on the choice of g and h, and $C_\gamma(U) = C_{\gamma,\gamma}(U)$;*

(b) *$C_{r,s}(U)$ does not depend on the choice of γ;*

(c) *the space of matrix coefficients of $\mathfrak{R}_{\mathscr{O}}(U)$ is exactly*

$$\bigoplus_{r,s\in\mathscr{O}} u_r C_{r,s}(U) \subseteq \mathscr{A}, \tag{I.4.10}$$

where $u_r C_{r,s} = \{u_r\psi \ : \ \psi \in C_{r,s}\}$.

Proof. Since both $\sum_{\gamma\in\mathscr{O}} e_{\gamma,\gamma} \otimes \mathrm{id}_{\mathscr{H}} \otimes u_\gamma$ and U are unitary operators, the operator $\mathfrak{R}_{\mathscr{O}}(U)$ is unitary. We now check that $\mathfrak{R}_{\mathscr{O}}(U)$ is a representation of $\mathbb{G}$. Since $\mathrm{id}\otimes\Delta : \mathcal{B}(\mathscr{H})\otimes\mathscr{A} \to \mathcal{B}(\mathscr{H})\otimes\mathscr{A}\otimes\mathscr{A}$ is a $*$-morphism of $*$-algebras, we have

$$\begin{aligned}
(\mathrm{id}\otimes\Delta)[(\mathrm{id}_{\mathscr{H}}\otimes u_r)(u_{r,s})] &= [(\mathrm{id}\otimes\Delta)(\mathrm{id}_{\mathscr{H}}\otimes u_r)][(\mathrm{id}\otimes\Delta)u_{r,s}] \\
&= [\mathrm{id}_{\mathscr{H}}\otimes\Delta(u_r)] \sum_{\gamma\in\mathscr{O}} (u_{r,\gamma})_{12}(u_{\gamma,s})_{13} \\
&= \left\{\sum_{t\in\mathscr{O}} \mathrm{id}_{\mathscr{H}}\otimes u_r v_{r,t}\otimes u_t\right\} \sum_{\gamma\in\mathscr{O}} (u_{r,\gamma})_{12}(u_{\gamma,s})_{13} \\
&= \sum_{t,\gamma\in\mathscr{O}} \left[(\mathrm{id}_{\mathscr{H}}\otimes u_r v_{r,t})u_{r,\gamma}\right]_{12}\left[(\mathrm{id}_{\mathscr{H}}\otimes u_t)u_{\gamma,s}\right]_{13} \\
&= \sum_{\gamma\in\mathscr{O}} \left[(\mathrm{id}_{\mathscr{H}}\otimes u_r v_{r,\gamma})u_{r,\gamma}\right]_{12}\left[(\mathrm{id}_{\mathscr{H}}\otimes u_\gamma)u_{\gamma,s}\right]_{13} \\
&\quad (\text{since } (1\otimes v_{r,t})u_{r,\gamma} = \delta_{t,\gamma}u_{r,\gamma}).
\end{aligned}$$

This implies that $\mathfrak{R}_{\mathscr{O}}(U)$ is indeed a unitary representation of $\mathbb{G}$. The statement about the character follows immediately.

We now prove the second half of the lemma. For all $r, s \in \mathscr{O}$, by definition, one checks immediately that $\varphi \in C_{r,s}(U)$ if and only if $\operatorname{supp}\varphi \subseteq G_{r,s}$, and $\varphi(g\cdot h)|_{G_\gamma}$ is a matrix coefficient of $u_{\gamma,\gamma}|_{G_\gamma}$. This implies (a) by a simple computation and the fact that $u_{\gamma,\gamma}|_{G_\gamma}$ is a representation. (b) follows from (a) and Proposition I.4.2. Finally, combining (I.4.6), (I.4.8), (I.4.9) and Proposition I.4.2 yields (c). □

Lemma I.4.5. *Let $\mathscr{O}$ be a β-orbit, $\mathscr{H}$, $\mathscr{K}$ finite dimensional Hilbert spaces, and suppose $U = \sum_{r,s\in\mathscr{O}} e_{r,s}\otimes u_{r,s}$ and $V = \sum_{r,s\in\mathscr{O}} e_{r,s}\otimes v_{r,s}$ are $\mathscr{O}$-representations on $\ell^2(\mathscr{O})\otimes\mathscr{H}$ and $\ell^2(\mathscr{O})\otimes\mathscr{K}$ respectively. If $T\in\mathcal{B}(\mathscr{H},\mathscr{K})$, then the following are equivalent:*

(a) *$T \in \mathrm{Mor}_{G_\gamma}\left(u_{\gamma,\gamma}|_{G_\gamma}, v_{\gamma,\gamma}|_{G_\gamma}\right)$ for some $\gamma\in\mathscr{O}$;*

(b) *$T \in \mathrm{Mor}_{G_\gamma}\left(u_{\gamma,\gamma}|_{G_\gamma}, v_{\gamma,\gamma}|_{G_\gamma}\right)$ for all $\gamma\in\mathscr{O}$.*

In particular, $u_{\gamma,\gamma}|_{G_\gamma} \simeq v_{\gamma,\gamma}|_{G_\gamma}$ for some $\gamma\in\mathscr{O}$ if and only if $u_{\gamma,\gamma}|_{G_\gamma}\simeq v_{\gamma,\gamma}|_{G_\gamma}$ for all $\gamma\in\mathscr{O}$.

Proof. This is a direct consequence of the part (d) of Proposition I.4.2. □

Definition I.4.6. Let $\mathscr{O}$ be a β-orbit, an $\mathscr{O}$-representation $U = \sum_{r,s\in\mathscr{O}} e_{r,s}\otimes u_{r,s}$ is said to be $\mathscr{O}$-irreducible if any[4] of the representations $u_{\gamma,\gamma}|_{G_\gamma}$, $\gamma\in\mathscr{O}$ is irreducible.

Two $\mathscr{O}$-representations U and V are said to be $\mathscr{O}$-equivalent, denoted by $U\sim_{\mathscr{O}} V$, if there is a bijective (equivalently, unitary) T satisfying the equivalent conditions in Lemma I.4.5.

[4]Hence all by the point (d) of Proposition I.4.2

Notations I.4.7. *It is clear that all $\mathscr{O}$-representations that are $\mathscr{O}$-equivalent to an $\mathscr{O}$-irreducible one remain $\mathscr{O}$-irreducible. We denote the set of $\mathscr{O}$-equivalence classes of $\mathscr{O}$-irreducible $\mathscr{O}$-representations by $\mathrm{Irr}_{\mathscr{O}}(G)$. By Proposition I.4.2 and Proposition I.4.3, we see that for all $\gamma \in \mathscr{O}$, there is a canonical bijection*

$$\begin{aligned} \Phi_\gamma : \mathrm{Irr}(G_\gamma) &\longrightarrow \mathrm{Irr}_{\mathscr{O}}(G) \\ [u] &\mapsto \left[\sum_{r,s \in \mathscr{O}} e_{r,s} \otimes u_{r,s} \right], \end{aligned} \tag{I.4.11}$$

where $u_{r,s}$, $r, s \in \mathscr{O}$ are defined as in Proposition I.4.3. In particular, Φ_{e_Γ} is a bijection from $\mathrm{Irr}(G)$ onto $\mathrm{Irr}_{\{e_\Gamma\}}(G)$, which we will use later (§ I.6) to identify $\mathrm{Irr}(G)$ with $\mathrm{Irr}_{\{e_\Gamma\}}(G)$.

Remark I.4.8. Intuitively speaking, the bijection Φ_γ can be seen as a parameterization of $\mathrm{Irr}_{\mathscr{O}}(G)$ with the relatively more concrete data $\mathrm{Irr}(G_\gamma)$.

We can finally state and prove the classification of irreducible representations of $\mathbb{G}$.

Theorem I.4.9. *Using the above notations, the following hold:*

(a) *Let $\mathscr{O}$ be a β-orbit, U an $\mathscr{O}$-representation, then the representation $\mathfrak{R}_{\mathscr{O}}(U)$ of the bicrossed product $\mathbb{G}$ is irreducible if and only if U is $\mathscr{O}$-irreducible.*

(b) *We have the following decomposition of $\mathscr{A}$ as a vector space*

$$\mathscr{A} = \bigoplus_{\mathscr{O} \in \mathrm{Orb}_\beta} \bigoplus_{\substack{r,s \in O, \\ [U] \in \mathrm{Irr}_{\mathscr{O}(G)}}} u_r C_{r,s}(U). \tag{I.4.12}$$

(c) *(Classification of irreducible representations of $\mathbb{G}$) The mapping*

$$\begin{aligned} \mathfrak{R} : \coprod_{\mathscr{O} \in \mathrm{Orb}_\beta} \mathrm{Irr}_{\mathscr{O}}(G) &\to \mathrm{Irr}(\mathbb{G}) \\ [U] \in \mathrm{Irr}_{\mathscr{O}}(G) &\mapsto [\mathfrak{R}_{\mathscr{O}}(U)] \end{aligned}$$

is a well-defined bijection.

Proof. (a). Recall that the functional $\widetilde{\tau}$ in Theorem I.2.11 is the Haar state on $\mathbb{G}$. By the characteristic formula (I.4.2) in Proposition I.4.2, fixing an arbitrary $\gamma \in \mathscr{O}$, we have

$$\begin{aligned} \dim \mathrm{End}_{\mathbb{G}} \left(\mathfrak{R}_{\mathscr{O}}(U)\right) &= \widetilde{\tau}(\chi_U{}^* \chi_U) \\ &= \sum_{r \in \mathscr{O}} \tau \left(\iota_r \left(\chi_{u_{r,r}|_{G_r}}{}^* \chi_{u_{r,r}|_{G_r}} \right)\right) \\ &\quad \left(\text{only } (\chi_U{}^* \chi_U)(e) \text{ matters}\right) \\ &= |\mathscr{O}| \tau \left(\iota_\gamma \left(\chi_{u_{\gamma,\gamma}|_{G_\gamma}}{}^* \chi_{u_{\gamma,\gamma}|_{G_\gamma}} \right)\right) \\ &\quad \left(\text{by Proposition I.4.2, (d) and } \tau \text{ is a Haar state}\right) \\ &= \frac{|\mathscr{O}|}{[G : G_\gamma]} \dim \mathrm{End}_{G_\gamma} \left(u_{\gamma,\gamma}|_{G_\gamma} \right) = \dim \mathrm{End}_{G_\gamma} \left(u_{\gamma,\gamma}|_{G_\gamma} \right). \end{aligned}$$

This proves (a).

(b). It is clear by Lemma I.4.4 that the sum on the right side of (I.4.12) is a direct sum. Moreover, by Lemma I.2.3, Lemma I.4.5, the orthogonality relations for representations, and Lemma I.4.4 again, we have

$$\bigoplus_{[U]\in \mathrm{Irr}_{\mathscr{O}}(G)} C_{r,s}(U) = v_{r,s}\,\mathrm{Pol}(G). \tag{I.4.13}$$

Since $\mathscr{A} = \bigoplus_{r\in\Gamma} u_r \,\mathrm{Pol}(G)$, equation (I.4.13) implies the decomposition (I.4.12), hence proves (b).

(c) follows from (a) and (b). ◻

Remark I.4.10. One can even show that the direct sum decomposition (I.4.12) is orthogonal with respect to the inner product on $\mathscr{A}$ induced by the faithful Haar state $\widetilde{\tau}$ with a bit more calculation, but this fact is not needed and we leave it to the reader.

For the purpose of studying property (RD) of $\widehat{\mathbb{G}}$, we also need to understand how the conjugate representation of the irreducible representations of $\mathbb{G}$ is expressed using the above classification, as well as the fusion rules of $\mathbb{G}$. The problem of identifying the conjugate operation on $\mathrm{Irr}(\mathbb{G})$ using our classification result is almost trivial (Theorem I.4.13), while the fusion rules of $\mathbb{G}$ requires some further work (twisted tensor products).

Since $\mathbb{G}$ is of Kac type (Theorem I.2.11), the conjugate and the contragredient representations of any unitary representation of $\mathbb{G}$ coincide. For all finite dimensional Hilbert space $\mathscr{H}$, let $j_{\mathscr{H}} : \mathcal{B}(\mathscr{H}) \to \mathcal{B}(\overline{\mathscr{H}})$ be the $*$-antihomomorphism $T \mapsto \overline{T^*}$, where $\overline{T^*}$ is the mapping $\overline{x} \mapsto \overline{T^*x}$. We often omit the subscript $\mathscr{H}$ and write $j_{\mathscr{H}}$ simply as j when there is no risk of confusion.

Lemma I.4.11. *Let $\mathscr{O}$ be a β-orbit, $\mathscr{H}$ a finite dimensional Hilbert space. Suppose*

$$U = \sum_{r,s\in\mathscr{O}} e_{r,s} \otimes u_{r,s}$$

is an $\mathscr{O}$-representation on $\ell^2(\mathscr{O}) \otimes \mathscr{H}$. Let $\overline{e_{r,s}}$ denote the unitary operator $\overline{\ell^2(\mathscr{O})} \to \overline{\ell^2(\mathscr{O})}$ sending $\overline{\eta}$ to $\overline{e_{r,s}\eta}$. Then the conjugate representation $\overline{\mathfrak{R}_{\mathscr{O}}(U)}$ of $\mathfrak{R}_{\mathscr{O}}(U)$ is given by

$$\overline{\mathfrak{R}_{\mathscr{O}}(U)} = \sum_{r,s\in\mathscr{O}} \overline{e_{s,r}} \otimes \left\{(\mathrm{id}_{\mathscr{H}} \otimes u_{s^{-1}})\left((\mathrm{id} \otimes \alpha^*_{s^{-1}})\left[(j\otimes S)u_{r,s}\right]\right)\right\}. \tag{I.4.14}$$

Furthermore, posing

$$w_{s^{-1},r^{-1}} := (\mathrm{id} \otimes \alpha^*_{s^{-1}})\left[(j\otimes S)u_{r,s}\right] \in \mathcal{B}(\overline{\mathscr{H}}) \otimes \mathrm{Pol}(G), \tag{I.4.15}$$

the unitary operator

$$\begin{aligned} W := \sum_{r,s\in\mathscr{O}} e_{s^{-1},r^{-1}} \otimes w_{s^{-1},r^{-1}} &= \sum_{s^{-1},r^{-1}\in\mathscr{O}} e_{s^{-1},r^{-1}} \otimes w_{s^{-1},r^{-1}} \\ &\in \mathcal{B}\big(\ell^2(\mathscr{O}^{-1})\big) \otimes \mathcal{B}(\mathscr{H}) \otimes \mathrm{Pol}(G) \end{aligned} \tag{I.4.16}$$

is an $\mathscr{O}^{-1}$-representation, and $\overline{\mathfrak{R}_{\mathscr{O}}(U)} \simeq \mathfrak{R}_{\mathscr{O}^{-1}}(W)$.

Proof. Since $\mathbb{G}$ is of Kac type, we have

$$\begin{aligned}
&\overline{\mathfrak{R}_{\mathcal{O}}(U)} = (j \otimes j \otimes \widetilde{S})\left(\sum_{r,s\in\mathcal{O}} e_{r,s} \otimes (1 \otimes u_r)u_{r,s}\right) \\
&= \sum_{r,s\in\mathcal{O}} \overline{e_{s,r}} \otimes \sum_{t\in\mathcal{O}} [(j \otimes S)u_{r,s}]\left(\mathrm{id}_{\mathcal{H}} \otimes v_{t,r}u_{t^{-1}}\right) \\
&= \sum_{r,s\in\mathcal{O}} \overline{e_{s,r}} \otimes \sum_{t\in\mathcal{O}} \delta_{s,t}[(j \otimes S)u_{r,s}]\left(\mathrm{id}_{\mathcal{H}} \otimes v_{t,r}u_{t^{-1}}\right) \\
&\quad \left(\text{since the support of } (j \otimes S)u_{r,s} \text{ is exactly } G_{r,s}^{-1} = G_{s,r}, \text{ and supp}\, v_{t,r} = G_{t,r}\right) \\
&= \sum_{r,s\in\mathcal{O}} \overline{e_{s,r}} \otimes [(j \otimes S)u_{r,s}](\mathrm{id}_{\mathcal{H}} \otimes u_{s^{-1}}) \\
&= \sum_{r,s\in\mathcal{O}} \overline{e_{s,r}} \otimes \left\{(\mathrm{id}_{\mathcal{H}} \otimes u_{s^{-1}})\left(\alpha^*_{s^{-1}}\left[(j \otimes S)u_{r,s}\right]\right)\right\} \\
&= \sum_{r,s\in\mathcal{O}} \overline{e_{s,r}} \otimes w_{s^{-1},r^{-1}}.
\end{aligned}$$

Since $\overline{\mathfrak{R}_{\mathcal{O}}(U)}$ is a unitary representation of $\mathbb{G}$, by the above calculation, and the fact that $\left(\overline{e_{s,r}} : r, s \in \mathcal{O}\right)$ is the matrix unit corresponding to the dual basis

$$\left(\overline{\delta_\gamma} : \gamma \in \mathcal{O}\right)$$

of the canonical basis $(\delta_\gamma : \gamma \in \mathcal{O})$ of $\ell^2(\mathcal{O})$, we deduce that the operator V is unitary, and for all $r, s \in \mathcal{O}$, we have

$$(\mathrm{id} \otimes \Delta)(v_{s^{-1},r^{-1}}) = \sum_{t\in\mathcal{O}} \left(w_{s^{-1},t^{-1}}\right)_{12}\left(w_{t^{-1},r^{-1}}\right)_{13}. \tag{I.4.17}$$

It is trivial to check that

$$w_{s^{-1},r^{-1}} w^*_{s^{-1},r^{-1}} = w^*_{s^{-1},r^{-1}} w_{s^{-1},r^{-1}} = \mathrm{id}_{\overline{\mathcal{H}}} \otimes v_{s^{-1},r^{-1}},$$

which together with (I.4.17), implies that W is indeed an $\mathcal{O}^{-1}$-representation.

Finally, the fact that $\overline{\mathfrak{R}_{\mathcal{O}}(U)} \simeq \mathfrak{R}_{\mathcal{O}^{-1}}(W)$ can be seen by directly comparing their characters, both of which are given by

$$\sum_{\gamma\in\mathcal{O}} u_{\gamma^{-1}}(\mathrm{Tr}_{\mathcal{B}(\overline{\mathcal{H}})} \otimes \mathrm{id})(w_{\gamma^{-1},\gamma^{-1}}).$$

Alternatively, one can see this equivalence more concretely by noting that the unitary operator $\Upsilon \otimes \mathrm{id}_{\overline{\mathcal{H}}}$ from $\overline{\ell^2(\mathcal{O})} \otimes \overline{\mathcal{H}}$ onto $\ell^2(\mathcal{O}^{-1}) \otimes \overline{\mathcal{H}}$ is an isomorphism of representations from $\overline{\mathfrak{R}_{\mathcal{O}}(U)}$ to $\mathfrak{R}_{\mathcal{O}^{-1}}(W)$, where $\Upsilon : \overline{\ell^2(\mathcal{O})} \to \ell^2(\mathcal{O})$ is the unique linear(unitary) operator determined by $\overline{\delta_\gamma} \mapsto \delta_{\gamma^{-1}}$ for all $\gamma \in \mathcal{O}$. □

Definition I.4.12. Using the notations in Lemma I.4.11, the $\mathcal{O}^{-1}$-representation W is called *the orbital conjugate* of the $\mathcal{O}$-representation U, and will be denoted by $U^\dagger$ in the following.

Using (I.4.15) and (I.4.16) (and the fact that the antipode $S : \mathrm{Pol}(G) \to \mathrm{Pol}(G)$ is of order 2), it is easy to see that $U^{\dagger\dagger} = U$, i.e. $(\cdot)^\dagger$ is an involution on the class of all $\mathcal{O}$-representations, where $\mathcal{O}$ runs through all β-orbits, and passes to a well-defined involution, still denoted by $\dagger$, on the set $\coprod_{\mathcal{O}\in\mathrm{Orb}_\beta} \mathrm{Irr}_{\mathcal{O}}(G)$ with $[U]^\dagger := [U^\dagger]$.

It is clear that † restricts to a bijection between $\mathrm{Irr}_{\mathscr{O}}(G)$ and $\mathrm{Irr}_{\mathscr{O}^{-1}}(G)$, and when $\mathscr{O} = \{e_\Gamma\}$, so $\mathscr{O} = \mathscr{O}^{-1}$, it reduces to the conjugate operation on the class of finite dimensional unitary representations $\mathrm{Rep}(G)$ of G, modulo the obvious identifications of course. Now the conjugate representation in terms of our classification (Theorem I.4.9) can be neatly summarized as the following theorem.

Theorem I.4.13. *The classification mapping*

$$\mathfrak{R} : \coprod_{\mathscr{O} \in \mathrm{Orb}_\beta} \mathrm{Irr}_{\mathscr{O}}(G) \to \mathrm{Irr}(\mathbb{G})$$

preserves involution.

Proof. This is merely a restatement of Lemma I.4.11 using Definition I.4.12. □

The following proposition relates the orbital conjugation presented above to the parallel treatment in (Fima and Wang, 2018, Theorem 3.1 (4)).

Proposition I.4.14. *Let $\gamma \in \Gamma$ and $\mathscr{O} = \gamma \cdot G$. If $u : G_\gamma \to \mathcal{B}(\mathscr{H})$ is a finite dimensional unitary representation of G_γ and U is the $\mathscr{O}$-representation determined by u as in Proposition I.4.3, then $U^\dagger \simeq \mathrm{Ind}(u \circ \alpha_{\gamma^{-1}}|_{G_{\gamma^{-1}}})$.*

Proof. This follows from Lemma I.1.8, Proposition I.4.2, and Lemma I.4.11. □

We now turn our attention to the fusion rules of $\mathbb{G}$. For $i = 1, 2, 3$, let $\mathscr{O}_i$ be an β-orbit, $U_i = \sum_{r,s \in \mathscr{O}_i} e_{r,s} \otimes u^{(i)}_{r,s}$ an $\mathscr{O}_i$-representation on $\ell^2(\mathscr{O}_i) \otimes \mathscr{H}_i$. To simplify the notations of our discussion, we denote the representation $\mathfrak{R}_{\mathscr{O}_i}(U_i)$ of $\mathbb{G}$ by W_i, and its character by χ_{W_i}. By Lemma I.1.10, we know that $\mathscr{O}_1\mathscr{O}_2$ is the disjoint union of β-orbits. For each $\gamma \in \mathscr{O}_3$, we define $\mathscr{K}^\gamma_{\mathscr{O}_1,\mathscr{O}_2}$ to be the subspace of $\ell^2(\mathscr{O}_1) \otimes \ell^2(\mathscr{O}_2)$ spanned by

$$\mathbf{B}^\gamma_{\mathscr{O}_1,\mathscr{O}_2} := \left\{\delta_{\gamma_1} \otimes \delta_{\gamma_2} \ : \ (\gamma_1, \gamma_2) \in \mathscr{O}_1 \times \mathscr{O}_2 \text{ and } \gamma_1\gamma_2 = \gamma\right\}. \tag{I.4.18}$$

If $\gamma \notin \mathscr{O}_1\mathscr{O}_2$ (which is equivalent to $\mathscr{O}_3 \cap \mathscr{O}_1\mathscr{O}_2 = \emptyset$), then $\mathbf{B}^\gamma_{\mathscr{O}_1,\mathscr{O}_2} = \emptyset$ and $K^\gamma_{\mathscr{O}_1,\mathscr{O}_2} = 0$.

Theorem I.4.15. *Using the above notations, and posing*

$$\mathsf{F}^\gamma_{\mathscr{O}_1,\mathscr{O}_2} := \{(\gamma_1, \gamma_2) \in \mathscr{O}_1 \times \mathscr{O}_2 \ : \ \gamma_1\gamma_2 = \gamma\} \tag{I.4.19}$$

for all $\gamma \in \Gamma$, then the following hold:

(a) *The mapping*

$$\begin{aligned} U_1 \times_\gamma U_2 : G_\gamma &\to \mathcal{B}\left(\mathscr{K}^\gamma_{\mathscr{O}_1,\mathscr{O}_2}\right) \otimes \mathcal{B}(\mathscr{H}_1) \otimes \mathcal{B}(\mathscr{H}_2) \\ g &\mapsto \sum_{(r_1,r_2),(s_1,s_2) \in \mathsf{F}^\gamma_{\mathscr{O}_1,\mathscr{O}_2}} \left(e_{r_1,s_1} \otimes e_{r_2,s_2}\right) \\ &\qquad\qquad \otimes u^{(1)}_{r_1,s_1}\left(\alpha_{r_2}(g)\right) \otimes u^{(2)}_{r_2,s_2}(g) \end{aligned} \tag{I.4.20}$$

is a unitary representation of G_γ.

(b) *The character of* $U_1 \times_\gamma U_2$ *is*

$$\begin{aligned}&\chi\left(U_1 \times_\gamma U_2\right)\\ &= \sum_{(r_1,r_2)\in \mathsf{F}^{\gamma}_{\mathcal{O}_1,\mathcal{O}_2}} \left\{\left[\iota_{r_1}\left(\chi\left(u^{(1)}_{r_1,r_1}|_{G_{r_1}}\right)\right)\circ \alpha_{r_2}\right]\left[\iota_{r_2}\left(\chi\left(u^{(2)}_{r_2,r_2}|_{G_{r_2}}\right)\right)\right]\right\}_{|G_\gamma}.\end{aligned} \tag{I.4.21}$$

Or equivalently, for all $g \in G_\gamma$, *we have*

$$\begin{aligned}&\left[\chi\left(U_1 \times_\gamma U_2\right)\right](g)\\ &= \sum_{(r_1,r_2)\in \mathsf{F}^{\gamma}_{\mathcal{O}_1,\mathcal{O}_2}} \left[\mathrm{Tr}_{\mathcal{B}(\mathscr{H}_1)}\left(u^{(1)}_{r_1,r_1}\left(\alpha_{r_2}(g)\right)\right)\right]\left[\mathrm{Tr}_{\mathcal{B}(\mathscr{H}_2)}\left(u^{(2)}_{r_2,r_2}(g)\right)\right].\end{aligned} \tag{I.4.22}$$

(c) *We have*

$$\begin{aligned}&\dim \mathrm{Mor}_{\mathbb{G}}(W_3, W_1 \times W_2)\\ &= \frac{1}{|\mathcal{O}_3|}\sum_{\gamma_3\in\mathcal{O}_3} \dim \mathrm{Mor}_{G_{\gamma_3}}\left(u^{(3)}_{\gamma_3,\gamma_3}|_{G_{\gamma_3}}, U_1 \times_{\gamma_3} U_2\right).\end{aligned} \tag{I.4.23}$$

In particular, if $\mathcal{O}_3 \cap \mathcal{O}_1\mathcal{O}_2 = \emptyset$, *then* $\dim \mathrm{Mor}_{\mathbb{G}}(W_3, W_1 \times W_2) = 0$.

Proof. (a). It is clear that $\left(U_1 \times_\gamma U_2\right)(e_G) = \mathrm{id}_{\mathscr{K}^{\gamma}_{\mathcal{O}_1,\mathcal{O}_2}} \otimes \mathrm{id}_{\mathscr{H}_1} \otimes \mathrm{id}_{\mathscr{H}_2}$. For all $g, h \in G_\gamma$, we have

$$\begin{aligned}&\left(U_1 \times_\gamma U_2\right)(g)\left(U_1 \times_\gamma U_2\right)(h)\\ &= \sum_{(r_1,r_2),(s_1,s_2),(t_1,t_2)\in \mathsf{F}^{\gamma}_{\mathcal{O}_1,\mathcal{O}_2}} e_{r_1,t_1}\otimes e_{r_2,t_2}\\ &\qquad \otimes \left[u^{(1)}_{r_1,s_1}\left(\alpha_{r_2}(g)\right)u^{(1)}_{s_1,t_1}\left(\alpha_{s_2}(h)\right)\right]\otimes\left[u^{(2)}_{r_2,s_2}(g)u^{(2)}_{s_2,t_2}(h)\right]\\ &= \sum_{\substack{(r_1,r_2),(s_1,s_2),(t_1,t_2)\in \mathsf{F}^{\gamma}_{\mathcal{O}_1,\mathcal{O}_2},\\ r_1\cdot\alpha_{r_2}(g)=s_1,\ s_1.\alpha_{s_2}(h)=t_1,\\ r_2\cdot g=s_2,\ s_2\cdot h=t_2}} e_{r_1,t_1}\otimes e_{r_2,t_2}\\ &\qquad \otimes \left[u^{(1)}_{r_1,s_1}\left(\alpha_{r_2}(g)\right)u^{(1)}_{s_1,t_1}\left(\alpha_{s_2}(h)\right)\right]\otimes\left[u^{(2)}_{r_2,s_2}(g)u^{(2)}_{s_2,t_2}(h)\right]\\ &\quad \left(\text{consider the support of the components } u^{(1)}_{r_1,s_1}, u^{(1)}_{s_1,t_1}, u^{(2)}_{r_2,s_2} \text{ and } u^{(2)}_{s_2,t_2}\right)\\ &= \sum_{\substack{(r_1,r_2),(s_1,s_2),(t_1,t_2)\in \mathsf{F}^{\gamma}_{\mathcal{O}_1,\mathcal{O}_2},\\ r_1\cdot\alpha_{r_2}(g)=s_1,\ s_1.\alpha_{s_2}(h)=t_1,\\ r_2\cdot g=s_2,\ s_2\cdot h=t_2}} e_{r_1,t_1}\otimes e_{r_2,t_2}\otimes\left[u^{(1)}_{r_1,t_1}\left(\alpha_{r_2}(g)\alpha_{s_2}(h)\right)\right]\otimes\left[u^{(2)}_{r_2,t_2}(gh)\right]\\ &\quad (\text{Proposition I.4.2})\\ &= \sum_{(r_1,r_2),(t_1,t_2)\in \mathsf{F}^{\gamma}_{\mathcal{O}_1,\mathcal{O}_2}} e_{r_1,t_1}\otimes e_{r_2,t_2}\otimes u^{(1)}_{r_1,t_1}\left(\alpha_{r_2}(gh)\right)\otimes u^{(2)}_{r_2,t_2}(gh)\\ &\quad \left(\text{since } \alpha_{r_2}(gh) = \alpha_{r_2}(g)\alpha_{r_2\cdot g}(h)\right)\\ &= \left(U_1 \times_\gamma U_2\right)(gh).\end{aligned}$$

Hence $U_1 \times_\gamma U_2$ is indeed a representation of G_γ. The fact that this representation is unitary follows from (I.4.20) and the conditions that

$$\left(u^{(1)}_{r_1,s_1}\right)\left(u^{(1)}_{r_1,s_1}\right)^* = \left(u^{(1)}_{r_1,s_1}\right)^*\left(u^{(1)}_{r_1,s_1}\right) = \mathrm{id}_{\mathscr{H}_1}\otimes v_{r_1,s_1}$$

and that

$$\left(u^{(2)}_{r_2,s_2}\right)\left(u^{(2)}_{r_2,s_2}\right)^* = \left(u^{(2)}_{r_2,s_2}\right)^*\left(u^{(2)}_{r_2,s_2}\right) = \mathrm{id}_{\mathscr{H}_2} \otimes v_{r_2,s_2}.$$

(b). This follows directly from (I.4.20).

(c). Using the character formula (I.4.7), we have

$$\begin{aligned}\dim \mathrm{Mor}_{\mathbb{G}}(W_3, W_1 \times W_2) &= \widetilde{\tau}\left(\chi^*_{W_3}\chi_{W_1}\chi_{W_2}\right)\\ &= \tau\left(\sum_{\gamma_3\in\mathscr{O}_3}\left[\iota_{\gamma_3}\left(\chi(u^{(3)}_{\gamma_3,\gamma_3}|_{G_{\gamma_3}})\right)\right]^*\chi(U_1 \times_{\gamma_3} U_2)\right)\\ &= \sum_{\gamma_3\in\mathscr{O}_3}\frac{1}{[G:G_{\gamma_3}]}\dim\mathrm{Mor}_{G_{\gamma_3}}\left(u^{(3)}_{\gamma_3,\gamma_3}|_{G_{\gamma_3}}, U_1\times_\gamma U_2\right)\\ &= \frac{1}{|\mathscr{O}_3|}\sum_{\gamma_3\in\mathscr{O}_3}\dim\mathrm{Mor}_{G_{\gamma_3}}\left(u^{(3)}_{\gamma_3,\gamma_3}|_{G_{\gamma_3}}, U_1\times_\gamma U_2\right).\end{aligned}$$

The case when $\mathscr{O}_3 \cap \mathscr{O}_1\mathscr{O}_2 = \emptyset$ is already covered in the above formula, as $U_1 \times_{\gamma_3} U_2$ is the zero representation of G_{γ_3} in this case. □

Definition I.4.16. Using the above notations, we call the unitary representation $U_1 \times_\gamma U_2$ of G_γ the *γ-twisted tensor product of U_1 and U_2*.

Theorem I.4.15 (which is a reformulation of Theorem 3.2 of (Fima and Wang, 2018)), together with Theorem I.4.13, gives the fusion rules of $\mathbb{G}$. In preparing this thesis, the author finds that the formula (I.4.23) for calculating the fusion rules can in fact be simplified (Theorem I.4.19). We present this simplification in the rest of this section.

For all $\gamma \in \mathscr{O}_3$, we define

$$\mathsf{F}^{\gamma}_{\mathscr{O}_1,\mathscr{O}_2} := \{(r_1, r_2) \in \mathscr{O}_1 \times \mathscr{O}_2 \ : \ r_1r_2 = \gamma\}. \tag{I.4.24}$$

Lemma I.4.17. *Using the above notations, suppose $\gamma, \gamma' \in \mathscr{O}_3$, and $g \in G_{\gamma,\gamma'}$, then*

$$\begin{aligned}\Phi_{\gamma,g} : \mathsf{F}^{\gamma}_{\mathscr{O}_1,\mathscr{O}_2} &\to \mathsf{F}^{\gamma'}_{\mathscr{O}_1,\mathscr{O}_2}\\ (r_1, r_2) &\mapsto \left(r_1 \cdot \alpha_{r_2}(g), r_2 \cdot g\right)\end{aligned}$$

is a well-defined bijection, whose inverse is

$$\begin{aligned}\Phi_{\gamma',g^{-1}} : \mathsf{F}^{\gamma'}_{\mathscr{O}_1,\mathscr{O}_2} &\to \mathsf{F}^{\gamma}_{\mathscr{O}_1,\mathscr{O}_2}\\ (s_1, s_2) &\mapsto \left(s_1 \cdot \alpha_{s_2}(g^{-1}), s_2 \cdot g^{-1}\right).\end{aligned}$$

Proof. For all $(r_1, r_2) \in \mathsf{F}^{\gamma}_{\mathscr{O}_1,\mathscr{O}_2}$, by the matched pair relations (I.1.5), we have

$$\gamma' = \gamma \cdot g = \beta_g(r_1r_2) = \beta_{\alpha_{r_2}(g)}(r_1)\beta_g(r_2).$$

Thus $\Phi_{\gamma,g}$ is well-defined. Similarly, $\Phi_{\gamma',g^{-1}}$ is well-defined too.

We now show that $\Phi_{\gamma',g^{-1}} \circ \Phi_{\gamma,g}$ is the identity on $\mathsf{F}^{\gamma}_{\mathscr{O}_1,\mathscr{O}_2}$. Indeed, for all $(r_1, r_2) \in \mathsf{F}^{\gamma}_{\mathscr{O}_1,\mathscr{O}_2}$, we have

$$\begin{aligned}\left(\Phi_{\gamma',g^{-1}} \circ \Phi_{\gamma,g}\right)(r_1, r_2) &= \Phi_{\gamma',g^{-1}}\left(r_1 \cdot \alpha_{r_2}(g), r_2 \cdot g\right) = \left(r_1 \cdot \alpha_{r_2}(g)\alpha_{r_2\cdot g}(g^{-1}), r_2 \cdot gg^{-1}\right)\\ &\qquad \text{(by the matched pair relations again)}\\ &= \left(r_1 \cdot \alpha_{r_2}(gg^{-1}), r_2\right) = \left(r_1 \cdot \alpha_{r_2}(e_G), r_2\right) = (r_1, r_2).\end{aligned}$$

By symmetry, we have $\Phi_{\gamma,g} \circ \Phi_{\gamma',g^{-1}}$ is the identity on $\mathsf{F}^{\gamma'}_{\mathscr{O}_1,\mathscr{O}_2}$. □

Lemma I.4.18. *Suppose $\gamma, \gamma' \in \mathscr{O}_3$ and take any $g \in G_{\gamma,\gamma'}$. For all $(r_1, r_2) \in \mathsf{F}^{\gamma}_{\mathscr{O}_1,\mathscr{O}_2}$, put*

$$(s_1, s_2) := \Phi_{\gamma,g}(r_1, r_2) = \left(r_1 \cdot \alpha_{r_2}(g), r_2 \cdot g\right) \in \mathsf{F}^{\gamma'}_{\mathscr{O}_1,\mathscr{O}_2}.$$

Then for all $x \in G_{\gamma'}$, we have

$$\begin{aligned} &\operatorname{Tr}\left(u^{(1)}_{r_1,r_1}(\alpha_{r_2}(gxg^{-1}))\right)\operatorname{Tr}\left(u^{(2)}_{r_2,r_2}(gxg^{-1})\right) \\ &= \operatorname{Tr}\left(u^{(1)}_{s_1,s_1}(\alpha_{s_2}(x))\right)\operatorname{Tr}\left(u^{(2)}_{s_2,s_2}(x)\right). \end{aligned} \tag{I.4.25}$$

In particular, we have

$$\chi\left(U_1 \times_{\gamma'} U_2\right) = \chi\left(U_1 \times_{\gamma} U_2\right) \circ \operatorname{Ad}_g|_{G_{\gamma'}}, \tag{I.4.26}$$

where $\operatorname{Ad}_g : G \to G$ is the automorphism sending x to gxg^{-1}, and

$$\dim \operatorname{Mor}_{G_\gamma}\left(u^{(3)}_{\gamma,\gamma}|_{G_\gamma}, U_1 \times_\gamma U_2\right) = \dim \operatorname{Mor}_{G_{\gamma'}}\left(u^{(3)}_{\gamma',\gamma'}|_{G_{\gamma'}}, U_1 \times_{\gamma'} U_2\right). \tag{I.4.27}$$

Proof. First notice that by assertion (a) of Proposition I.4.2, we have $u^{(i)}_{r,r\cdot(yz)}(yz) = u^{(i)}_{r,r\cdot y}(y)u^{(i)}_{r\cdot y,r\cdot(yz)}(z)$ for all $y, z \in G$ and $i = 1, 2$. By the definition of $\Phi_{\gamma,g}$, we have $s_1 = r_1 \cdot \alpha_{r_2}(g)$ and $s_2 = r_2 \cdot g$. We also notice that $\alpha_{s_2}(g^{-1}) = \left[\alpha_{r_2}(g)\right]^{-1}$, because the matched pair relations imply

$$e_G = \alpha_{r_2}(gg^{-1}) = \alpha_{r_2}(g)\alpha_{r_2\cdot g}(g^{-1}) = \alpha_{r_2}(g)\alpha_{s_2}(g^{-1}).$$

To prove (I.4.25), we distinguish the following different cases.

<u>**Case I.** $x \notin G_{s_2}$</u>

Since for all $y \in G$, we have

$$y \in G_{s_2} \iff r_2 \cdot g = s_2 = s_2 \cdot y = r_2 \cdot (gy) \iff gyg^{-1} \in G_{r_2},$$

we have $gxg^{-1} \notin G_{r_2}$. As $\operatorname{supp} u^{(1)}_{r_2,r_2} = G_{r_2}$ and $\operatorname{supp} u^{(2)}_{s_2,s_2} = G_{s_2}$, both sides of (I.4.26) is 0.

<u>**Case II.** $x \in G_{s_2}$ and $\alpha_{s_2}(x) \notin G_{s_1}$</u>

The matched pair relations imply

$$y \in G_{s_2} \implies \left\{ \begin{aligned} \alpha_{r_2}(gyg^{-1}) = \alpha_{r_2}(g)\alpha_{r_2\cdot g}(yg^{-1}) &= \alpha_{r_2}(g)\alpha_{s_2}(y)\alpha_{s_2}(g^{-1}) \\ &= \alpha_{r_2}(g)\alpha_{s_2}(y)\left[\alpha_{r_2}(g)\right]^{-1}. \end{aligned} \right. \tag{I.4.28}$$

Hence for all $y \in G_{\gamma'} \cap G_{s_2}$, by (I.4.28), we have

$$\begin{aligned} \alpha_{r_2}(gyg^{-1}) \in G_{r_1} &\iff r_1 \cdot \alpha_{r_2}(g)\alpha_{s_2}(y) = r_1 \cdot a_{r_2}(g) \\ &\iff s_1 \cdot \alpha_{s_2}(y) = s_1 \iff \alpha_{s_2}(y) \in G_{s_1}. \end{aligned}$$

Thus $\alpha_{r_2}(gxg^{-1}) \notin G_{r_1}$ in this case. Consequently, the operators $u^{(1)}_{r_1,r_1}\left(\alpha_{r_2}(gxg^{-1})\right)$ and $u^{(2)}_{s_1,s_1}\left(\alpha_{s_2}(x)\right)$ are both zero, and both sides of (I.4.25) are 0.

<u>**Case III.** $x \in G_{s_2}$ and $\alpha_{s_2}(x) \in G_{s_1}$</u>

Using (I.4.28) in the previous case, we have $\alpha_{r_2}(gxg^{-1}) \in G_{r_1}$, and

$$u^{(1)}_{r_1,r_1}\left(\alpha_{r_2}(gxg^{-1})\right) = \left\{u^{(1)}_{r_1,s_1}\left(\alpha_{r_2}(g)\right)\right\}\left\{u^{(1)}_{s_1,s_1}\left(\alpha_{s_2}(x)\right)\right\}\left\{u^{(1)}_{s_1,r_1}\left(\left[\alpha_{r_2}(g)\right]^{-1}\right)\right\}. \tag{I.4.29}$$

By Proposition I.4.2 again, the operator $u^{(1)}_{r_1,s_1}(\alpha_{r_2}(g))$ is unitary and

$$u^{(1)}_{s_1,r_1}\left(\left[\alpha_{r_2}(g)\right]^{-1}\right) = \left\{u^{(1)}_{r_1,s_1}(\alpha_{r_2}(g))\right\}^{-1}. \tag{I.4.30}$$

Combining (I.4.29) and (I.4.30) yields

$$\mathrm{Tr}\left(u^{(1)}_{r_1,r_1}\left(\alpha_{r_2}(gxg^{-1})\right)\right) = \mathrm{Tr}\left(u^{(1)}_{s_1,s_1}\left(\alpha_{s_2}(x)\right)\right). \tag{I.4.31}$$

Similarly (which is even easier), we have

$$\mathrm{Tr}\left(u^{(2)}_{r_2,r_2}(gxg^{-1})\right) = \mathrm{Tr}\left(u^{(2)}_{s_2,s_2}(x)\right). \tag{I.4.32}$$

It is clear that (I.4.28) follows from (I.4.31) and (I.4.32).

Combining the cases above concludes the proof of (I.4.25).

Using the formula (I.4.22) in assertion (b) of Theorem I.4.15, we see that (I.4.25) implies (in fact is equivalent to) (I.4.26).

Finally, (I.4.27) follows from the above by a simple calculation of characters using the invariance of the Haar state. □

Theorem I.4.19 (Fusion rules of $\mathbb{G}$—simplified version)**.** *Using the same notations as in Theorem I.4.15, the formula* (I.4.23) *is reduced to*

$$\dim \mathrm{Mor}_{\mathbb{G}}(W_3, W_1 \times W_2) = \dim \mathrm{Mor}_{G_\gamma}\left(u^{(3)}_{\gamma,\gamma}|_{G_\gamma}, U_1 \times_\gamma U_2\right), \tag{I.4.33}$$

where γ is an arbitrary *element in $\mathscr{O}_3$.*

Proof. This follows immediately from of Theorem I.4.15 (c) and Lemma I.4.18. □

I.5 Generalities on property (RD) and polynomial growth

We aim to present some generalities on property (RD) and the closely related property of polynomial growth in this section. The results proved here will be vitally important in the proofs of our characterization of these properties for the discrete quantum group $\widehat{\mathbb{G}}$. As $\mathbb{G}$ is of Kac-type, we present only the theory for unimodular discrete quantum groups, which we view as the dual of the compact quantum groups that are of Kac type. The treatment here are adapted using the more systematic study in (Vergnioux, 2007), with some simplifications in the unimodular case of course, as we don't need multiplicative unitary for the compact-discrete duality of quantum groups, which play a key technical role in Vergnioux's general theory on the subject). For the non-unimodular case, we refer our readers to the article (Bhowmick et al., 2015).

As a warm up, we prove a simple well-known result in polynomial algebra over field of characteristic 0.

Lemma I.5.1. *For all $P(X) \in \mathbb{R}[X]$, there exists a unique $Q(X) \in \mathbb{R}[X]$, such that*

$$\forall k \in \mathbb{N}, \qquad Q(k) = \sum_{j=0}^{k} P(j). \tag{I.5.1}$$

Proof. Put $\binom{X}{m} := \frac{1}{m!}\prod_{j=0}^{m-1}(X-j) \in \mathbb{R}[X]$ for every $m \in \mathbb{N}$. Then we have $\deg\binom{X}{m} = m$, and $\binom{X}{m} = \binom{X+1}{m+1} - \binom{X+1}{m}$. It is easy to see that $\left(\binom{X}{m} \ : \ m \in \mathbb{N}\right)$ is a basis for the real vector space $\mathbb{R}[X]$. Thus there exists $a_0, \ldots, a_n \in \mathbb{R}$, such that $P(X) = \sum_{j=0}^{n} a_j \binom{X}{j}$. Hence $Q(X) = \sum_{j=0}^{n} a_j \binom{X+1}{j+1}$ satisfies (I.5.1). Uniqueness of Q is obvious as a non-zero polynomial admits only finitely many roots and $\mathbb{N}$ is infinite. □

It is an important idea in geometric group theory to use length functions to control the growth of a discrete group. The same idea also applies to discrete quantum groups, which we viewed as the dual of compact quantum groups. Let $\mathbb{H}$ be a compact quantum group. Recall that $\mathrm{Irr}(\mathbb{H})$ denotes the set of equivalency classes of irreducible unitary representation of $\mathbb{H}$.

Definition I.5.2. A length function on the discrete quantum group $\widehat{\mathbb{H}}$ is a mapping $l : \mathrm{Irr}(\mathbb{H}) \longrightarrow \mathbb{R}_{\geq 0}$, such that (i) $l([\varepsilon_{\mathbb{H}}]) = 0$, where $\varepsilon_{\mathbb{H}}$ is the trivial representation of $\mathbb{H}$; (ii) $l(\overline{x}) = l(x)$ for all $x \in \mathrm{Irr}(\mathbb{H})$; (iii) $l(z) \leq l(x) + l(y)$ for all $x, y, z \in \mathrm{Irr}(\mathbb{H})$ such that $z \subseteq x \otimes y$.

Proposition I.5.3. *Let $\mathbb{H}$ be a compact quantum group, l a length function on $\widehat{\mathbb{H}}$. The following are equivalent.*

(a) *There exists a polynomial $P(X) \in \mathbb{R}[X]$, such that for all $k \in \mathbb{N}$, we have*

$$\sum_{x \in \mathrm{Irr}(\mathbb{H})\,,\, k \leq l(x) < k+1} (\dim x)^2 \leq P(k).$$

(b) *There exists a polynomial $Q(X) \in \mathbb{R}[X]$, such that for all $k \in \mathbb{N}$, we have*

$$\sum_{x \in \mathrm{Irr}(\mathbb{H})\,,\, l(x) < k+1} (\dim x)^2 \leq Q(k).$$

Proof. Clearly (b) implies (a), while the reverse implication follows from Lemma I.5.1 and the fact that

$$\{x \in \mathrm{Irr}(\mathbb{H}) \ : \ l(x) < k+1\} = \bigcup_{j=0}^{k} \{x \in \mathrm{Irr}(\mathbb{H}) \ : \ j \leq l(x) < j+1\}.$$ □

Definition I.5.4. The pair $\left(\widehat{\mathbb{H}}, l\right)$ is said to have *polynomial growth*, it any of the equivalent conditions in Proposition I.5.3 is satisfied.

The discrete quantum group $\widehat{\mathbb{H}}$ is said to have polynomial growth, if there is a length function l on it, such that the pair $\left(\widehat{\mathbb{H}}, l\right)$ has polynomial growth.

Our formulation of the rapid decay property (property (RD)) in the quantum setting requires more work—we need to define the Fourier transform and the Sobolev-0-norm.

There are several important algebras associated to $\widehat{\mathbb{H}}$, which we introduce now. For every class $x \in \mathrm{Irr}(\mathbb{H})$, we choose and fix a unitary representation u^x of $\mathbb{H}$ on some finite dimensional Hilbert space $\mathscr{H}_x$ such that $u^x \in x$. With these choices fixed, $\ell^\infty(\widehat{\mathbb{H}})$, $c_0(\widehat{\mathbb{H}})$ and $c_c(\widehat{\mathbb{H}})$ denote respectively the ℓ^∞-direct sum, c_0-direct sum, and c_c-direct sum of the "block algebras" $\mathcal{B}(\mathscr{H}_x)$, i.e.

$$\begin{aligned} \ell^\infty(\widehat{\mathbb{H}}) &= \bigoplus_{x\in\mathrm{Irr}(\mathbb{H})}^{\ell^\infty} \mathcal{B}(\mathscr{H}_x), \\ c_0(\widehat{\mathbb{H}}) &= \bigoplus_{x\in\mathrm{Irr}(\mathbb{H})}^{c_0} \mathcal{B}(\mathscr{H}_x), \\ c_c(\widehat{\mathbb{H}}) &= \bigoplus_{x\in\mathrm{Irr}(\mathbb{H})}^{\mathrm{alg}} \mathcal{B}(\mathscr{H}_x). \end{aligned} \tag{I.5.2}$$

Of course, $c_c(\widehat{\mathbb{H}})$ is an involutive dense subalgebra[5] of the C^*-algebra $c_0(\widehat{\mathbb{H}})$, and $c_0(\widehat{\mathbb{H}})$ is weakly dense in the von Neumann algebra $\ell^\infty(\widehat{\mathbb{H}})$. In particular, $c_c(\widehat{\mathbb{H}})$ is a weakly dense ideal in $\ell^\infty(\mathbb{H})$. To fix the notations, for all $x \in \mathrm{Irr}(\mathbb{H})$, the symbol p_x denotes the central projection in $c_c(\widehat{\mathbb{H}})$ whose component at $y \in \mathrm{Irr}(\mathbb{H})$ is $\mathrm{id}_{\mathscr{H}_x}$ if $y = x$ and is 0 otherwise. Hence for all $a \in \ell^\infty(\widehat{\mathbb{H}})$, the element $ap_x \in c_c(\widehat{\mathbb{H}})$ is supported at $x \in \mathrm{Irr}(\mathscr{H}_x)$, and we often abuse notation by letting ap_x also denote its x-component (so $ap_x \in \mathcal{B}(\mathscr{H}_x)$).

Remark I.5.5. Although we don't need this, we mention in passing here that the comultiplication on $\widehat{\mathbb{H}}$ is a bit tricky if we adopt the c_0 or the algebraic c_c picture of $\widehat{\mathbb{H}}$ (one has to consider their multiplier algebras and use non-degenerate maps as comultiplication). In the spirit of treating the bicrossed product $\mathbb{G}$ as an algebraic compact quantum group, the algebraic dual of $\mathbb{H}$ viewed as an algebraic compact quantum group is $c_c(\mathbb{H})$, and this duality can be nicely treated in the framework of van Daele's multiplier Hopf algebras, see Part I of (Timmermann, 2008), or the original papers of van Daele (Van Daele, 1994; 1996; 1998).

We can now introduce the Fourier transform and Sobolev norms on these quantum objects. As we've mentioned earlier, since the bicrossed product $\mathbb{G}$ is of Kac type (Theorem I.2.11), it is enough for us to treat the unimodular case, for which we only need the Sobolev-0-norm instead of all possible Sobolev norms.

Definition I.5.6. Suppose $\mathbb{H}$ is of Kac type, and $a \in c_c(\widehat{\mathbb{H}})$.

- The Fourier transform of a, denoted by $\mathcal{F}_{\mathbb{H}}(a)$ or simply $\mathcal{F}(a)$, is the element in $\mathrm{Pol}(\mathbb{H})$ defined by

$$\mathcal{F}_{\mathbb{H}}(a) := \sum_{x\in\mathrm{Irr}(\mathbb{H})} (\dim x)\left[(\mathrm{Tr}_{\mathscr{H}_x}\otimes\mathrm{id})\big(u^x(ap_x\otimes 1)\big)\right]. \tag{I.5.3}$$

- The Sobolev-0-norm of a, denoted by $\|a\|_{\mathbb{H},0}$, is determined by

$$\|a\|_{\mathbb{H},0}^2 = \sum_{x\in\mathrm{Irr}(\mathbb{H})} \mathrm{Tr}_{\mathscr{H}_x}\big((a^*a)p_x\big). \tag{I.5.4}$$

Here, $\dim x = \dim\mathscr{H}_x$, which is independent of the choice of the representative u^x (hence $\mathscr{H}_x$), and we've adopted the abuse of notation as explained right before Remark I.5.5.

[5] which is non-unital if the quantum group $\mathbb{H}$ is not finite.

Remark I.5.7. At first glance, it seems that the Fourier transform and the Sobolev-0-norm both depend on our choice of a complete set of representatives of the set of classes $\mathrm{Irr}(\mathbb{H})$. This dependence is in fact superfluous in the sense as we will now explain. For each $x \in \mathrm{Irr}(\mathbb{H})$, choose $u^x, v^x \in x$, and denote the finite dimensional Hilbert space on which u^x (resp. v^x) acts by $\mathscr{H}_x$ (resp. $\mathscr{K}_x$). Since $[u^x] = x = [v^x]$, there exists a unitary $T_x \in \mathrm{Mor}_{\mathbb{H}}(u^x, v^x)$, which is unique up to a multiple by a scalar in $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Hence there is a canonical isomorphism of C^*-algebras $\Theta_x : \mathcal{B}(\mathscr{H}_x) \to \mathcal{B}(\mathscr{K}_x)$, $S \mapsto T_x S T_x^*$. Here, when we say Θ_x is canonical, we mean Θ_x does not depend on the choice of T_x, which is true since T_x is unique up to a rescaling by a constant in $\mathbb{T}$. Let $\mathscr{U} := \{u_x : x \in \mathrm{Irr}(\mathbb{H})\}$, $\mathscr{V} := \{v_x : x \in \mathrm{Irr}(\mathbb{H})\}$. To emphasize the dependence on the choice of the complete set of representatives of $\mathrm{Irr}(\mathbb{H})$, we use $c_c^{\mathscr{U}}(\widehat{\mathbb{H}})$ (resp. $c_c^{\mathscr{V}}(\widehat{\mathbb{H}})$) to denote the copy of $c_c(\widehat{\mathbb{H}})$ as defined above with respect to the choice $\mathscr{U}$ (resp. $\mathscr{V}$), and we denote the resulting Fourier transform by $\mathcal{F}_{\mathbb{H}}^{\mathscr{U}}$ (resp. $\mathcal{F}_{\mathbb{H}}^{\mathscr{V}}$), and the Sobolev-0-norm by $\|\cdot\|_{\mathbb{H},0,\mathscr{U}}$ (resp. $\|\cdot\|_{\mathbb{H},0,\mathscr{V}}$). It is clear that we now have a canonical isomorphism $\Theta : c_c^{\mathscr{U}}(\widehat{\mathbb{H}}) \to c_c^{\mathscr{V}}(\mathbb{H})$ that restricts to Θ_x on each block $\mathcal{B}(\mathscr{H}_x)$. Now for every $x \in \mathrm{Irr}(\mathbb{H})$ and $a_x \in \mathcal{B}(\mathscr{H}_x)$, we have

$$\begin{aligned}
&(\mathrm{Tr}_{\mathscr{K}_x} \otimes \mathrm{id})\big[\Theta_x(u^x)\big(\Theta_x(a_x) \otimes 1\big)\big] \\
&= (\mathrm{Tr}_{\mathscr{K}_x} \otimes \mathrm{id})\, \big[(T_x \otimes 1) u^x (T_x^* \otimes 1)\big((T_x a T_x^*) \otimes 1\big)\big] \\
&= (\mathrm{Tr}_{\mathscr{K}_x} \otimes \mathrm{id})\, \big[(T_x \otimes 1) u^x (a_x \otimes 1)(T_x^* \otimes 1)\big] \\
&= (\mathrm{Tr}_{\mathscr{H}_x} \otimes \mathrm{id})\, \big[\big(u^x(a_x \otimes 1)\big)\big((T_x^* T_x) \otimes 1\big)\big] \\
&= (\mathrm{Tr}_{\mathscr{H}_x} \otimes \mathrm{id})[u^x(a_x \otimes 1)],
\end{aligned}$$

which implies

$$\forall a \in c_c^{\mathscr{U}}(\widehat{\mathbb{H}}), \qquad \mathcal{F}_{\mathbb{H}}^{\mathscr{V}}\big(\Theta(a)\big) = \mathcal{F}_{\mathbb{H}}^{\mathscr{U}}(a). \tag{I.5.5}$$

Similarly,

$$\forall a_x \in \mathcal{B}(\mathscr{H}_x), \qquad \mathrm{Tr}_{\mathscr{K}_x}\big([\Theta_x(a_x)]^*[\Theta_x(a_x)]\big) = \mathrm{Tr}_{\mathscr{K}_x}(T_x a_x^* a_x T_x^*) = \mathrm{Tr}_{\mathscr{H}_x}\big(a_x^* a_x\big),$$

hence

$$\forall a \in c_c^{\mathscr{U}}(\widehat{\mathbb{H}}), \qquad \|\Theta_x(a)\|_{\mathbb{H},0,\mathscr{V}} = \|a\|_{\mathbb{H},0,\mathscr{U}}. \tag{I.5.6}$$

By (I.5.5) and (I.5.6), we see that as far as the Fourier transform and the Sobolev-0-norm are concerned, all possible choices of complete set of representatives of $\mathrm{Irr}(\mathbb{H})$ behave coherently via the canonical ismorphisms of between their corresponding copies of $c_c(\widehat{\mathbb{H}})$. We shall therefore write simply $\mathcal{F}_{\mathbb{H}}$ and $\|\cdot\|_{\mathbb{H}}$, with the obvious adaptations if different choices of complete sets of representatives of $\mathrm{Irr}(\mathbb{H})$ are chosen, as indicated above.

The following useful result can be easily obtained using the methods presented in (Vergnioux, 2007, Proposition 4.4, assertion 2) or (Bhowmick et al., 2015, Proposition 4.2, assertion b)). We include a detailed proof of it for our reader's convenience.

Lemma I.5.8. *Using the above notations. Suppose F is a finite subset of* $\mathrm{Irr}(\mathbb{H})$ *and put $p_F := \sum_{x \in F} p_x \in c_c(\widehat{\mathbb{H}})$. Then*

$$\forall a \in p_F c_c(\widehat{\mathbb{H}}) \implies \|\mathcal{F}_{\mathbb{H}}(a)\| \le 2\left(\sqrt{\sum_{x \in F} (\dim x)^2}\right) \|a\|_{\mathbb{H},0}. \tag{I.5.7}$$

Proof. We first give a neat reformulation of the Fourier transform as in the article of Vergnioux (Vergnioux, 2007). Let $\widehat{\tau}$ be the unique linear form on $c_c(\widehat{\mathbb{H}})$, such that

$$\forall x \in \operatorname{Irr}(\mathbb{H}),\ \forall A_x \in \mathcal{B}(\mathscr{H}_x), \qquad \widehat{\tau}(A_x) = (\dim x)\operatorname{Tr}_{\mathcal{B}(\mathscr{H}_x)}(A_x).$$

It is clear that $\widehat{\tau}$ is *positive and tracial*, and restricts to a positive form on every finite dimensional $*$-subalgebra (which is automatically a C^*-algebra) of $c_c(\widehat{\mathbb{H}})$ ($\widehat{\tau}$ is actually the Haar weight of the unimodular discrete quantum group $\widehat{\mathbb{H}}$, but we don't need this). For every $a \in c_c(\widehat{\mathbb{H}})$, we denote the linear form $\widehat{\tau}(\cdot\, a)$ on $c_c(\widehat{\mathbb{H}})$ by $a \cdot \widehat{\tau}$. Note that if $a \in c_c(\widehat{\mathbb{H}})$ is positive, so is $a \cdot \widehat{\tau}$, since

$$(a \cdot \widehat{\tau})(b^*b) = \widehat{\tau}(b^*ba) = \widehat{\tau}(a^{1/2}b^*ba^{1/2}) \geq 0$$

for all $b \in c_c(\widehat{\mathbb{H}})$. Using the commutativity of algebraic direct sum and tensor product, we identify the tensor product $c_c(\widehat{\mathbb{H}}) \otimes \operatorname{Pol}(\mathbb{H})$ with

$$\bigoplus_{x\in\operatorname{Irr}(\mathbb{H})}^{\text{alg}} \mathcal{B}(\mathscr{H}_x) \otimes \operatorname{Pol}(\mathbb{H}),$$

and we define

$$U := (u^x)_{x\in\operatorname{Irr}(\mathbb{H})} \in \bigoplus_{x\in\operatorname{Irr}(\mathbb{H})}^{\ell^\infty} \mathcal{B}(\mathscr{H}_x) \otimes \operatorname{Pol}(\mathbb{H}).$$

It is clear that U is unitary, and for every b in the finite dimensional C^*-algebra $p_F c_c(\widehat{\mathbb{H}}) = \sum_{x\in F} \mathcal{B}(\mathscr{H}_x)$, we have Ub lies in the unital C^*-algebra

$$\bigoplus_{x\in F} \mathcal{B}(\mathscr{H}_x) \otimes C_r(\mathbb{H}), \tag{I.5.8}$$

where $C_r(\mathbb{H})$ is the C^*-completion of $\operatorname{Pol}(\mathbb{H})$ using the C^*-norm induced by the GNS construction with respect to the Haar state on $\operatorname{Pol}(\mathbb{H})$.

With these notations fixed, formula (I.5.3) can be rewritten as

$$\mathcal{F}_{\mathbb{H}}(a) = \big[(a \cdot \widehat{\tau}) \otimes \mathrm{id}\big](U), \tag{I.5.9}$$

while formula (I.5.4) becomes

$$\|a\|_{\mathbb{H},0}^2 = \widehat{\tau}(a^*a). \tag{I.5.10}$$

Suppose $a \in p_F c_c(\widehat{\mathbb{H}})$ from now on. We first treat the case where a is *positive*. In this case, it is clear that the mapping $(a \cdot \widehat{\tau}) \otimes \mathrm{id}$ is positive, and using (I.5.9), we have

$$\|\mathcal{F}_{\mathbb{H}}(a)\| = \big\|\big[(a \cdot \widehat{\tau}) \otimes \mathrm{id}\big](U)\big\| = \big\|\big[(a \cdot \widehat{\tau}) \otimes \mathrm{id}\big](U_F)\big\| \leq \|a \cdot \widehat{\tau}\| = \widehat{\tau}_F(a), \tag{I.5.11}$$

where U_F is the partial unitary $p_F U$, and $\widehat{\tau}_F$ is the restriction of $\widehat{\tau}$ onto the finite dimensional C^*-algebra $p_F c_c(\widehat{\mathbb{H}})$. Denote the unit $\sum_{x\in F} \mathrm{id}_{\mathscr{H}_x}$ of this last C^*-algebra by e_F, then by the Cauchy-Schwartz inequality and (I.5.10), we have

$$[\widehat{\tau}_F(a)]^2 \leq \widehat{\tau}(e_F)\widehat{\tau}(a^*a) = \left(\sum_{x\in F} (\dim x)^2\right) \|a\|_{\mathbb{H},0}^2. \tag{I.5.12}$$

Combining (I.5.11) and (I.5.12) proves

$$\forall a \in \left(p_F c_c(\widehat{\mathbb{H}})\right)_+, \qquad \|\mathcal{F}_{\mathbb{H}}(a)\| \leq \left(\sqrt{\sum_{x\in F} (\dim x)^2}\right) \|a\|_{\mathbb{H},0}. \tag{I.5.13}$$

Finally, suppose a is an arbitrary element in the C^*-algebra $p_F c_c(\widehat{\mathbb{H}})$. Decomposing the real and imaginary parts of a onto the sum of their positive and negative parts, we can find $a_k \in \left(p_F c_c(\widehat{\mathbb{H}})\right)_+$, $k = 1, 2, 3, 4$, such that $a = \sum_{k=1}^4 i^k a_k$. By (I.5.10), we have

$$\|a\|^2_{\mathbb{H},0} = \sum_{k=1}^4 \|a_k\|^2_{\mathbb{H},0}.$$

Hence by (I.5.13) and the Cauchy-Schwartz inequality, we have

$$\|\mathcal{F}_{\mathbb{H}}(a)\|^2 \leq \left(\sum_{k=1}^4 \|\mathcal{F}_{\mathbb{H}}(a_k)\|\right)^2 \leq 4 \sum_{k=1}^4 \|\mathcal{F}_{\mathbb{H}}(a_k)\|^2$$
$$\leq 4 \left(\sum_{x\in F} (\dim x)^2\right) \left(\sum_{k=1}^2 \|a_k\|^2_{\mathbb{H},0}\right) = 4 \left(\sum_{x\in F} (\dim x)^2\right) \|a\|^2_{\mathbb{H},0},$$

which proves (I.5.7). □

We now define property (RD) and polynomial growth of $\widehat{\mathbb{H}}$ in terms of the Fourier transform and the Sobolev-0-norm, with the help of length functions.

Notations I.5.9. *Let l be a length function on $\widehat{\mathbb{H}}$, for all $n \in \mathbb{N}$, we pose the following central projections*

$$q_{l,n} := \sum_{x\in \mathrm{Irr}(\mathbb{H}),\, n\leq l(x)<n+1} p_x \in \ell^\infty(\widehat{\mathbb{H}}),$$
$$Q_{l,n} := \sum_{j=0}^n q_{l,j} = \sum_{x\in \mathrm{Irr}(\mathbb{H}),\, l(x)<n+1} p_x \in \ell^\infty(\widehat{\mathbb{H}}).$$

To use the more succinct language of (Woronowicz, 1991)*, $L := \sum_{x\in\mathrm{Irr}(\mathbb{H})} l(x)p_x$ defines an unbounded element affiliated with the C^*-algebra $c_0(\widehat{\mathbb{H}})$, and $q_{l,n}$ (resp. $Q_{l,n}$) is the spectral projection of L associated with the interval $[n, n+1[$ (resp. $[0, n+1[$).*

We have a similar result as Proposition I.5.3 concerning the control of the norm of the Fourier transform using the Sobolev-0-norm with the help of a length function.

Proposition I.5.10. *Let $\mathbb{H}$ be a compact quantum group, $l : \mathrm{Irr}(\mathbb{H}) \to \mathbb{R}_{\geq 0}$ a length function on $\widehat{\mathbb{H}}$. The following are equivalent.*

(a) *There is a polynomial $P(X) \in \mathbb{R}[X]$, such that for all $n \in \mathbb{N}$,*

$$a \in q_{l,n} c_c(\widehat{\mathbb{H}}) \implies \|\mathcal{F}_{\mathbb{G}}(a)\| \leq P(n)\|a\|_{\mathbb{G},0}.$$

(b) *There is a polynomial $Q(X) \in \mathbb{R}[X]$, such that for all $n \in \mathbb{N}$,*

$$a \in Q_{l,n} c_c(\widehat{\mathbb{H}}) \implies \|\mathcal{F}_{\mathbb{G}}(a)\| \leq Q(n)\|a\|_{\mathbb{G},0}.$$

Proof. It is clear that (b) implies (a). Suppose (a) holds, and let's prove (b). Take $a \in Q_{l,n} c_c(\widehat{\mathbb{H}})$. Put $a_j := p_{l,j} a$ for $j = 0, \ldots, n$, then $a_0, a_1, \ldots, a_n$ are mutually orthogonal in $c_c(\widehat{\mathbb{H}})$. Hence by (a), we have

$$\begin{aligned}\|\mathcal{F}_{\mathbb{G}}(a)\| &\le \sum_{j=0}^{n}\left\|\mathcal{F}_{\mathbb{G}}(a_j)\right\| \le \sum_{j=0}^{n} P(j)\left\|a_j\right\|_{\mathbb{G},0}\\ &= \left(\sum_{0}^{n}[P(j)]^2\right)^{1/2}\left(\sum_{j=0}^{n}\left\|a_j\right\|^2_{\mathbb{G},0}\right) = \left(\sum_{j=0}^{n}[P(j)]^2\right)^{1/2}\|a\|_{\mathbb{G},0}.\end{aligned}$$

This, together with Lemma I.5.1 applied to the polynomial $[P(X)]^2$, implies (b). □

Definition I.5.11. Suppose $\mathbb{H}$ is of Kac type, and $l : \mathrm{Irr}(\mathbb{H}) \longrightarrow \mathbb{R}_{\geq 0}$ is a length function on $\widehat{\mathbb{H}}$. We say the pair $\left(\widehat{\mathbb{H}}, l\right)$ has the *rapid decay* property, or property (RD) (or even simply (RD)), if any of the equivalent conditions in Proposition I.5.10 is satisfied.

We say the discrete quantum group $\widehat{\mathbb{H}}$, where $\mathbb{H}$ is a compact quantum group of Kac type, has property (RD) if there exists a length function l on $\widehat{\mathbb{H}}$ such that the pair $(\widehat{\mathbb{H}}, l)$ has property (RD).

If H is a classical discrete group, then the convolution algebra $C_c(H)$ of finitely supported complex-valued functions on H is a Hopf $*$-algebra, with the comultiplication Δ_H determined by $\Delta_H(\delta_x) = \delta_x \otimes \delta_x$ for all $x \in H$. In fact, $\widehat{H} := \left(C_c(H), \Delta_H\right)$ an algebraic compact quantum group (so $C_c(H) = \mathrm{Pol}(\widehat{H})$), with the Haar state being the tracial state sending δ_x to $\delta_{x,e_H} \in \mathbb{C}$, where e_H is the neutral element of H. It is easy to see that $\mathrm{Irr}(\widehat{H})$ is in canonical bijective correspondence with H. Indeed, every $x \in H$ determines a one-dimensional unitary representation $u^x \in \mathcal{B}(\mathbb{C}) \otimes C_c(H)$ of $\widehat{H}$ given by $u^x = \mathrm{id}_{\mathbb{C}} \otimes \delta_x$, and $x \mapsto [u^x]$ is this canonical bijection from H onto $\mathrm{Irr}(\widehat{H})$. Now apply the above discussion to the compact quantum group $\widehat{H}$, we get the notions of length functions, polynomial growth, (RD) for $\widehat{\widehat{H}}$. But $\widehat{\widehat{H}}$ is just H viewed as a discrete quantum group, with $c_c\left(\widehat{\widehat{H}}\right) = C_c(H)$ if we identify H with $\mathrm{Irr}(\widehat{H})$ as we've just explained, thus Definitions I.5.2, I.5.4, I.5.6 and I.5.11 give a corresponding notion of length functions on H, Fourier transform on H and Sobolev-0-norm on H, and polynomial growth and (RD) for H, which coincide with their classical counterparts in the group case (see (Jolissaint, 1990)). We record the precise form of these notions for discrete groups in the following.

Definition I.5.12. Let H be a (discrete) group. A length function on H is a mapping $l : H \longrightarrow \mathbb{R}_{\geq 0}$ such that (i) $l(e_H) = 0$; (ii) $l(x) = l(x^{-1})$ for all $x \in H$; (iii) $l(xy) \leq l(x) + l(y)$ for all $x, y \in H$.

Suppose l is length function on H.

- We say (H, l) has *polynomial growth*, if there is a polynomial $P(X) \in \mathbb{R}[X]$, such that for all $n \in \mathbb{N}$, we have $|\{x \in H : l(x) \in [n, n+1[\}| \leq P(n)$.
- We say (H, l) has *rapid decay* (*property* (RD)), if there is a polynomial $P(X) \in \mathbb{R}[X]$, such that for all $n \in \mathbb{N}$, and all $a \in C_c(H)$ such that $l(x) \in [n, n+1[$ whenever $a(x) \neq 0$, we have $\|a\|_\lambda \leq P(n)\|a\|_2$, where $\|a\|_\lambda$ is the operator norm of the convolution[6] $a * (\cdot) : \ell^2(H) \longrightarrow \ell^2(H)$, and $\|a\|_2$ is the ℓ^2-norm[7] of a viewed as in $\ell^2(H)$.

[6] This is exactly the norm of the Fourier transform of a.

[7] This is exactly $\|a\|_{\widehat{\widehat{H}},0}$.

We remark that the interval $[n, n+1[$ can also be replaced by the interval $[0, n+1[$.

Finally, H is said to have polynomial growth (resp. (RD)) if there is a length function l on H such that the pair (H, l) has polynomial growth (resp. (RD)).

We refer the reader to the survey paper (Chatterji, 2017) and the references there for many (non)examples, as well as a nice survey of our current knowledge and open questions, concerning property (RD) for discrete groups. We mention in passing some results here.

- (Gromov (Gromov, 1981)) Finitely generated a group has polynomial growth if and only if it is virtually nilpotent (i.e. containing a nilpotent subgroup of finite index).
- (Jolissaint (Jolissaint, 1990)) For all discrete group H, polynomial growth of H implies that H has (RD), and the converse also holds if H is amenable.
- (Jolissaint (Jolissaint, 1990)) The group $\mathrm{SL}_n(\mathbb{Z})$ has (RD) if and only if $n \leq 2$. The free group $\mathbb{F}_2$ has (RD) (none of these groups has polynomial growth).

In the quantum case, Vergnioux showed in (Vergnioux, 2007) the following results:

- As in the classical case, a discrete quantum group $\widehat{\mathbb{H}}$ has polynomial growth implies that $\widehat{\mathbb{H}}$ has (RD) (which can also be proved directly by Lemma I.5.8), and the converse holds if $\widehat{\mathbb{H}}$ is coamenable (we won't prove this result here, but see Proposition I.5.13 for the proof of a special case).
- If H is a real compact connected Lie group, then the dual $\widehat{H}$ has polynomial growth (hence (RD)).

We track here a quick proof of the result of Vergnioux in the special case of the dual of classical compact groups.

Proposition I.5.13. *Suppose H is a classical compact group, and l is a length function on $\widehat{H}$. If $\left(\widehat{H}, l\right)$ has (RD), then it has polynomial growth.*

Proof. Let $P(X) \in \mathbb{R}[X]$ satisfies

$$a \in q_{l,k} c_c(\widehat{H}) \implies \|\mathcal{F}_H(a)\| \leq P(k)\|a\|_{H,0}$$

for all $k \in \mathbb{N}$. Let $\epsilon : C(H) \to \mathbb{C}$ be the character of the C^*-algebra $C(H)$ given by $\varphi \mapsto \varphi(e_H)$. Since $u(e_H) = \mathrm{id}$ for any representation u of H, we have

$$\begin{aligned}&\forall x \in \mathrm{Irr}(H),\ a_x \in \mathcal{B}(\mathscr{H}_x),\\ &\epsilon\big((\mathrm{Tr}_{\mathscr{H}_x} \otimes \mathrm{id})\left[u^x(a_x \otimes 1)\right]\big) = (\mathrm{Tr}_{\mathscr{H}_x} \otimes \epsilon)\left[u^x(a_x \otimes 1)\right] = \mathrm{Tr}_{\mathscr{H}}(a_x).\end{aligned} \tag{I.5.14}$$

Put $p_F := \sum_{x\in F} p_x \in q_{l,k} c_c(\widehat{H})$. Take an arbitrary $k \in \mathbb{N}$, and a finite subset F of $\{x \in \mathrm{Irr}(H) \ : \ k \leq l(x) < k+1\}$. Then by (I.5.14), we have

$$\epsilon\big(\mathcal{F}_H(p_F)\big) = \sum_{x\in F} (\dim x)\,\mathrm{Tr}_{\mathscr{H}_x}(p_x) = \sum_{x\in F} (\dim x)^2,$$

and

$$\|p_F\|_{H,0}^2 = \sum_{x\in F} (\dim x)\,\mathrm{Tr}_{\mathscr{H}}(p_x) = \sum_{x\in F} (\dim x)^2.$$

Thus

$$\sum_{x\in F}(\dim x)^2 = \epsilon\big(\mathcal{F}_{\mathscr{H}}(p_F)\big) \leq \|\mathcal{F}_{\mathscr{H}}(p_F)\| \leq P(k)\|p_F\|_{H,0} = P(k)\sqrt{\sum_{x\in F}(\dim x)^2}.$$

Hence

$$\sum_{x\in F}(\dim x)^2 \leq [P(k)]^2.$$

As F is taken arbitrarily, this forces

$$\sum_{x\in \mathrm{Irr}(H),\, k\leq l(x)<k+1}(\dim x)^2 \leq Q(k),$$

where $Q(X) = [P(X)]^2 \in \mathbb{R}[X]$. This proves that $\left(\widehat{H}, l\right)$ indeed has polynomial growth. □

Remark I.5.14. The quantum case of Proposition I.5.13 is proved in the same way assuming the underlying compact quantum group is coamenable, thus the counit is everywhere defined and hence is a character of the underlying C^*-algebra, whose norm is thus bounded by 1 (we always have $(\mathrm{id}\otimes\epsilon)(u) = \mathrm{id}$ for any finite dimensional unitary representation u of a compact quantum group). This approach is due to Vergnioux (cf. (Vergnioux, 2007, Proposition 4.4)).

Corollary I.5.15. *Suppose H is a classical compact group, l is a length function on $\widehat{H}$,, then $\left(\widehat{H}, l\right)$ has (RD) if and only if it has polynomial growth. In particular, $\widehat{H}$ has (RD) if and only if it has polynomial growth.*

Proof. This is clear by Proposition I.5.13 and Lemma I.5.8. □

We terminate this section with a technical result for classical compact groups, which will play a vital role in the proof our result on (RD) for $\widehat{\mathbb{G}}$.

Lemma I.5.16. *Suppose H is a compact group and K is an open subgroup of H. Let $e_K \in C(H)$ be the characteristic function of K. For all $a \in c_c(\widehat{K})$, there exists $\widetilde{a} \in c_c(\widehat{H})$, such that* (i) *if $\widetilde{a}p_y \neq 0$ for some $y \in \mathrm{Irr}(H)$, then $ap_x \neq 0$ for some $x \in \mathrm{Irr}(K)$ with $y \subseteq \mathrm{Ind}_K^H(x)$;* (ii) *$e_K\mathcal{F}_H(\widetilde{a}) = \mathfrak{E}_K(\mathcal{F}_K(a))$, where $\mathfrak{E}_K : C(K) \to C(H)$ is the unique extension of functions by making the extension vanish outside K;* (iii) *$\|\widetilde{a}\|_{H,0} \leq \|a\|_{K,0}$.*

Proof. For every $x \in \mathrm{Irr}(K)$, $y \in \mathrm{Irr}(H)$, define

$$d_{x,y} := \dim \mathrm{Mor}_H\big(y, \mathrm{Ind}_K^H(x)\big) = \dim \mathrm{Mor}_H\big(\mathrm{Res}_K^H(y), x\big) = \dim \mathrm{Mor}_H\big(x, \mathrm{Res}_K^H(y)\big).$$

By definition $d_{x,y} \neq 0$ if and only if $y \subseteq \mathrm{Ind}_K^H(x)$. Put

$$\mathrm{supp}_H(x) := \big\{y \in \mathrm{Irr}(H) \;:\; y \subseteq \mathrm{Ind}_K^H(x)\big\} = \big\{y \in \mathrm{Irr}(H) \;:\; x \subseteq \mathrm{Res}_K^H(y)\big\},$$

and

$$\mathrm{supp}_K(y) := \big\{x \in \mathrm{Irr}(K) \;:\; y \subseteq \mathrm{Ind}_K^H(x)\big\} = \big\{x \in \mathrm{Irr}(K) \;:\; x \subseteq \mathrm{Res}_K^H(y)\big\}.$$

Using the semisimplicity of the additive category $\mathrm{Rep}(H)$, for each $x \in \mathrm{Irr}(K)$, one can choose a family of pairwise orthogonal isometries $s^i_{x,y} \in \mathrm{Mor}_H\big(u^y, \mathrm{Ind}^H_K(u^x)\big)$, $y \in \mathrm{supp}_H(x)$, $i = 1, \dots, d_{x,y}$ such that

$$\sum_{y \in \mathrm{supp}_H(x)} \sum_{i=1}^{d_{x,y}} \left(s^i_{x,y}\right)\left(s^i_{x,y}\right)^* = \mathrm{id}_{\mathscr{K}_x}, \tag{I.5.15}$$

where $\mathscr{K}_x$ is the finite dimensional Hilbert space on which $\mathrm{Ind}^H_K(u^x)$ acts. For all $x \in \mathrm{Irr}(K)$, since

$$\dim \mathrm{Mor}_K\Big(x, \mathrm{Res}^H_K\big(\mathrm{Ind}^H_K(x)\big)\Big) = \dim \mathrm{End}_G\big(\mathrm{Ind}^H_K(x)\big) \neq 0,$$

by Schur's lemma, we are able to choose and fix an isometry

$$r_x \in \mathrm{Mor}_K\Big(u^x, \mathrm{Res}^H_K\big(\mathrm{Ind}^H_K(u^x)\big)\Big).$$

With these notations, let $\widetilde{a}$ be the unique element in $\ell^\infty(\widehat{H})$ such that

$$\forall y \in \mathrm{Irr}(H), \qquad \widetilde{a}p_y = \sum_{x \in \mathrm{supp}_K(y)} \frac{\dim x}{\dim y} \sum_{i=1}^{d_{x,y}} \left(s^i_{x,y}\right)^* r_x(ap_x) r_x^* \left(s^i_{x,y}\right). \tag{I.5.16}$$

By definition,

$$\forall x \in \mathrm{Irr}(K), y \in \mathrm{Irr}(H), \qquad x \in \mathrm{supp}_K(y) \iff y \in \mathrm{supp}_K(x), \tag{I.5.17}$$

thus

$$\widetilde{a}p_y \neq 0 \implies \exists x \in \mathrm{supp}(a), y \in \mathrm{supp}_K(x).$$

This shows that the set

$$\big\{y \in \mathrm{Irr}(H) \ : \ \widetilde{a}p_y \neq 0\big\} \subseteq \bigcup_{x \in \mathrm{supp}(a)} \mathrm{supp}_H(x)$$

is finite, and we in fact have $\widetilde{a} \in c_c(\widetilde{G})$. We now show that $\widetilde{a}$ has the desired Fourier transform and Sobolev-0-norm.

By our choices of $s^i_{x,y}$ and rx, we have

$$\begin{gathered} \forall x \in \mathrm{Irr}(K), y \in \mathrm{supp}_H(x), i = 1, \dots, d_{x,y}, \\ u^y\left[\left(s^i_{x,y}\right)^* \otimes 1\right] = \left[\left(s^i_{x,y}\right)^* \otimes 1\right] \mathrm{Ind}^H_K(u^x), \end{gathered} \tag{I.5.18}$$

and note that $C(H)$ is *commutative*, we also have

$$\forall x \in \mathrm{Irr}(K), \qquad (\mathrm{id} \otimes e_K)\, \mathrm{Ind}^H_K(u^x) = \mathrm{Ind}^H_K(u^x)(r_x \otimes e_K) = (r_x \otimes 1)u^x. \tag{I.5.19}$$

Combining (I.5.18) and (I.5.19), we have

$$\begin{aligned} & e_K\left\{(\mathrm{Tr}_{\mathscr{H}_y} \otimes \mathrm{id})\left(u^y\left\{\left[\left(s^i_{x,y}\right)^* r_x(ap_x) r_x^*\left(s^i_{x,y}\right)\right] \otimes 1\right\}\right)\right\} \\ &= e_K\left\{(\mathrm{Tr}_{\mathscr{H}_y} \otimes \mathrm{id})\left(\left\{\left(s^i_{x,y}\right)^* \otimes 1\right\} \mathrm{Ind}^H_K(x)\left\{\left[r_x(ap_x) r_x^*\left(s^i_{x,y}\right)\right] \otimes 1\right\}\right)\right\} \\ &= (\mathrm{Tr}_{\mathscr{H}_y} \otimes \mathrm{id})\left(\left\{\left(s^i_{x,y}\right)^* \otimes 1\right\}\left\{\mathrm{Ind}^H_K(x)(r_x \otimes e_K)\right\}\left\{\left[(ap_x) r_x^*\left(s^i_{x,y}\right)\right] \otimes 1\right\}\right) \\ &= (\mathrm{Tr}_{\mathscr{H}_y} \otimes \mathrm{id})\left(\left\{\left[\left(s^i_{x,y}\right)^* r_x\right] \otimes 1\right\}\left\{u^x[(ap_x) \otimes 1]\right\}\left\{\left[r_x^*\left(s^i_{x,y}\right)\right] \otimes 1\right\}\right) \\ &= (\mathrm{Tr}_{\mathscr{H}_y} \otimes \mathrm{id})\left(\left\{u^x[(ap_x) \otimes 1]\right\}\left\{\left[r_x^*\left(s^i_{x,y}\right)\left(s^i_{x,y}\right)^* r_x\right] \otimes 1\right\}\right). \end{aligned} \tag{I.5.20}$$

By definition of the Fourier transform, (I.5.15), (I.5.16), (I.5.17) and (I.5.20), we have

$$\begin{aligned}
e_K\mathcal{F}_H(\widetilde{a}) &= e_K \sum_{y\in\mathrm{Irr}(H)} (\dim y) \left[(\mathrm{Tr}_{\mathscr{H}_y}\otimes\mathrm{id})\Big(u^y(\widetilde{a}p_y\otimes 1)\Big)\right] \\
&= e_K \sum_{y\in\mathrm{Irr}(H)} \sum_{x\in\mathrm{supp}_K(y)} \sum_{i=1}^{d_{x,y}} (\dim x) \\
&\qquad \left\{(\mathrm{Tr}_{\mathscr{H}_y}\otimes\mathrm{id})\Big(u^y\left\{\left[\left(s^i_{x,y}\right)^* r_x(ap_x)r_x^*\left(s^i_{x,y}\right)\right]\otimes 1\right\}\Big)\right\} \\
&= \sum_{x\in\mathrm{supp}(a)} \sum_{y\in\mathrm{supp}_H(x)} \sum_{i=1}^{d_{x,y}} (\dim x) \\
&\qquad \left\{(\mathrm{Tr}_{\mathscr{H}_y}\otimes\mathrm{id})\left(\left\{\left[\left(s^i_{x,y}\right)^* r_x\right]\otimes 1\right\}\right.\right. \\
&\qquad\qquad \left.\left.\{u^x[(ap_x)\otimes 1]\}\left\{\left[r_x^*\left(s^i_{x,y}\right)\right]\otimes 1\right\}\right)\right\} \\
&= \sum_{x\in\mathrm{supp}(a)} (\dim x) \sum_{y\in\mathrm{supp}_H(x)} \sum_{i=1}^{d_{x,y}} \\
&\qquad \left\{(\mathrm{Tr}_{\mathscr{H}_y}\otimes\mathrm{id})\left(\{u^x[(ap_x)\otimes 1]\}\left\{\left[r_x^*\left(s^i_{x,y}\right)\left(s^i_{x,y}\right)^* r_x\right]\otimes 1\right\}\right)\right\} \\
&= \sum_{x\in\mathrm{supp}(a)} (\dim x)\left\{(\mathrm{Tr}_{\mathscr{H}_x}\otimes\mathrm{id})\left(\{u^x[(ap_x)\otimes 1]\}\{(r_x^* r_x)\otimes 1\}\right)\right\} \\
&= \sum_{x\in\mathrm{supp}(a)} (\dim x)\cdot \mathfrak{E}_K\left[(\mathrm{Tr}_{\mathscr{H}_x}\otimes\mathrm{id})\left(u^x[(ap_x)\otimes 1]\right)\right] = \mathfrak{E}_K(\mathcal{F}_K(a)).
\end{aligned} \tag{I.5.21}$$

Finally, since $s^{(i)}_{x,y}$, r_x are all isometries, we have

$$r_x^*\left(s^i_{x,y}\right)\left(s^i_{x,y}\right)^* r_x \le \mathrm{id}_{\mathscr{H}_x},$$

so by (I.5.16), and the mutual orthogonality of $s^i_{x,y}$, $i = 1, \ldots, d$, we have

$$\begin{aligned}
&\mathrm{Tr}_{\mathscr{H}_y}\left(\widetilde{a}^*\widetilde{a}p_y\right) \\
&= \sum_{x\in\mathrm{supp}_K(y)} \sum_{i=1}^{d_{x,y}} \left(\frac{\dim x}{\dim y}\right)^2 \\
&\qquad \mathrm{Tr}_{\mathscr{H}_x}\left(\left(s^i_{x,y}\right)^* r_x(a^*p_x)r_x^*\left(s^i_{x,y}\right)^*\left(s^i_{x,y}\right)^* r_x(ap_x)r_x^*\left(s^i_{x,y}\right)^*\right) \\
&\le \sum_{x\in\mathrm{supp}_K(y)} \sum_{i=1}^{d_{x,y}} \left(\frac{\dim x}{\dim y}\right)^2 \mathrm{Tr}_{\mathscr{H}_x}\left(\left(s^i_{x,y}\right)^* r_x(a^*ap_x)r_x^*\left(s^i_{x,y}\right)\right) \\
&= \sum_{x\in\mathrm{supp}_K(y)} \sum_{i=1}^{d_{x,y}} \left(\frac{\dim x}{\dim y}\right)^2 \mathrm{Tr}_{\mathscr{H}_x}\left((a^*ap_x)r_x^*\left(s^i_{x,y}\right)\left(s^i_{x,y}\right)^* r_x\right).
\end{aligned} \tag{I.5.22}$$

It follows from (I.5.17) and (I.5.22) that

$$
\begin{aligned}
&\|\widetilde{a}\|_{H,0}^2\\
&= \sum_{y\in\mathrm{Irr}(H)} (\dim y)\,\mathrm{Tr}_{\mathscr{H}_y}\left(\widetilde{a}^*\widetilde{a}p_y\right)\\
&\le \sum_{y\in\mathrm{Irr}(H)} (\dim y) \sum_{x\in\mathrm{supp}_K(y)} \sum_{i=1}^{d_{x,y}} \left(\frac{\dim x}{\dim y}\right)^2\\
&\qquad \mathrm{Tr}_{\mathscr{H}_x}\left(\left(s_{x,y}^i\right)^* r_x(a^*ap_x)r_x^*\left(s_{x,y}^i\right)\right)\\
&= \sum_{x\in\mathrm{supp}(a)} (\dim x) \sum_{y\in\mathrm{supp}_H(x)} \sum_{i=1}^{d_{x,y}} \left(\frac{\dim x}{\dim y}\right)\\
&\qquad \mathrm{Tr}_{\mathscr{H}_x}\left((a^*ap_x)r_x^*\left(s_{x,y}^i\right)\left(s_{x,y}^i\right)^* r_x\right)\\
&\le \sum_{x\in\mathrm{supp}(a)} (\dim x) \sum_{y\in\mathrm{supp}_H(x)} \sum_{i=1}^{d_{x,y}} \mathrm{Tr}_{\mathscr{H}_x}\left((a^*ap_x)r_x^*\left(s_{x,y}^i\right)\left(s_{x,y}^i\right)^* r_x\right)\\
&\quad (\text{since } y\in\mathrm{supp}_H(x) \implies x\subseteq \mathrm{Res}_K^H(y) \implies \dim x\le\dim y)\\
&= \sum_{x\in\mathrm{supp}(a)} (\dim x)\,\mathrm{Tr}_{\mathscr{H}_x}\left(a^*ap_xr_x^*r_x\right)\\
&= \sum_{x\in\mathrm{supp}(a)} (\dim x)\,\mathrm{Tr}_{\mathscr{H}_x}\left(a^*ap_x\right) = \|a\|_{K,0}^2.
\end{aligned}
\tag{I.5.23}
$$

The lemma is now established by (I.5.21) and (I.5.23). □

I.6 Macthed pair of length functions

We now study the length functions on $\widehat{\mathbb{G}}$. Naturally, this is closely related to the representation theory of $\mathbb{G}$ as presented in § I.4. Recall that for any β-orbit $\mathscr{O}$, the notation $\mathrm{Irr}_{\mathscr{O}}(G)$ denotes the set of equivalency classes of $\mathscr{O}$-representations of G, and there is a dagger operation $(\cdot)^\dagger$ on $\coprod_{\mathscr{O}\in\mathrm{Orb}_\beta}\mathrm{Irr}_{\mathscr{O}}(G)$ given by Definition I.4.12. We also recall our classification bijection (Theorem I.4.9)

$$
\begin{aligned}
\mathfrak{R} : \coprod_{\mathscr{O}\in\mathrm{Orb}_\beta} &\mathrm{Irr}_{\mathscr{O}}(G) \to \mathrm{Irr}(\mathbb{G})\\
[U]\in\mathrm{Irr}_{\mathscr{O}}(G) &\mapsto [\mathfrak{R}_{\mathscr{O}}(U)]
\end{aligned}
$$

preserves involution (Theorem I.4.13), where $\mathfrak{R}_{\mathscr{O}}(U)$ is given by the formula (I.4.6) in Lemma I.4.4.

Suppose $l : \mathrm{Irr}(\mathbb{G}) \to \mathbb{R}_{\ge 0}$ is a length function. For each $\mathscr{O}\in\mathrm{Orb}_\beta$, let $l_{\mathscr{O}} : \mathrm{Irr}_{\mathscr{O}}(G)\to\mathbb{R}_{\ge0}$ be the composition of l, $\mathfrak{R}$ and the inclusion

$$
\mathrm{Irr}_{\mathscr{O}}(G) \hookrightarrow \coprod_{\mathscr{O}\in\mathrm{Orb}_\beta}\mathrm{Irr}_{\mathscr{O}}(G).
$$

We also adopt Notations I.4.7, so in particular, the bijection $\Phi_{\{e_\Gamma\}}$ given there allows us to identify $\mathrm{Irr}_{\{e_\Gamma\}}(G)$ with $\mathrm{Irr}(G)$, and we denote $l_{\{e_\Gamma\}}$ by $l_{\widehat{G}} : \mathrm{Irr}(G)\to\mathbb{R}_{\ge0}$ using this identification. On the other hand, for all $\mathscr{O}\in\mathrm{Orb}_\beta$, let $\varepsilon_{\mathscr{O}}$ denote the trivial $\mathscr{O}$-representation of G, i.e. $\varepsilon_{\mathscr{O}} = \sum_{r,s\in\mathscr{O}} e_{r,s}\otimes v_{r,s}$, so that $[\varepsilon_{\mathscr{O}}] = \Phi_\gamma([\varepsilon_{G_\gamma}])$ for every $\gamma\in\mathscr{O}$. We define $l_\Gamma : \Gamma\to\mathbb{R}_{\ge0}$, $\gamma\mapsto l_{\gamma\cdot G}([\varepsilon_{\gamma\cdot G}])$.

Lemma I.6.1. *Using the above notations, the following hold:*

(a) $l_{\widehat{G}}$ *is a length function on* $\widehat{G}$ *and* l_Γ *is a* β*-invariant length function on* Γ*;*

(b) $l_{\{e_\Gamma\}}([\varepsilon_G]) = 0$*;*

(c) *for all* $\mathscr{O} \in \mathrm{Orb}_\beta$ *and* $[U] \in \mathrm{Irr}_{\mathscr{O}}(G)$*, we have* $l_{\mathscr{O}}([U]) = l_{\mathscr{O}^{-1}}([U^\dagger])$*;*

(d) *for* $i = 1, 2, 3$*, let* $\mathscr{O}_i \in \mathrm{Orb}_\beta$*, and* $[U_i] \in \mathrm{Irr}_{\mathscr{O}_i}(G)$*, with* $U_i = \sum_{r,s\in\mathscr{O}_i} e_{r,s} \otimes u^{(i)}_{r,s}$ *being an* $\mathscr{O}_i$*-irreducible* $\mathscr{O}_i$*-representation of* G *(see Definition I.4.6) on* $\ell^2(\mathscr{O}_i) \otimes \mathscr{H}_i$*, if* $\dim \mathrm{Mor}_{G_\gamma}\left(u^{(3)}_{\gamma,\gamma}|_{G_\gamma}, U_1 \times_\gamma U_2\right) \neq 0$ *for some (hence for all, by Lemma I.4.18)* $\gamma \in \mathscr{O}_3$*, then* $l_{\mathscr{O}_3}([U_3]) \le l_{\mathscr{O}_1}([U_1]) + l_{\mathscr{O}_2}([U_2])$*.*

Proof. Since l is a length function on $\widehat{G}$, (a) and (c) are easy consequences of the definitions of $l_{\widehat{G}}$, l_Γ, $l_{\mathscr{O}}$ and the fact that the classification bijection $\mathfrak{R}$ is involution preserving. (b) is a corollary of (a) (l_Γ is a length function on Γ). Assertion (d) is a consequence of Theorem I.4.19 and the fact that l is a length function. □

The above discussion motivates the following definition.

Definition I.6.2. A family of mappings $(l_{\mathscr{O}} : \mathrm{Irr}_{\mathscr{O}}(G) \to \mathbb{R}_{\ge 0})_{\mathscr{O}\in\mathrm{Orb}_\beta}$ indexed by Orb_β is called *affording*, if conditions (b), (c) and (d) in Lemma I.6.1 are satisfied.

Proposition I.6.3. *Let* $\mathfrak{L}$ *be the set of length functions on* $\widehat{\mathbb{G}}$*, and* $\mathfrak{A}$ *be the set of affording family of mappings. Then*

$$\begin{aligned} \Phi : \mathfrak{L} &\to \mathfrak{A} \\ l &\mapsto (l_{\mathscr{O}})_{\mathscr{O}\in\mathrm{Orb}_\beta} \end{aligned}$$

is a well-defined bijection, where $l_{\mathscr{O}} := l \circ \mathfrak{R} \circ \iota_{\mathscr{O}}$ *with* $\iota_{\mathscr{O}}$ *being the inclusion* $\mathrm{Irr}_{\mathscr{O}}(G) \hookrightarrow \coprod_{\mathscr{O}\in\mathrm{Orb}_\beta} \mathrm{Irr}_{\mathscr{O}}(G)$*.*

Proof. That Φ is well-defined follows directly from Lemma I.6.1 and Definition I.6.2. Note that

$$\bigcup_{\mathscr{O}\in\mathrm{Orb}_\beta} \mathrm{Image}(\mathfrak{R} \circ \iota_{\mathscr{O}}) = \coprod_{\mathscr{O}\in\mathrm{Orb}_\beta} \mathrm{Irr}_{\mathscr{O}}(G),$$

since

$$\forall \mathscr{O} \in \mathrm{Orb}_\beta, \qquad \mathrm{Image}(\mathfrak{R} \circ \iota_{\mathscr{O}}) = \mathrm{Irr}_{\mathscr{O}}(G).$$

By the definition of $l_{\mathscr{O}}$, this implies that that Φ is injective.

It remains to show that Φ is surjective. By the definition of Φ, this amounts to prove that for every affording family $(l_{\mathscr{O}})_{\mathscr{O}\in\mathrm{Orb}_\beta} \in \mathfrak{A}$, the mapping $l : \mathrm{Irr}(\mathbb{G}) \to \mathbb{R}_{\ge 0}$ defined by $[\mathfrak{R}_{\mathscr{O}}(U)] \mapsto l_{\mathscr{O}}([U])$ is a length function on $\widehat{\mathbb{G}}$. With the representation theory (Theorem I.4.9, Theorem I.4.13 and Theorem I.4.19) of $\mathbb{G}$ in mind, it is clear that the conditions in Definition I.5.2 correspond exactly to the conditions in Definition I.6.2, hence l is indeed a length function on $\widehat{\mathbb{G}}$. □

Corollary I.6.4. *If* $(l_{\mathscr{O}})_{\mathscr{O}\in\mathrm{Orb}_\beta}$ *is an affording family of mappings, then* $l_\Gamma : \gamma \mapsto l_{\gamma\cdot G}([\varepsilon_{\gamma\cdot G}])$ *is a* β*-invariant (i.e.* $l_\Gamma(\mathscr{O})$ *is a singleton for every* $\mathscr{O} \in \mathrm{Orb}_\beta$*) length function on* $\widehat{G}$*, and* $l_{\widehat{G}} := l_{\{e_\Gamma\}}$ *is a length-function on* $\widehat{G}$*.*

Proof. This follows from Proposition I.6.3 and assertion (a) of Lemma I.6.1. □

There is also a way of producing length functions on $\widehat{\mathbb{G}}$ from length functions on Γ that are β-invariant.

Lemma I.6.5. *Let l_Γ be a length function on Γ. If l_Γ is β-invariant, then the mapping $l_1 : \mathrm{Irr}(\mathbb{G}) \to \mathbb{R}_{\geq 0}$, $[\mathfrak{R}_{\mathscr{O}}(U)] \mapsto l_\Gamma(\gamma)$ where $\gamma \in \mathscr{O}$ is a well-defined length function on $\widehat{\mathbb{G}}$.*

Proof. That l_1 is well-defined follows from the β-invariance of l. The fact that l_1 is a length function follows immediately from the representation theory (Theorem I.4.9, Theorem I.4.13 and Theorem I.4.19) of $\mathbb{G}$. ☐

To facilitate our discussion, we introduce some terminologies in the following definitions, which will be useful later in the proofs of our results on (*RD*) and polynomial growth of $\widehat{\mathbb{G}}$.

Definition I.6.6. Let $\mathscr{F} = (l_{\mathscr{O}})_{\mathscr{O}\in\mathrm{Orb}_\beta}$ be an affording family of mappings. Suppose Φ is defined as in Proposition I.6.3. The length function $l := \Phi^{-1}(\mathscr{F})$ on $\widehat{\mathbb{G}}$ is called *the standard length function* associated with $\mathscr{F}$.

Definition I.6.7. Let l_Γ be the β-invariant length function on Γ as in Corollary I.6.4, and l_1 the length function on $\widehat{\mathbb{G}}$ as in Lemma I.6.5. We say that l_1 is *induced by* l_Γ.

Definition I.6.8. Let $\mathscr{F} = (l_{\mathscr{O}})_{\mathscr{O}\in\mathrm{Orb}_\beta}$ be an affording family of mappings and $(l_\Gamma, l_{\widehat{G}})$ is the matched pair of length functions afforded by $\mathscr{F}$. Let l be the standard length function associated with $\mathscr{F}$ and l_1 be the length function induced by l_Γ. The length function $l + l_1$ on $\widehat{\mathbb{G}}$ is called *the inflated length function* associated with $\mathscr{F}$.

We want to study the permanence of (*RD*) and polynomial growth of the dual of the bicrossed product $\mathbb{G}$. For this purpose, the affording families of mappings contain a little too much information, as we want to relate a length function on $\widehat{\mathbb{G}}$ to only two length functions—one on $\widehat{G}$ and one on Γ—instead of a family of mappings indexed by Orb_β (but the affording families of mappings are still very relevant as they are equivalent to length functions on $\widehat{\mathbb{G}}$ via Proposition I.6.3). To address this problem, we introduce the notion of matched pair of length functions.

Definition I.6.9. Let $l_{\widehat{G}}$ be a length function on $\widehat{G}$, and l_Γ be a length function on Γ, we say that the pair $(l_\Gamma, l_{\widehat{G}})$ is *matched*, if there exists an affording family of mappings $(l_{\mathscr{O}} : \mathrm{Irr}_{\mathscr{O}}(G) \to \mathbb{R}_{\geq 0})_{\mathscr{O}\in\mathrm{Orb}_\beta}$ such that

- for all $[U] \in \mathrm{Irr}_{\{e_\Gamma\}}(G) = \mathrm{Irr}(G)$, we have $l_{\widehat{G}}([U]) = l_{\{e_\Gamma\}}([U])$;
- for all $\mathscr{O} \in \mathrm{Orb}_\beta$, the image $l_\Gamma(\mathscr{O})$ is the singleton $l_{\mathscr{O}}([\varepsilon_{\mathscr{O}}])$.

If this is the case, we say that the family $\{l_{\mathscr{O}} : \mathscr{O} \in \mathrm{Orb}_\beta\}$ *affords the matching of* $(l_\Gamma, l_{\widehat{G}})$.

Definition I.6.10. Let Φ be as in Proposition I.6.3. Suppose l is a length function on $\widehat{\mathbb{G}}$, both the affording family $\Phi(l)$ and the matched pair $(l_\Gamma, l_{\widehat{G}})$ of length functions afforded by $\Phi(l)$ are said to be *induced by* l.

Remark I.6.11. One sees immediately that every length function l on $\widehat{\mathbb{G}}$ is the standard length associated with the matched pair (of length functions) induced by l. On the other hand, it is possible that a given matched pair of length functions can be afforded by more than one affording family of mappings, i.e induced by different length functions on $\widehat{\mathbb{G}}$. Intuitively speaking, some information is lost when we pass from affording family of mappings to matched pair of length functions.

We terminate our discussion of matched pairs of length functions with the following technical result, which is important in our characterization of both polynomial growth and (RD) for $\widehat{\mathbb{G}}$.

Lemma I.6.12. *Let $\left(l_\Gamma, l_{\widehat{G}}\right)$ be a matched pair of length functions that is afforded by some affording family $(l_{\mathscr{O}})_{\mathscr{O}\in \mathrm{Orb}_\beta}$. Let l_1 denote length function on $\widehat{\mathbb{G}}$ induced by l_Γ (Definition I.6.7). Let $\widetilde{l}$ denote the inflated length function on $\widehat{\mathbb{G}}$ associated with $(l_{\mathscr{O}})_{\mathscr{O}\in \mathrm{Orb}_\beta}$ (Definition I.6.8). For every $k \in \mathbb{N}$, put $q_k := \sum_{x\in F_k} p_x \in \ell^\infty(\widehat{G})$ where F_k denotes the set $\left\{x \in \mathrm{Irr}(G) \ : \ l_{\widehat{G}}(x) < k+1\right\}$.*

If $\left(\widehat{G}, l_{\widehat{G}}\right)$ has polynomial growth with $Q(X) \in \mathbb{R}[X]$ such that

$$\forall k \in \mathbb{N}, \qquad \sum_{x\in \mathrm{Irr}(G), l(x)<k+1} (\dim x)^2 \le Q(k),$$

then the following hold:

(a) *for all $k \in \mathbb{N}$ and $a \in Q_{l,k} c_c(\widehat{G})$, we have*

$$\|\mathcal{F}_G(a)\| \le \left(\sqrt{Q(k)}\right) \|a\|_{G,0}; \tag{I.6.1}$$

(b) *for all $y \in \mathrm{Irr}(\mathbb{G})$, we have*

$$\dim y \le Q\left(\lfloor \widetilde{l}(y)\rfloor\right); \tag{I.6.2}$$

(c) *for all $\gamma \in \mathscr{O} \in \mathrm{Orb}_\beta$, let $\Phi_\gamma : \mathrm{Irr}(G_\gamma) \to \mathrm{Irr}_{\mathscr{O}}(G)$ be the bijection given in Notations I.4.7. For each $k \in \mathbb{N}$, put $F_{\gamma,k} := \left\{w \in \mathrm{Irr}(G_\gamma) \ : \ l_{\mathscr{O}}\left(\Phi_\gamma(w)\right) < k+1\right\}$, and define*

$$F_{\mathscr{O},k} := \Phi(F_{\gamma,k}) = \{z \in \mathrm{Irr}_{\mathscr{O}}(G) \ : \ l_{\mathscr{O}}(z) < k+1\},$$

and

$$\mathbb{F}_{\mathscr{O},k} := \mathfrak{R}\left(F_{\mathscr{O},k}\right) = \left\{y \in \mathfrak{R}^{-1}\left(\mathrm{Irr}_{\mathscr{O}}(G)\right) \ : \ l(y) < k+1\right\} \subseteq \mathrm{Irr}(\mathbb{G}),$$

we have

$$\begin{aligned} &\sum_{y\in\mathbb{F}_{\mathscr{O},k}} (\dim y)^2 = \sum_{z\in F_{\mathscr{O},k}} (\dim z)^2 \\ &= |\mathscr{O}|^2 \sum_{w\in F_{\gamma,k}} (\dim w)^2 \le |\mathscr{O}|^2 \cdot Q\left(\lfloor l_\Gamma(\gamma)\rfloor + k + 1\right) \end{aligned} \tag{I.6.3}$$

Proof. For every $k \in \mathbb{N}$, put $F_k = \{x \in \mathrm{Irr}(G) \ : \ l(x) < k+1\}$, then by Lemma I.5.8, we have

$$a \in Q_{l,k} c_c(\widehat{G}) \implies \|\mathcal{F}_G(a)\|^2 \le \left(\sum_{x\in F_k} \dim x^2\right) \|a\|^2_{G,0} = Q(k)\|a\|^2_{G,0}. \tag{I.6.4}$$

This proves (a).

For all $y \in \mathrm{Irr}(\mathbb{G})$, there is a unique $\mathscr{O} \in \mathrm{Orb}_\beta$, and an $\mathscr{O}$-irreducible representation $\mathscr{O}$-representation $U = \sum_{r,s\in\mathscr{O}} e_{r,s} \otimes u_{r,s}$ on $\ell^2(\mathscr{O}) \otimes \mathscr{H}$, such that $y = [\mathfrak{R}_{\mathscr{O}}(U)] = \mathfrak{R}\big([U]\big)$. Take a $\gamma \in \mathscr{O}$, and denote the irreducible representation $u_{\gamma,\gamma}|_{G_\gamma}$ of G_γ by u_γ. We know $\gamma^{-1} \in \mathscr{O}^{-1} \in \mathrm{Orb}_\beta$, hence $e_\Gamma \in \mathscr{O}^{-1}\mathscr{O}$. By Proposition I.4.2 assertion (e), and Theorem I.4.15, assertion (b), the characters of the $\mathscr{O}$-representation U and the twisted product $\varepsilon_{\mathscr{O}^{-1}} \times_{e_\Gamma} U$ (which is also a representation of $G_{e_\Gamma} = G$) coincide, hence U and $\varepsilon_{\mathscr{O}^{-1}} \times_{e_\Gamma} U$ are equivalent as representations of G. Recall that U is also equivalent to $\mathrm{Ind}_{G_\gamma}^G(u_\gamma)$ (Proposition I.4.2, assertion (e)), we have

$$U \simeq \mathrm{Ind}(u_\gamma) \simeq \varepsilon_{\mathscr{O}^{-1}} \times_{e_\Gamma} U. \tag{I.6.5}$$

By (I.6.5), we see that

$$\begin{aligned} \forall x \in \mathrm{Irr}(G), \qquad \dim \mathrm{Mor}_G(x, [U]) &= \dim \mathrm{Mor}_{G_\gamma}\left(\mathrm{Res}_{G_\gamma}^G(x)\,,\,[u_\gamma]\right) \\ &= \dim \mathrm{Mor}_{G_\gamma}\left([u_\gamma]\,,\,\mathrm{Res}_{G_\gamma}^G(x)\right) \le \dim x. \end{aligned} \tag{I.6.6}$$

Define

$$\mathrm{supp}(U) := \{x \in \mathrm{Irr}(G) \ : \ x \subseteq [U]\} = \{x \in \mathrm{Irr}(G) \ : \ \dim \mathrm{Mor}_G(x, [U]) \neq 0\}.$$

Since the family $(l_{\mathscr{O}})_{\mathscr{O}\in\mathrm{Orb}_\beta}$ is affording, and it affords $\big(l_\Gamma\,,\,l_{\widehat{G}}\big)$, by (I.6.5) again, we have (recall the identification of $\mathrm{Irr}_{\{e_\Gamma\}}(G)$ with $\mathrm{Irr}(G)$)

$$\begin{aligned} \forall x \in \mathrm{supp}(U), \qquad l_{\widehat{G}}(x) = l_{\{G\}}(x) &\le l_{\mathscr{O}^{-1}}([\varepsilon_{\mathscr{O}^{-1}}]) + l_{\mathscr{O}}([U]) \\ &= l_\Gamma(\gamma^{-1}) + l(y) = l_\Gamma(\gamma) + l(y) = \widetilde{l}(y). \end{aligned} \tag{I.6.7}$$

Combining (I.6.6) and (I.6.7), we have

$$\begin{aligned} \dim y = \dim U &= \sum_{x\in\mathrm{supp}(U)} (\dim x)\cdot\big(\dim \mathrm{Mor}_G(x,[U])\big) \\ &\le \sum_{x\in\mathrm{supp}(U)} (\dim x)^2 \le \sum_{x\in\mathrm{Irr}(G), l_{\widehat{G}}(x)\le\widetilde{l}(y)} (\dim x)^2 \\ &\le \sum_{j=0}^{\lfloor\widetilde{l}(y)\rfloor} P(j) = Q\left(\lfloor\widetilde{l}(y)\rfloor\right). \end{aligned} \tag{I.6.8}$$

This proves (b).

We now establish (c). Recall that for any two $\mathscr{O}$-representations U_1, U_2, the relation $U_1 \sim_{\mathscr{O}} U_2$ implies $U_1 \simeq U_2$ as representations of G. Thus let $\mathrm{Rep}(G)$ denote the set of equivalency classes of finite dimensional unitary representations of G, the mapping

$$\begin{aligned} \rho_{\mathscr{O}} : \mathrm{Irr}_{\mathscr{O}}(G) &\longrightarrow \mathrm{Rep}(G) \\ [U] &\mapsto [U] \end{aligned}$$

is well-defined[8]. It is clear from the proof of (a) that for every $w \in \mathrm{Irr}(G_\gamma)$, we have $(\rho_{\mathscr{O}} \circ \Phi_\gamma)(w) = \mathrm{Ind}_{G_\gamma}^G(w)$. Thus

$$\begin{aligned} & w \in F_{\gamma,k} \implies \Phi_\gamma(w) \in F_{\mathscr{O},k} \\ & \xRightarrow{\text{by (I.6.7)}} \forall x \in \mathrm{supp}\big(\mathrm{Ind}_{G_\gamma}^G(w)\big),\ l_{\widehat{G}}(x) < l_\Gamma(\gamma) + k + 1. \end{aligned} \tag{I.6.9}$$

[8]However, in general $\rho_{\mathscr{O}}$ is neither injective nor surjective

But for all $x \in \mathrm{Irr}(G)$, we have $x \subseteq \mathrm{Ind}_{G_\gamma}^G(w)$ if and only if $w \subseteq \mathrm{Res}_{G_\gamma}^G(x)$, thus as subsets of $\mathrm{Irr}(G_\gamma)$, we have

$$F_{\gamma,k} \subseteq \bigcup_{\substack{x \in \mathrm{Irr}(G),\\ l_{\widehat{G}}(x) < l_\Gamma(\gamma)+k+1}} \mathrm{supp}\big(\mathrm{Res}_{G_\gamma}^G(x)\big).$$

Hence $F_{\gamma,k}$ is finite, and $q_{\gamma,k} := \sum_{w \in F_{\gamma,k}} p_w$ is a central projection in $c_c(\widehat{G_\gamma})$. A simple calculation shows that

$$\mathcal{F}_{G_\gamma}(q_{\gamma,k}) = \sum_{w \in F_{\gamma,k}} (\dim w)\chi(w). \tag{I.6.10}$$

By Lemma I.5.16 and (I.6.9), there exists a $\widetilde{q_{\gamma,k}} \in c_c(\widehat{G})$ with $v_{\gamma,\gamma}\mathcal{F}_G(\widetilde{q_{\gamma,k}}) = \mathcal{F}_{G_\gamma}(q_{\gamma,k})$, $\|\widetilde{q_{\gamma,k}}\|_{G,0} \leq \|q_{\gamma,k}\|_{G_\gamma,0}$, and $l_{\widehat{G}}(x) < l_\Gamma(\gamma)+k+1$ whenever $\widetilde{q_{\gamma,k}}p_x \neq 0$. Since $[\chi(w)](e_G) = \dim w$, by (I.6.10) and (a), we have

$$\begin{aligned}
\left[\sum_{w \in F_{\gamma,k}} (\dim w)^2\right]^2 &= \left[\sum_{w \in F_{\gamma,k}} (\dim w)\,\{[\chi(w)](e_G)\}\right]^2 \\
&\leq \left\|\sum_{w \in F_{\gamma,k}} (\dim w)\chi(w)\right\|^2 = \left\|\mathcal{F}_{G_\gamma}(q_{\gamma,k})\right\|^2 \\
&= \left\|v_{\gamma,\gamma}\mathcal{F}_G(\widetilde{q_{\gamma,k}})\right\|^2 \\
&\leq \left\|\mathcal{F}_G(\widetilde{q_{\gamma,k}})\right\|^2 \leq Q\big(\lfloor l_\Gamma(\gamma)\rfloor + k+1\big)\left\|\widetilde{q_{\gamma,k}}\right\|_{G,0}^2 \\
&\leq Q\big(\lfloor l_\Gamma(\gamma)\rfloor + k+1\big)\left\|q_{\gamma,k}\right\|_{G_\gamma,0}^2 \\
&= Q\big(\lfloor l_\Gamma(\gamma)\rfloor + k+1\big) \sum_{w \in F_{\gamma,k}} (\dim w)^2.
\end{aligned} \tag{I.6.11}$$

Now (c) follows from (I.6.11) by noting that $\dim \mathfrak{R}\big(\Phi(w)\big) = \dim \Phi_\gamma(w) = |\mathscr{O}| \cdot \dim w$. □

I.7 Polynomial growth of $\widehat{\mathbb{G}}$

We begin by giving a necessary condition for a pair $\big(\widehat{\mathbb{G}}, l\big)$ to have polynomial growth.

Proposition I.7.1. *Suppose l is a length function on $\widehat{\mathbb{G}}$. Let $(l_\Gamma, l_{\widehat{G}})$ be matched pair of length functions induced by l. If $P(X) \in \mathbb{R}[X]$ and*

$$\forall k \in \mathbb{N}, \qquad a \in q_{l,k}c_c(\widehat{\mathbb{G}}) \implies \big|\{y \in \mathrm{Irr}(\mathbb{G}) \;:\; k \leq l(y) < k+1\}\big| \leq P(k),$$

then for all $k \in \mathbb{N}$, we have

$$\{\gamma \in \Gamma \;:\; k \leq l_\Gamma(\gamma) < k+1\} \leq P(k), \tag{I.7.1}$$

and

$$\sum_{x \in \mathrm{Irr}(G),\, k \leq l_{\widehat{G}}(x) < k+1} (\dim x)^2 \leq P(k). \tag{I.7.2}$$

Proof. Let $(l_{\mathscr{O}})_{\mathscr{O}\in\mathrm{Orb}_\beta}$ be the affording family induced by l. For every class x of $\{e_\Gamma\}$-representation of G (i.e. $x \in \mathrm{Irr}_{\{e_\Gamma\}}(G)$), we have $l_{\{e_\Gamma\}}(x) = l\big(\mathfrak{R}(x)\big)$ as well as $\dim x = \dim \mathfrak{R}(x)$. We identify $\mathrm{Irr}_{\{e_\Gamma\}}(G)$ with $\mathrm{Irr}(G)$ using Φ_{e_Γ} as in Notations I.4.7. Note that the classification bijection $\mathfrak{R}$ preserves dimensions, we have

$$\begin{aligned}\forall n \in \mathbb{N}, \qquad &\mathfrak{R}\Big(\big\{x \in \mathrm{Irr}(G) \ : \ l_{\widehat{G}}(x) = l_{\{e_\Gamma\}}(x) = l\big(\mathfrak{R}(x)\big) \in [n, n+1[\big\}\Big) \\ &\subseteq \big\{y \in \mathrm{Irr}(\mathbb{G}) \ : \ l(y) \in [n, n+1[\big\},\end{aligned}$$

which clearly implies (I.7.2). Moreover, for all $n \in \mathbb{N}$, we have

$$\begin{aligned}&\Big|\Big\{\gamma \in \Gamma \ : \ l_\Gamma(\gamma) = l_{\gamma\cdot G}([\varepsilon_{\gamma\cdot G}]) = l\big(\mathfrak{R}([\varepsilon_{\gamma\cdot G}])\big) \in [n, n+1[\Big\}\Big| \\ &= \sum_{\substack{\gamma\in\Gamma, \\ l(\mathfrak{R}([\varepsilon_{\gamma\cdot G}]))\in[n,n+1[}} 1 = \sum_{\substack{\mathscr{O}\in\mathrm{Orb}_\beta, \\ l(\mathfrak{R}([\varepsilon_{\mathscr{O}}]))=l_{\mathscr{O}}([\varepsilon_{\mathscr{O}}])\in[n,n+1[}} |\mathscr{O}| \\ &\le \sum_{\substack{y\in\mathrm{Irr}(\mathbb{G}), \\ n\le l(y)<n+1}} (\dim y)^2 \le P(n),\end{aligned}$$

where the first inequality follows from the fact that $\dim y \ge |\mathscr{O}| \ge 1$ if y lies in $\mathfrak{R}\big(\mathrm{Irr}_{\mathscr{O}}(G)\big)$, and the second inequality holds because of our choice of the polynomial $P(X)$. This establishes (I.7.1). ◻

The following result is a close converse[9] to Proposition I.7.1.

Proposition I.7.2. *Suppose $\big(l_\Gamma, l_{\widehat{G}}\big)$ is a matched pair of length functions. If*

$$P(X), Q(X) \in \mathbb{R}[X]$$

satisfy

$$\big|\big\{\gamma \in \Gamma \ : \ l_\Gamma(\gamma) < k+1\big\}\big| \le P(k),$$

and (see Notations I.5.9)

$$b \in Q_{l_{\widehat{G}},k} c_c(\widehat{G}) \implies \|\mathcal{F}_G(b)\| \le Q(k)\|b\|_{G,0}$$

for all $k \in \mathbb{N}$, then for all $k \in \mathbb{N}$, we have

$$\sum_{y\in\mathrm{Irr}(\mathbb{G}), \widetilde{l}(y)<k+1} (\dim y)^2 \le [P(k)]^3 Q(2k+2), \tag{I.7.3}$$

where $\widetilde{l}$ is the inflated *length function associated with any affording family that affords $\big(l_\Gamma, l_{\widehat{G}}\big)$.*

Proof. Let $(l_{\mathscr{O}})_{\mathscr{O}\in\mathrm{Orb}_\beta}$ be an affording family that affords $\big(l_\Gamma, l_{\widehat{G}}\big)$ and $\widetilde{l}$ is the associated inflated length function on $\widehat{\mathbb{G}}$. Take any $k \in \mathbb{N}$. For all $\mathscr{O} \in \mathrm{Orb}_\beta$, define

$$R_{\mathscr{O},k} := \Big\{z \in \mathrm{Irr}_{\mathscr{O}}(G) \ : \ \widetilde{l}\big(\mathfrak{R}(z)\big) < k+1\Big\} \subseteq \mathrm{Irr}_{\mathscr{O}}(G),$$

[9] Note that we will use the inflated length function instead of the standard length function associated with the corresponding affording family.

and

$$F_k := \left\{\mathscr{O} \in \mathrm{Orb}_\beta \ : \ R_{\mathscr{O},k} \neq \emptyset\right\}.$$

Using the classification bijection $\mathfrak{R} : \coprod_{\mathscr{O}\in\mathrm{Orb}_\beta} \mathrm{Irr}_{\mathscr{O}}(G) \longrightarrow \mathrm{Irr}(\mathbb{G})$, we obtain

$$\left\{y \in \mathrm{Irr}(\mathbb{G}) \ : \ \widetilde{l}(y) < k+1\right\} = \coprod_{\mathscr{O}\in F_k} \mathfrak{R}\big(R_{\mathscr{O},k}\big). \tag{I.7.4}$$

If $R_{\mathscr{O},k} \neq \emptyset$, then for all $\gamma \in \mathscr{O}$, $z \in R_{\mathscr{O},k}$, we have

$$l_\Gamma(\gamma) \leq l_{\mathscr{O}}(z) + l_\Gamma(\gamma) = \widetilde{l}\big(\mathfrak{R}(z)\big) < k+1. \tag{I.7.5}$$

Hence

$$z \in \coprod_{\mathscr{O}\in F_k} R_{\mathscr{O},k} \implies l_{\mathscr{O}}(z) < k+1, \tag{I.7.6}$$

and

$$\coprod_{\mathscr{O}\in F_k} \mathscr{O} \subseteq \{\gamma \in \Gamma \ : \ l_\Gamma(\gamma) < k+1\}.$$

Consequently,

$$\sum_{\mathscr{O}\in F_k} |\mathscr{O}| \leq P(k).$$

In particular, $|F_k| \leq Q(k)$ and $|\mathscr{O}| \leq P(k)$ whenever $\mathscr{O} \in F_k$. Hence, by Lemma I.6.12, (I.7.5) and (I.7.6), we have

$$\begin{aligned}
\sum_{\substack{y\in\mathrm{Irr}(\mathbb{G}),\\ \widetilde{l}(y)<k+1}} (\dim y)^2 &= \sum_{\mathscr{O}\in F_k} \sum_{z\in R_{\mathscr{O},k}} (\dim \mathfrak{R}(z))^2 = \sum_{\mathscr{O}\in F_k} \sum_{z\in R_{\mathscr{O},k}} (\dim z)^2 \\
&\leq \sum_{\mathscr{O}\in F_k} \sum_{\substack{z\in\mathrm{Irr}_{\mathscr{O}}(G),\\ l_{\mathscr{O}}(z)<k+1}} (\dim z)^2 \leq \sum_{\mathscr{O}\in F_k} |\mathscr{O}|^2 Q(2k+2) \\
&\leq \sum_{\mathscr{O}\in F_k} [P(k)]^2 Q(2k+2) \leq [P(k)]^3 Q(2k+2).
\end{aligned}$$

This proves (I.7.3). ◻

We have the following characterization of the polynomial growth of $\widehat{\mathbb{G}}$.

Theorem I.7.3 (Permanence of polynomial growth)**.** *The following are equivalent:*

(a) *$\widehat{\mathbb{G}}$ has polynomial growth;*

(b) *there exists a matched pair of length functions $\big(l_{\widehat{G}}, l_\Gamma\big)$, such that both $\big(\widehat{G}, l_{\widehat{G}}\big)$ and (Γ, l_Γ) have polynomial growth.*

Proof. That (b) implies (a) follows from Proposition I.7.1, and the reverse implication follows from Proposition I.7.2. ◻

I.8 Rapid decay of $\widehat{\mathbb{G}}$

Obviously the study of (RD) of $\widehat{\mathbb{G}}$ requires a more detailed study of the Fourier transform and the Sobolev-0-norm as defined in § I.5, as well as their interplay with the bicrossed product construction. To facilitate our discussion, let's fix some notations and then prove some preparatory results. In the following, we will freely use the results in § I.4, § I.5 and § I.6 without further explanations.

We first fix a choice function $\theta : \mathrm{Orb}_\beta \to \Gamma$ such that $\theta(\mathscr{O}) \in \mathscr{O}$ for all $\mathscr{O} \in \mathrm{Orb}_\beta$. Then for all $\gamma \in \Gamma$, we write $\theta_\gamma := \theta(\gamma \cdot G)$, then choose and fix a $\sigma_\theta(\gamma) \in G_{\gamma,\theta_\gamma}$. It is clear that whenever $r, s \in \mathscr{O} \in \mathrm{Orb}_\beta$, the mapping $g \mapsto \sigma_\theta(r)g[\sigma_\theta(s)]^{-1}$ is a well-defined homeomorphism from $G_{r,s}$ onto $G_{\theta(\mathscr{O})} = G_{\theta_\gamma}$ for every $\gamma \in \mathscr{O}$, which we denoted by $\psi^\theta_{r,s}$. Thus

$$v_{\theta(\mathscr{O}),\theta(\mathscr{O})} \circ \psi^\theta_{r,s} = v_{\theta_\gamma,\theta_\gamma} \circ \psi^\theta_{r,s} = v_{r,s}.$$

Now for every $\mathscr{O} \in \mathrm{Orb}_\beta$, choose and fix a complete set of representatives

$$\big(u_z : G_{\theta(\mathscr{O})} \to \mathcal{B}(\mathscr{H}_z)\big)_{z\in \mathrm{Irr}_{\mathscr{O}}(G)}$$

of $\mathrm{Irr}(G_{\theta(\mathscr{O})})$, such that for all $z \in \mathrm{Irr}_{\mathscr{O}}(G)$, the $\mathscr{O}$-irreducible $\mathscr{O}$-representation

$$U^z := \sum_{r,s\in\mathscr{O}} e_{r,s} \otimes u^z_{r,s} \in \mathcal{B}\big(\ell^2(\mathscr{O})\big) \otimes \mathcal{B}(\mathscr{H}_z) \otimes \mathrm{Pol}(G) \tag{I.8.1}$$

lies in z, where $u^z_{r,s} \in \mathcal{B}(\mathscr{H}_z) \otimes \mathrm{Pol}(G) \subseteq C(G, \mathcal{B}(\mathscr{H}_z))$ is the unique extension of $u_z \circ \psi^\theta_{r,s} : G_{r,s} \to \mathcal{B}(\mathscr{H}_z)$ by letting $u^z_{r,s}(g) = 0$ when $g \notin G_{r,s}$. We denote such extensions using $\iota_{r,s} : C(G_{r,s}) \to C(G)$, hence

$$\iota_{r,s}(u_z \circ \psi^\theta_{r,s}) = u^z_{r,s}.$$

Thus

$$\coprod_{\mathscr{O}\in\mathrm{Orb}_\beta} \{\mathfrak{R}_{\mathscr{O}}(U^z) \ : \ z \in \mathrm{Irr}_{\mathscr{O}}(G)\}$$

is a complete set of representatives for $\mathrm{Irr}(\mathbb{H})$.

For convenience, we may and do suppose that $\{u_{z_1} \ : \ z_1 \in \mathscr{O}_1\} = \{u_{z_2} \ : \ z_2 \in \mathscr{O}_2\}$ if $G_{\theta(\mathscr{O}_1)} = G_{\theta(\mathscr{O}_2)}$, whenever $\mathscr{O}_1, \mathscr{O}_2 \in \mathrm{Orb}_\beta$, as well as $u_{[\varepsilon_{\mathscr{O}}]} = \varepsilon_{\mathscr{O}}$ in the choices above.

In the following, when we talk about the Fourier transform and the Sobolev-0-norm on $c_c(\widehat{G_{\theta(\mathscr{O})}})$ for all $\mathscr{O} \in \mathrm{Orb}_\beta$, we always mean the corresponding constructions with respect to the complete set of representatives $\{u_z \ : \ z \in \mathrm{Irr}_{\mathscr{O}}(G)\}$ of $\mathrm{Irr}(G_{\theta(\mathscr{O})})$. We recall that $G = G_{\theta(\{e_\Gamma\})}$. And of course, the Fourier transform and the Sobolev-0-norm on $c_c(\widehat{\mathbb{G}})$ is taken with respect to the complete set of representatives

$$\coprod_{\mathscr{O}\in\mathrm{Orb}_\beta} \{\mathfrak{R}_{\mathscr{O}}(U^z) \ : \ z \in \mathrm{Irr}_{\mathscr{O}}(G)\}$$

of $\mathrm{Irr}(\mathbb{G})$.

Using these notations, for all $\mathscr{O} \in \mathrm{Orb}_\beta$, we have

$$c_c(\widehat{G_{\theta(\mathscr{O})}}) = \bigoplus_{z\in\mathrm{Irr}_{\mathscr{O}}}^{\mathrm{alg}} \mathcal{B}(\mathscr{H}_z), \tag{I.8.2}$$

$$\ell^\infty(\widehat{G_{\theta(\mathscr{O})}}) = \bigoplus_{z\in\mathrm{Irr}_{\mathscr{O}}}^{\ell^\infty} \mathcal{B}(\mathscr{H}_z). \tag{I.8.3}$$

Note that $\mathrm{Irr}(\mathbb{G})$ is parameterized by $z \in \mathrm{Irr}_{\mathscr{O}}(G) \mapsto [\mathfrak{R}_{\mathscr{O}}(U^z)]$ where $\mathscr{O}$ runs through Orb_β, we have

$$\begin{aligned} c_c(\widehat{\mathbb{G}}) &= \bigoplus_{\mathscr{O}\in\mathrm{Orb}_\beta}^{\mathrm{alg}} \bigoplus_{z\in\mathrm{Irr}_{\mathscr{O}}(G)}^{\mathrm{alg}} \mathcal{B}\big(\ell^2(\mathscr{O})\big)\otimes\mathcal{B}(\mathscr{H}_z) \\ &= \bigoplus_{\mathscr{O}\in\mathrm{Orb}(\beta)}^{\mathrm{alg}} \mathcal{B}\big(\ell^2(\mathscr{O})\big)\otimes c_c(\widehat{G_{\theta(\mathscr{O})}}), \end{aligned} \tag{I.8.4}$$

$$\begin{aligned} \ell^\infty(\widehat{\mathbb{G}}) &= \bigoplus_{\mathscr{O}\in\mathrm{Orb}_\beta}^{\ell^\infty} \bigoplus_{z\in\mathrm{Irr}_{\mathscr{O}}(G)}^{\ell^\infty} \mathcal{B}\big(\ell^2(\mathscr{O})\big)\otimes\mathcal{B}(\mathscr{H}_z) \\ &= \bigoplus_{\mathscr{O}\in\mathrm{Orb}(\beta)}^{\ell^\infty} \mathcal{B}\big(\ell^2(\mathscr{O})\big)\otimes \ell^\infty(\widehat{G_{\theta(\mathscr{O})}}), \end{aligned} \tag{I.8.5}$$

where we've freely used some canonical identifications.

Each $a \in c_c(\widehat{\mathbb{G}})$ has a unique decomposition

$$a = \sum_{\mathscr{O}\in\mathrm{Orb}_\beta}\sum_{z\in\mathrm{Irr}_{\mathscr{O}}(G)}\sum_{r,s\in\mathscr{O}} e_{r,s}\otimes a^z_{r,s}, \tag{I.8.6}$$

where each $a^z_{r,s} \in \mathcal{B}(\mathscr{H}_z)$, and all but finitely many of them are 0. For each $r,s \in \mathscr{O} \in \mathrm{Orb}_\beta$, we put

$$a_{r,s} := \sum_{z\in\mathrm{Irr}_{\mathscr{O}}(G)} a^z_{r,s} \in c_{c,\mathscr{O}}(\widehat{\mathbb{G}}). \tag{I.8.7}$$

Lemma I.8.1. *Using the above notations, for each $a \in c_c(\widehat{\mathbb{G}})$, we have*

$$\mathcal{F}_{\mathbb{G}}(a) = \sum_{\mathscr{O}\in\mathrm{Orb}_\beta} |\mathscr{O}| \sum_{r,s\in\mathscr{O}} u_r\iota_{r,s}\left(\mathcal{F}_{G_{\theta(\mathscr{O})}}(a_{s,r})\circ\psi^\theta_{r,s}\right), \tag{I.8.8}$$

and

$$\|a\|^2_{\mathbb{G},0} = \sum_{\mathscr{O}\in\mathrm{Orb}_\beta} |\mathscr{O}| \sum_{r,s\in\mathscr{O}} \left\|a_{r,s}\right\|^2_{G_{\theta(\mathscr{O})},0}. \tag{I.8.9}$$

Proof. By definition, for all $r,s \in \mathscr{O} \in \mathrm{Orb}_\beta$ and $z \in \mathrm{Irr}_{\mathscr{O}}(G)$, we have

$$\begin{aligned} \mathfrak{R}_{\mathscr{O}}(U^z)\sum_{r,s\in\mathscr{O}} e_{r,s}\otimes a^z_{r,s}\otimes 1 &= \left(\sum_{r,s\in\mathscr{O}} e_{r,s}\otimes(\mathrm{id}\otimes u_r)u_{r,s}\right)\left(\sum_{r,s\in\mathscr{O}} e_{r,s}\otimes a^z_{r,s}\otimes 1\right) \\ &= \sum_{r,s} e_{r,s}\otimes\sum_{t\in\mathscr{O}}(\mathrm{id}\otimes u_r)u_{r,t}(a^z_{t,s}\otimes 1), \end{aligned}$$

and

$$\begin{aligned} &(\mathrm{Tr}_{\mathscr{H}_z}\otimes\mathrm{id})\left[u^z_{r,s}(a^z_{s,r}\otimes 1)\right] \\ &= (\mathrm{Tr}_{\mathscr{H}_z}\otimes\mathrm{id})\left[\iota_{r,s}(u_z\circ\psi^\theta_{r,s})(a^z_{s,r}\otimes 1)\right] \\ &= \iota_{r,s}\Big((\mathrm{Tr}_{\mathscr{H}_z}\otimes\mathrm{id})\left[(u_z\circ\psi^\theta_{r,s})(a^z_{s,r}\otimes 1)\right]\Big) \\ &= \iota_{r,s}\Big(\Big\{(\mathrm{Tr}_{\mathscr{H}_z}\otimes\mathrm{id})\left[u_z(a^z_{s,r}\otimes 1)\right]\Big\}\circ\psi^\theta_{r,s}\Big). \end{aligned}$$

Hence

$$
\begin{aligned}
&\mathcal{F}_{\mathbb{G}}(a)\\
&= \sum_{\mathcal{O}\in\mathrm{Orb}_\beta}\sum_{z\in\mathrm{Irr}_{\mathcal{O}}(G)}|\mathcal{O}|(\dim\mathscr{H}_z)\\
&\qquad\left\{(\mathrm{Tr}_{\ell^2(\mathcal{O})}\otimes\mathrm{Tr}_{\mathscr{H}_z}\otimes\mathrm{id})\left[\mathfrak{R}_{\mathcal{O}}(U^z)\sum_{r,s\in\mathcal{O}}(e_{r,s}\otimes a^z_{r,s}\otimes 1)\right]\right\}\\
&= \sum_{\mathcal{O}\in\mathrm{Orb}_\beta}\sum_{z\in\mathrm{Irr}_{\mathcal{O}}(G)}|\mathcal{O}|(\dim\mathscr{H}_z)\\
&\qquad\sum_{r,s\in\mathcal{O}}(\mathrm{Tr}_{\ell^2(\mathcal{O})}\otimes\mathrm{Tr}_{\mathscr{H}_z}\otimes\mathrm{id})\left[e_{r,s}\otimes\sum_{t\in\mathcal{O}}(\mathrm{id}\otimes u_r)u^z_{r,t}(a^z_{t,s}\otimes 1)\right]\\
&= \sum_{\mathcal{O}\in\mathrm{Orb}_\beta}\sum_{z\in\mathrm{Irr}_{\mathcal{O}}(G)}|\mathcal{O}|(\dim\mathscr{H}_z)\sum_{r,t\in\mathcal{O}}(\mathrm{Tr}_{\mathscr{H}_z}\otimes\mathrm{id})\left[(\mathrm{id}\otimes u_r)u^z_{r,t}(a^z_{t,r}\otimes 1)\right]\\
&= \sum_{\mathcal{O}\in\mathrm{Orb}_\beta}|\mathcal{O}|\sum_{r,s\in\mathcal{O}}u_r\sum_{z\in\mathrm{Orb}_{\mathcal{O}}(G)}(\dim\mathscr{H}_z)\left\{(\mathrm{Tr}_{\mathscr{H}_z}\otimes\mathrm{id})\left[u^z_{r,s}(a^z_{s,r}\otimes 1)\right]\right\}\\
&= \sum_{\mathcal{O}\in\mathrm{Orb}_\beta}|\mathcal{O}|\sum_{r,s\in\mathcal{O}}u_r\\
&\qquad\sum_{z\in\mathrm{Orb}_{\mathcal{O}}(G)}(\dim\mathscr{H}_z)\iota_{r,s}\left(\left\{(\mathrm{Tr}_{\mathscr{H}_z}\otimes\mathrm{id})\left[u_z(a^z_{s,r}\otimes 1)\right]\right\}\circ\psi^\theta_{r,s}\right)\\
&= \sum_{\mathcal{O}\in\mathrm{Orb}_\beta}|\mathcal{O}|\sum_{r,s\in\mathcal{O}}u_r\\
&\qquad\iota_{r,s}\left(\left\{\sum_{z\in\mathrm{Orb}_{\mathcal{O}}(G)}(\dim\mathscr{H}_z)(\mathrm{Tr}_{\mathscr{H}_z}\otimes\mathrm{id})\left[u_z(a^z_{s,r}\otimes 1)\right]\right\}\circ\psi^\theta_{r,s}\right)\\
&= \sum_{\mathcal{O}\in\mathrm{Orb}_\beta}|\mathcal{O}|\sum_{r,s\in\mathcal{O}}u_r\iota_{r,s}\left(\mathcal{F}_{G_{\theta(\mathcal{O})}}(a_{s,r})\circ\psi^\theta_{r,s}\right).
\end{aligned}
$$

This proves (I.8.8).

We also have

$$
\begin{aligned}
\|a\|^2_{\mathbb{G},0} &= \sum_{\mathcal{O}\in\mathrm{Orb}_\beta}\sum_{z\in\mathrm{Irr}_{\mathcal{O}}(G)}|\mathcal{O}|(\dim\mathscr{H}_z)\\
&\qquad(\mathrm{Tr}_{\ell^2(\mathcal{O})}\otimes\mathrm{Tr}_{\mathscr{H}_z})\left(\left[\sum_{r,s\in\mathcal{O}}e_{r,s}\otimes a^z_{r,s}\right]^*\left[\sum_{r,s\in\mathcal{O}}e_{r,s}\otimes a^z_{r,s}\right]\right)\\
&= \sum_{\mathcal{O}\in\mathrm{Orb}_\beta}\sum_{z\in\mathrm{Irr}_{\mathcal{O}}(G)}|\mathcal{O}|(\dim\mathscr{H}_z)\\
&\qquad(\mathrm{Tr}_{\ell^2(\mathcal{O})}\otimes\mathrm{Tr}_{\mathscr{H}_z})\left(\sum_{r,s\in\mathcal{O}}e_{r,s}\otimes\sum_{t\in\mathcal{O}}(a^z_{t,r})^*(a^z_{t,s})\right)\\
&= \sum_{\mathcal{O}\in\mathrm{Orb}_\beta}|\mathcal{O}|\sum_{r,t\in\mathcal{O}}\sum_{z\in\mathrm{Irr}_{\mathcal{O}}(G)}(\dim\mathscr{H}_z)\,\mathrm{Tr}_{\mathscr{H}_z}\left((a^z_{t,r})^*(a^z_{t,r})\right)\\
&= \sum_{\mathcal{O}\in\mathrm{Orb}_\beta}|\mathcal{O}|\sum_{r,t\in\mathcal{O}}\left\|a_{t,r}\right\|^2_{G_{\theta(\mathcal{O})}},
\end{aligned}
$$

which proves (I.8.9). ◻

Recall Notations I.5.9, and we are ready to prove the following result.

Proposition I.8.2. *Let* $(l_\Gamma, l_{\widehat{G}})$ *be a matched pair of length functions. Suppose*

$$P(X), Q(X) \in \mathbb{R}[X]$$

such that

$$\forall k \in \mathbb{N}, \qquad \xi \in Q_{l_\Gamma,k}C_c(\Gamma) \implies \|\xi\|_\lambda \leq P(k)\|a\|_2, \tag{I.8.10}$$

$$\forall k \in \mathbb{N}, \qquad \sum_{x \in \mathrm{Irr}(G),\, l_{\widehat{G}}(x)<k+1} (\dim x)^2 \leq Q(k). \tag{I.8.11}$$

Then

$$\forall k \in \mathbb{N},\ a \in Q_{\widetilde{l},k}c_c(\widehat{G}) \implies \|\mathcal{F}_{\mathbb{G}}(a)\| \leq R(k)\|a\|_{\mathbb{G},0}, \tag{I.8.12}$$

where $\widetilde{l}$ *is the inflated length function associated with any affording family that affords* $(l_\Gamma, l_{\widehat{G}})$*, and*

$$R(k) = P(k)\sqrt{Q(k)Q(k+1)}. \tag{I.8.13}$$

Proof. Take any affording family $(l_{\mathscr{O}})_{\mathscr{O}\in \mathrm{Orb}_\beta}$ that affords $(l_\Gamma, l_{\widehat{G}})$ and let $\widetilde{l}$ (resp. l) be the inflated (resp. standard) length function associated with $(l_{\mathscr{O}})_{\mathscr{O}\in \mathrm{Orb}_\beta}$.

Recall that $\widetilde{\tau}$ is the Haar state on $\mathscr{A} = \mathrm{Pol}(\mathbb{G})$. Let $\|\cdot\|_{\widetilde{\tau},2}$ be the norm on $\mathscr{A}$ when $\mathscr{A}$ is viewed as an inner-product space with the inner product induced by $\widetilde{\tau}$. By the GNS construction for compact quantum groups with respect to the Haar state, (I.8.12) is equivalent to

$$\forall a \in Q_{\widetilde{l},k}c_c(\widehat{\mathbb{G}}), b \in \mathscr{A} \implies \|\mathcal{F}_{\mathbb{G}}(a)b\|_{\widetilde{\tau},2} \leq R(k)\|a\|_{\mathbb{G},0}\|b\|_{\widetilde{\tau},2}. \tag{I.8.14}$$

Now fix arbitrarily a and b in (I.8.14). We pose

$$\forall r, s \in \mathscr{O} \in \mathrm{Orb}_\beta, \qquad \varphi_{r,s} := |\mathscr{O}| \cdot \left[\iota_{r,s}\left(\mathcal{F}_{G_{\theta(\mathscr{O})}}(a_{s,r}) \circ \psi^\theta_{r,s}\right)\right] \in v_{r,s}\,\mathrm{Pol}(G). \tag{I.8.15}$$

Then by Lemma I.8.1, we have

$$\mathcal{F}_{\mathbb{G}}(a) = \sum_{\mathscr{O}\in\mathrm{Orb}_\beta} \sum_{r,s\in\mathscr{O}} u_r\varphi_{r,s} \in \mathrm{Pol}(\mathbb{G}) = \mathscr{A}. \tag{I.8.16}$$

On the other hand, using the direct sum decomposition (I.4.12) in Theorem I.4.9, there exists $\psi_{r,s} \in v_{r,s}\,\mathrm{Pol}(G)$ for all $r, s \in \mathscr{O} \in \mathrm{Orb}_\beta$, such that

$$b = \sum_{\mathscr{O}\in\mathrm{Orb}_\beta} \sum_{r,s\in\mathscr{O}} u_r\psi_{r,s} \in \mathscr{A}. \tag{I.8.17}$$

Using the decomposition (I.4.12) again, we can find a finite subset $F \subseteq \mathrm{Orb}_\beta$, such that (I.8.16) and (I.8.17) can be rewritten respectively as

$$\mathcal{F}_{\mathbb{G}}(a) = \sum_{r\in\theta(F)} \sum_{s\in r\cdot G} u_r\varphi_{r,s} = \sum_{r\in\theta(F)} u_r\varphi_r, \tag{I.8.18}$$

and

$$b = \sum_{r\in\theta(F)} \sum_{s\in r\cdot G} u_r\psi_{r,s} = \sum_{r\in\theta(F)} u_r\psi_r, \tag{I.8.19}$$

where

$$\varphi_r := \sum_{s\in r\cdot G} \varphi_{r,s} \in \mathrm{Pol}(G), \tag{I.8.20}$$

$$\psi_r := \sum_{s\in r\cdot G} \psi_{r,s} \in \mathrm{Pol}(G). \tag{I.8.21}$$

Hence by (I.8.18) and (I.8.19), we have

$$\mathcal{F}_{\mathbb{G}}(a)b = \sum_{r,s\in\theta(F)} u_{rs}[\alpha_s^*(\varphi_r)]\psi_s = \sum_{t\in\theta(F)^2} u_t \sum_{r,s\in\theta(F),\, rs=t} \alpha_s^*(\varphi_r)\psi_s. \tag{I.8.22}$$

Recall τ is the Haar state on $(C(G),\Delta)$, so we use $\|\cdot\|_{\tau,2}$ to denote the L^2-norm on G with respect to the Haar integral τ. We also use $\|\cdot\|_\infty$ to denote the L^∞ norm on $C(G)$. To simplify our calculation, we introduce

$$\varphi := \sum_{r\in\theta(F)} \|\varphi_r\|_\infty \delta_r \in C_c(\Gamma) \tag{I.8.23}$$

$$\psi := \sum_{r\in\theta(F)} \|\psi_r\|_{\tau,2} \delta_r \in C_c(\Gamma). \tag{I.8.24}$$

For each fixed $r \in \theta(F)$, the clopen sets $G_{r,s}$, $s \in r\cdot G$ are disjoint. It follows that the functions $\psi_{r,s} \in v_{r,s}\operatorname{Pol}(G)$, $s \in r\cdot G$ are mutually orthogonal with respect to the Haar integral τ. By definition,

$$\|\psi\|_2^2 = \sum_{r\in\theta(F)} \|\psi_r\|_{\tau,2}^2 = \sum_{r\in\theta(F)} \sum_{s\in r\cdot G} \left\|\psi_{r,s}\right\|_{\tau,2}^2 = \|b\|_{\widetilde{\tau},2}^2. \tag{I.8.25}$$

As $\alpha_s^*(\varphi_r) = \varphi_r \circ \alpha_s$, we always have $\left\|\alpha_s^*(\varphi_r)\right\| = \|\varphi_r\|$. Using $a \in Q_{\widetilde{l},k}c_c(\widehat{\mathbb{G}})$, we have

$$\begin{aligned} &\operatorname{supp}(a) := \left\{y \in \operatorname{Irr}(\mathbb{G}) \;:\; ap_y \neq 0\right\} \\ &= \coprod_{\mathscr{O}\in F} \left\{[\mathfrak{R}_{\mathscr{O}}(z)] \;:\; l_\Gamma\big(\theta(\mathscr{O})\big) + l\big([\mathfrak{R}_{\mathscr{O}}(z)]\big) = \widetilde{l}\big([\mathfrak{R}_{\mathscr{O}}(z)]\big) < k+1\right\}, \end{aligned} \tag{I.8.26}$$

which implies that

$$\forall \mathscr{O} \in F, \qquad l_\Gamma\big(\theta(\mathscr{O})\big) < k+1. \tag{I.8.27}$$

Combining (I.8.25), (I.8.27) and (I.8.10), we have

$$\|\phi * \psi\|_2 \le \|\varphi\|_\lambda \|\psi\|_2 \le P(k)\|\varphi\|_2 \cdot \|\psi\|_2 = P(k)\|\varphi\|_2 \cdot \|b\|_{\widetilde{\tau},2}. \tag{I.8.28}$$

Since $\mathscr{A}_\gamma = u_\gamma \operatorname{Pol}(G)$ are pairwise orthogonal as γ runs through Γ, it follows from (I.8.28), (I.8.22) and (I.8.10) that

$$\begin{aligned} \|\mathcal{F}_{\mathbb{G}}(a)b\|_{\widetilde{\tau},2}^2 &= \sum_{t\in\theta(F)^2} \left\| \sum_{r,s\in\theta(F),\, rs=t} \alpha_s^*(\varphi_r)\psi_s \right\|_{\tau,2}^2 \\ &\le \sum_{t\in\theta(F)^2} \left\{ \sum_{r,s\in\theta(F), rs=t} \left\|\alpha_s^*(\varphi_r)\right\|_\infty \|\psi_s\|_2 \right\}^2 \\ &= \sum_{t\in\theta(F)^2} \left\{ \sum_{r,s\in\theta(F), rs=t} \|\varphi_r\|_\infty \|\psi_s\|_2 \right\}^2 = \|\varphi * \psi\|_2^2 \\ &\le [P(k)]^2 \|\varphi\|_2^2 \|b\|_{\widetilde{\tau},2}^2. \end{aligned} \tag{I.8.29}$$

We now estimate $\|\varphi\|_2$. Recall that for all $\mathscr{O} \in \operatorname{Orb}_\beta$, and

$$\big(u_z : G_{\theta(\mathscr{O})} \to \mathcal{B}(\mathscr{H}_z)\big)_{z\in\operatorname{Irr}_{\mathscr{O}}(G)}$$

is a complete set of representatives for $\mathrm{Irr}(G_{\theta(\mathscr{O})})$. Using (I.8.26) and Lemma I.5.16 again, we can find a $\widetilde{a_{s,t}} \in c_c(\widehat{G})$ whenever $r \in \theta(F)$ and $s \in r \cdot G$, such that

$$v_{r,r}\mathcal{F}_G(\widetilde{a_{s,r}}) = \iota_{r,r}\big(\mathcal{F}_{G_r}(a_{s,r})\big), \qquad \big\|\widetilde{a_{s,r}}\big\|_{G,0} \le \big\|a_{s,r}\big\|_{G_r,0}; \tag{I.8.30}$$

and for all $x \in \mathrm{Irr}(G)$, if $\widetilde{a_{s,r}}p_x \neq 0$, then there exists $\mathscr{O} \in F$ and $z \in \mathrm{Irr}_{\mathscr{O}}(G)$, such that

$$a_{s,t}p_{[u_z]} \neq 0, \qquad \text{and} \qquad x \subseteq \mathrm{Ind}_{G_r}^{G}([u_z]) = [U^z] = [\varepsilon_{\mathscr{O}^{-1}} \times_{e_G} U^z].$$

In particular, $\widetilde{l}(\mathfrak{R}(z)) < k+1$ since $a \in Q_{\widetilde{l},k}c_c(\widehat{\mathbb{G}})$. Hence the family $(l_{\mathscr{O}})_{\mathscr{O}\in\mathrm{Orb}_\beta}$being affording implies that for all $x \in \mathrm{Irr}(G)$, we have

$$\begin{aligned}\widetilde{a_{s,t}}p_x \neq 0 \implies l_{\widehat{G}}(x) &\le l_\Gamma\big([\theta(\mathscr{O})]^{-1}\big) + l_{\mathscr{O}}(z)\\ &= l_\Gamma\big(\theta(\mathscr{O})\big) + l\big([\mathfrak{R}(z)]\big) = \widetilde{l}\big(\mathfrak{R}(z)\big) < k+1.\end{aligned} \tag{I.8.31}$$

Using the disjointness of $G_{r,s}$, $s \in r\cdot G$ for every fixed $r \in \theta(F)$, as well as (I.8.31), (I.8.30), (I.8.15) and Lemma I.6.12 point (a), we have

$$\begin{aligned}\forall r \in \theta(F), \qquad \|\varphi_r\|_\infty &= \Big\|\sum_{s\in r\cdot G}\varphi_{r,s}\Big\|_\infty = \max_{s\in r\cdot G}\big\|\phi_{r,s}\big\|_\infty\\ &= |r\cdot G|\max_{s\in r\cdot G}\big\|\mathcal{F}_{G_r}(a_{s,r})\circ\psi^\theta_{r,s}\big\| = |r\cdot G|\max_{s\in r\cdot G}\big\|\mathcal{F}_{G_r}(a_{s,r})\big\|\\ &= |r\cdot G|\max_{s\in r\cdot G}\big\|v_{r,r}\mathcal{F}_G(\widetilde{a_{s,r}})\big\| \le |r\cdot G|\max_{s\in r\cdot G}\big\|\mathcal{F}_G(\widetilde{a_{s,r}})\big\|\\ &\le |r\cdot G|\sqrt{Q(k)}\left(\max_{s\in r\cdot G}\big\|\widetilde{a_{s,r}}\big\|_{G_r,0}\right)\\ &\le |r\cdot G|\sqrt{Q(k)}\left(\max_{s\in r\cdot G}\big\|a_{s,r}\big\|_{G_r,0}\right)\end{aligned} \tag{I.8.32}$$

Now for each fixed $r \in \theta(F)$, either $a_{s,r} = 0$ for all $s \in r\cdot G$, in which case $\max_{s\in r\cdot G}\big\|a_{s,r}\big\|_{G_r,0} = 0$; or there is some $s \in r\cdot G$ with $a_{s,r} \neq 0$, in which case there exists $z_r \in \mathrm{Irr}_{r\cdot G}(G)$ with $a_{s,r}^{z_r} \neq 0$, hence

$$l_\Gamma(r) + l_{r\cdot G}(z_r) = \widetilde{l}\big(\mathfrak{R}([U^{z_r}])\big) < k+1,$$

and Lemma I.6.12 point (b) implies that

$$|r\cdot G| \le |r\cdot G|\dim z_r = \dim\mathfrak{R}([U^{z_r}]) \le Q\left(\Big\lfloor\widetilde{l}\big(\mathfrak{R}([U^{z_r}])\big)\Big\rfloor\right) \le Q(k+1).$$

Thus by (I.8.32), we always have

$$\begin{aligned}\forall r \in \theta(F), \qquad \|\varphi_r\|_\infty^2 &\le |r\cdot G|^2 Q(k)\left(\max_{s\in r\cdot G}\big\|a_{s,r}\big\|^2_{G_r,0}\right)\\ &\le Q(k)Q(k+1)\left(\max_{s\in r\cdot G}\left(|r\cdot G|\cdot\big\|a_{s,r}\big\|^2_{G_r,0}\right)\right).\end{aligned} \tag{I.8.33}$$

It follows from Lemma I.8.1 and (I.8.33) that

$$\begin{aligned}\|\varphi\|_2^2 &= \sum_{r\in\theta(F)} \|\varphi_r\|_\infty^2 \leq \sum_{r\in\theta(F)} Q(k)Q(k+1)\left(\max_{s\in r\cdot G}\left(|r\cdot G|\cdot\|a_{s,r}\|_{G_r,0}^2\right)\right)\\ &\leq Q(k)Q(k+1)\sum_{\mathscr{O}\in F}|\mathscr{O}|\sum_{r,s\in\mathscr{O}}\|a_{r,s}\|_{G_{\theta(\mathscr{O})},0}^2\\ &= Q(k)Q(k+1)\sum_{\mathscr{O}\in\mathrm{Orb}_\beta}|\mathscr{O}|\sum_{r,s\in\mathscr{O}}\|a_{r,s}\|_{G_{\theta(\mathscr{O})},0}^2\\ &= Q(k)Q(k+1)\|a\|_{\mathbb{G},0}^2.\end{aligned} \tag{I.8.34}$$

Finally, it follows from (I.8.34), (I.8.25) and (I.8.29) that

$$\forall b\in\mathscr{A},\qquad \|\mathcal{F}_{\mathbb{G}}(a)b\|_{\widetilde{\tau},2} \leq P(k)\sqrt{Q(k)Q(k+1)}\|a\|_{\mathbb{G},0}^2\|b\|_{\widetilde{\tau},2}.$$

This establishes (I.8.14) with $R(k)$ given by (I.8.13), hence finishes the proof. □

We also have a necessary condition for $\widehat{\mathbb{G}}$ to have (RD).

Proposition I.8.3. *Let l be a length function on $\widehat{\mathbb{G}}$. Let $(l_\Gamma, l_{\widehat{G}})$ the matched pair of length functions induced by l. If $P(X)\in\mathbb{R}[X]$ satisfies*

$$\forall k\in\mathbb{N},\quad a\in q_{l,k}c_c(\widehat{\mathbb{G}}) \implies \|\mathcal{F}_{\mathbb{G}}(a)\|\leq P(k)\|a\|_{\mathbb{G},0}, \tag{I.8.35}$$

then

$$\forall k\in\mathbb{N},\quad b\in q_{l_{\widehat{G}},k}c_c(\widehat{G}) \implies \|\mathcal{F}_G(a)\|\leq P(k)\|a\|_{G,0}, \tag{I.8.36}$$

and

$$\forall k\in\mathbb{N},\quad \xi\in q_{l_\Gamma,k}C_c(\Gamma) \implies \|\xi\|_\lambda\leq P(k)\|\xi\|_2, \tag{I.8.37}$$

where

$$q_{l_\Gamma,k} := \sum_{\gamma\in\Gamma,\, k\leq l_\Gamma(\gamma)<k+1}\delta_\gamma\in\ell^\infty(\Gamma).$$

Proof. Let $(l_\mathscr{O})_{\mathscr{O}\in\mathrm{Orb}_\beta}$ be the affording family induced by l, and let $(l_\Gamma, l_{\widehat{G}})$ be the matched pair of length functions induced by l, so $(l_\Gamma, l_{\widehat{G}})$ is afforded by $(l_\mathscr{O})_{\mathscr{O}\in\mathrm{Orb}_\beta}$. As usual, we identify $\mathrm{Irr}_{\{e_\Gamma\}}(G)$ with $\mathrm{Irr}(G)$ using the bijection Φ_{e_Γ} in Notations I.4.7. Since $l\big(\mathfrak{R}(x)\big) = l_{\widehat{G}}(x)$ and $\dim\mathfrak{R}(x) = \dim x$ for all $x\in\mathrm{Irr}(G)$, it is clear (I.8.36) follows directly from (I.8.35).

It suffices now to show (I.8.37). Indeed, take any $k\in\mathbb{N}$, and any $\xi\in q_{l_\Gamma,k}C_c(\Gamma)$, which amounts to say that ξ is supported in $\{\gamma\in\Gamma\ :\ l_\Gamma(\gamma)\in[k,k+1[\}$. We pose $\widetilde{\xi}$ to be the unique element in $\ell^\infty(\widehat{\mathbb{G}})$ with $\widetilde{\xi}p_y = 0$ unless $y = \mathfrak{R}([\varepsilon_\mathscr{O}])$ for some $\mathscr{O}\in\mathrm{Orb}_\beta$, in which case

$$\widetilde{\xi}p_y = \widetilde{\xi}p_{\mathfrak{R}([\varepsilon_\mathscr{O}])} = \frac{1}{|\mathscr{O}|}\sum_{\gamma\in\mathscr{O}}\xi(\gamma)e_{\gamma,\gamma}\in\mathcal{B}\big(\ell^2(\mathscr{O})\big).$$

Since ξ is finitely supported, we have in fact $\widetilde{\xi} \in c_c(\widehat{\mathbb{G}})$. We now have (recall that $\varepsilon_{\mathscr{O}} = \sum_{r,s\in\mathscr{O}} e_{r,s} \otimes v_{r,s}$)

$$\begin{aligned}\mathcal{F}_{\mathbb{G}}(\widetilde{\xi}) = \sum_{\mathscr{O}\in\mathrm{Orb}_\beta} \frac{\dim \mathfrak{R}([\varepsilon_{\mathscr{O}}])}{|\mathscr{O}|} \\ \left\{ (\mathrm{Tr}_{\ell^2(\mathscr{O})} \otimes \mathrm{id}) \left(\left[\sum_{r,s\in\mathscr{O}} e_{r,s} \otimes u_r v_{r,s} \right] \left[\sum_{\gamma\in\mathscr{O}} \xi(\gamma) e_{\gamma,\gamma} \otimes 1 \right] \right) \right\} \\ = \sum_{\mathscr{O}\in\mathrm{Orb}_\beta} \xi(\gamma) u_\gamma v_{\gamma,\gamma} = \sum_{\gamma\in\mathrm{supp}(\xi)} \xi(\gamma) u_\gamma v_{\gamma,\gamma},\end{aligned} \tag{I.8.38}$$

and

$$\begin{aligned}\left\| \widetilde{\xi} \right\|^2_{\mathbb{G},0} &= \sum_{\mathscr{O}\in\mathrm{Orb}_\beta} \frac{\dim \mathfrak{R}([\varepsilon_{\mathscr{O}}])}{|\mathscr{O}|^2} \mathrm{Tr}_{\ell^2(\mathscr{O})} \left(\sum_{\gamma\in\mathscr{O}} \overline{\xi(\gamma)} \xi(\gamma) e_{\gamma,\gamma} \right) \\ &= \sum_{\gamma\in\mathrm{supp}(\xi)} \frac{\overline{\xi(\gamma)}\xi(\gamma)}{|\gamma\cdot G|} \le \sum_{\gamma\in\mathrm{supp}(\xi)} |\xi(\gamma)|^2 = \|\xi\|_2^2.\end{aligned} \tag{I.8.39}$$

Moreover, we have

$$\gamma \in \mathscr{O} \in \mathrm{Orb}_\beta \implies l\big(\mathfrak{R}([\varepsilon_{\mathscr{O}}])\big) = l_{\mathscr{O}}([\varepsilon_{\mathscr{O}}]) = l_\Gamma(\gamma).$$

Thus $\widetilde{\xi}$ is supported in $\{y \in \mathrm{Irr}(\mathbb{G}) \ : \ l(y) \in [k, k+1[\}$. It follows from (I.8.38), (I.8.39), and our choice of $P(X)$ that

$$\left\| \sum_{\gamma\in\mathrm{supp}(\xi)} \xi(\gamma) u_\gamma v_{\gamma,\gamma} \right\| = \left\| \mathcal{F}_{\mathbb{G}}(\widetilde{\xi}) \right\| \le P(k) \left\| \widetilde{\xi} \right\|_{\mathbb{G},0} \le P(k)\|\xi\|_2. \tag{I.8.40}$$

Let $C_r^*(\Gamma)$ be the C^*-algebra in $\mathcal{B}\big(\ell^2(\Gamma)\big)$ generated by $\lambda_\gamma \ : \ \ell^2(\Gamma) \to \ell^2(\Gamma)$, $\delta_r \mapsto \delta_{\gamma r}$, where γ runs through Γ. Then $\mathrm{Pol}(\widehat{\Gamma}) = \mathrm{Vect}\{\lambda_\gamma \ : \ \gamma \in \Gamma\}$ is a dense $*$-subalgebra of $C_r^*(\Gamma)$. One checks immediately that

$$\begin{aligned}\mathrm{Pol}(\mathbb{G}) = \mathscr{A} &\to \mathrm{Pol}(\widehat{\Gamma}) \\ u_\gamma \varphi &\mapsto \varphi(e_G)\lambda_\gamma\end{aligned}$$

is a morphism from the algebraic compact quantum group $\widehat{\Gamma}$ to $\mathbb{G}$, hence extends uniquely to a unital $*$-morphism of C^*-algebras

$$\Psi : C_r(\mathbb{G}) = \Gamma \ltimes_{\widetilde{\alpha},\mathrm{red}} C(G) \to C_r^*(\Gamma).$$

In particular, $\|\Psi\| \le 1$. Posing $\lambda(\xi) := \sum_{\gamma\in\mathrm{supp}(\xi)} \xi(\gamma)\lambda_\gamma$, we have

$$\lambda(\xi) = \Psi\left(\sum_{\gamma\in\mathrm{supp}(\xi)} \xi(\gamma) u_\gamma e_{\gamma,\gamma} \right),$$

which, by (I.8.40), implies that

$$\|\xi\|_\lambda = \|\lambda(\xi)\| \le \left\| \sum_{\gamma\in\mathrm{supp}(\xi)} \xi(\gamma) u_\gamma e_{\gamma,\gamma} \right\| \le P(k)\|\xi\|_2.$$

This proves (I.8.37) and finishes the proof. □

We finally have the following characterization of property (RD) for $\widehat{\mathbb{G}}$.

Theorem I.8.4 (Permanence of rapid decay)**.** *The following are equivalent:*

(a) $\widehat{\mathbb{G}}$ *has* (RD)*;*

(b) *there exists a matched pair of length functions* $\left(l_{\widehat{G}}, l_\Gamma\right)$*, such that* $\left(\widehat{G}, l_{\widehat{G}}\right)$ *has polynomial growth and* (Γ, l_Γ) *has* (RD)*;*

(c) *there exists a matched pair of length functions* $\left(l_{\widehat{G}}, l_\Gamma\right)$*, such that both* $\left(\widehat{G}, l_{\widehat{G}}\right)$ *and* (Γ, l_Γ) *has* (RD)*.*

Proof. By Corollary I.5.15, (b) and (c) are equivalent. By Proposition I.8.2, we have (b) implies (a). By Proposition I.8.3, we have (a) implies (c). □

Chapter II

Representation theory of semidirect products of a compact quantum group with a finite group

Introduction

It is often the case that one can retrieve significant information about representations of a group G from representations of some subgroups of G. As a trivial example, the study of representations of a direct product $G \times H$ of groups of G and H can be easily reduced to the study of representations of G and H separately. However, when one replaces direct products with the more ubiquitous semidirect products, the situation quickly becomes complicated. To get a taste of this complication, the classic (Serre, 1977, §8.2) treats representations of a semidirect product $G \rtimes H$ in the special case where G, H are both finite and G is abelian.

In the setting of locally compact groups and their unitary representations, via the theories of systems of imprimitivity, induced representations, projective representations (a.k.a. ray representations), etc., George Mackey developed a heavy machinery of techniques, often referred as Mackey's analysis, Mackey's machine or the little group method (which is also due to Wigner), to attack such kind of problems. Subsequent works based on Mackey's analysis emerge rapidly, making it one of the most powerful tools to study unitary representations of locally compact groups. For an introduction of this development, we refer the reader to (Mackey, 1958; 1952; 1949; Fell and Doran, 1988; Kaniuth and Taylor, 2013) among the large volumes of literature on this subject.

The author's own interest of this subject comes from the joint work (Fima and Wang, 2018) with Pierre Fima. In (Fima and Wang, 2018), we systematically studied the permanence of property (RD) and polynomial growth of the dual of a bicrossed product of a matched pair consisting of a second countable compact group and a countable discrete group. The natural subsequent question of constructing examples of nontrivial bicrossed products with or without (RD) leads one to study closely the representation theory of semidirect products $G \rtimes \Lambda$ of a compact group G with a finite group Λ. More precisely, as required by the study of length functions relevant to these properties, we need a classification of all irreducible unitary representations of $G \rtimes \Lambda$, the conjugate (which, when we adopt the point of view of topological quantum groups as in this chapter, is also the contragredient since classic groups are of Kac type) of irreducible representations in terms of this classification, and most importantly, the fusion rules of $G \rtimes \Lambda$, i.e. how the tensor product of two irreducible representations decomposes into a direct sum of irreducible representations. While the first two questions can be settled using Mackey's machine as mentioned above, the fusion rules, however, to the best of the author's limited knowledge, are never calculated in the literature.

This chapter treats these questions in the more general setting of semidirect products of the form $\mathbb{G} \rtimes \Lambda$, where $\mathbb{G}$ is a compact quantum group and Λ a finite group. However, instead of using systems of imprimitivity, we introduce the notion of representation parameters (see Definition II.9.8), which appears naturally when we try to analyze the rigid C^*-tensor category $\operatorname{Rep}(\mathbb{G} \rtimes \Lambda)$. Roughly speaking, a representation parameter is a triple (u, V, v), where u is an irreducible representation of $\mathbb{G}$ on some finite dimensional Hilbert space $\mathscr{H}$, V is a unitary projective representation of a certain subgroup Λ_0 of Λ on the same space $\mathscr{H}$, and v is a unitary projective representation of the same Λ_0 on some other finite dimensional Hilbert space, such that V is covariant with u in a certain sense, and V and v have opposing cocycles. Here, the subgroup Λ_0 arises as an isotropy subgroup of a natural action $\Lambda \curvearrowright \operatorname{Irr}(\mathbb{G})$, and the projective representation V is then determined by Schur's lemma on irreducible representations.

As the precise formulation of our main results are long and complicated, we give here only a crude summary of these results in terms of representation parameters (see Definition II.9.8) mentioned above.

(A) Up to equivalence, irreducible unitary representations of $\mathbb{G} \rtimes \Lambda$ are classified by (equivalence classes of) representation parameters (see Theorem II.12.1 for the precise formulation);

(B) The classification in (A) is compatible with the conjugate operation— the conjugate[1] of an irreducible representation of $\mathbb{G} \rtimes \Lambda$ parameterized by some representation parameter (u, V, v) is itself parameterized by the conjugate of (u, V, v) (see Theorem II.13.5 for the precise formulation);

(C) The fusion rules of $\mathbb{G} \rtimes \Lambda$ is calculated by summing a series of incidence numbers, where all of these numbers can be calculated using unitary projective representations of some suitable subgroup of Λ through an explicit reduction procedure (see Theorem II.15.1 for the precise formulation), where the reduction procedure itself is determined by the representation theory of $\mathbb{G}$ and the action of Λ acting on $\mathbb{G}$, with respect to which we form the semidirect product.

While (A) and (B) may well be regarded as the quantum analogue of the corresponding results of Mackey's analysis in the classical case of groups, our result (C) is new, even in the case where $\mathbb{G}$ is another finite group. We should mention that our main idea of this chapter starts with reformulating $\operatorname{Rep}(\mathbb{G})$ as a semisimple rigid C^*-tensor category for an arbitrary compact quantum group $\mathbb{G}$, which is the modern point of view; however, Mackey's ingenious ideas, such as studying the dynamics of a naturally appeared group action on the representations of a normal subgroup, and using projective representations of the isotropy subgroups of this action, still play an essential part in the development of this theory.

[1] The conjugate should *not* be confused with the contragredient, with the contragredient not necessarily unitary if the quantum group is not unimodular (of Kac-type).

We now describe the organization of this chapter. The numerous sections of this chapter are roughly divided into the following four parts. In the first part (§ II.1 and § II.2), we lay out the basic properties and constructions of the objects to be studied in this chapter—semidirect products of a compact quantum group by a finite group and their unitary representations. In § II.2, the problem of describing unitary representations of these semidirect products is reduced to the study of the so-called covariant pair of representations for each of the factors. The second part (§§ II.3- II.6) gives a self-contained treatment of induced representation which will be used later in this chapter. We are aware that there are already much more general theory for induced representations in the quantum setting, e.g. (Kustermans, 2002) based on the classic work (Rieffel, 1974). Moreover S. Vaes has generalizes a large part of Mackey's theory of imprimitivity to locally compact quantum groups in (Vaes, 2005). Besides the obvious reason for fixing the notations, the treatment of the induced representation here is specially tailored to the various calculations in the later half of this chapter. The third part (§§ II.7-II.11) is the technical core of this chapter. The treatment here is largely inspired by Woronowicz's Krein-Tannaka reconstruction (Woronowicz, 1988) of a compact quantum group from its representation category. Here instead of directly attacking the representation category Rep $\mathbb{G} \rtimes \Lambda$ of the semidirect product, we introduce and study a family of rigid C^*-tensor categories (called the category of covariant systems of representations and denoted by $\mathcal{CSR}_{\Lambda_0}$ with respect to some suitable subgroup of Λ), each of which has a simpler structure. Combining the information we have on these simpler C^*-tensor categories allows us not only to classify the irreducible unitary representations of $\mathbb{G} \rtimes \Lambda$, but also to calculate the fusion rules of $\mathbb{G} \rtimes \Lambda$. The details of this classification and calculation are given in the fourth part (§§ II.12-II.15).

Before we proceed further, we feel that we should say a little more about § II.1 for the experts. We emphasize our construction of semidirect products as the axiomatically more elaborate *algebraic* compact quantum groups, the theory of which is developed by van Daele (Van Daele, 1998; 1996; 1994), instead of the more modern and standard formulation, due to Woronowicz (Woronowicz, 1998; 1988; 1987), using C^*-algebras. Of course, these two approaches are essentially equivalent—one passes from Woronowicz's approach to van Daele's via the Peter-Weyl theory for compact quantum groups, and from van Daele's approach to Woronowicz's via the famous GNS construction with respect to the Haar integral. The reasons we prefer van Daele's algebraic theory here are two-fold: on the one hand, one has the advantage of having direct access to the Haar state and the antipode, as well as the polynomial algebra, which are powerful tools for our purposes of studying the representations of these objects (or corepresentations if one insists on viewing these essentially analytic objects as Hopf algebras); on the other hand, when one tries to restrict representations to certain (quantum) subgroups of these semidirect products, as we will do later, one will need to use the counit, which is always everywhere defined in the more elaborate algebraic approach of van Daele, but is merely densely defined if the compact quantum group in the sense of Woronowicz is not universal. We also point out here that the term *semidirect product* in the quantum setting has an unfortunate ambiguity. Nowadays many use this term to refer to the crossed product, as first defined and studied by S. Wang[2] (Wang, 1995). In the case of classical groups, it is long known that this crossed product construction yields the *convolution algebra* of the semidirect of groups. So if we believe classic compact groups are

[2]who has no direct relation to the author, as Wang is a very common Chinese surname.

exactly compact quantum groups whose algebra is commutative, then this is not the correct formulation for semidirect products, even though these are closely related via the convolution operation (which is a manifestation of the quantum version of Pontryagin's duality as developed in (Kustermans and Vaes, 2000), preceded by many important works along the lines of the Kac's program, a history of which is described in the introduction of the above article). To be more precise, this crossed product of S. Wang is in fact a special case of the bicrossed product as described in (Vaes and Vainerman, 2003) where one of the actions for the matched pair is trivial; what we call semidirect in this chapter is a special case of the double crossed product as described in (Baaj and Vaes, 2005) where again one of the actions for the matched pair is trivial. We don't pursue the full generality of the bicrossed product construction and double crossed product construction here, but merely point out that they are all based on the notion of matched pair of (quantum) groups ((Majid, 1990b; 1991),(Takeuchi, 1981)). We also mention in passing works such as (Baaj and Skandalis, 1993), (Majid, 1990a), (Singer, 1972), (Yamanouchi, 2000), and (Vaes and Vainerman, 2003) in the direction of bicrossed products, and works such as (Baaj and Skandalis, 1993), (Majid, 1990a) and (Baaj and Vaes, 2005) in the direction of double crossed products. We hope these backgrounds provide some justification of our choice of terminology for semidirect products by its consistency with the classical group case. We also note that representation theory for compact bicrossed products (which includes the crossed product as a special case) of a matched pair of classical groups are thoroughly investigated in the author's joint work with P. Fima (Fima and Wang, 2018), and as one can see by comparing the results there and the results of this chapter, the representation theory for semidirect products are significantly more delicate than crossed products, even for classical finite groups.

We conclude this introduction by making some conventions. All representations and projective representations in this chapter are finite dimensional. All of them are unitary, except the contragredient of a unitary representation, which may not be unitary when the compact quantum group is not of Kac-type. We also assume all (projective) representations are over a finite dimensional Hilbert space instead of a mere complex vector space. Terminologies and notations concerning compact quantum groups and C^*-tensor categories are largely in consistent with those in (Neshveyev and Tuset, 2013). We also use freely the Peter-Weyl theory for projective representations of finite groups as presented in (Cheng, 2015). We also freely use the Heyenmann-Sweedler notation in performing calculations on comultiplications. The unitary group of unitary transformations from a Hilbert space $\mathscr{H}$ to itself is denoted by $\mathcal{U}(\mathscr{H})$. From § II.8 on, $\mathbb{T}$ denotes the circle group, i.e. the abelian compact group $\{z \in \mathbb{C} \ : \ |z| = 1\}$ viewed as a subgroup of $\mathbb{C}^\times$. Since we often view a representation of compact quantum groups as an operator, we denote the tensor product of representations using $\times$ instead of $\otimes$, as the latter is reserved to denote tensor products of spaces, algebras, linear operators, etc. Finally, throughout this chapter, we fix a compact quantum group $\mathbb{G} = (A, \Delta)$, a finite group Λ, and an antihomomorphism of groups $\alpha^*\colon \Lambda \to \mathrm{Aut}\big(C(\mathbb{G}), \Delta\big)$, where $\mathrm{Aut}\big(C(\mathbb{G}), \Delta\big)$ is the subgroup[3] of $\mathrm{Aut}\big(C(\mathbb{G})\big)$ consisting of automorphisms of the C^*-algebra $C(\mathbb{G})$ that intertwines the comulti-

[3]Note that the notation $\mathrm{Aut}(\mathbb{G})$ has a certain ambiguity which we try to avoid: one the one hand, if we let $\mathbb{G}$ to be a classical compact group, then elements of $\mathrm{Aut}(\mathbb{G})$ are group automorphisms, and the group law of $\mathrm{Aut}(\mathbb{G})$ is given by composition of *set-theoretic* mappings; on the other hand, if we view $\mathbb{G}$ as a Hopf-C^*-algebra, say $\big(C(\mathbb{G}), \Delta\big)$, then $\mathrm{Aut}(\mathbb{G})$ can also be mean the automorphism group of this Hopf-C^*-algebra, whose group law is given by composition of *Hopf algebraic*-morphisms. This is the reason we prefer the more cumbersome notation $\mathrm{Aut}\big(C(\mathbb{G}), \Delta\big)$ instead of the ambiguous but more succinct $\mathrm{Aut}(\mathbb{G})$.

plication Δ.

II.1 Semidirect product of a compact quantum group with a finite group

Let $\mathbb{G} = (A, \Delta)$ be a compact quantum group, Λ a finite group. An action of Λ on $\mathbb{G}$ via quantum automorphisms is an antihomomorphism $\alpha^*\colon \Lambda \longrightarrow \mathrm{Aut}\big(C(\mathbb{G}), \Delta\big)$. One can then form the semi-direct $\mathbb{G} \rtimes_{\alpha^*} \Lambda$, or simply $\mathbb{G} \rtimes \Lambda$ if the action α^* is clear from the context, which is again a compact quantum group. The underlying C^*-algebra $\mathscr{A}$ of $\mathbb{G} \rtimes \Lambda$ is $A \otimes C(\Lambda)$, and the comultiplication $\widetilde{\Delta}$ on $\mathscr{A}$ is determined by

$$\widetilde{\Delta}(a \otimes \delta_r) = \sum_{s\in\Lambda} \big[(\mathrm{id}_A \otimes \alpha_s^*)\Delta(a)\big]_{13} (\delta_s \otimes \delta_{s^{-1}r})_{24} \in A \otimes C(\Lambda) \otimes A \otimes C(\Lambda) \tag{II.1.1}$$

for any $a \in A$ and $r \in \Lambda$. As we've mentioned at the end of the introduction, from now on, $\mathbb{G}$, Λ and the action α^* are fixed until the end of the chapter.

It is clear that $\widetilde{\Delta}$ is a unital $*$-morphism. We now check that in the six-fold tensor product $A \otimes C(\Lambda) \otimes A \otimes C(\Lambda) \otimes A \otimes C(\Lambda)$, we have

$$\forall a \in A, r \in \Lambda, \qquad (\mathrm{id} \otimes \mathrm{id} \otimes \widetilde{\Delta})[\widetilde{\Delta}(a \otimes \delta_r)] = (\widetilde{\Delta} \otimes \mathrm{id} \otimes \mathrm{id})[\widetilde{\Delta}(a \otimes \delta_r)], \tag{II.1.2}$$

i.e. our new comultiplication $\widetilde{\Delta}$ is coassociative. Indeed, put $\Delta^{(2)} := (\mathrm{id} \otimes \Delta)\Delta = (\Delta \otimes \mathrm{id})\Delta$, since $\alpha_s^* \in \mathrm{Aut}\big(C(\mathbb{G}), \Delta\big)$ for all $s \in \Lambda$, we have $(\alpha_s^* \otimes \alpha_s^*) \circ \Delta = \Delta \circ \alpha_s^*$,

$$\begin{aligned}
&(\mathrm{id} \otimes \mathrm{id} \otimes \widetilde{\Delta})[\widetilde{\Delta}(a \otimes \delta_r)] \\
&= (\mathrm{id} \otimes \mathrm{id} \otimes \widetilde{\Delta}) \left(\sum_{s\in\Lambda} [(\mathrm{id} \otimes \alpha_s^*)\Delta(a)]_{13} (\delta_s \otimes \delta_{s^{-1}r})_{24} \right) \\
&= \sum_{s,t\in\Lambda} \Big\{ \big[\big(\mathrm{id} \otimes \alpha_s^* \otimes (\alpha_t^* \circ \alpha_s^*)\big)(\mathrm{id} \otimes \Delta)\Delta\big](a) \Big\}_{135} (\delta_s \otimes \delta_t \otimes \delta_{t^{-1}s^{-1}r})_{246} \\
&= \sum_{s,t\in\Lambda} \Big\{ \big[(\mathrm{id} \otimes \alpha_s^* \otimes \alpha_{st}^*)\Delta^{(2)}\big](a) \Big\}_{135} (\delta_s \otimes \delta_t \otimes \delta_{t^{-1}s^{-1}r})_{246}.
\end{aligned} \tag{II.1.3}$$

On the other hand,

$$\begin{aligned}
&(\widetilde{\Delta} \otimes \mathrm{id} \otimes \mathrm{id})[\widetilde{\Delta}(a \otimes \delta_r)] \\
&= (\widetilde{\Delta} \otimes \mathrm{id} \otimes \mathrm{id}) \left(\sum_{s\in\Lambda} [(\mathrm{id} \otimes \alpha_s^*)\Delta(a)]_{13} (\delta_s \otimes \delta_{s^{-1}r})_{24} \right) \\
&= \sum_{s,t\in\Lambda} \Big\{ \big[(\mathrm{id} \otimes \alpha_t^* \otimes \alpha_s^*)(\Delta \otimes \mathrm{id})\Delta\big](a) \Big\}_{135} (\delta_t \otimes \delta_{t^{-1}s} \otimes \delta_{s^{-1}r})_{246} \\
&\quad (s' = t, t' = t^{-1}s \iff s = s't', t = s') \\
&= \sum_{s',t'\in\Lambda} \Big\{ \big[(\mathrm{id} \otimes \alpha_{s'}^* \otimes \alpha_{s't'}^*)\Delta^{(2)}\big](a) \Big\}_{135} (\delta_{s'} \otimes \delta_{t'} \otimes \delta_{t'^{-1}s'^{-1}r})_{246}.
\end{aligned} \tag{II.1.4}$$

Now (II.1.2) follows from (II.1.3) and (II.1.4).

Since $\mathrm{Pol}(\mathbb{G}) \otimes C(\Lambda)$ is dense in $A \otimes C(\Lambda)$, in order to prove that $\mathbb{G} \rtimes \Lambda$ is indeed a compact quantum group, it suffices to show that $(\mathrm{Pol}(\mathbb{G}) \otimes C(\Lambda), \widetilde{\Delta})$ is an algebraic compact quantum group, i.e. a Hopf $*$-algebra with an invariant state (called the Haar state or Haar integral).

First of all, since $\alpha_s^* \in \mathrm{Aut}\big(C(\mathbb{G}), \Delta\big)$ for all $s \in \Lambda$, we have $\alpha_s^*\big(\mathrm{Pol}(\mathbb{G})\big) = \mathrm{Pol}(\mathbb{G})$, and $\widetilde{\Delta}$ indeed restricts to a well-defined comultiplication on $\mathrm{Pol}(\mathbb{G}) \otimes C(\Lambda)$.

Let ϵ, S be the counit and the antipode respectively for the Hopf $*$-algebra $\mathrm{Pol}(\mathbb{G})$. Denoting the neutral element of the group Λ by e, we define

$$\begin{aligned} \widetilde{\epsilon}\colon \mathrm{Pol}(\mathbb{G}) \otimes C(\Lambda) &\to \mathbb{C} \\ \sum_{r\in\Lambda} x_r \otimes \delta_r &\mapsto \epsilon(x_e), \end{aligned} \tag{II.1.5}$$

and

$$\begin{aligned} \widetilde{S}\colon \mathrm{Pol}(\mathbb{G}) \otimes C(\Lambda) &\to \mathrm{Pol}(\mathbb{G}) \otimes C(\Lambda) \\ \sum_{r\in\Lambda} x_r \otimes \delta_r &\mapsto \sum_{r\in\Lambda} \alpha_r^*(S(x_r)) \otimes \delta_{r^{-1}} = \sum_{r\in\Lambda} S(\alpha_{r^{-1}}^*(x_{r^{-1}})) \otimes \delta_r. \end{aligned} \tag{II.1.6}$$

Since ϵ is a $*$-morphism of algebras, so is $\widetilde{\epsilon}$. Moreover, for any $x \in \mathrm{Pol}(\mathbb{G})$ and $r \in \Lambda$, we have

$$\begin{aligned} (\widetilde{\epsilon} \otimes \mathrm{id})\widetilde{\Delta}(x \otimes \delta_r) &= (\widetilde{\epsilon} \otimes \mathrm{id}) \sum_{s\in\Lambda} \sum x_{(1)} \otimes \delta_s \otimes \alpha_s^*(x_{(2)}) \otimes \delta_{s^{-1}r} \\ &= \sum \epsilon(x_{(1)})\alpha_e^*(x_{(2)}) \otimes \delta_r = \sum \epsilon(x_{(1)})x_{(2)} \otimes \delta_r = x \otimes \delta_r \\ &= \sum x_{(1)}\epsilon(x_{(2)}) \otimes \delta_r = \sum x_{(1)}\epsilon(\alpha_r^*(x_{(2)})) \otimes \delta_r \\ &= (\mathrm{id} \otimes \widetilde{\epsilon}) \sum_{s\in\Lambda} \sum x_{(1)} \otimes \delta_s \otimes \alpha_s^*(x_{(2)}) \otimes \delta_{s^{-1}r} = (\mathrm{id} \otimes \widetilde{\epsilon})\widetilde{\Delta}(x \otimes \delta_r). \end{aligned}$$

Hence $\widetilde{\epsilon}$ is the counit for $\widetilde{\Delta}$. Let $m\colon \mathrm{Pol}(G) \otimes \mathrm{Pol}(G) \to \mathrm{Pol}(G)$ be the multiplication map, and $\widetilde{m}$ the multiplication map on $\mathrm{Pol}(\mathbb{G}) \otimes C(\Lambda)$, then

$$\begin{aligned} \widetilde{m}(\widetilde{S} \otimes \mathrm{id})\widetilde{\Delta}(x \otimes \delta_r) &= \widetilde{m}(\widetilde{S} \otimes \mathrm{id}) \sum_{s\in\Lambda} \sum x_{(1)} \otimes \delta_s \otimes \alpha_s^*(x_{(2)}) \otimes \delta_{s^{-1}r} \\ &= \widetilde{m} \sum_{s\in\Lambda} \sum \alpha_s^*(S(x_{(1)})) \otimes \delta_{s^{-1}} \otimes \alpha_s^*(x_{(2)}) \otimes \delta_{s^{-1}r} \\ &= \sum_{s\in\Lambda} \big[m(S \otimes \mathrm{id})(\alpha_s^* \otimes \alpha_s^*)\Delta(x)\big] \otimes \delta_{s^{-1}} \cdot \delta_{s^{-1}r} \\ &= \delta_{e,r} \sum_{s\in\Lambda} \big[m(S \otimes \mathrm{id})\Delta(\alpha_s^*(x))\big] \otimes \delta_{s^{-1}} \\ &= \delta_{e,r} \sum_{s\in\Lambda} \epsilon(\alpha_s^*(x))1_A \otimes \delta_{s^{-1}} \\ &= \delta_{e,r}\epsilon(x)1_A \otimes \sum_{s\in\Lambda} \delta_{s^{-1}} \\ &= \delta_{e,r}\epsilon(x)1_A \otimes 1_{C(\Lambda)} = \widetilde{\epsilon}(x \otimes \delta_r)1_A \otimes 1_{C(\Lambda)}. \end{aligned}$$

Similarly, since for any $s \in \Lambda$,

$$\alpha_{s^{-1}r}^* S \alpha_s^* = \alpha_{s^{-1}r}^* \alpha_s^* S = \alpha_r^* S,$$

we have

$$\begin{aligned}
\widetilde{m}(\mathrm{id}\otimes\widetilde{S})\widetilde{\Delta}(x\otimes\delta_r) &= \widetilde{m}(\mathrm{id}\otimes\widetilde{S})\sum_{s\in\Lambda}\sum x_{(1)}\otimes\delta_s\otimes\alpha_s^*(x_{(2)})\otimes\delta_{s^{-1}r}\\
&= \widetilde{m}\sum_{s\in\Lambda}\sum x_{(1)}\otimes\delta_s\otimes(\alpha_{s^{-1}r}^*S\alpha_s^*)(x_{(2)})\otimes\delta_{r^{-1}s}\\
&= \widetilde{m}\sum_{s\in\Lambda}\sum x_{(1)}\otimes\delta_s\otimes(\alpha_r^*S)(x_2)\otimes\delta_{r^{-1}s}\\
&= \delta_{e,r}\sum_{s\in\Lambda}\sum x_{(1)}[S(x_{(2)})]\otimes\delta_s = \delta_{e,r}\sum_{s\in\Lambda}\epsilon(x)1_A\otimes\delta_s\\
&= \delta_{e,r}\epsilon(x)1_A\otimes 1_{C(\Lambda)} = \widetilde{\epsilon}(x\otimes\delta_r)1_A\otimes 1_{C(\Lambda)}.
\end{aligned}$$

Therefore, $\widetilde{S}$ is the antipode for $(\mathrm{Pol}(\mathbb{G})\otimes C(\Lambda),\widetilde{\Delta})$.

It remains to construct the Haar state on the Hopf $*$-algebra $\mathrm{Pol}(\mathbb{G})\otimes C(\Lambda)$. Suppose $h\colon A\to\mathbb{C}$ is the Haar state on $\mathbb{G}$, define

$$\begin{aligned}
\widetilde{h}\colon\ \mathrm{Pol}(\mathbb{G})\otimes C(\Lambda) &\to \mathbb{C}\\
\sum_r x_r\otimes\delta_r &\mapsto |\Lambda|^{-1}\sum_{r\in\Lambda}h(x_r).
\end{aligned}\tag{II.1.7}$$

It is obvious that $\widetilde{h}$ is a state. For any $x\in\mathrm{Pol}(\mathbb{G})$, $r\in\Lambda$,

$$\begin{aligned}
(\widetilde{h}\otimes\mathrm{id})\widetilde{\Delta}(x\otimes\delta_r) &= |\Lambda|^{-1}\sum_{s\in\Lambda}\sum h(x_{(1)})\alpha_s^*(x_{(2)})\otimes\delta_{s^{-1}r}\\
&= |\Lambda|^{-1}\sum_{s\in\Lambda}\alpha_s^*\Big(\sum h(x_{(1)})x_{(2)}\Big)\otimes\delta_{s^{-1}r}\\
&= |\Lambda|^{-1}\sum_{s\in\Lambda}\alpha_s^*(h(x)1_A)\otimes\delta_{s^{-1}r}\\
&= |\Lambda|^{-1}h(x)\sum_{s\in\Lambda}1_A\otimes\delta_{s^{-1}r}\\
&= \widetilde{h}(x\otimes\delta_r)1_A\otimes 1_{C(\Lambda)}.
\end{aligned}$$

The uniqueness of the Haar state implies that $h\circ\alpha_s^*=h$ for any $s\in\Lambda$, hence

$$\begin{aligned}
(\mathrm{id}\otimes\widetilde{h})\widetilde{\Delta}(x\otimes\delta_r) &= |\Lambda|^{-1}\sum_{s\in\Lambda}\sum x_{(1)}h\big(\alpha_s^*(x_{(2)})\big)\otimes\delta_s\\
&= |\Lambda|^{-1}\sum_{s\in\Lambda}\sum x_{(1)}h(x_{(2)})\otimes\delta_s\\
&= |\Lambda|^{-1}h(x)1_A\otimes\sum_{s\in\Lambda}\delta_s\\
&= \widetilde{h}(x\otimes\delta_r)1_A\otimes 1_{C(\Lambda)}.
\end{aligned}$$

Therefore, $\widetilde{h}$ is indeed the Haar state on $(\mathrm{Pol}(\mathbb{G})\otimes C(\Lambda),\widetilde{\Delta})$. So far, we've established that $(\mathrm{Pol}(\mathbb{G})\otimes C(\Lambda),\widetilde{\Delta})$ is an algebraic compact quantum group (cf. (Timmermann, 2008, chapter 3)).

Now the density of $\mathrm{Pol}(\mathbb{G})\otimes C(\Lambda)$ in $A\otimes C(\Lambda)$ implies that $(A\otimes C(\Lambda),\widetilde{\Delta})$ is indeed a compact quantum group, with

$$\mathrm{Pol}(\mathbb{G}\rtimes\Lambda)=\mathrm{Pol}(\mathbb{G})\otimes C(\Lambda),\tag{II.1.8}$$

and Haar state (which we still denote by $\widetilde{h}$)

$$\begin{aligned}
\widetilde{h}: A\otimes C(\Lambda) &\to \mathbb{C}\\
\sum x_r\otimes\delta_r &\mapsto |\Lambda|^{-1}\sum_{r\in\Lambda}h(x_r).
\end{aligned}\tag{II.1.9}$$

Furthermore, we've seen that the counit $\widetilde{\epsilon}$ and the antipode $\widetilde{S}$ of the Hopf $*$-algebra $\mathrm{Pol}(\mathbb{G} \rtimes \Lambda)$ are given by (II.1.5) and (II.1.6) respectively (cf. (Timmermann, 2008, §5.4.2)).

Definition II.1.1. Using the above notations, it is well-known that the analytic compact quantum group $(\mathscr{A}, \widetilde{\Delta})$ and the algebraic compact quantum group $(A \otimes C(\Lambda), \widetilde{\Delta})$ are equivalent descriptions of the same object, which we call **the semidirect product** of $\mathbb{G}$ and Λ with respect to the action α^*, and is denoted by $\mathbb{G} \rtimes_{\alpha^*} \Lambda$, or simply $\mathbb{G} \rtimes \Lambda$ if the underlying action α^* is clear from context.

Remark II.1.2. There is a faster way of establishing $\mathbb{G} \rtimes \Lambda$ as a compact quantum group, which we refer to as the analytic approach. Namely, one might use (II.1.1) directly to define a comultiplication on the C^*-algebra $A \otimes C(\Lambda)$ and show that this comultiplication satisfy the density condition in the definition of a compact quantum group in the sense of Woronowicz (cf. (Woronowicz, 1998)). We prefer the more algebraic approach presented above as it provides more insight for our purpose of studying representations of $\mathbb{G} \rtimes \Lambda$. As an illustration, from our treatment, one knows immediately that $\mathrm{Pol}(\mathbb{G} \rtimes \Lambda) = \mathrm{Pol}(\mathbb{G}) \rtimes \Lambda$, a fact that is not clear from the faster analytic approach.

Remark II.1.3. When $\mathbb{G}$ comes from a genuine compact group G, it is easy to check via Gelfand theory that the antihomomorphism $\alpha^* : \Lambda \to \mathrm{Aut}\big(C(\mathbb{G}), \Delta\big)$ comes from the pull-back of a group morphism $\alpha : \Lambda \to \mathrm{Aut}\big(C(\mathbb{G}), \Delta\big)$, and $\mathbb{G} \rtimes \Lambda$ is exactly the compact group $G \rtimes_\alpha \Lambda$ viewed as a compact quantum group, where the group law on $G \times \Lambda$ is defined by

$$\forall g, h \in G,\ r, s \in \Lambda, \qquad (g, r)(h, s) = \big(g\alpha_r(h), rs\big).q \tag{II.1.10}$$

In treating the dual objects of some rigid C^*-tensor to be presented later, the following result will be useful.

Proposition II.1.4. *The compact quantum group $\mathbb{G} \rtimes \Lambda$ is of Kac type if and only if $\mathbb{G}$ is of Kac type.*

Proof. Of the many equivalent characterization for a compact quantum group to be of Kac type[4], we use the fact that such a quantum group is of Kac type if and only if the antipode of its polynomial algebra preserves adjoints. The proposition now becomes trivial in view of (II.1.6). □

II.2 A first look at unitary representations of $\mathbb{G} \rtimes \Lambda$

A unitary representation U of a classic compact semidirect product $G \rtimes \Lambda$ is determined by the restrictions U_G and U_Λ on the subgroups $G \times 1_\Lambda \simeq G$ and $1_G \times \Lambda \simeq \Lambda$ respectively. It is easy to see that (cf. (II.1.10))

$$\begin{aligned} \forall g \in G, r \in \Lambda, \quad U_G(\alpha_r(g))U_\Lambda(r) &= U(\alpha_r(g), r) \\ &= U((1, r)(g, 1)) = U_\Lambda(r)U_G(g). \end{aligned} \tag{II.2.1}$$

Conversely, suppose U_G, U_Λ are unitary representations on the same Hilbert space of G and Λ respectively, if (II.2.1) is satisfied, then $U(g, r) := U_G(g)U_\Lambda(r)$ defines a

[4] see e.g. (Neshveyev and Tuset, 2013, §1.7)

unitary representation of $G \rtimes \Lambda$. When G is replaced by a general compact quantum group $\mathbb{G}$, even though the "elements" of $\mathbb{G}$ are no longer available, one can still establish a reasonable quantum analogue. We begin with a simple lemma.

Lemma II.2.1. *Let ϵ be the counit for $\mathrm{Pol}(\mathbb{G})$, ϵ_Λ the counit for $C(\Lambda)$, then $\epsilon \otimes \mathrm{id}_{C(\Lambda)}$ is a Hopf $*$-algebra morphism from $\mathrm{Pol}(\mathbb{G}) \otimes C(\Lambda)$ onto $C(\Lambda)$, and $\mathrm{id}_{\mathrm{Pol}(\mathbb{G})} \otimes \epsilon_\Lambda$ is a Hopf $*$-algebra morphism from $\mathrm{Pol}(\mathbb{G}) \otimes C(\Lambda)$ onto $\mathrm{Pol}(\mathbb{G})$.*

Proof. Since the antipodes are $*$-morphisms of involutive algebras, it suffices to check that both morphisms preserve comultiplication.

Take any $a \in \mathrm{Pol}(\mathbb{G})$, $r \in \Lambda$, we have

$$\begin{aligned}
&[(\epsilon \otimes \mathrm{id}) \otimes (\epsilon \otimes \mathrm{id})]\Delta_{\mathbb{G}\rtimes\Lambda}(a \otimes \delta_r) \\
=& \sum_{s\in\Lambda} \sum \epsilon(a_{(1)})\epsilon(\alpha_s^*(a_{(2)}))\delta_s \otimes \delta_{s^{-1}r} \\
=& \sum_{s\in\Lambda} \sum \epsilon(a_{(1)})\epsilon(a_{(2)})\delta_s \otimes \delta_{s^{-1}r} \\
=& \sum_{s\in\Lambda} \epsilon(a)\delta_s \otimes \delta_{s^{-1}t} = \Delta_\Lambda(\epsilon \otimes \mathrm{id})(a \otimes \delta_r),
\end{aligned}$$

where Δ_Λ is the comultiplication for Λ viewed as a compact quantum group. Thus $\epsilon \otimes \mathrm{id}$ preserves comultiplication. On the other hand, note that $\epsilon_\Lambda(\delta_r) = \delta_{r,1_\Lambda}$, we have

$$\begin{aligned}
&[(\mathrm{id} \otimes \epsilon_\Lambda) \otimes (\mathrm{id} \otimes \epsilon_\Lambda)]\Delta_{\mathbb{G}\otimes\Lambda}(a \otimes \delta_r) \\
=& \sum_{s\in\Lambda} \delta_{s,1_\Lambda}\delta_{s^{-1}r,1_\Lambda} \sum a_{(1)} \otimes \alpha_s^*(a_{(2)}) \\
=& \delta_{r,1_\Lambda} \sum a_{(1)} \otimes a_{(2)} \\
=& \delta_{r,1_\Lambda}\Delta(a) = \Delta[(\mathrm{id} \otimes \epsilon_\Lambda)(a \otimes \delta_r)].
\end{aligned}$$

Thus $\mathrm{id} \otimes \epsilon_\Lambda$ preserves comultiplication too. ◻

Let $U \in \mathcal{B}(\mathscr{H}) \otimes \mathrm{Pol}(\mathbb{G}) \otimes C(\Lambda)$ be a finite dimensional unitary representation of $\mathbb{G} \rtimes \Lambda$. Define the unitaries

$$\mathrm{Res}_{\mathbb{G}}(U) := (\mathrm{id}_{\mathcal{B}(\mathscr{H})} \otimes \mathrm{id}_{\mathrm{Pol}(\mathbb{G})} \otimes \epsilon_\Lambda)(U) \in \mathcal{B}(\mathscr{H}) \otimes \mathrm{Pol}(\mathbb{G}),$$

and

$$\mathrm{Res}_\Lambda(U) := (\mathrm{id}_{\mathcal{B}(\mathscr{H})} \otimes \epsilon_{\mathbb{G}} \otimes \mathrm{id}_{C(\Lambda)})(U) \in \mathcal{B}(\mathscr{H}) \otimes C(\Lambda).$$

Then by Lemma II.2.1, we see that $\mathrm{Res}_{\mathbb{G}}(U)$ is a finite dimensional unitary representation of $\mathbb{G}$ and $\mathrm{Res}_\Lambda(U)$ a finite dimensional unitary representation of Λ. We call $\mathrm{Res}_{\mathbb{G}}(U)$ (resp. $\mathrm{Res}_\Lambda(U)$) the restriction of U to $\mathbb{G}$ (resp. Λ). For reasons to be explained presently, we also write $U_{\mathbb{G}}$ for $\mathrm{Res}_{\mathbb{G}}(U)$ and U_Λ for $\mathrm{Res}_\Lambda(U)$.

Proposition II.2.2. *Using the above notations, we have*

$$\forall r_0 \in \Lambda, \quad (U_\Lambda(r_0) \otimes 1_A)U_{\mathbb{G}} = [(\mathrm{id}_{\mathcal{B}(\mathscr{H})} \otimes \alpha_{r_0}^*)(U_{\mathbb{G}})](U_\Lambda(r_0) \otimes 1_A) \tag{II.2.2}$$

in $\mathcal{B}(\mathscr{H}) \otimes \mathrm{Pol}(\mathbb{G})$. Moreover,

$$U = (U_{\mathbb{G}})_{12}(U_\Lambda)_{13} \in \mathcal{B}(\mathscr{H}) \otimes \mathrm{Pol}(\mathbb{G}) \otimes C(\Lambda). \tag{II.2.3}$$

Conversely, suppose $U_{\mathbb{G}}$ and U_Λ are finite dimensional unitary representations of $\mathbb{G}$ and Λ respectively on the same Hilbert space $\mathscr{H}$, if $U_{\mathbb{G}}$ and U_Λ satisfy (II.2.2), then (II.2.3) defines a finite dimensional unitary representation U of $\mathbb{G} \rtimes \Lambda$ on $\mathscr{H}$. Moreover,

$$U_{\mathbb{G}} = (\mathrm{id}_{\mathcal{B}(\mathscr{H})} \otimes \mathrm{id}_{\mathrm{Pol}(\mathbb{G})} \otimes \epsilon_\Lambda)(U) \in \mathcal{B}(\mathscr{H}) \otimes \mathrm{Pol}(\mathbb{G}), \tag{II.2.4a}$$

$$U_\Lambda = (\mathrm{id}_{\mathcal{B}(\mathscr{H})} \otimes \epsilon_{\mathbb{G}} \otimes \mathrm{id}_{C(\Lambda)})(U) \in \mathcal{B}(\mathscr{H}) \otimes C(\Lambda). \tag{II.2.4b}$$

Proof. Let $d = \dim \mathscr{H}$, and fix a Hilbert basis $(e_1, \dots, e_d)$ for $\mathscr{H}$. Let $(e_{ij}, i, j = 1, \dots, d)$ be the corresponding matrix units (i.e. $e_{ij} \in \mathcal{B}(\mathscr{H})$ is characterized by $e_{ij}(e_k) = \delta_{j,k} e_i$). Then there is a unique $U_{ij} \in \mathrm{Pol}(\mathbb{G}) \otimes C(\Lambda)$ for each pair of i, j, such that

$$U = \sum_{i,j} e_{ij} \otimes U_{ij},$$

with each U_{ij} decomposed further as $U_{ij} = \sum_{r \in \Lambda} U_{ij,r} \otimes \delta_r$, where each $U_{ij,r} \in \mathrm{Pol}(\mathbb{G})$. Since U is a finite dimensional unitary representation of $\mathbb{G} \rtimes \Lambda$, for any $i, j \in \{1, \dots, d\}$, we have

$$\Delta_{\mathbb{G} \rtimes \Lambda}(U_{ij}) = \sum_{k=1}^{d} U_{ik} \otimes U_{kj}, \tag{II.2.5}$$

where in $\mathrm{Pol}(\mathbb{G}) \otimes C(\Lambda) \otimes \mathrm{Pol}(\mathbb{G}) \otimes C(\Lambda)$, we have

$$\begin{aligned} \Delta_{\mathbb{G} \rtimes \Lambda}(U_{ij}) &= \sum_{\substack{r,s,t \in \Lambda, \\ r = st}} \left[(\mathrm{id}_A \otimes \alpha_s^*) \Delta(U_{ij,r}) \right]_{13} (1_A \otimes \delta_s \otimes 1_A \otimes \delta_t) \\ &= \sum_{s,t \in \Lambda} \left[(\mathrm{id}_A \otimes \alpha_s^*) \Delta(U_{ij,st}) \right]_{13} (1_A \otimes \delta_s \otimes 1_A \otimes \delta_t) \end{aligned} \tag{II.2.6}$$

and

$$\sum_{k=1}^{d} U_{ik} \otimes U_{kj} = \sum_{k=1}^{d} \sum_{s,t \in \Lambda} U_{ik,s} \otimes \delta_s \otimes U_{kj,t} \otimes \delta_t. \tag{II.2.7}$$

Comparing (II.2.5), (II.2.6) and (II.2.7), we get

$$(\mathrm{id}_A \otimes \alpha_s^*) \Delta(U_{ij,st}) = \sum_{k=1}^{d} U_{ik,s} \otimes U_{kj,t} \in A \otimes A \tag{II.2.8}$$

or equivalently (by applying $(\mathrm{id}_A \otimes \alpha_{s^{-1}}^*)$ on both sides)

$$\Delta(U_{ij,st}) = \sum_{k=1}^{d} U_{ik,s} \otimes \alpha_{s^{-1}}^*(U_{kj,t}) \tag{II.2.9}$$

for every $s, t \in \Lambda$. Since $(\mathrm{id} \otimes \epsilon)\Delta = \mathrm{id} = (\epsilon \otimes \mathrm{id})\Delta$, we have

$$U_{ij,st} = \sum_{k=1}^{d} \epsilon(U_{ik,s}) \alpha_{s^{-1}}^*(U_{kj,t}) = \sum_{k=1}^{d} \epsilon(U_{kj,s}) U_{ik,t} \tag{II.2.10}$$

for any $i, j \in \{1, \dots, d\}$, $s, t \in \Lambda$.

We have $\epsilon_\Lambda(\delta_r) = \delta_{r,1_\Lambda}$, thus by definition

$$U_{\mathbb{G}} = \sum_{i,j=1}^{d} e_{ij} \otimes U_{ij,1_\Lambda} \in \mathcal{B}(\mathscr{H}) \otimes \mathrm{Pol}(\mathbb{G}). \tag{II.2.11}$$

Similarly,

$$U_\Lambda = \sum_{r\in\Lambda} \sum_{i,j=1}^{d} \epsilon(U_{ij,r}) e_{ij} \otimes \delta_r \in \mathcal{B}(\mathscr{H}) \otimes C(\Lambda). \tag{II.2.12}$$

Thus

$$U_\Lambda(r_0) = \sum_{i,j=1}^{d} \epsilon(U_{ij,r_0}) e_{ij} \in \mathcal{B}(\mathscr{H}). \tag{II.2.13}$$

Hence,

$$\begin{aligned}(U_\Lambda(r_0) \otimes 1_A)U_{\mathbb{G}} &= \sum_{i,j,k,l=1}^{d} \delta_{j,k} \epsilon(U_{ij,r_0}) e_{il} \otimes U_{kl,1_\Lambda} \\ &= \sum_{i,l=1}^{d} e_{il} \otimes \sum_{k=1}^{d} \epsilon(U_{ik,r_0}) U_{kl,1_\Lambda} \\ &= \sum_{i,l=1}^{d} e_{il} \otimes \alpha^*_{r_0}\left(\sum_{k=1}^{d} \epsilon(U_{ik,r_0}) \alpha^*_{r_0^{-1}}(U_{kl,1_\Lambda})\right) \\ &= \sum_{i,l=1}^{d} e_{il} \otimes \alpha^*_{r_0}(U_{il,r_0})\end{aligned} \tag{II.2.14}$$

where the last equality follows from (II.2.10); and

$$\begin{aligned}[(\mathrm{id}\otimes\alpha^*_{r_0})U_{\mathbb{G}}](U_\Lambda(r_0) \otimes 1_A) &= \sum_{i,j,k,l=1}^{d} \delta_{j,k} \epsilon(U_{kl,r_0}) e_{il} \otimes \alpha^*_{r_0}(U_{ik,1_\Lambda}) \\ &= \sum_{i,l=1} e_{il} \otimes \sum_{k=1}^{d} \epsilon(U_{kl,r_0}) \alpha^*_{r_0}(U_{ik,1_\Lambda}) \\ &= \sum_{i,l=1} e_{il} \otimes \alpha^*_{r_0}\left(\sum_{k=1}^{d} \epsilon(U_{ik,r_0}) U_{kj,1_\Lambda}\right) \\ &= \sum_{i,l=1} e_{il} \otimes \alpha^*_{r_0}(U_{il,r_0})\end{aligned} \tag{II.2.15}$$

where (II.2.10) is used again in the last equality.

Combining (II.2.14) and (II.2.15) finishes the proof of (II.2.2).

By (II.2.11), (II.2.12) and (II.2.10), one has

$$\begin{aligned}(U_{\mathbb{G}})_{12}(U_\Lambda)_{13} &= \sum_{i,j,k,l=1}^{d} \sum_{r\in\Lambda} \delta_{j,k} \epsilon(U_{kl,r}) e_{il} \otimes U_{ij,1_\Lambda} \otimes \delta_r \\ &= \sum_{i,l=1}^{d} e_{il} \otimes \sum_{r\in\Lambda}\left(\sum_{k=1}^{d} \epsilon(U_{kl,r}) U_{ik,1_\Lambda}\right) \otimes \delta_r \\ &= \sum_{i,l=1}^{d} e_{il} \otimes \sum_{r\in\Lambda} U_{il,r} \otimes \delta_r = U\end{aligned} \tag{II.2.16}$$

in $\mathcal{B}(\mathscr{H}) \otimes \mathrm{Pol}(\mathbb{G}) \otimes C(\Lambda)$. This proves (II.2.3).

Conversely, suppose $U_{\mathbb{G}}$ and U_Λ are unitary representations on some finite dimensional Hilbert space $\mathscr{H}$. We still use $(e_1, \ldots, e_d)$ to denote a Hilbert basis for $\mathscr{H}$, where $d = \dim \mathscr{H}$, and $(e_{ij}, i, j = 1, \ldots, d)$ the corresponding matrix unit of $\mathcal{B}(\mathscr{H})$.

Then for each pair i, j, one has a unique $u_{ij} \in \mathrm{Pol}(\mathbb{G})$ and a unique $f_{ij} \in C(\Lambda)$, such that $U_{\mathbb{G}} = \sum_{i,j} e_{ij} \otimes u_{ij}$, $U_\Lambda = \sum_{ij} e_{ij} \otimes f_{ij}$. By suitably choosing the basis $(e_1, \ldots, e_d)$, we may and do assume $\epsilon(u_{ij}) = \delta_{i,j}$. Since these are representations, we have

$$\Delta(u_{ij}) = \sum_{k=1}^{d} u_{ik} \otimes u_{kj}, \tag{II.2.17a}$$

$$\Delta_\Lambda(f_{ij}) = \sum_{k=1}^{d} f_{ik} \otimes f_{kj}. \tag{II.2.17b}$$

By definition,

$$U = \sum_{i,j,k,l=1}^{d} \delta_{jk} e_{il} \otimes u_{ij} \otimes f_{kl} = \sum_{i,j=1}^{d} e_{ij} \otimes U_{ij} \tag{II.2.18}$$

with

$$U_{ij} = \sum_{k=1}^{d} u_{ik} \otimes f_{kj} = \sum_{r \in \Lambda} \sum_{k=1}^{d} f_{kj}(r) u_{ik} \otimes \delta_r. \tag{II.2.19}$$

Since $U_{\mathbb{G}}$ and U_Λ are unitary, so is U. Using $\epsilon(u_{ij}) = \delta_{i,j}$, one has

$$(\mathrm{id}_{\mathcal{B}(\mathscr{H})} \otimes \epsilon \otimes \mathrm{id}_{C(\Lambda)})(U) = \sum_{i,j=1}^{d} e_{ij} \otimes \sum_{k=1}^{d} \delta_{i,k} f_{kj} = \sum_{i,j}^{d} e_{ij} \otimes f_{ij} = U_\Lambda. \tag{II.2.20}$$

This proves (II.2.4b). The proof of (II.2.4a) is more involved and must resort to condition (II.2.2), which using the above notations, translates to

$$\forall r \in \Lambda, \quad \sum_{i,j} e_{ij} \otimes \sum_{k} f_{ik}(r) u_{kj} = \sum_{i,j} e_{ij} \otimes \sum_{k} f_{kj}(r) \alpha_r^*(u_{ik}), \tag{II.2.21}$$

or equivalently,

$$\forall r \in \Lambda, i, j \in \{1, \ldots, d\}, \quad \sum_{k=1}^{d} f_{ik}(r) u_{kj} = \sum_{k=1}^{d} f_{kj}(r) \alpha_r^*(u_{ik}). \tag{II.2.22}$$

Since $U_\Lambda(1_\Lambda) = \mathrm{id}_{\mathscr{H}}$, one has $f_{ij}(1_\Lambda) = \delta_{i,j}$. Taking $r = 1_\Lambda$ in (II.2.22) yields

$$\begin{aligned}(\mathrm{id}_{\mathcal{B}(\mathscr{H})} \otimes \mathrm{id}_{\mathrm{Pol}(\mathbb{G})} \otimes \epsilon_\Lambda)(U) &= \sum_{i,j=1}^{d} e_{ij} \otimes \sum_{k=1}^{d} f_{kj}(1_\Lambda) u_{ik} \\ = \sum_{i,j=1}^{d} e_{ij} \otimes \sum_{k=1}^{d} \delta_{k,j} u_{ik} &= \sum_{i,j=1}^{d} e_{ij} \otimes u_{ij} = U_{\mathbb{G}},\end{aligned} \tag{II.2.23}$$

which proves (II.2.4a). To finishes the proof of the proposition, it remains to check that the unitary U is indeed a representation of $\mathbb{G} \rtimes \Lambda$.

Using (II.2.17a), (II.2.19) and (II.2.22), one has

$$
\begin{aligned}
\Delta_{\mathbb{G}\rtimes\Lambda}(U_{ij}) &= \sum_{r\in\Lambda}\sum_{s\in\Lambda}\left[(\mathrm{id}_{\mathrm{Pol}(\mathbb{G})}\otimes\alpha_s^*)\Delta\left(\sum_{k=1}^{d} f_{kj}(r)u_{ik}\right)\right]_{13}(\delta_s\otimes\delta_{s^{-1}1})_{24}\\
&= \sum_{r,s\in\Lambda}\sum_{k,l=1}^{d} f_{kj}(r)u_{il}\otimes\delta_s\otimes\alpha_s^*(u_{lk})\otimes\delta_{s^{-1}r}\\
&= \sum_{r,s\in\Lambda}\sum_{l=1}^{d} u_{il}\otimes\delta_s\otimes\left[\sum_{k=1}^{d} f_{kj}(r)\alpha_s^*(u_{lk})\right]\otimes\delta_{s^{-1}r}\\
&= \sum_{s,t\in\Lambda}\sum_{k,l=1}^{d} u_{il}\otimes\delta_s\otimes\left[f_{kj}(st)\alpha_s^*(u_{lk})\right]\otimes\delta_t\\
&= \sum_{s,t\in\Lambda}\sum_{k,l=1}^{d} u_{il}\otimes\delta_s\otimes\left[\sum_{h=1}^{d} f_{hj}(t)f_{kh}(s)\alpha_s^*(u_{lk})\right]\otimes\delta_t \qquad\qquad \text{(II.2.24)}\\
&= \sum_{s,t\in\Lambda}\sum_{h,l=1}^{d} u_{il}\otimes\delta_s\otimes\left[f_{hj}(t)\sum_{k=1}^{d} f_{kh}(s)\alpha_s^*(u_{lk})\right]\otimes\delta_t\\
&= \sum_{s,t\in\Lambda}\sum_{h,l=1}^{d} u_{il}\otimes\delta_s\otimes\left[f_{hj}(t)\sum_{k=1}^{d} f_{lk}(s)u_{kh}\right]\otimes\delta_t\\
&= \sum_{h,k,l=1}^{d} u_{il}\otimes f_{lk}\otimes u_{kh}\otimes f_{hj}\\
&= \sum_{h=1}^{d}\left(\sum_{l=1}^{d} u_{il}\otimes f_{lk}\right)\otimes\left(\sum_{k=1}^{d} u_{kh}\otimes f_{hj}\right) = \sum_{h=1}^{d} U_{ih}\otimes U_{hj}.
\end{aligned}
$$

Thus U is indeed a (unitary) representation. □

Definition II.2.3. Let $U_{\mathbb{G}} \in \mathcal{B}(\mathscr{H}) \otimes \mathrm{Pol}(\mathbb{G})$ be a finite dimensional unitary representation of $\mathbb{G}$, $U_\Lambda \in \mathcal{B}(\mathscr{H}) \otimes C(\Lambda)$ a finite dimensional unitary representation of Λ on the same space $\mathscr{H}$, we say $U_{\mathbb{G}}$ and U_Λ are **covariant** if they satisfy condition (II.2.2).

We track here a simple criterion for two representations to be covariant using matrix units and matrix coefficients.

Proposition II.2.4. *Let $U_{\mathbb{G}} \in \mathcal{B}(\mathscr{H}) \otimes \mathrm{Pol}(\mathbb{G})$, $U_\Lambda \in \mathcal{B}(\mathscr{H}) \otimes C(\Lambda)$ be finite-dimensional unitary representations of $\mathbb{G}$ and Λ respectively. Let $(e_1, \dots, e_d)$ be a Hilbert basis of $\mathscr{H}$, $e_{ij} \in \mathcal{B}(\mathscr{H})$ the operator with $e_{ij}(e_k) = \delta_{j,k}e_i$, and $U_{\mathbb{G}} = \sum_{i,j} e_{ij} \otimes u_{ij}$, $U_\Lambda = \sum_{i,j} e_{ij} \otimes f_{ij}$, then $U_{\mathbb{G}}$ and U_Λ are covariant if and only if*

$$
\forall r \in \Lambda, i, j \in \{1, \dots, d\}, \quad \sum_{k=1}^{d} f_{ik}(r)u_{kj} = \sum_{k=1}^{d} f_{kj}(r)\alpha_r^*(u_{ik}). \qquad \text{(II.2.25)}
$$

Proof. This is just a restatement of condition (II.2.2). □

By Proposition II.2.2, unitary representations of $\mathbb{G} \rtimes \Lambda$, at least the finite dimensional ones, correspond bijectively to pairs of covariant unitary representations of $\mathbb{G}$ and Λ.

II.3 Principal subgroups of $\mathbb{G} \rtimes \Lambda$

Definition II.3.1. Let $\mathbb{H} = (B, \Delta_B)$, $\mathbb{K} = (C, \Delta_C)$ be compact quantum groups, we say $\mathbb{K}$ is isomorphic to a closed quantum subgroup of $\mathbb{H}$, or simply $\mathbb{K}$ is a closed subgroup of $\mathbb{H}$, if there exists a *surjective* mapping $\varphi : \mathrm{Pol}(\mathbb{H}) \to \mathrm{Pol}(\mathbb{K})$ such that φ is a morphism of Hopf $*$-algebras.

When $\mathbb{H}$ is universal, then Definition II.3.1 can be reformulated as the existence of a surjective unital C^*-algebra morphism $\varphi : B \to C$ such that $(\varphi \otimes \varphi)\Delta_B = \Delta_C \varphi$.

In the context of compact quantum groups, we will use the terms "quantum closed subgroup" and "closed subgroup", sometimes even "subgroup", interchangeably without further explanation.

Remark II.3.2. If $\mathbb{H}$ and $\mathbb{K}$ are commutative, i.e. they come from genuine compact groups, then $\mathbb{K}$ being isomorphic to a closed subgroup, says exactly that there exists a continuous injective map φ_* from $\mathrm{Spec}(C)$, the underlying space of the compact group $\mathbb{K}$, into $\mathrm{Spec}(B)$, the underlying space of the compact group $\mathbb{H}$, such that φ_* preserves multiplication. Thus the above definition for closed (quantum) subgroups is consistent with the classical case of compact groups.

Recall that $\mathbb{G} = (A, \Delta)$, $C(\mathbb{G} \rtimes \Lambda) = A \otimes C(\Lambda)$, and $\mathrm{Pol}(\mathbb{G} \rtimes \Lambda) = C(\mathbb{G}) \rtimes C(\Lambda)$.

Proposition II.3.3. *Let Λ_0 be a subgroup of Λ, then the mapping*

$$\begin{aligned}\varphi \colon A \otimes C(\Lambda) &\to A \otimes C(\Lambda_0)\\ \sum_{r\in\Lambda} a_r \otimes \delta_r &\mapsto \sum_{r\in\Lambda_0} a_r \otimes \delta_r\end{aligned}$$

is a unital surjective morphism[5] *of C^*-algebras that also intertwines the comultiplications on $\mathbb{G} \rtimes \Lambda_0$ and $\mathbb{G} \rtimes \Lambda$. In particular, $\mathbb{G} \rtimes \Lambda_0$ is a closed subgroup of $\mathbb{G} \rtimes \Lambda$.*

Proof. Obviously φ is a unital surjective morphism of C^*-algebras. We need to show that φ intertwines the comultiplication $\widetilde{\Delta}$ on $\mathbb{G} \rtimes \Lambda$ and the comultiplication $\widetilde{\Delta}_0$ on $\mathbb{G} \rtimes \Lambda_0$. For this, by density, it suffices to prove that the restriction

$$\begin{aligned}\varphi \colon \ \mathrm{Pol}(\mathbb{G}) \otimes C(\Lambda) &\to \mathrm{Pol}(\mathbb{G}) \otimes C(\Lambda_0)\\ \sum_{r\in\Lambda} a_r \otimes \delta_r &\mapsto \sum_{r\in\Lambda_0} a_r \otimes \delta_r\end{aligned} \tag{II.3.1}$$

intertwines the comultiplications. Indeed, given an arbitrary $a_r \in \mathrm{Pol}(G)$ for any $r \in \Lambda$, note that for any $a \in \mathrm{Pol}(\mathbb{G})$ and $\lambda \in \Lambda$, $\varphi(a \otimes \delta_\lambda) = 0$ whenever $\lambda \notin \Lambda_0$, we have

$$\begin{aligned}&(\varphi \otimes \varphi)\widetilde{\Delta}\left(\sum_{r\in\Lambda} a_r \otimes \delta_r\right)\\ &= (\varphi \otimes \varphi) \sum_{r\in\Lambda}\sum_{s\in\Lambda}\sum (a_r)_{(1)} \otimes \delta_s \otimes \alpha_s^*\big((a_r)_{(2)}\big) \otimes \delta_{s^{-1}r}\\ &= \sum_{r\in\Lambda_0}\sum_{s\in\Lambda_0}\sum (a_r)_{(1)} \otimes \delta_s \otimes \alpha_s^*\big((a_r)_{(2)}\big) \otimes \delta_{s^{-1}r} \qquad \text{(II.3.2)}\\ &\text{(Since } s, s^{-1}r \in \Lambda_0 \text{ implies } r = s(s^{-1}r) \in \Lambda_0)\\ &= \widetilde{\Delta}_0\left(\sum_{r\in\Lambda_0} a_r \otimes \delta_r\right) = \widetilde{\Delta}_0\varphi\left(\sum_{r\in\Lambda} a_r \otimes \delta_r\right).\end{aligned}$$

[5] Note that δ_r has different meanings when viewed as functions in $C(\Lambda)$ and in $C(\Lambda_0)$

This shows that φ indeed intertwines comultiplications and finishes the proof. $\square$

Definition II.3.4. A closed subgroup of $\mathbb{G} \rtimes \Lambda$ of the form $\mathbb{G} \rtimes \Lambda_0$, where Λ_0 is a subgroup Λ, is called a **principal subgroup** of $\mathbb{G} \rtimes \Lambda$.

Remark II.3.5. If we let $p_0 = \sum_{r \in \Lambda_0} \delta_r \in C(\Lambda)$, then p_0 is a projection in $C(\Lambda)$, thus $1 \otimes p_0$ is a central projection in $A \otimes C(\Lambda)$. The morphism φ is in fact given by the "compression" map $(1 \otimes p_0)(\cdot)(1 \otimes p_0)$. Essentially, these data says that the principal subgroup $\mathbb{G} \rtimes \Lambda_0$ is in fact an **open** subgroup of $\mathbb{G} \rtimes \Lambda$. As we don't really need the general theory of open subgroups of topological quantum groups in this chapter, we won't recall the relevant notions here and refer the interested reader to the articles (Daws et al., 2012; Kalantar et al., 2016) for a treatment in the more general setting of locally compact quantum groups.

Corollary II.3.6. *Using the notations in Proposition II.3.3, if $U \in \mathcal{B}(\mathscr{H}) \otimes A \otimes C(\Lambda)$ is a (unitary) representation of $\mathbb{G} \rtimes \Lambda$, then $(\mathrm{id} \otimes \varphi)(U)$ is a (unitary) representation of $\mathbb{G} \rtimes \Lambda_0$.*

Proof. This follows directly from the fact that the restriction of the mapping φ as specified in (II.3.1) is a morphism of Hopf $*$-algebras. $\square$

Definition II.3.7. Using the above notations, the representation $(\mathrm{id} \otimes \varphi)(U)$ is called the restriction of U to $\mathbb{G} \rtimes \Lambda_0$, and is denoted by $U|_{\mathbb{G} \rtimes \Lambda_0}$.

Remark II.3.8. Again, when $\mathbb{G}$ is an classical compact group G, we recover the classical notion of restriction of a representation of $G \rtimes \Lambda$ to the subgroup $G \rtimes \Lambda_0$.

There is a natural "conjugate" relation between principal subgroups of the form $\mathbb{G} \rtimes \Lambda_0$ where Λ_0 is a subgroup of Λ, which will be used to simplify some calculations in our later treatment of representations. This relation is described in the following proposition.

Proposition II.3.9. *Let Λ_0 be a subgroup of Λ, $r \in \Lambda$, $\mathrm{Ad}_r \colon \Lambda_0 \to r\Lambda_0 r^{-1}$ the isomorphism $s \mapsto rsr^{-1}$. Then $\alpha_r^* \otimes \mathrm{Ad}_r^*$ is an isomorphism of compact quantum groups from $\mathbb{G} \rtimes \Lambda_0$ to $\mathbb{G} \rtimes r\Lambda_0 r^{-1}$.*

Proof. By density, it suffices to prove that the unital $*$-isomorphism

$$\alpha_r^* \otimes \mathrm{Ad}_r^* \colon \mathrm{Pol}(\mathbb{G}) \otimes C(r\Lambda_0 r^{-1}) \to \mathrm{Pol}(\mathbb{G}) \otimes C(\Lambda_0)$$

of involutive algebras preserves comultiplication. To fix the notations, let Δ_0 (resp. Δ_r) be the comultiplication on $\mathrm{Pol}(\mathbb{G}) \otimes C(\Lambda_0)$ (resp. $\mathrm{Pol}(\mathbb{G}) \otimes C(r\Lambda_0 r^{-1})$). For any $x \in \mathrm{Pol}(\mathbb{G})$, $\lambda \in \Lambda_0$, we have

$$\begin{aligned}
&(\alpha_r^* \otimes \mathrm{Ad}_r^* \otimes \alpha_r^* \otimes \mathrm{Ad}_r^*)\Delta_r(x \otimes \delta_{r\lambda r^{-1}}) \\
&= \sum_{\mu \in \Lambda_0} \left[\left(\alpha_r^* \otimes (\alpha_r^* \alpha_{r\mu r^{-1}}^*)\right)\Delta(x)\right]_{13} \left((\mathrm{Ad}_r^* \delta_{r\mu r^{-1}}) \otimes (\mathrm{Ad}_r^* \delta_{r\mu^{-1}\lambda r^{-1}})\right)_{24} \\
&= \sum_{\mu \in \Lambda_0} \left[\left(\alpha_r^* \otimes \alpha_{r\mu}^*\right)\Delta(x)\right]_{13} \left((\mathrm{Ad}_r^* \delta_{r\mu r^{-1}}) \otimes (\mathrm{Ad}_r^* \delta_{r\mu^{-1}\lambda r^{-1}})\right)_{24} \\
&= \sum_{\mu \in \Lambda_0} \left[\left((\mathrm{id} \otimes \alpha_\mu^*)[(\alpha_r^* \otimes \alpha_r^*)\Delta(x)]\right)\right]_{13} (\delta_\mu \otimes \delta_{\mu^{-1}\lambda})_{24} \\
&= \sum_{\mu \in \Lambda_0} \left[\left((\mathrm{id} \otimes \alpha_\mu^*)\Delta(\alpha_r^*(x))\right)\right]_{13} (\delta_\mu \otimes \delta_{\mu^{-1}\lambda})_{24} \\
&= \Delta_0\left(\alpha_r^*(x) \otimes \delta_\lambda\right) = [\Delta_0(\alpha_r^* \otimes \mathrm{Ad}_r^*)](x \otimes \delta_{r\lambda r^{-1}}).
\end{aligned} \tag{II.3.3}$$

Thus $\alpha_r^* \otimes \mathrm{Ad}_r^*$ indeed preserves comultiplication. □

II.4 Induced representations of principal subgroups

We begin by describing an outline of our approach to induced representations of principal subgroups of $\mathbb{G}\rtimes\Lambda$. Let Λ_0 be a subgroup of Λ, $U \in \mathcal{B}(\mathscr{H})\otimes\mathrm{Pol}(\mathbb{G})\otimes C(\Lambda_0)$ a finite dimensional unitary representation of $\mathbb{G} \rtimes \Lambda_0$. We want to construct the induced representation $\mathrm{Ind}_{\mathbb{G}\rtimes\Lambda_0}^{\mathbb{G}\rtimes\Lambda}(U)$ of the larger quantum group $\mathbb{G} \rtimes \Lambda$. The idea of the construction goes as follows: by the results in § II.2, we know U is determined by its restrictions $U_{\mathbb{G}} = \mathrm{Res}_{\mathbb{G}}(U)$ and $U_{\Lambda_0} = \mathrm{Res}_{\Lambda_0}(U)$. While one may not be able to directly extend the representation U_{Λ_0} of Λ_0 to a representation of Λ on the same space $\mathscr{H}$, we do have the right-regular representation $\widetilde{W}_\Lambda$ of Λ on $\ell^2(\Lambda) \otimes \mathscr{H}$ using the group structure of Λ. On the other hand, the direct sum $\widetilde{W}_{\mathbb{G}}$ of various copies of $U_{\mathbb{G}}$ placed suitably in $\ell^2(\Lambda) \otimes \mathscr{H}$ will give a representation of $\mathbb{G}$ on $\ell^2(\Lambda) \otimes \mathscr{H}$. It is then easy to check that $\widetilde{W}_{\mathbb{G}}$ and $\widetilde{W}_\Lambda$ are covariant, thus determine a representation $\widetilde{W}$ of $\mathbb{G} \rtimes \Lambda$ on $\ell^2(\Lambda) \otimes \mathscr{H}$. To retrieve the information of U_{Λ_0}, which is implicitly encoded in the $\mathscr{H}$ factor of $\ell^2(\Lambda) \otimes \mathscr{H}$, we consider the subspace $\mathscr{K}$ of $\ell^2(\Lambda) \otimes \mathscr{H}$ consisting of vectors which behave in a covariant way with the representation U_{Λ_0} on $\mathscr{H}$. More precisely, $\mathscr{K}$ is defined by

$$\mathscr{K} = \left\{ \sum_{r\in\Lambda} \delta_r \otimes \xi_r \ : \ \forall r_0 \in \Lambda_0, \forall r \in \Lambda,\ \xi_{r_0 r} = U_{\Lambda_0}(r_0)\xi_r \right\}. \tag{II.4.1}$$

One checks that $\mathscr{K}$ is an invariant subspace for both $\widetilde{W}_\Lambda$ and $\widetilde{W}_{\mathbb{G}}$, hence $\mathscr{K}$ is a subrepresentation W of $\widetilde{W}$, and we define W to be the induced representation $\mathrm{Ind}(U)$. We now proceed to carry out this idea precisely.

Definition II.4.1. Let U, $\mathscr{H}$, Λ_0 retain their meanings as above, and let $(e_{r,s};\ r, s \in \Lambda)$ be the matrix unit of $\mathcal{B}(\ell^2(\Lambda))$ associated with the standard Hilbert basis $(\delta_r;\ r \in \Lambda)$ for $\ell^2(\Lambda)$, i.e. $e_{r,s}\delta_t = \delta_{s,t}\delta_r$ for all $r, s, t \in \Lambda$. The right regular representation $\widetilde{W}_\Lambda$ of Λ on $\ell^2 \otimes \mathscr{H}$ is an operator in $\mathcal{B}(\ell^2(\Lambda)) \otimes \mathcal{B}(\mathscr{H}) \otimes C(\Lambda)$ defined by

$$\widetilde{W}_\Lambda = \sum_{r,s\in\Lambda} e_{rs^{-1},r} \otimes \mathrm{id}_{\mathscr{H}} \otimes \delta_s. \tag{II.4.2}$$

It is easy to see that if we regard $\ell^2(\Lambda)\otimes\mathscr{H}$ as $\ell^2(\Lambda, \mathscr{H})$, then for any $s \in \Lambda$, $\widetilde{W}_\Lambda(s)$ is the operator in $\mathcal{B}(\ell^2(\Lambda, \mathscr{H}))$ sending each $F\colon \Lambda \to \mathscr{H}$ to $F \circ R_s$, where $R_s\colon \Lambda \to \Lambda$ is the right multiplication by s. Hence $\widetilde{W}_\Lambda$ is indeed a unitary representation of Λ on $\ell^2(\Lambda)\otimes\mathscr{H}$. By definition, for any $s \in \Lambda$, the unitary operator $\widetilde{W}_\Lambda(s) \in \mathcal{U}(\ell^2(\Lambda)\otimes\mathscr{H})$ is characterized by

$$\begin{aligned} \widetilde{W}_\Lambda(s)\colon \ell^2(\Lambda) \otimes \mathscr{H} &\to \ell^2(\Lambda) \otimes \mathscr{H} \\ \delta_r \otimes \xi &\mapsto \delta_{rs^{-1}} \otimes \xi, \end{aligned} \tag{II.4.3}$$

or equivalently

$$\begin{aligned} \widetilde{W}_\Lambda(s)\colon \ell^2(\Lambda) \otimes \mathscr{H} &\to \ell^2(\Lambda) \otimes \mathscr{H} \\ \sum_{r\in\Lambda} \delta_r \otimes \xi_r &\mapsto \sum_{r\in\Lambda} \delta_{rs^{-1}} \otimes \xi_r = \sum_{r\in\Lambda} \delta_r \otimes \xi_{rs}. \end{aligned} \tag{II.4.4}$$

Proposition II.4.2. *Using the above notations, the unitary operator*

$$\widetilde{W}_{\mathbb{G}} := \sum_{s\in\Lambda} e_{s,s} \otimes [(\mathrm{id} \otimes \alpha_s^*)(U_{\mathbb{G}})] \in \mathcal{B}(\ell^2(\Lambda)) \otimes \mathcal{B}(\mathscr{H}) \otimes \mathrm{Pol}(\mathbb{G}) \tag{II.4.5}$$

is a unitary representation of $\mathbb{G}$ *on* $\ell^2(\Lambda)\otimes\mathscr{H}$*. Furthermore, for every* $s \in \Lambda$*,* $\delta_s \otimes \mathscr{H}$ *is invariant under* $\widetilde{W}_{\mathbb{G}}$*, and the subrepresentation* $\delta_s \otimes \mathscr{H}$ *of* $\widetilde{W}_{\mathbb{G}}$ *is unitarily equivalent to the unitary representation* $(\mathrm{id}\otimes\alpha_s^*)(U_{\mathbb{G}})$ *of* $\mathbb{G}$*. In particular,* $\widetilde{W}_{\mathbb{G}} \simeq \bigoplus_{s\in\Lambda}(\mathrm{id}\otimes\alpha_s^*)(U_{\mathbb{G}})$*.*

Proof. For each $s \in \Lambda$, since $\alpha_s^* \in \mathrm{Aut}\big(C(\mathbb{G}), \Delta\big)$, the unitary operator $(\mathrm{id}\otimes\alpha_s^*)(U_{\mathbb{G}})$ is indeed a representation of $\mathbb{G}$ on $\mathscr{H}$. It is easy to see that

$$e_{s,s}(\ell^2(\Lambda)) \otimes \mathscr{H} = \mathbb{C}\delta_s \otimes \mathscr{H} = \delta_s \otimes \mathscr{H},$$

hence $e_{s,s} \otimes \mathrm{id}_{\mathscr{H}}$ is the orthogonal projection in $\mathcal{B}(\ell^2(\Lambda) \otimes \mathscr{H})$ onto the subspace $\delta_s \otimes \mathscr{H}$ of $\ell^2(\Lambda)\otimes\mathscr{H}$ (Here and below, we abuse the notation $\delta_s\otimes\mathscr{H}$ to denote the subspace $\{\delta_s \otimes \xi \ : \ \xi \in \mathscr{H}\}$ of $\ell^2(\Lambda) \otimes \mathscr{H}$). We also have the intertwining relation

$$(e_{s,s} \otimes \mathrm{id}_{\mathscr{H}} \otimes 1_A)\widetilde{W}_{\mathbb{G}} = e_{s,s} \otimes [(\mathrm{id}\otimes\alpha_s^*)(U_{\mathbb{G}})] = \widetilde{W}_{\mathbb{G}}(e_{s,s} \otimes \mathrm{id}_{\mathscr{H}} \otimes 1_A). \tag{II.4.6}$$

Now the theorem follows from (II.4.6), the direct sum decomposition

$$\ell^2(\Lambda) \otimes \mathscr{H} = \bigoplus_{s\in\Lambda} e_{s,s}(\ell^2(\Lambda)) \otimes \mathscr{H} = \bigoplus_{s\in\Lambda} \delta_s \otimes \mathscr{H}, \tag{II.4.7}$$

and the obvious fact that the unitary operator $\delta_s\otimes\mathscr{H} \to \mathscr{H}$, $\delta_s\otimes\xi \mapsto \xi$ intertwines the representation $(\mathrm{id}\otimes\delta_s)(U_{\mathbb{G}})$ and the subrepresentation of $\widetilde{W}_{\mathbb{G}}$ determined by the subspace $\delta_s \otimes \mathscr{H}$ of $\ell^2(\Lambda) \otimes \mathscr{H}$. □

Proposition II.4.3. *The representations* $\widetilde{W}_{\mathbb{G}}$ *and* $\widetilde{W}_{\Lambda}$ *are covariant.*

Proof. For any $s \in \Lambda$, by definition,

$$\widetilde{W}_{\Lambda}(s) = \sum_{r\in\Lambda} e_{rs^{-1},r} \otimes \mathrm{id}_{\mathscr{H}} \in \mathcal{B}(\ell^2(\Lambda)) \otimes \mathcal{B}(\mathscr{H}) = \mathcal{B}(\ell^2(\Lambda)\otimes\mathscr{H}). \tag{II.4.8}$$

Thus

$$\begin{aligned}
&\Big(\widetilde{W}_{\Lambda}(s) \otimes 1\Big)\,\widetilde{W}_{\mathbb{G}} \\
&= \left(\sum_{r\in\Lambda} e_{rs^{-1},r} \otimes \mathrm{id}_{\mathscr{H}} \otimes 1_A\right) \sum_{t\in\Lambda} e_{t,t} \otimes [(\mathrm{id}\otimes\alpha_t^*)(U_{\mathbb{G}})] \\
&= \sum_{r,t\in\Lambda} \delta_{r,t} e_{rs^{-1},t} \otimes [(\mathrm{id}\otimes\alpha_t^*)(U_{\mathbb{G}})] = \sum_{r\in\Lambda} e_{rs^{-1},r} \otimes [(\mathrm{id}\otimes\alpha_r^*)(U_{\mathbb{G}})] \\
&= (\mathrm{id}\otimes\mathrm{id}\otimes\alpha_s^*) \sum_{r\in\Lambda} e_{rs^{-1},r} \otimes [(\mathrm{id}\otimes\alpha_{rs^{-1}}^*)(U_{\mathbb{G}})] \\
&= (\mathrm{id}\otimes\mathrm{id}\otimes\alpha_s^*)\left[\left(\sum_{t\in\Lambda} e_{t,t} \otimes [(\mathrm{id}_{\mathscr{H}}\otimes\alpha_t^*)(U_{\mathbb{G}})]\right)\left(\sum_{r\in\Lambda} e_{rs^{-1},r} \otimes \mathrm{id}_{\mathscr{H}} \otimes 1_A\right)\right] \\
&= [(\mathrm{id}\otimes\mathrm{id}\otimes\alpha_s^*)(\widetilde{W}_{\mathbb{G}})]\big(\widetilde{W}_{\Lambda}(s) \otimes 1\big).
\end{aligned} \tag{II.4.9}$$

This proves that $\widetilde{W}_{\mathbb{G}}$ and $\widetilde{W}_{\Lambda}$ are indeed covariant. □

Corollary II.4.4. *The unitary operator*

$$\begin{aligned}\widetilde{W} := (\widetilde{W}_{\mathbb{G}})_{123}(\widetilde{W}_{\Lambda})_{124} &= \sum_{r,s,t\in\Lambda} e_{t,t}e_{rs^{-1},r} \otimes [(\mathrm{id}\otimes\alpha_t^*)(U_{\mathbb{G}})] \otimes \delta_s \\ &= \sum_{r,s\in\Lambda} e_{rs^{-1},r} \otimes [(\mathrm{id}\otimes\alpha_{rs^{-1}}^*)(U_{\mathbb{G}})] \otimes \delta_s \\ &\in \mathcal{B}(\ell^2(\Lambda)) \otimes \mathcal{B}(\mathscr{H}) \otimes \mathrm{Pol}(\mathbb{G}) \otimes C(\Lambda)\end{aligned} \tag{II.4.10}$$

is a representation of $\mathbb{G} \rtimes \Lambda$ *on* $\ell^2(\Lambda) \otimes \mathscr{H}$.

Proof. This follows from Proposition II.2.2 and Proposition II.4.3. □

We now proceed to prove the invariance of the subspace $\mathscr{K}$ defined in (II.4.1) under $\widetilde{W}_{\mathbb{G}}$ and $\widetilde{W}_{\Lambda}$.

Lemma II.4.5. *Using the above notations, the following hold:*

(a) *the orthogonal projection* $\pi \in \mathcal{B}(\ell^2(\Lambda) \otimes \mathscr{H})$ *with range* $\mathscr{K}$ *is given by*[6] *the following formula:*

$$\begin{aligned}\pi\colon \ell^2(\Lambda) \otimes \mathscr{H} &\to \ell^2(\Lambda) \otimes \mathscr{H} \\ \delta_r \otimes \xi &\mapsto |\Lambda_0|^{-1} \sum_{r_0\in\Lambda_0} \delta_{r_0 r} \otimes U_{\Lambda_0}(r_0)\xi.\end{aligned} \tag{II.4.11}$$

In other words,

$$\pi = |\Lambda_0|^{-1} \sum_{r_0\in\Lambda_0} \sum_{s\in\Lambda} e_{r_0 s,s} \otimes U_{\Lambda_0}(r_0); \tag{II.4.12}$$

(b) $\mathscr{K}$ *is invariant under both* $\widetilde{W}_{\mathbb{G}}$ *and* $\widetilde{W}_{\Lambda}$, *i.e.*

$$(\pi \otimes 1)\widetilde{W}_{\mathbb{G}} = \widetilde{W}_{\mathbb{G}}(\pi \otimes 1) = (\pi \otimes 1)\widetilde{W}_{\mathbb{G}}(\pi \otimes 1), \tag{II.4.13a}$$

$$(\pi \otimes 1)\widetilde{W}_{\Lambda} = \widetilde{W}_{\Lambda}(\pi \otimes 1) = (\pi \otimes 1)\widetilde{W}_{\Lambda}(\pi \otimes 1). \tag{II.4.13b}$$

In particular, we have

$$(\pi \otimes 1 \otimes 1)\widetilde{W} = \widetilde{W}(\pi \otimes 1 \otimes 1) = (\pi \otimes 1 \otimes 1)\widetilde{W}(\pi \otimes 1 \otimes 1). \tag{II.4.14}$$

Proof. It is easy to see that $\pi(\ell^2(\Lambda) \otimes \mathscr{H})$ is precisely $\mathscr{K}$ and $\pi_{\mathscr{K}} = \mathrm{id}_{\mathscr{K}}$. To finish the proof of (a), it suffices to check that π is self-adjoint (or even stronger, positive). Since

$$\begin{aligned}(\pi(\delta_r \otimes \xi_r), \delta_r \otimes \xi_r) &= |\Lambda_0|^{-1} \sum_{r_0\in\Lambda_0} \left(\delta_{r_0 r} \otimes U_{\Lambda_0}(r_0)\xi, \delta_r \otimes \xi\right) \\ &= |\Lambda_0|^{-1}\|\xi\|^2 \geq 0,\end{aligned} \tag{II.4.15}$$

π is indeed positive.

[6] Recall that we've identified $\mathcal{B}(\ell^2(\Lambda) \otimes \mathscr{H})$ with $\mathcal{B}(\ell^2(\Lambda)) \otimes \mathcal{B}(\mathscr{H})$

We now prove (b). The invariance of $\mathscr{K}$ under $\widetilde{W}_\Lambda$ (equation (II.4.13b)) follows from (II.4.1) and (II.4.4). We now prove the invariance of $\mathscr{K}$ under $\widetilde{W}_{\mathbb{G}}$ (equation (II.4.13a)). By the definitions of π and $\widetilde{W}_{\mathbb{G}}$, we have

$$\begin{aligned}
&|\Lambda_0|(\pi\otimes 1)\widetilde{W}_{\mathbb{G}}\\
&=\sum_{r_0\in\Lambda_0}\sum_{r,s\in\Lambda}\left(e_{r_0s,s}\otimes U_{\Lambda_0}(r_0)\otimes 1\right)\left(e_{r,r}\otimes[(\mathrm{id}\otimes\alpha_r^*)(U_{\mathbb{G}})]\right)\\
&=\sum_{r\in\Lambda}(\mathrm{id}\otimes\mathrm{id}\otimes\alpha_r^*)\left(\sum_{r_0\in\Lambda_0}\sum_{s\in\Lambda}e_{r_0s,s}e_{r,r}\otimes\left[\left(U_{\Lambda_0}(r_0)\otimes 1\right)U_{\mathbb{G}}\right]\right)\\
&=\sum_{r\in\Lambda}(\mathrm{id}\otimes\mathrm{id}\otimes\alpha_r^*)\left(\sum_{r_0\in\Lambda_0}e_{r_0r,r}\otimes\left[\left(U_{\Lambda_0}(r_0)\otimes 1\right)U_{\mathbb{G}}\right]\right);
\end{aligned}\tag{II.4.16}$$

and

$$\begin{aligned}
&|\Lambda_0|\widetilde{W}_{\mathbb{G}}(\pi\otimes 1)\\
&=\sum_{r_0\in\Lambda_0}\sum_{r,s\in\Lambda}\left(e_{r,r}\otimes\left[(\mathrm{id}\otimes\alpha_r^*)(U_{\mathbb{G}})\right]\right)\left(e_{r_0s,s}\otimes U_{\Lambda_0}(r_0)\otimes 1\right)\\
&=\sum_{r_0\in\Lambda_0}\sum_{r,s\in\Lambda}(\mathrm{id}\otimes\mathrm{id}\otimes\alpha_r^*)\left(e_{r,r}e_{r_0s,s}\otimes\left[U_{\mathbb{G}}(U_{\Lambda_0}(r_0)\otimes 1)\right]\right)\\
&=\sum_{r_0\in\Lambda_0}\sum_{s\in\Lambda}(\mathrm{id}\otimes\mathrm{id}\otimes\alpha_{r_0s}^*)\left(e_{r_0s,s}\otimes\left[U_{\mathbb{G}}(U_{\Lambda_0}(r_0)\otimes 1)\right]\right)\\
&=\sum_{s\in\Lambda}(\mathrm{id}\otimes\mathrm{id}\otimes\alpha_s^*)\left[(\mathrm{id}\otimes\mathrm{id}\otimes\alpha_{r_0}^*)\left(\sum_{r_0\in\Lambda_0}e_{r_0s,s}\otimes\left[U_{\mathbb{G}}(U_{\Lambda_0}(r_0)\otimes 1)\right]\right)\right]\\
&=\sum_{s\in\Lambda}(\mathrm{id}\otimes\mathrm{id}\otimes\alpha_s^*)\left(\sum_{r_0\in\Lambda_0}e_{r_0s,s}\otimes\left(\left[(\mathrm{id}\otimes\alpha_{r_0}^*)(U_{\mathbb{G}})\right](U_{\Lambda_0}(r_0)\otimes 1)\right)\right)\\
&=\sum_{s\in\Lambda}(\mathrm{id}\otimes\mathrm{id}\otimes\alpha_s^*)\left(\sum_{r_0\in\Lambda_0}e_{r_0s,s}\otimes\left[\left(U_{\Lambda_0}(r_0)\otimes 1\right)(U_{\mathbb{G}})\right]\right),
\end{aligned}\tag{II.4.17}$$

where the last equality used the covariance of $U_{\mathbb{G}}$ and U_Λ. Combining (II.4.16) and (II.4.17) proves

$$(\pi\otimes 1)\widetilde{W}_{\mathbb{G}}=\widetilde{W}_{\mathbb{G}}(\pi\otimes 1),\tag{II.4.18}$$

from which (II.4.13a) follows by noting that π is a projection. Now (II.4.14) follows from (II.4.13a), (II.4.13b) and (II.4.10). This proves (b). □

Proposition II.4.6. *Using the above notations, let $c_\pi\colon\mathcal{B}(\ell^2(\Lambda)\otimes\mathscr{H})\to\mathcal{B}(\mathscr{K})$ be the compression by the projection π (i.e. the graph of $c_\pi(A)$ is the intersection of the graph of $\pi A\pi$ with $\mathscr{K}\times\mathscr{K}$), then the following holds:*

(a) *the unitary operator*

$$W=\left(c_\pi\otimes\mathrm{id}_{\mathrm{Pol}(\mathbb{G})}\otimes\mathrm{id}_{C(\Lambda)}\right)\left(\widetilde{W}_{\mathbb{G}}\right)\in\mathcal{B}(\mathscr{K})\otimes\mathrm{Pol}(\mathbb{G})\otimes C(\Lambda)$$

is a unitary representation of $\mathbb{G}\rtimes\Lambda$ on $\mathscr{K}$;

(b) *The subrepresentation $\mathscr{K}$ of $\widetilde{W}_{\mathbb{G}}$ (resp. $\widetilde{W}_\Lambda$) is given by $W_{\mathbb{G}}=(c_\pi\otimes\mathrm{id})\left(\widetilde{W}_{\mathbb{G}}\right)$ (resp. $W_\Lambda=(c_\pi\otimes\mathrm{id})\left(\widetilde{W}_\Lambda\right)$), and*

$$W_{\mathbb{G}}=\mathrm{Res}_{\mathbb{G}}(W),\quad W_\Lambda=\mathrm{Res}_\Lambda(W).\tag{II.4.19}$$

Proof. This follows from Proposition II.2.2, Corollary II.4.4, Lemma II.4.5 and the definition of subrepresentations. □

Definition II.4.7. Using the above notations, we call W the induced representation of U, and denote it by $\mathrm{Ind}_{\mathbb{G}\rtimes\Lambda_0}^{\mathbb{G}\rtimes\Lambda}(U)$, or simply $\mathrm{Ind}(U)$ when the underlying compact quantum groups $\mathbb{G}\rtimes\Lambda_0$ and $\mathbb{G}\rtimes\Lambda$ are clear from context.

II.5 Some character formulae

Let Λ_0 be a subgroup of Λ, U a finite dimensional unitary representation of $\mathbb{G}\rtimes\Lambda_0$, $\mathrm{Ind}_{\mathbb{G}\rtimes\Lambda_0}^{\mathbb{G}\rtimes\Lambda}(U)$ the induced representation of the global compact quantum group $\mathbb{G}\rtimes\Lambda$. In this section, we aim to calculate the character of the induced representation $\mathrm{Ind}_{\mathbb{G}\rtimes\Lambda_0}^{\mathbb{G}\rtimes\Lambda}(U)$. The approach adopted here emphasizes the underlying group action of Λ on the characters of the conjugacy class of the principal subgroup $\mathbb{G}\rtimes\Lambda_0$ as described in Proposition II.3.9.

For any subgroup Λ_1 and any $f_0\in C(\Lambda_1)$, we use $E_{\Lambda_1}(f_0)$ to denote the function in $C(\Lambda)$ with $[E_{\Lambda_1}(f_0)](r)=0$ if $r\notin\Lambda_1$ and $[E_{\Lambda_1}(f_0)](r)=f_0(r)$ if $r\in\Lambda_1$. Then $E_{\Lambda_1}: C(\Lambda_1)\to C(\Lambda)$ is a morphism of C^*-algebras, which is not unital unless $\Lambda_1=\Lambda$, in which case $E_{\Lambda_1}=\mathrm{id}_{C(\Lambda)}$. By Proposition II.3.9, we have an action

$$\begin{aligned}\Lambda \curvearrowright &\left\{\mathbb{G}\rtimes r\Lambda_0 r^{-1} \;:\; r\in\Lambda\right\}\\ s\mapsto &\left\{\mathbb{G}\rtimes r\Lambda_0 r^{-1}\mapsto \mathbb{G}\rtimes sr\Lambda_0(sr)^{-1}\right\}\end{aligned}\tag{II.5.1}$$

of Λ on the set of subgroups of $\mathbb{G}\rtimes\Lambda$ conjugate to $\mathbb{G}\rtimes\Lambda_0$ via elements in Λ (the term conjugate is justified by considering the case when $\mathbb{G}$ is a genuine compact group).

Our main result in this section is the following proposition.

Proposition II.5.1. *Let Λ_0 be a subgroup of Λ, $U\in\mathcal{B}(\mathscr{H})\otimes\mathrm{Pol}(\mathbb{G})\otimes C(\Lambda_0)$ a finite dimensional unitary representation of $\mathbb{G}\rtimes\Lambda_0$, W the induced representation $\mathrm{Ind}_{\mathbb{G}\rtimes\Lambda_0}^{\mathbb{G}\rtimes\Lambda}(U)$. Suppose χ is the character of the unitary representation U of $\mathbb{G}\rtimes\Lambda_0$, and for each r, define*

$$r\cdot U := (\mathrm{id}_{\mathscr{H}}\otimes\alpha^*_{r^{-1}}\otimes\mathrm{Ad}^*_{r^{-1}})(U)\in\mathcal{B}(\mathscr{H})\otimes\mathrm{Pol}(\mathbb{G})\otimes C(r\Lambda_0 r^{-1}).\tag{II.5.2}$$

Then $r\cdot U$ is a unitary representation of $\mathbb{G}\rtimes r\Lambda_0 r^{-1}$ with $1\cdot U=U$, and $(rs)\cdot U=r\cdot(s\cdot U)$ for all $r,s\in\Lambda$. Denote the character of $r\cdot U$ by χ_r (so $\chi_{1_\Lambda}=\chi$), then

$$\chi_W = |\Lambda_0|^{-1}\sum_{r\in\Lambda}(\mathrm{id}_A\otimes E_{r\Lambda_0 r^{-1}})\chi_r,\tag{II.5.3}$$

where χ_W is the character of W.

Proof. That $r\cdot U$ is a finite dimensional unitary representation of $\mathbb{G}\rtimes r\Lambda_0 r^{-1}$ follows from the fact (Proposition II.3.9) that

$$\alpha_r^*\otimes\mathrm{Ad}_r^*\colon A\otimes r^{-1}\Lambda_0 r\to A\otimes\Lambda_0$$

is an isomorphism of compact quantum groups for any $r\in\Lambda$. The identities $1_\Lambda\cdot U=U$ and $r\cdot(s\cdot U)=(rs)\cdot U$ follows directly from definitions. We proceed to prove the character formula (II.5.3).

For any $r\in\Lambda$, let $(r\cdot U)_{\mathbb{G}}$ be the restriction of $r\cdot U$ to $\mathbb{G}$, and $(r\cdot U)_{r\Lambda_0 r^{-1}}$ the restriction of $r\cdot U$ to $r\Lambda_0 r^{-1}$. We denote the character of $(r\cdot U)_{\mathbb{G}}$ (resp. $(r\cdot U)_{r\Lambda_0 r^{-1}}$)

by $\chi_{r,\mathbb{G}}$ (resp. $\chi_{r,r\Lambda_0 r^{-1}}$). One easily checks that $\chi_{r,\mathbb{G}} = \alpha^*_{r^{-1}}(\chi_{1_\Lambda,\mathbb{G}})$ and $\chi_{r,r\Lambda_0 r^{-1}} = \mathrm{Ad}^*_{r^{-1}}(\chi_{1_\Lambda,\Lambda_0})$. Fix a Hilbert basis $(e_1,\ldots,e_d)$ for $\mathscr{H}$, and let $(e_{ij}, i,j = 1,\ldots,d)$ be the corresponding matrix unit for $\mathcal{B}(\mathscr{H})$. Using this matrix unit, we can write

$$U_{\mathbb{G}} = \sum_{i,j=1}^{d} e_{i,j} \otimes u_{ij},\ u_{ij} \in \mathrm{Pol}(\mathbb{G}); \tag{II.5.4a}$$

$$U_{\Lambda_0} = \sum_{r_0 \in \Lambda_0} U_{\Lambda_0}(r_0) \otimes \delta_{r_0}. \tag{II.5.4b}$$

Let $e_{r,s}$, π, $\mathscr{K}$, $\widetilde{W}_{\mathbb{G}}$, $\widetilde{W}_\Lambda$, $W_{\mathbb{G}}$ and W_Λ have the same meaning as in § II.4, then the construction in § II.4 tells us that

$$\chi_W = (\mathrm{Tr}_{\ell^2(\Lambda)} \otimes \mathrm{Tr}_{\mathscr{H}} \otimes \mathrm{id}_A \otimes \mathrm{id}_{C(\Lambda)}) \left[\pi_{12} \cdot (\widetilde{W}_{\mathbb{G}})_{123} \cdot \pi_{12} \cdot (\widetilde{W}_\Lambda)_{124} \cdot \pi_{12}\right]. \tag{II.5.5}$$

In the following calculations, we often omit the subscripts of the trace functions Tr on $\ell^2(\Lambda)$ or on $\mathscr{H}$, and also the subscripts for the multiplicative neutral element 1 of various algebras, whenever it is a trivial task to decipher to which trace and multiplicative neutral element we are referring. The same goes with id without subscripts.

Note that for any $r,s \in \Lambda$, $\mathrm{Ad}^*_r(\delta_s) = \delta_{r^{-1}sr}$. With these preparations, we now have

$$\begin{aligned}
\chi_r &= (\alpha^*_{r^{-1}} \otimes \mathrm{Ad}^*_{r^{-1}})(\chi) \\
&= (\alpha^*_{r^{-1}} \otimes \mathrm{Ad}^*_{r^{-1}}) \left(\sum_{i,j=1}^{d} \sum_{r_0 \in \Lambda_0} \mathrm{Tr}\Big(e_{i,j} U_{\Lambda_0}(r_0)\Big) u_{ij} \otimes \delta_{r_0}\right) \\
&= \sum_{i,j=1}^{d} \sum_{r_0 \in \Lambda_0} \mathrm{Tr}\Big(e_{i,j} U_{\Lambda_0}(r_0)\Big) \alpha^*_{r^{-1}}(u_{ij}) \otimes \delta_{r r_0 r^{-1}}.
\end{aligned} \tag{II.5.6}$$

By (II.4.4), (II.4.10) and (II.4.12), we deduce from (II.5.5) that

$$\begin{aligned}
|\Lambda_0|^3 \chi_W = \sum_{a_0,b_0,c_0 \in \Lambda_0} \sum_{a,b,c \in \Lambda} \sum_{r,s,t \in \Lambda} \sum_{i,j=1}^{d} &\mathrm{Tr}(e_{a_0 a,a} e_{r,r} e_{b_0 b,b} e_{st^{-1},s} e_{c_0 c,c}) \\
&\mathrm{Tr}\Big(U_{\Lambda_0}(a_0) e_{i,j} U_{\Lambda_0}(b_0) U_{\Lambda_0}(c_0)\Big) \alpha^*_r(u_{ij}) \otimes \delta_t.
\end{aligned} \tag{II.5.7}$$

On the right side of the above sum, the first trace doesn't vanish if and only if it is 1, which happens exactly when

$$\begin{aligned}
&a = r = b_0 b,\ b = st^{-1},\ s = c_0 c,\ a_0 a = c \\
\iff\ & b = b_0^{-1} a,\ c = a_0 a,\ r = a,\ s = c_0 a_0 a,\ t = b^{-1} s = a^{-1} b_0 c_0 a_0 a.
\end{aligned} \tag{II.5.8}$$

Using this condition in (II.5.7), we get

$$
\begin{aligned}
&|\Lambda_0|^3 \chi_W \\
&= \sum_{a_0,b_0,c_0\in\Lambda_0} \sum_{a\in\Lambda} \sum_{i,j=1}^{d} \mathrm{Tr}\Big(U_{\Lambda_0}(a_0)e_{i,j}U_{\Lambda_0}(b_0)U_{\Lambda_0}(c_0)\Big)\alpha_a^*(u_{ij}) \otimes \delta_{a^{-1}b_0c_0a_0a} \\
&= \sum_{a_0,b_0,c_0\in\Lambda_0} \sum_{a\in\Lambda} \sum_{i,j=1}^{d} \mathrm{Tr}\Big(e_{i,j}U_{\Lambda_0}(b_0)U_{\Lambda_0}(c_0)U_{\Lambda_0}(a_0)\Big)\alpha_a^*(u_{ij}) \otimes \delta_{a^{-1}b_0c_0a_0a} \\
&= \sum_{a_0,b_0,c_0\in\Lambda_0} \sum_{a\in\Lambda} \sum_{i,j=1}^{d} \mathrm{Tr}\Big(e_{i,j}U_{\Lambda_0}(b_0c_0a_0)\Big)\alpha_a^*(u_{ij}) \otimes \delta_{a^{-1}b_0c_0a_0a} \\
&= |\Lambda_0|^2 \sum_{a\in\Lambda} \sum_{r_0\in\Lambda_0} \sum_{i,j=1}^{d} \mathrm{Tr}\Big(e_{i,j}U_{\Lambda_0}(r_0)\Big)\alpha_a^*(u_{ij}) \otimes \delta_{a^{-1}r_0a} \\
&= |\Lambda_0|^2 \sum_{r\in\Lambda} (\mathrm{id}\otimes E_{r^{-1}\Lambda_0 r})(\chi_r),
\end{aligned}
\tag{II.5.9}
$$

where the last line uses (II.5.6) and the change of variable $r = a^{-1}$. Dividing $|\Lambda_0|^3$ on both sides of (II.5.9) proves (II.5.3). □

Corollary II.5.2. *Using the notations in Proposition II.5.1, U and $r\cdot U$ induce equivalent unitary representations of $\mathbb{G} \rtimes \Lambda$ for all $r \cdot U$.*

Proof. By Proposition II.5.1, we see that $\mathrm{Ind}(U)$ and $\mathrm{Ind}(r \cdot U)$ have the same character. □

It is worth pointing out that there are in fact many repetitions in the terms of the right side of formula (II.5.3), as is shown by the following lemma.

Lemma II.5.3. *Using the notations of Proposition II.5.1, the following holds:*

(a) *for any $r \in \Lambda$, we have*

$$(\mathrm{id}\otimes E_{r\Lambda_0 r^{-1}})\chi_r = \left(\alpha_{r^{-1}}^* \otimes \mathrm{Ad}_{r^{-1}}^*\right)\left[(\mathrm{id}\otimes E_{\Lambda_0})(\chi)\right]; \tag{II.5.10}$$

in $\mathrm{Pol}(\mathbb{G}) \otimes C(\Lambda)$*;*

(b) *for any $r, s \in \Lambda$, if $r^{-1}s \in \Lambda_0$, i.e. $r\Lambda_0 = s\Lambda_0$ and $r\Lambda_0 r^{-1} = s\Lambda_0 s^{-1}$, then*

$$(\mathrm{id}\otimes E_{r\Lambda_0 r^{-1}})\chi_r = (\mathrm{id}\otimes E_{s\Lambda_0 s^{-1}})\chi_s \tag{II.5.11}$$

in $\mathrm{Pol}(\mathbb{G}) \otimes C(\Lambda)$*. In particular,*

$$\chi_r = \chi_s, \tag{II.5.12}$$

or equivalently, $r \cdot U$ and $s \cdot U$ are unitarily equivalent unitary representations of the same compact quantum group $\mathbb{G} \rtimes r\Lambda_0 r^{-1}$.

Proof. Using the same notations as in the proof of Proposition II.5.1, it is clear that

$$(r \cdot U)_{\mathbb{G}} = \sum_{i,j=1}^{d} e_{i,j} \otimes \alpha_{r^{-1}}^*(u_{ij}), \tag{II.5.13a}$$

$$(r \cdot U)_{r\Lambda_0 r^{-1}} = \sum_{r_0 \in \Lambda_0} U_{\Lambda_0}(r_0) \otimes \delta_{r r_0 r^{-1}}. \tag{II.5.13b}$$

Calculating in $\mathrm{Pol}(\mathbb{G}) \otimes C(\Lambda)$, we have

$$\begin{aligned}(\mathrm{id} \otimes E_{r\Lambda_0 r^{-1}})\chi_r &= \sum_{i,j=1}^{d} \sum_{r_0 \in \Lambda_0} \mathrm{Tr}\big(e_{i,j} U_{\Lambda_0}(r_0)\big) \otimes \alpha^*_{r^{-1}}(u_{ij}) \otimes \delta_{r r_0 r^{-1}} \\ &= \big(\alpha^*_{r^{-1}} \otimes \mathrm{Ad}^*_{r^{-1}}\big) \sum_{i,j=1}^{d} \sum_{r_0 \in \Lambda_0} \mathrm{Tr}\big(e_{i,j} U_{\Lambda_0}(r_0)\big) \otimes u_{ij} \otimes \delta_{r_0} \\ &= \big(\alpha^*_{r^{-1}} \otimes \mathrm{Ad}_{r^{-1}}\big)\left[(\mathrm{id} \otimes E_{\Lambda_0})\chi\right].\end{aligned} \tag{II.5.14}$$

This proves (a).

By (a), to establish (b), it suffices to show that

$$\forall s_0 \in \Lambda_0, \quad (\mathrm{id} \otimes E_{\Lambda_0})\chi = \big(\alpha^*_{s_0} \otimes \mathrm{Ad}^*_{s_0}\big)\left[(\mathrm{id} \otimes E_{\Lambda_0})\chi\right]. \tag{II.5.15}$$

Calculating the right side gives

$$\begin{aligned}&\big(\alpha^*_{s_0} \otimes \mathrm{Ad}^*_{s_0}\big)\left[(\mathrm{id} \otimes E_{\Lambda_0})\chi\right] \\ &= \sum_{i,j=1}^{d} \sum_{r_0 \in \Lambda_0} \mathrm{Tr}\big(e_{i,j} U_{\Lambda_0}(r_0)\big) \otimes \alpha^*_{s_0}(u_{ij}) \otimes \delta_{s_0^{-1} r_0 s_0} \\ &= \sum_{i,j=1}^{d} \sum_{r_0 \in \Lambda_0} \mathrm{Tr}\big(e_{i,j} U_{\Lambda_0}(s_0 r_0 s_0^{-1})\big) \otimes \alpha^*_{s_0}(u_{ij}) \otimes \delta_{r_0} \\ &= \sum_{i,j=1}^{d} \sum_{r_0 \in \Lambda_0} \mathrm{Tr}\Big(U(r_0) U_{\Lambda_0}(s_0^{-1}) e_{i,j} U(s_0)\Big) \otimes \alpha^*_{s_0}(u_{ij}) \otimes \delta_{r_0}.\end{aligned} \tag{II.5.16}$$

Since U_{Λ_0} and $U_{\mathbb{G}}$ are covariant, we have

$$\sum_{i,j}^{d} U(s_0) e_{i,j} \otimes u_{ij} = \sum_{i,j=1}^{d} e_{i,j} U(s_0) \otimes \alpha^*_{s_0}(u_{ij}). \tag{II.5.17}$$

Combining (II.5.16) and (II.5.17), we have

$$\begin{aligned}&\big(\alpha^*_{s_0} \otimes \mathrm{Ad}^*_{s_0}\big)\left[(\mathrm{id} \otimes E_{\Lambda_0})\chi\right] \\ &= \sum_{i,j=1}^{d} \sum_{r_0 \in \Lambda_0} \mathrm{Tr}\Big(U(r_0) U_{\Lambda_0}(s_0^{-1}) e_{i,j} U(s_0)\Big) \otimes \alpha^*_{s_0}(u_{ij}) \otimes \delta_{r_0} \\ &= \sum_{i,j=1}^{d} \sum_{r_0 \in \Lambda_0} \mathrm{Tr}\Big(U(r_0) U_{\Lambda_0}(s_0^{-1}) U(s_0) e_{i,j}\Big) \otimes u_{ij} \otimes \delta_{r_0} \\ &= \sum_{i,j=1}^{d} \sum_{r_0 \in \Lambda_0} \mathrm{Tr}\Big(U(r_0) e_{i,j}\Big) \otimes u_{ij} \otimes \delta_{r_0} \\ &= \sum_{i,j=1}^{d} \sum_{r_0 \in \Lambda_0} \mathrm{Tr}\Big(e_{i,j} U(r_0)\Big) \otimes u_{ij} \otimes \delta_{r_0} \\ &= (\mathrm{id} \otimes E_{\Lambda_0})\chi.\end{aligned} \tag{II.5.18}$$

This establishes (II.5.15) and proves (b). □

Remark II.5.4. By Lemma II.5.3 (b) and Proposition II.5.1, one can in fact choose any complete set $L \subseteq \Lambda$ of representatives of the left coset space Λ/Λ_0, and the character formula (II.5.3) can then be written more concisely as

$$\chi_W = \sum_{r\in L} (\mathrm{id}_A \otimes E_{r\Lambda_0 r^{-1}})\chi_r. \tag{II.5.19}$$

In the classical case where $\mathbb{G}$ is a genuine compact group, one can easily check that the usual character formula for the representation induced by a representation of an open subgroup takes the form (II.5.19). The reason we prefer (II.5.3) is that it does not involve a seemingly arbitrary choice of a complete set of representatives L for Λ/Λ_0, and thus, in the author's opinion, is more aesthetically pleasing. One might also use this choice of left coset representatives to fabric the induced representation. However, in our more symmetric approach (cf. § II.4), everything seems more natural, and the underlying group action of Λ on the various characters $\chi_r, r \in \Lambda$ becomes more transparent in (II.5.3), and we hope this hidden symmetry will keep the reader from losing himself/herself in the details of the tedious calculations to be presented later.

II.6 Dimension of the intertwiner space of induced representations

Let Θ, Ξ be subgroups of Λ, $U \in \mathcal{B}(\mathscr{H}) \otimes \mathrm{Pol}(\mathbb{G}) \otimes C(\Theta)$ a finite dimensional unitary representation of $\mathbb{G} \rtimes \Theta$, $W \in \mathcal{B}(\mathscr{H}) \otimes \mathrm{Pol}(\mathbb{G}) \otimes C(\Xi)$ a finite dimensional unitary representation of $\mathbb{G} \rtimes \Xi$. For the sake of brevity, we denote the induced representation $\mathrm{Ind}_{\mathbb{G}\rtimes\Theta}^{\mathbb{G}\rtimes\Lambda}(U)$ simply by $\mathrm{Ind}(U)$, and $\mathrm{Ind}(W)$ has the similar obvious meaning. Equipped with the character formula established in § II.5, one naturally wonders how can we calculate $\dim \mathrm{Mor}_{\mathbb{G}\rtimes\Lambda}\big(\mathrm{Ind}(U), \mathrm{Ind}(W)\big)$ in terms of some simpler data. This section focuses on this calculation, and the result here will play an important role in proving the irreducibility of some induced representations (as it turns out, these are all irreducible representations of $\mathbb{G} \rtimes \Lambda$ up to equivalence) as well as our later calculation of the fusion rules.

For any representation ρ, we use $\chi(\rho)$ to denote the character of the representation. We denote the Haar state on $\mathbb{G}$ by h, and the Haar state on $\mathbb{G} \rtimes \Lambda_0$ by h^{Λ_0} whenever Λ_0 is a subgroup of Λ.

By the general representation theory of compact quantum groups, we have

$$\dim \mathrm{Mor}_{\mathbb{G}\rtimes\Lambda}\big(\mathrm{Ind}(U), \mathrm{Ind}(W)\big) = h^{\Lambda}\big([\chi(\mathrm{Ind}(U))]^*[\chi(\mathrm{Ind}(W))]\big). \tag{II.6.1}$$

By Proposition II.5.1, for each $r \in \Lambda$, we have a representation $r \cdot U$ (resp. $r \cdot W$) of $\mathbb{G} \rtimes r\Theta r^{-1}$ (resp. $\mathbb{G} \rtimes r\Xi r^{-1}$), and combined with (II.6.1), we have

$$\begin{aligned}&\dim \mathrm{Mor}_{\mathbb{G}}(\mathrm{Ind}(U), \mathrm{Ind}(W))\\ =& \frac{1}{|\Theta|\cdot|\Xi|}\sum_{r,s\in\Lambda} h^{\Lambda}\left([(\mathrm{id}\otimes E_{r\Theta r^{-1}})\chi(r\cdot U)]^*[(\mathrm{id}\otimes E_{s\Xi s^{-1}})\chi(s\cdot W)]\right).\end{aligned} \tag{II.6.2}$$

Notations II.6.1. *To simplify our notations, let $\Lambda(r,s) := r\Theta r^{-1} \cap s\Xi s^{-1}$ for any $r, s \in \Lambda$.*

Lemma II.6.2. *Using the above notations, for any $r, s \in \Lambda$, we have*

$$\begin{aligned}&h^{\Lambda}\left([(\mathrm{id}\otimes E_{r\Theta r^{-1}})\chi(r\cdot U)]^*[(\mathrm{id}\otimes E_{s\Xi s^{-1}})\chi(s\cdot W)]\right)\\ =& \frac{1}{[\Lambda\colon \Lambda(r,s)]}\dim \mathrm{Mor}_{\mathbb{G}\rtimes\Lambda(r,s)}\left((r\cdot U)|_{\mathbb{G}\rtimes\Lambda(r,s)}, (s\cdot W)|_{\mathbb{G}\rtimes\Lambda(r,s)}\right).\end{aligned} \tag{II.6.3}$$

Proof. For any subgroup Λ_0 of Λ, whenever $f \in \mathrm{Pol}(\mathbb{G})$, $r_0 \in \Lambda_0$, by (II.1.7) in § II.1, we have

$$h^\Lambda(f \otimes \delta_{r_0}) = \frac{1}{|\Lambda|} h(f) = \frac{1}{[\Lambda\colon \Lambda_0]} h^{\Lambda_0}(f \otimes \delta_{r_0}). \tag{II.6.4}$$

Hence,

$$h^\Lambda \circ (\mathrm{id} \otimes E_{\Lambda_0}) = \frac{1}{[\Lambda\colon \Lambda_0]} h^{\Lambda_0}. \tag{II.6.5}$$

By definition and a straightforward calculation, we have

$$(\mathrm{id} \otimes E_{r\Theta r^{-1}})\chi(r \cdot U) = \sum_{t \in r\Theta r^{-1}} (\mathrm{Tr} \otimes \mathrm{id}) \left((r \cdot U)_{\mathbb{G}}\left((r \cdot U)_{r\Theta r^{-1}}(t) \otimes 1\right)\right) \otimes \delta_t, \tag{II.6.6a}$$

$$(\mathrm{id} \otimes E_{s\Xi s^{-1}})\chi(s \cdot W) = \sum_{t \in s\Xi s^{-1}} (\mathrm{Tr} \otimes \mathrm{id}) \left((s \cdot W)_{\mathbb{G}}\left((s \cdot W)_{s\Xi s^{-1}}(t) \otimes 1\right)\right) \otimes \delta_t. \tag{II.6.6b}$$

It follows from (II.6.6a) and (II.6.6b) that

$$\begin{aligned}
&[(\mathrm{id} \otimes E_{r\Theta r^{-1}})\chi(r \cdot U)]^*[(\mathrm{id} \otimes E_{s\Xi s^{-1}})\chi(s \cdot W)] \\
&= \sum_{t \in \Lambda(r,s)} \Big\{\left[(\mathrm{Tr} \otimes \mathrm{id}) \left((r \cdot U)_{\mathbb{G}}\left((r \cdot U)_{r\Theta r^{-1}}(t) \otimes 1\right)\right)\right]^* \\
&\qquad \left[(\mathrm{Tr} \otimes \mathrm{id}) \left((s \cdot W)_{\mathbb{G}}\left((s \cdot W)_{s\Xi s^{-1}}(t) \otimes 1\right)\right)\right] \Big\} \otimes \delta_t \\
&= (\mathrm{id} \otimes E_{\Lambda(r,s)}) \left(\left[\chi\left((r \cdot U)|_{\mathbb{G} \rtimes \Lambda(r,s)}\right)\right]^* \left[\chi\left((s \cdot W)|_{\mathbb{G} \rtimes \Lambda(r,s)}\right)\right]\right).
\end{aligned} \tag{II.6.7}$$

Taking $\Lambda_0 = \Lambda(r,s)$ in (II.6.5) and combining with (II.6.7) proves (II.6.3). □

Proposition II.6.3. *Using the above notations, we have*

$$\begin{aligned}
&\dim \mathrm{Mor}_{\mathbb{G} \rtimes \Lambda}\left(\mathrm{Ind}(U), \mathrm{Ind}(W)\right) \\
&= \frac{1}{|\Theta| \cdot |\Xi|} \sum_{r,s \in \Lambda} \frac{1}{[\Lambda\colon \Lambda(r,s)]} \\
&\qquad \dim \mathrm{Mor}_{\mathbb{G} \rtimes \Lambda(r,s)} \left((r \cdot U)|_{\mathbb{G} \rtimes \Lambda(r,s)}, (s \cdot W)|_{\mathbb{G} \rtimes \Lambda(r,s)}\right).
\end{aligned} \tag{II.6.8}$$

Proof. This follows directly from the formula (II.6.2) and Lemma II.6.2. □

Corollary II.6.4. *Let Λ_0 be a subgroup of Λ, U a unitary representation of $\mathbb{G} \rtimes \Lambda_0$, then the following are equivalent:*

(a) *the unitary representation* $\mathrm{Ind}(U)$ *of* $\mathbb{G} \rtimes \Lambda$ *is irreducible;*

(b) *for any $r, s \in \Lambda$, posing $\Lambda(r,s) = r\Lambda_0 r^{-1} \cap s\Lambda_0 s^{-1}$, we have*

$$\dim \mathrm{Mor}_{\mathbb{G} \rtimes \Lambda(r,s)}\left((r \cdot U)|_{\mathbb{G} \rtimes \Lambda(r,s)}, (s \cdot U)|_{\mathbb{G} \rtimes \Lambda(r,s)}\right) = \delta_{r\Lambda_0, s\Lambda_0}; \tag{II.6.9}$$

(c) *U is irreducible, and*

$$\begin{aligned}
&\forall r, s \in \Lambda, \quad r^{-1}s \notin \Lambda_0 \\
&\implies \dim \mathrm{Mor}_{\mathbb{G} \rtimes \Lambda(r,s)}\left((r \cdot U)|_{\mathbb{G} \rtimes \Lambda(r,s)}, (s \cdot U)|_{\mathbb{G} \rtimes \Lambda(r,s)}\right) = 0.
\end{aligned} \tag{II.6.10}$$

In particular, if any of the above conditions holds, then U itself is irreducible.

Proof. If $r^{-1}s \in \Lambda_0$, then $r\Lambda_0 r^{-1} = s\Lambda_0 s^{-1}$, so $\Lambda(r,s) = r\Lambda_0 r^{-1} = s\Lambda_0 s^{-1}$. By Proposition II.3.9, we see that

$$\dim \mathrm{Mor}_{\mathbb{G} \rtimes r\Lambda_0 r^{-1}}\big(r \cdot U, r \cdot U\big) = \dim \mathrm{Mor}_{\mathbb{G} \rtimes \Lambda_0}(U, U). \tag{II.6.11}$$

By Proposition II.6.3, Lemma II.5.3, and the above, we have

$$\begin{aligned}
&\dim \mathrm{Mor}_{\mathbb{G}}\big(\mathrm{Ind}(U), \mathrm{Ind}(U)\big)\\
&= \frac{1}{|\Lambda_0|^2} \sum_{\substack{r,s\in\Lambda,\\ r^{-1}s\in\Lambda_0}} \frac{1}{[\Lambda\colon r\Lambda_0 r^{-1}]} \dim \mathrm{Mor}_{\mathbb{G}\rtimes r\Lambda_0 r^{-1}}\big(r \cdot U, s \cdot U\big)\\
&\qquad + \frac{1}{|\Lambda_0|^2} \sum_{\substack{r,s\in\Lambda,\\ r^{-1}s\notin\Lambda_0}} \frac{1}{[\Lambda\colon \Lambda(r,s)]} \dim \mathrm{Mor}_{\mathbb{G}\rtimes\Lambda(r,s)}\Big((r\cdot U)|_{\mathbb{G}\rtimes\Lambda(r,s)}, (s\cdot U)|_{\mathbb{G}\rtimes\Lambda(r,s)}\Big)\\
&= \frac{1}{|\Lambda_0|^2} \sum_{\substack{r,s\in\Lambda,\\ r^{-1}s\in\Lambda_0}} \dim \mathrm{Mor}_{\mathbb{G}\rtimes r\Lambda_0 r^{-1}}\big(r \cdot U, r \cdot U\big)\\
&\qquad + \frac{1}{|\Lambda_0|^2} \sum_{\substack{r,s\in\Lambda,\\ r^{-1}s\notin\Lambda_0}} \frac{1}{[\Lambda\colon \Lambda(r,s)]} \dim \mathrm{Mor}_{\mathbb{G}\rtimes\Lambda(r,s)}\Big((r\cdot U)|_{\mathbb{G}\rtimes\Lambda(r,s)}, (s\cdot U)|_{\mathbb{G}\rtimes\Lambda(r,s)}\Big)\\
&= \frac{1}{|\Lambda_0|^2} \sum_{\substack{r,s\in\Lambda,\\ r^{-1}s\in\Lambda_0}} \frac{1}{[\Lambda\colon \Lambda_0]} \dim_{\mathbb{G}\rtimes\Lambda_0}(U, U)\\
&\qquad + \frac{1}{|\Lambda_0|^2} \sum_{\substack{r,s\in\Lambda,\\ r^{-1}s\notin\Lambda_0}} \frac{1}{[\Lambda\colon \Lambda(r,s)]} \dim \mathrm{Mor}_{\mathbb{G}\rtimes\Lambda(r,s)}\Big((r\cdot U)|_{\mathbb{G}\rtimes\Lambda(r,s)}, (s\cdot U)|_{\mathbb{G}\rtimes\Lambda(r,s)}\Big)\\
&= \frac{|\Lambda|\cdot|\Lambda_0|}{|\Lambda_0|^2\cdot[\Lambda\colon\Lambda_0]} \dim \mathrm{Mor}_{\mathbb{G}\rtimes\Lambda_0}(U, U)\\
&\qquad + \frac{1}{|\Lambda_0|^2} \sum_{\substack{r,s\in\Lambda,\\ r^{-1}s\notin\Lambda_0}} \frac{1}{[\Lambda\colon \Lambda(r,s)]} \dim \mathrm{Mor}_{\mathbb{G}\rtimes\Lambda(r,s)}\Big((r\cdot U)|_{\mathbb{G}\rtimes\Lambda(r,s)}, (s\cdot U)|_{\mathbb{G}\rtimes\Lambda(r,s)}\Big).
\end{aligned}$$

Since $|\Lambda|\cdot|\Lambda_0| = |\Lambda_0|^2\cdot[\Lambda\colon\Lambda_0]$ and

$$\begin{aligned}
\dim \mathrm{Mor}_{\mathbb{G}\rtimes r\Lambda_0 r^{-1}}\big(r\cdot U, s\cdot U\big) &= \dim \mathrm{Mor}_{\mathbb{G}\rtimes r\Lambda_0 r^{-1}}\big(r\cdot U, r\cdot U\big)\\
&= \dim \mathrm{Mor}_{\mathbb{G}\rtimes\Lambda_0}(U,U) = \dim \mathrm{End}_{\mathbb{G}\rtimes\Lambda_0}(U)
\end{aligned} \tag{II.6.12}$$

whenever $r^{-1}s \in \Lambda_0$ by Lemma II.5.3 and Proposition II.3.9, the above calculation yields

$$\dim \mathrm{End}_{\mathbb{G}\rtimes\Lambda}\left(\mathrm{Ind}(U)\right) = \dim \mathrm{End}_{\mathbb{G}\rtimes\Lambda_0}(U) + \frac{1}{|\Lambda_0|^2} \sum_{\substack{r,s\in\Lambda,\\ r^{-1}s\notin\Lambda_0}} \frac{d(r,s)}{[\Lambda : \Lambda(r,s)]}, \tag{II.6.13}$$

where

$$d(r,s) := \dim \mathrm{Mor}_{\mathbb{G}\rtimes\Lambda(r,s)}\Big((r\cdot U)|_{\mathbb{G}\rtimes\Lambda(r,s)}, (s\cdot U)|_{\mathbb{G}\rtimes\Lambda(r,s)}\Big). \tag{II.6.14}$$

The corollary follows from (II.6.12) (II.6.13), (II.6.14) and the fact that a representation is irreducible if and only if the dimension of the space of its self-intertwiners is 1. □

Remark II.6.5. Corollary II.6.4 is the quantum analogue for Mackey's criterion for irreducibility.

II.7 The C^*-tensor category $\mathcal{CSR}_{\Lambda_0}$

We begin by recalling the notations in Proposition II.5.1: for any unitary representation $U_{\mathbb{G}} \in \mathcal{B}(\mathscr{H}) \otimes \mathrm{Pol}(\mathbb{G})$ of $\mathbb{G}$ on some finite dimensional Hilbert space $\mathscr{H}$, and any $r \in \Lambda$, let $r \cdot U_{\mathbb{G}}$ be the unitary representation $(\mathrm{id}_{\mathscr{H}} \otimes \alpha^*_{r^{-1}})(U_{\mathbb{G}})$ of $\mathbb{G}$ on the same space $\mathscr{H}$. It is easy to see that this defines a left group action of Λ on the proper class of all unitary representations of $\mathbb{G}$, and by passing to quotients, this representation induces an action of Λ on $\mathrm{Irr}(\mathbb{G})$. From now on, whenever we talk about Λ acting on a unitary representation $U_{\mathbb{G}}$ of $\mathbb{G}$, or on some class $x \in \mathrm{Irr}(\mathbb{G})$, we always mean these actions.

Definition II.7.1. A subgroup Λ_0 of Λ is called a general isotropy subgroup if there is some $n \in \mathbb{N}$, such that Λ_0 is an isotropy subgroup (subgroup of stabilizer for some point) for the n-fold product $[\mathrm{Irr}(\mathbb{G})]^n$ as a Λ-set; in other words, if there exists an n-tuple $(x_1, \ldots, x_n)$ with all $x_i \in \mathrm{Irr}(\mathbb{G})$, such that

$$\Lambda_0 = \{r \in \Lambda \ : \ \forall i = 1, \ldots, n, \quad r \cdot x_i = x_i\} = \cap_{i=1}^n \Lambda_{x_i}.$$

The finite (recall that Λ is finite) family of all general isotropy subgroups of Λ is denoted by $\mathcal{G}_{\mathrm{iso}}(\Lambda)$.

The following proposition is an immediate consequence of properties of Λ-sets and Definition II.7.1.

Proposition II.7.2. *The family $\mathcal{G}_{\mathrm{iso}}(\Lambda)$ is stable under intersection and conjugation by elements of Λ.* □

Definition II.7.3. Let Λ_0 be a general isotropy subgroup of Λ. A covariant system of representations (or CSR for short) subordinate to Λ_0 is a triple $(\mathscr{H}, u, w)$, where

- $\mathscr{H}$ is a finite dimensional Hilbert space;
- u is a unitary representation of $\mathbb{G}$ on $\mathscr{H}$;
- w is a unitary representation of Λ_0 on $\mathscr{H}$,

such that u and w are covariant. In this chapter, CSRs are often denoted by bold faced uppercase letters like $\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots$(mostly $\mathbf{S}$) with possible subscripts.

By Proposition II.2.2, the covariant systems of representations subordinate to a general isotropy subgroup Λ_0 correspond bijectively to the class of unitary representations of $\mathbb{G} \rtimes \Lambda_0$, via

$$(\mathscr{H}, u, w) \mapsto u_{12} w_{13}$$

in one direction, and

$$U_{\mathscr{H}} \mapsto (\mathscr{H}, U_{\mathscr{H},\mathbb{G}}, U_{\mathscr{H},\Lambda_0})$$

in the other, where $\mathscr{H}$ is the underlying space of the representation $U_{\mathscr{H}}$ of $\mathbb{G} \rtimes \Lambda_0$, and $U_{\mathscr{H},\mathbb{G}}$, $U_{\mathscr{H},\Lambda_0}$ are the restrictions of $U_{\mathscr{H}}$ to $\mathbb{G}$ and Λ_0 respectively. Using this bijection, we can transport the rigid C^*-tensor category structure on $\mathrm{Rep}(\mathbb{G} \rtimes \Lambda_0)$– the category of all finite dimensional unitary representations of $\mathbb{G} \rtimes \Lambda_0$, to the class of covariant systems of representations subordinate to Λ_0, thereby getting a rigid C^*-tensor category $\mathcal{CSR}_{\Lambda_0}$ whose objects are CSRs subordinate to Λ_0.

To make this transport of categorical structures less tautological, we make a convenient characterization of the morphisms in $\mathcal{CSR}_{\Lambda_0}$.

Proposition II.7.4. *Fix a general isotropy subgroup Λ_0 of Λ. For $i = 1, 2$, let $\mathrm{S}_i = (\mathscr{H}_i, u_i, w_i)$ be a CSR subordinate to Λ_0, $U_i = (u_i)_{12}(w_i)_{13}$ the corresponding unitary representation of $\mathbb{G} \rtimes \Lambda_0$, $S \in \mathcal{B}(\mathscr{H}_1, \mathscr{H}_2)$. Then $S \in \mathrm{Mor}_{\mathbb{G}\rtimes\Lambda_0}(U_1, U_2)$ if and only if*

$$S \in \mathrm{Mor}_{\mathbb{G}}(u_1, u_2) \cap \mathrm{Mor}_{\Lambda_0}(w_1, w_2). \tag{II.7.1}$$

Proof. The condition is easily seen to be sufficient. Indeed, if condition (II.7.1) holds, then

$$(S \otimes 1)w_1 = w_2(S \otimes 1), (S \otimes 1)u_1 = u_2(S \otimes 1). \tag{II.7.2}$$

Thus

$$\begin{aligned}(S \otimes 1 \otimes 1)U_1 &= (S \otimes 1 \otimes 1)(u_1)_{12}(w_1)_{13} = (u_2)_{12}(S \otimes 1 \otimes 1)(w_1)_{13} \\ &= (u_2)_{12}(w_2)_{13}(S \otimes 1 \otimes 1) = U_2(S \otimes 1 \otimes 1).\end{aligned} \tag{II.7.3}$$

This means exactly $S \in \mathrm{Mor}_{\mathbb{G}}(U_1, U_2)$.

To show the necessity of this condition, let $\epsilon_{\mathbb{G}} \colon \mathrm{Pol}(\mathbb{G}) \to \mathbb{C}$ be the counit of the Hopf-$*$-algebra, $\epsilon_{\Lambda_0} \colon C(\Lambda_0) \to \mathbb{C}$ the counit for the Hopf $*$-algebra $C(\Lambda_0)$. Since $U_i \in \mathcal{B}(\mathscr{H}_i) \otimes \mathrm{Pol}(\mathbb{G}) \otimes C(\Lambda_0)$ for $i = 1, 2$ and $S \in \mathrm{Mor}_{\mathbb{G}\rtimes\Lambda_0}(U_1, U_2)$, we have

$$(S \otimes 1 \otimes 1)U_1 = U_2(S \otimes 1 \otimes 1). \tag{II.7.4}$$

Applying $\mathrm{id} \otimes \mathrm{id} \otimes \epsilon_{\Lambda_0}$ on both sides of (II.7.4) yields

$$(S \otimes 1)u_1 = u_2(S \otimes 1), \tag{II.7.5}$$

which means $S \in \mathrm{Mor}_{\mathbb{G}}(u_1, u_2)$. Applying $\mathrm{id} \otimes \epsilon_{\mathbb{G}} \otimes \mathrm{id}$ on both sides of (II.7.4) yields

$$(S \otimes 1)w_1 = w_1(S \otimes 1), \tag{II.7.6}$$

which means $S \in \mathrm{Mor}_{\Lambda_0}(w_1, w_2)$. □

We now define a pair of functors,

$$\mathscr{R}_{\Lambda_0} \colon \mathcal{CSR}_{\Lambda_0} \to \mathrm{Rep}(\mathbb{G} \rtimes \Lambda_0) \qquad \text{and} \qquad \mathscr{S}_{\Lambda_0} \colon \mathrm{Rep}(\mathbb{G} \rtimes \Lambda_0) \to \mathcal{CSR}_{\Lambda_0}$$

between $\mathcal{CSR}_{\Lambda_0}$ and $\mathrm{Rep}(\mathbb{G} \rtimes \Lambda_0)$ that reflects the transport of categorical structures discussed above. On the object level, for any $(\mathscr{H}, u, w) \in \mathcal{CSR}_{\Lambda_0}$, let $\mathscr{R}_{\Lambda_0}(u, w)$ be the representation $u_{12}w_{13}$ of $\mathbb{G} \rtimes \Lambda_0$ on $\mathscr{H}$; for any unitary representation $U \in \mathrm{Rep}(\mathbb{G} \rtimes \Lambda_0)$ on $\mathscr{H}_U$, let $\mathscr{S}_{\Lambda_0}(U)$ be the CSR $(\mathscr{H}_U, U_{\mathbb{G}}, U_{\Lambda_0})$ where $U_{\mathbb{G}}$ (resp. U_{Λ_0}) is the restriction of U onto $\mathbb{G}$ (resp. Λ_0). On the morphism level, both $\mathscr{R}_{\Lambda_0}$ and $\mathscr{S}_{\Lambda_0}$ act as identity. By Proposition II.7.4 and Proposition II.2.2, $\mathscr{R}_{\Lambda_0}$ and $\mathscr{S}_{\Lambda_0}$ are indeed well-defined functors inverses to each other, and they are fiber functors (exact unitary tensor functors (Neshveyev and Tuset, 2013, §§2.1, 2.2)) simply because the rigid C^*-tensor category structure on $\mathcal{CSR}_{\Lambda_0}$ is transported from that of $\mathrm{Rep}(\mathbb{G} \rtimes \Lambda_0)$ via $\mathscr{S}_{\Lambda_0}$.

Proposition II.7.5. *For $i = 1, 2$, let $\mathrm{S}_i = (\mathscr{H}_i, u_i, w_i) \in \mathcal{CSR}_{\Lambda_0}$, $U_i = \mathscr{R}_{\Lambda_0}(\mathrm{S}_i) \in \mathrm{Rep}(\mathbb{G} \rtimes \Lambda_0)$, then*

$$\mathscr{S}_{\Lambda_0}(U_1 \times U_2) = (\mathscr{H}_1 \otimes \mathscr{H}_2, u_1 \times u_2, w_1 \times w_2) = \mathrm{S}_1 \otimes \mathrm{S}_2. \tag{II.7.7}$$

Proof. By definition of the tensor product of representations, $U_1 \times U_2$ is the representation of $\mathbb{G} \rtimes \Lambda_0$ defined by

$$U_1 \times U_2 = (U_1)_{134}(U_2)_{234} \in \mathcal{B}(\mathscr{H}_1) \otimes \mathcal{B}(\mathscr{H}_2) \otimes \mathrm{Pol}(\mathbb{G}) \otimes C(\Lambda_0), \tag{II.7.8}$$

where we identified $\mathcal{B}(\mathscr{H}_1) \otimes \mathcal{B}(\mathscr{H}_2)$ with $\mathcal{B}(\mathscr{H}_1 \otimes \mathscr{H}_2)$ canonically.

The restriction of $U_1 \times U_2$ onto $\mathbb{G}$ is

$$\begin{aligned}
&(\mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id} \otimes \epsilon_{\Lambda_0})(U_1 \times U_2) \\
=\ &(\mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id} \otimes \epsilon_{\Lambda_0})((U_1)_{134})(\mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id} \otimes \epsilon_{\Lambda_0})((U_2)_{234}) \\
=\ &(u_1)_{13}(u_2)_{23} = u_1 \times u_2 \in \mathcal{B}(\mathscr{H}_1) \otimes \mathcal{B}(\mathscr{H}_2) \otimes \mathrm{Pol}(\mathbb{G}).
\end{aligned} \tag{II.7.9}$$

Similarly, the restriction of $U_1 \times U_2$ onto Λ_0 is

$$\begin{aligned}
&(\mathrm{id} \otimes \mathrm{id} \otimes \epsilon_{\mathbb{G}} \otimes \mathrm{id})(U_1 \times U_2) \\
=\ &(\mathrm{id} \otimes \mathrm{id} \otimes \epsilon_{\mathbb{G}} \otimes \mathrm{id})((U_1)_{134})(\mathrm{id} \otimes \mathrm{id} \otimes \epsilon_{\mathbb{G}} \otimes \mathrm{id})((U_2)_{234}) \\
=\ &(w_1)_{13}(w_{(2)})_{23} = w_1 \times w_2 \in \mathcal{B}(\mathscr{H}_1) \otimes \mathcal{B}(\mathscr{H}_2) \otimes C(\Lambda_0).
\end{aligned} \tag{II.7.10}$$

Now (II.7.7) follows from (II.7.9), (II.7.10) and the definition of the tensor product in $\mathcal{CSR}_{\Lambda_0}$. □

Proposition II.7.6. *For $i = 1, 2$, let $\mathrm{S}_i = (\mathscr{H}, u_i, w_i) \in \mathcal{CSR}_{\Lambda_0}$, $U_i := \mathscr{R}_{\Lambda_0}(\mathrm{S}_i) \in \mathrm{Rep}(\mathbb{G} \rtimes \Lambda_0)$, then*

$$\mathscr{S}_{\Lambda_0}(U_1 \oplus U_2) = (\mathscr{H}_1 \oplus \mathscr{H}_2, u_1 \oplus u_2, w_1 \oplus w_2) = \mathrm{S}_1 \oplus \mathrm{S}_2. \tag{II.7.11}$$

Proof. The proof use the same restriction technique as in the proofs of Proposition II.7.4 and Proposition II.7.5, which is even simpler in this case. □

Until now, we've shown that the morphisms, tensor products, and direct sums all behave as expected in $\mathcal{CSR}_{\Lambda_0}$. The description of the dual of a CSR when $\mathbb{G}$ is of non-Kac type requires a bit further work on the so-called modular operator, as we presently discuss.

Recall that the contragredient representation U^c of a *unitary* representation U of $\mathbb{G} \rtimes \Lambda_0$ on some finite dimensional Hilbert space $\mathscr{H}$ is defined as $U^c = (j \otimes \mathrm{id}_{\mathrm{Pol}(\mathbb{G}) \otimes C(\Lambda_0)})(U^*)$, where $j\colon \mathcal{B}(\mathscr{H}) \to \mathcal{B}(\overline{\mathscr{H}})$ is defined as $T \mapsto \overline{T^*}$, with $\overline{\mathscr{H}}$ being the conjugate Hilbert space of $\mathscr{H}$, and $\overline{T^*}$ meaning T^* viewed as a linear mapping from $\overline{\mathscr{H}}$ to $\overline{\mathscr{H}}$. Note that $j : \mathcal{B}(\mathscr{H}) \to \mathcal{B}(\overline{\mathscr{H}})$, $T \mapsto \overline{T^*}$ is linear, antimultiplicative and positive (in particular, it preserves adjoints). If $\mathbb{G}$ is of non-Kac type, so is $\mathbb{G} \rtimes \Lambda_0$ by Proposition II.1.4, in which case U^c might not be unitary, which is exactly why the "modular" operator ρ_U is necessary to express the dual object of $\mathscr{S}_{\Lambda_0}(U)$ in $\mathcal{CSR}_{\Lambda_0}$ as presented in Proposition II.7.7.

Proposition II.7.7. *Let $\mathrm{S} = (\mathscr{H}, u, w) \in \mathcal{CSR}_{\Lambda_0}$, $U = \mathscr{R}_{\Lambda_0}(\mathrm{S}) \in \mathrm{Rep}(\mathbb{G} \rtimes \Lambda_0)$, U^c the contragredient representation of U on the conjugate space $\overline{\mathscr{H}}$ of $\mathscr{H}$. If ρ_U is the unique invertible positive operator in $\mathrm{Mor}_{\mathbb{G} \rtimes \Lambda_0}(U, U^{cc})$ (which we call modular operator) such that $\mathrm{Tr}(\cdot\, \rho_U) = \mathrm{Tr}(\cdot\, \rho_U^{-1})$ on $\mathrm{End}_{\mathbb{G} \rtimes \Lambda_0}(U)$, so that*

$$\overline{U} = \left\{[j(\rho_U)]^{1/2} \otimes 1_{\mathrm{Pol}(\mathbb{G})} \otimes 1_{C(\Lambda_0)}\right\} U^c \left\{[j(\rho_U)]^{-1/2} \otimes 1_{\mathrm{Pol}(\mathbb{G})} \otimes 1_{C(\Lambda_0)}\right\} \tag{II.7.12}$$

is the conjugate representation of U*, then the dual of* S *is given by* $\overline{\mathrm{S}} = (\overline{\mathscr{H}}, u', w')$*, where*

$$\begin{aligned} u' &= (j(\rho_U)^{1/2} \otimes 1)u^c(j(\rho_U)^{-1/2} \otimes 1), \\ w' &= (j(\rho_U)^{1/2} \otimes 1)w^c(j(\rho_U)^{-1/2} \otimes 1). \end{aligned} \tag{II.7.13}$$

Note that $w^c = \overline{w}$ *as* Λ *is a finite (compact) group. In particular, if* $\mathbb{G}$ *is of Kac-type, then* $u^c = \overline{u}$*,* $\rho_U = 1$*, and* $\overline{\mathrm{S}} = (\overline{\mathscr{H}}, \overline{u}, \overline{w})$*.*

Proof. By definition, $\overline{S} = \mathscr{S}_{\Lambda_0}(\overline{U})$, thus

$$\begin{aligned} u' &= (\mathrm{id}_{\mathcal{B}(\mathscr{H})} \otimes \mathrm{id}_{\mathrm{Pol}(\mathbb{G})} \otimes \epsilon_{\Lambda_0})(\overline{U}) \\ &= (\mathrm{id} \otimes \mathrm{id} \otimes \epsilon_{\Lambda_0})\big[(j(\rho_U)^{1/2} \otimes 1 \otimes 1)U^c(j(\rho_U)^{-1/2} \otimes 1 \otimes 1)\big] \\ &= (\mathrm{id} \otimes \mathrm{id} \otimes \epsilon_{\Lambda_0})\Big((j(\rho_U)^{1/2} \otimes 1 \otimes 1) \\ &\qquad\qquad [(j \otimes \mathrm{id}_{\mathrm{Pol}(\mathbb{G})} \otimes \mathrm{id}_{C(\Lambda_0)})(U^*)](j(\rho_U)^{-1/2} \otimes 1 \otimes 1)\Big) \\ &= (j(\rho_U)^{1/2} \otimes 1)\big[(j \otimes \mathrm{id} \otimes \epsilon_{\Lambda_0})(U^*)\big](j(\rho_U)^{-1/2} \otimes 1) \\ &= (j(\rho_U)^{1/2} \otimes 1)\big[(j \otimes \mathrm{id} \otimes \epsilon_{\Lambda_0})(U)\big]^*(j(\rho_U)^{-1/2} \otimes 1) \\ &= (j(\rho_U)^{1/2} \otimes 1)\big[(j \otimes \mathrm{id})(u)\big]^*(j(\rho_U)^{-1/2} \otimes 1) \\ &= (j(\rho_U)^{1/2} \otimes 1)\big[(j \otimes \mathrm{id})(u^*)\big](j(\rho_U)^{-1/2} \otimes 1) \\ &= (j(\rho_U)^{1/2} \otimes 1)u^c(j(\rho_U)^{-1/2} \otimes 1). \end{aligned}$$

The expression for w' is proved analogously by applying $\mathrm{id}_{\mathcal{B}(\mathscr{H})} \otimes \epsilon_{\mathrm{Pol}(\mathbb{G})} \otimes \mathrm{id}_{C(\Lambda_0)}$ on (II.7.12). Finally, if $\mathbb{G}$ is of Kac-type, then $\rho_U = \mathrm{id}_{\mathscr{H}} = 1$. □

Remark II.7.8. The "modular" operator ρ_U of the representation U is derived from the representation theory of $\mathbb{G} \rtimes \Lambda_0$ instead of the representation theory of $\mathbb{G}$ and (projective) representation theory of Λ_0. This makes the description of $\overline{\mathrm{S}}$ in Proposition II.7.7 quite unsatisfactory in the non-unimodular case. This being said, we point out that as far as the fusion rules of $\mathbb{G} \rtimes \Lambda_0$ are concerned, the duals of a sufficiently large family of CSRs admit a much more satisfactory description (see Proposition II.13.3).

Of course, the description of the dual in $\mathcal{CSR}_{\Lambda_0}$ is much easier if $\mathbb{G}$ is of Kac-type, as is clearly seen from the last part of Proposition II.7.7.

II.8 Group actions and projective representations

Fix a $\Lambda_0 \in \mathcal{G}_{\mathrm{iso}}(\Lambda)$. Via the functors $\mathscr{S}_{\Lambda_0}$ and $\mathscr{R}_{\Lambda_0}$, we see that the problem classifying of irreducible representations of $\mathbb{G} \rtimes \Lambda_0$ are essentially the same as classifying simple CSRs in $\mathcal{CSR}_{\Lambda_0}$. Thus for the moment, it might be too much to hope there exists a satisfactory description of *all* simple CSRs in $\mathcal{CSR}_{\Lambda_0}$. However, as we will see in § II.9, if we restrict our attention to the so-called *stably pure* simple CSRs in $\mathcal{CSR}_{\Lambda_0}$, then such a description is indeed achievable via the theory of unitary projective representations of Λ_0. This section studies how such projective representations arise naturally from the action of Λ on irreducible representations of $\mathbb{G}$, as well as establishes some basic properties of these projective representations. The results here will be used in § II.9 to describe the structure of stably pure CSRs in $\mathcal{CSR}_{\Lambda_0}$ (Proposition II.9.5 and Proposition II.9.6).

We begin with a simple observation which is a trivial quantum analogue of one of the most basic ingredients of the Mackey analysis. Let $U_{\mathbb{G}}$ be a unitary representation of $\mathbb{G}$ on some finite dimensional Hilbert space $\mathscr{H}$. Since $\alpha^*: \Lambda \to \mathrm{Aut}\big(C(\mathbb{G}), \Delta\big)$ is an antihomomorphism of groups, we know that $(\mathrm{id}_{\mathcal{B}(\mathscr{H})} \otimes \alpha^*_{r^{-1}})(U_{\mathbb{G}})$ is again a unitary representation of $\mathbb{G}$ on the same space $\mathscr{H}$, and we denote this new representation by $r \cdot U_{\mathbb{G}}$ as we did in Proposition II.5.1. One checks that $(rs) \cdot U_{\mathbb{G}} = r \cdot (s \cdot U_{\mathbb{G}})$. Thus this defines a left action of the group Λ on the (proper) class of all unitary representation of $\mathbb{G}$, which is easily seen to preserve irreducibility and pass to a well-defined action of Λ on the *set* $\mathrm{Irr}(\mathbb{G})$ by letting $r \cdot [u] = [r \cdot u]$, where $r \in \Lambda$, u is an irreducible unitary representation of $\mathbb{G}$ and $[u]$ is the equivalence class of u in $\mathrm{Irr}(\mathbb{G})$. Take another unitary representation $W_{\mathbb{G}}$ of $\mathbb{G}$ on some other finite dimensional Hilbert space $\mathscr{K}$. For any $r, s \in \Lambda$ and any $T \in \mathcal{B}(\mathscr{H}, \mathscr{K})$, we have

$$\begin{aligned}
& T \in \mathrm{Mor}_{\mathbb{G}}(r \cdot U_{\mathbb{G}}, W_{\mathbb{G}}) \\
\Longleftrightarrow\ & W_{\mathbb{G}}(T \otimes 1) = (T \otimes 1)(\mathrm{id} \otimes \alpha^*_{r^{-1}})(U_{\mathbb{G}}) \\
\Longleftrightarrow\ & [(\mathrm{id} \otimes \alpha^*_{s^{-1}})(W_{\mathbb{G}})](T \otimes 1) = (T \otimes 1)(\mathrm{id} \otimes \alpha^*_{(sr)^{-1}})(U_{\mathbb{G}}) \\
\Longleftrightarrow\ & T \in \mathrm{Mor}_{\mathbb{G}}(sr \cdot U_{\mathbb{G}}, s \cdot W_{\mathbb{G}}).
\end{aligned} \tag{II.8.1}$$

Now take any irreducible unitary representation u of $\mathbb{G}$ on some finite dimensional Hilbert space $\mathscr{H}$. Let $x = [u] \in \mathrm{Irr}(\mathbb{G})$, and

$$\Lambda_x = \{r \in \Lambda \ : \ r \cdot x = x\},$$

i.e. Λ_x is the isotropy subgroup of Λ fixing x. Then for any $r_0 \in \Lambda_x$, u and $r_0 \cdot u$ are equivalent by definition, hence there exists a unitary $V(r_0) \in \mathcal{U}(\mathscr{H})$ intertwining $r_0 \cdot u$ and u, in other words,

$$\big(V(r_0) \otimes 1\big)(\mathrm{id} \otimes \alpha^*_{r_0^{-1}})(u) = u\big(V(r_0) \otimes 1\big), \tag{II.8.2}$$

which is clearly equivalent to

$$\forall r_0 \in \Lambda_x, \quad \big(V(r_0) \otimes 1\big)u = [(\mathrm{id} \otimes \alpha^*_{r_0})(u)]\big(V(r_0) \otimes 1\big). \tag{II.8.3}$$

It is remarkable that (II.8.3) takes exactly the same form as the covariance condition (II.2.2) when we define covariant representations in § II.2. Now if we choose a

$$V(r_0) \in \mathrm{Mor}_{\mathbb{G}}(r_0 \cdot u, u) \cap \mathcal{U}(\mathscr{H}) \tag{II.8.4}$$

for each $r_0 \in \Lambda_x$, then for any $s_0 \in \Lambda_x$, by (II.8.1), we have

$$V(r_0) \in \mathrm{Mor}_{\mathbb{G}}(s_0 r_0 \cdot u, s_0 \cdot u), \quad V(s_0) \in \mathrm{Mor}_{\mathbb{G}}(s_0 \cdot u, u), \quad V(s_0 r_0) \in \mathrm{Mor}_{\mathbb{G}}(s_0 r_0 \cdot u, u),$$

thus

$$\forall r_0, s_0 \in \Lambda_x, \quad V(s_0 r_0)[V(r_0)]^*[V(s_0)]^* \in \mathrm{Mor}_{\mathbb{G}}(u, u) \cap \mathcal{U}(\mathscr{H}) = \mathbb{T} \cdot \mathrm{id}_{\mathscr{H}}. \tag{II.8.5}$$

This means that $V: \Lambda_x \to \mathcal{U}(\mathscr{H})$ is a unitary **projective representation** (Definition II.8.1) of Λ_x on $\mathscr{H}$, which satisfies the covariant condition (II.8.3) for each $r_0 \in \Lambda_x$.

To facilitate our discussion, we digress now to give a brief summary of some basic terminologies of the theory of group cohomology which we will use (cf. (Brown, 1994)). We regard $\mathbb{T}$ as a trivial module over any finite group when considering unitary projective representations of finite groups. For any finite group Γ, an n-cochain on Γ with coefficients in $\mathbb{T}$, or simply an n-cochain (on Γ), as we won't consider coefficient module other that the trivial module $\mathbb{T}$, is a mapping from the n-fold product $\Gamma^n = \Gamma \times \cdots \times \Gamma$ to $\mathbb{T}$. Let $C^n(\Gamma, \mathbb{T})$ be the abelian group of n-cochains on Γ, $Z^2(\Gamma, \mathbb{T})$ the subgroup of 2-cocycles on Γ, i.e. mappings $\omega\colon \Gamma \times \Gamma \to \mathbb{T}$ satisfying the cocycle condition

$$\forall r, s, t \in \Gamma, \quad \omega(r, st)\omega(s, t) = \omega(r, s)\omega(rs, t). \tag{II.8.6}$$

The mapping

$$\begin{aligned} \delta\colon C^1(\Gamma, \mathbb{T}) &\to Z^2(\Gamma, \mathbb{T}) \\ \mathrm{b} &\mapsto \left\{(r, s) \in \Gamma \times \Gamma \mapsto \frac{\mathrm{b}(r)\mathrm{b}(s)}{\mathrm{b}(rs)}\right\} \end{aligned} \tag{II.8.7}$$

is easily checked to be a well-defined group morphism. We use $B^2(\Gamma, \mathbb{T})$ to denote the image of δ, and the 2-cocycles in $B^2(\Gamma, \mathbb{T})$ are called 2-coboundaries of Γ. The quotient group $Z^2(\Gamma, \mathbb{T})/B^2(\Gamma, \mathbb{T})$ is called the second cohomology group of Γ with coefficients in the trivial Γ-module $\mathbb{T}$, and is denoted by $H^2(\Gamma, \mathbb{T})$. Elements in $H^2(\Gamma, \mathbb{T})$ are called cohomology class. Note that $\ker(\delta)$ is exactly the group of characters on Γ, i.e. group morphisms from Γ to $\mathbb{T}$.

Definition II.8.1. Let Γ be a group, $\mathscr{H}$ a finite dimensional Hilbert space, a projective representation of Γ on $\mathscr{H}$ is a mapping $V : \Gamma \to \mathcal{U}(\mathscr{H})$ such that $V(e_\Gamma) = \mathrm{id}_{\mathscr{H}}$, and there exists a 2-cochain $\omega \in C^2(\Gamma, \mathbb{T})$, such that

$$\forall r, s \in \Gamma, \qquad \omega(r, s)V(r, s) = V(r)V(s). \tag{II.8.8}$$

It is easy to check that such ω is uniquely determined by V, and it is in fact a 2-cocylce, with the additional property (which follows from our assumption $V(e_\Gamma) = \mathrm{id}_{\mathscr{H}}$) that

$$\forall \gamma \in \Gamma, \qquad \omega(e_\Gamma, \gamma) = \omega(\gamma, e_\Gamma) = 1 \in \mathbb{T}. \tag{II.8.9}$$

We call ω the cocylce (or Schur multiplier after Schur who introduced them in his work on projective representations (Schur, 1904)) of the projective representation V.

We will freely use the character theory and the Peter Weyl theory of projective representations of finite groups, and we refer the reader to (Cheng, 2015) for the proofs.

We track here the following easy results for convenience of the reader.

Lemma II.8.2. *Let Γ be a finite group, $V\colon \Gamma \to \mathcal{U}(\mathscr{H})$ a finite dimensional unitary projective representation of Γ with cocycle ω. If $\omega' \in [\omega] \in H^2(\Gamma, \mathbb{T})$, then there exists a mapping $\mathrm{b}\colon \Gamma \to \mathbb{T}$, such that $\mathrm{b}V\colon \Gamma \to \mathcal{U}(\mathscr{H})$, $\gamma \mapsto \mathrm{b}(\gamma)V(\gamma)$ is a unitary projective representation with cocycle ω'.*

Proof. Since $\omega' \in [\omega]$, there is a mapping $\mathrm{b}\colon \Gamma \to \mathbb{T}$ such that $\omega' = (\delta\mathrm{b})\omega$, and obviously, $\mathrm{b}V$ is a unitary projective representation with $(\delta\mathrm{b})\omega = \omega'$ as its cocycle. $\square$

Lemma II.8.3. *Let Γ be a finite group, $V\colon \Gamma \to \mathcal{U}(\mathscr{H})$ a finite dimensional unitary projective representation of Γ with cocycle ω, and let $\mathrm{b}\colon \Gamma \to \mathbb{T}$ be an arbitrary mapping. The following hold:*

(a) $\mathrm{b}V\colon \Gamma \to \mathcal{U}(\mathscr{H})$, $\gamma \mapsto \mathrm{b}(\gamma)V(\gamma)$ *is a projective representation with cocycle* $(\delta\mathrm{b})\omega$*;*

(b) $\mathrm{b}V$ *and* V *have the same cocycle if and only if* $\mathrm{b} \in \ker(\delta)$*, i.e.* b *is a character of* Γ*;*

(c) $\mathrm{b}V$ *is irreducible if and only if* V *is irreducible.*

Proof. It is clear that (a) and (b) are direct consequences of the relevant definitions. We now prove (c). If we denote the character of V by χ_V, then the character of $\mathrm{b}V$ is $\mathrm{b}\chi_V$. Hence

$$\begin{aligned}\dim \mathrm{Mor}_\Gamma(\mathrm{b}V, \mathrm{b}V) &= \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma} \overline{\mathrm{b}(\gamma)\chi_V(\gamma)}\mathrm{b}(\gamma)\chi_V(\gamma)\\ &= \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma} \overline{\chi_V(\gamma)}\chi_V(\gamma) = \dim \mathrm{Mor}_\Gamma(V,V),\end{aligned} \tag{II.8.10}$$

and $\mathrm{b}V$ is irreducible if and only if V is. ◻

Remark II.8.4. If b is a character of Γ, and $V\colon \Gamma \to \mathcal{U}(\mathscr{H})$ an irreducible unitary projective representation, then $\mathrm{b}V$ is also an irreducible unitary projective representation with the same cocycle as that of V. Note that $|\mathrm{b}(\gamma)| = 1$ for all $\gamma \in \Gamma$, we have

$$\begin{aligned}\dim \mathrm{Mor}_\Gamma(\mathrm{b}V, V) &= \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma} \overline{\mathrm{b}(\gamma)}\chi_V(\gamma)\overline{\chi_V(\gamma)}\\ &\le \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma} \chi_V(\gamma)\overline{\chi_V(\gamma)} = \dim \mathrm{Mor}_\Gamma(V,V) = 1.\end{aligned} \tag{II.8.11}$$

with equality holds if and only if $\mathrm{b}(\gamma) = 1$ whenever $\chi_V(\gamma) \neq 0$. If equality doesn't hold in (II.8.11), then $\dim \mathrm{Mor}_\Gamma(\mathrm{b}V, V)$ must be 0 since it is a natural number. Therefore, whenever Γ is not trivial, it is possible that $\mathrm{b}V$ and V are irreducible unitary projective representations with the same cocycle but not equivalent. Thus one must be careful not to confuse our definition of projective representation with the more naive definition where one simply replaces $\mathrm{GL}(\mathscr{H})$ by $\mathrm{PGL}(\mathscr{H})$ as the target model group. For us, how we lift from $\mathrm{PGL}(\mathscr{H})$ to $\mathrm{GL}(\mathscr{H})$ *does* matter, even if we keep the cocycle in the process.

After this digression, we now resume our discussion. Using terminologies in the theory of group cohomology, and regarding $\mathbb{T}$ as the trivial Λ_x-module, we see that the 2-cocycle $\omega_x \in C^2(\Lambda_x, \mathbb{T})$ of the unitary projective representation V of Λ_x is determined up to a 2-boundary in $B^2(\Lambda_x, \mathbb{T})$, because each unitary operator $V(r_0)$, $r_0 \in \Lambda_x$ is uniquely determined up to a scalar multiple in $\mathbb{T}$ (Schur's lemma plus the unitarity of $V(r_0)$). In other words, $[\omega_x] \in H^2(\Lambda_x, \mathbb{T})$ is a well-defined cohomology class of Λ_x with coefficients in $\mathbb{T}$.

Conversely, let u be an *irreducible* unitary representation of $\mathbb{G}$ on some finite dimensional Hilbert space $\mathscr{H}$, and $x = [u] \in \mathrm{Irr}(\mathbb{G})$. If Λ_0 is a subgroup of Λ,

$V\colon \Lambda_0 \to \mathcal{U}(\mathscr{H})$ a unitary projection representation of Λ_0 such that u and V satisfy the covariance condition (II.8.3), then

$$\forall r_0 \in \Lambda_0, \quad V(r_0) \in \mathrm{Mor}_{\mathbb{G}}(r_0 \cdot u, u).$$

In particular, Λ_0 fixes $x = [u]$ under the action $\Lambda \curvearrowright \mathrm{Irr}(\mathbb{G})$. Repeat the above reasoning shows that (II.8.5) still holds.

We summarize the above discussion in the following proposition, which proves slightly more.

Proposition II.8.5. *Let u be an irreducible unitary representation of $\mathbb{G}$ on some finite dimensional Hilbert space $\mathscr{H}$, $x = [u] \in \mathrm{Irr}(\mathbb{G})$, Λ_x the isotropy group fixing x (under the action $\Lambda \curvearrowright \mathrm{Irr}(\mathbb{G})$). For any $r_0 \in \Lambda_x$, choose a unitary $V(r_0)$ according to (II.8.4). Then*

(a) *$V\colon \Lambda_x \to \mathcal{U}(\mathscr{H})$, $r_0 \mapsto V(r_0)$ is a unitary projective representation satisfying the covariance condition (II.8.3);*

(b) *let $\omega \in C^2(\Lambda_0, \mathbb{T})$ be the 2-cocycle of V, then the cohomology class $c_x\colon = [\omega] \in H^2(\Lambda_x, \mathbb{T})$ depends only on x, i.e. it does not depend on any particular choice of $u \in x$.*

Conversely, if $V_0\colon \Lambda_0 \to \mathcal{U}(\mathscr{H})$ is a unitary projective representation of some subgroup Λ_0 of Λ that satisfies the covariance condition (II.8.3), then

(c) *for every $r_0 \in \Lambda_0$, the condition (II.8.4) holds;*

(d) *$\Lambda_0 \subseteq \Lambda_x$;*

(e) *there is a choice of $V\colon \Lambda_x \to \mathcal{U}(\mathscr{H})$ satisfying (II.8.3) such that $V|_{\Lambda_0} = V_0$;*

(f) *let $\omega_0 \in C^2(\Lambda_0, \mathbb{T})$ be the 2-cocycle of V_0, then $[\omega_0]$ is the image of c_x under the morphism of groups*

$$H^2(\Lambda_0 \hookrightarrow \Lambda_x)\colon H^2(\Lambda_x, \mathbb{T}) \to H^2(\Lambda_0, \mathbb{T}).$$

Proof. The above discussion already establishes (a), (c) and (d). Assertion (e) follows from (a) and (c), while (f) follows from (e). Moreover, we've seen that $[\omega] \in H^2(\Lambda_x, \mathbb{T})$ does not depend on the choice of V. For any $w \in x$, there exists a unitary intertwiner $U \in \mathrm{Mor}_{\mathbb{G}}(u, w)$. It is trivial to check that $V_w(r_0) = UV(r_0)U^*$ defines a unitary projective representation of Λ_x such that

$$V_w(r_0) \in \mathrm{Mor}_{\mathbb{G}}(r_0 \cdot w, w).$$

Since V_w and V are unitarily equivalent projective representations of Λ_x, the 2-cocycle of V_w coincides with ω—the 2-cocycle of V. This proves that $c_x = [\omega] \in H^2(\Lambda_x, \mathbb{T})$ indeed depends only on x and not on any particular choice of $u \in x$. This proves (b) and finishes the proof of the proposition. □

Definition II.8.6. Using the notations in Proposition II.8.5, we call the cohomology class $[\omega] \in H^2(\Lambda_x, \mathbb{T})$ the cohomology class associated with $x = [u] \in \mathrm{Irr}(\mathbb{G})$, and we denote $[\omega]$ by c_x. If Λ_0 is a subgroup of Λ_x, the cohomology class $[\omega_0] \in H^2(\Lambda_0, \mathbb{T})$ is called the restriction of the cohomology class c_x on Λ_0, and is denoted by c_{x,Λ_0}.

Obviously, c_{x,Λ_0} depends on Λ_0 and x, and $c_{x,\Lambda_0} = c_x$ if $\Lambda_0 = \Lambda_x$. To apply the character theory of projective representations, we need to suitably rescale the projective representations in question so that they share the same cocycle (and not merely the same cohomology class for their cocycles). In the case where the representation $u \in \mathcal{B}(\mathscr{H}) \otimes \mathrm{Pol}(\mathbb{G})$ of $\mathbb{G}$ is irreducible, and $V : \Lambda_0 \to \mathcal{U}(\mathscr{H})$ is a unitary projective representation satisfying the covariance condition (II.8.3), such a rescaling is implicit in the choice of $V(r_0) \in \mathrm{Mor}_{\mathbb{G}}(r_0 \cdot u, u)$ for each $r_0 \in \Lambda_0$. However, Remark II.8.4 tells us we should take extra care if we want to talk about equivalence class of these projective representations once we do the rescaling.

We finish this section with an easy result.

Proposition II.8.7. *Let $x \in \mathrm{Irr}(\mathbb{G})$, $u \in x$, Λ_0 a subgroup of Λ_x, $c_0 \in H^2(\Lambda_0, \mathbb{T})$ is the image of the cohomology class $c_x \in H^2(\Lambda_x, \mathbb{T})$ associated with x under $H^2(\Lambda_0 \hookrightarrow \Lambda_x, \mathbb{T})$. Then for any 2-cocycle $\omega_0 \in c_0$, there exists a unitary projective representation V of the isotropy subgroup Λ_0 with cocycle ω_0, such that V and u are covariant, and such V is unique up to rescaling by a character of Λ_0.*

Proof. This is clear from Proposition II.8.5, Lemma II.8.2 and Lemma II.8.3. ◻

II.9 Pure, stable, distinguished CSRs and representation parameters

Recall that for any finite dimensional representation u of $\mathbb{G}$, the support of u, denoted by $\mathrm{supp}(u)$, is the set

$$\{x \in \mathrm{Irr}(\mathbb{G}) \ : \ \dim_{\mathbb{G}} \mathrm{Mor}_{\mathbb{G}}(x, [u]) \neq 0\}$$

where $[u]$ is the class of unitary representations of $\mathbb{G}$ equivalent to u. We call u *pure* if $\mathrm{supp}(u)$ is a singleton.

Definition II.9.1. Fix a $\Lambda_0 \in \mathcal{G}_{\mathrm{iso}}(\Lambda)$, $\mathrm{S} = (\mathscr{H}, u, w) \in \mathcal{CSR}_{\Lambda_0}$, we call S

- **pure**, if u is pure;
- **stable**, if $r \cdot [u] (= [r \cdot u]) = [u]$ for all $r \in \Lambda_0$;
- **stably pure**, if it is both pure and stable;
- **maximally stable**, if

$$\Lambda_0 = \{r \in \Lambda \ : \ r \cdot [u] = [u]\};$$

- **simple**, if S is a simple object in $\mathcal{CSR}_{\Lambda_0}$;
- **distinguished**, if it is maximally stable, pure and simple.

As remarked earlier, while it is not reasonable for the moment to hope for a satisfactory description of all simple CSRs in $\mathcal{CSR}_{\Lambda_0}$, it is possible to describe simple CSRs that are stably pure using unitary projective representations of Λ_0. Somewhat surprisingly, one can even describe all stably pure CSRs, even the non-simple ones, in this way. To achieve the latter, we introduce the following definitions, which are closely related to the materials in § II.8.

Definition II.9.2. Let $\Lambda_0 \in \mathcal{G}_{\mathrm{iso}}(\Lambda)$. Suppose u is a unitary representation of $\mathbb{G}$ on some finite dimensional Hilbert space $\mathscr{H}$, and $V\colon \Lambda_0 \to \mathcal{U}(\mathscr{H})$ is a unitary projective representation of Λ_0. We say u and V are covariant if they satisfy the covariance condition (II.8.3), or equivalently $V(r_0) \in \mathrm{Mor}_{\mathbb{G}}(r_0 \cdot u, u)$ for all $r_0 \in \Lambda_0$.

Definition II.9.3. Let $x \in \mathrm{Irr}(\mathbb{G})$, $\Lambda_0 \in \mathcal{G}_{\mathrm{iso}}(\Lambda)$ with $\Lambda_0 \subseteq \Lambda_x$, $u \in x$, $\omega_0 \in c_{x,\Lambda_0}$ (see Definition II.8.6), then a unitary projective representation V of Λ_0 that is covariant with u is said to be a covariant projective Λ_0-representation of u (with cocycle ω_0).

Remark II.9.4. In the setting of Definition II.9.3, fix any covariant projective Λ_0-representation V of u with cocycle ω_0, the set of covariant projective Λ_0-representations of u with multiplier ω_0 is in bijective correspondence with the group of characters of Λ_0, via $\mathrm{b} \mapsto \mathrm{b}V$ (see Lemma II.8.2 and Lemma II.8.3).

Proposition II.9.5 (Structure of stably pure CSR)**.** *Fix a $\Lambda_0 \in \mathcal{G}_{\mathrm{iso}}(\Lambda)$, let $\mathrm{S} = (\mathscr{H}, u, w)$ be a stably pure CSR in $\mathcal{CSR}_{\Lambda_0}$ $x \in \mathrm{Irr}(\mathbb{G})$ is the support point of u, $u_0 \in x$ a representation on some finite dimensional Hilbert space $\mathscr{H}_0$, n is the multiplicity of u_0 in u, V_0 a covariant projective Λ_0-representation of u, then there exists a unique unitary projective representation $v_0\colon \Lambda_0 \to \mathcal{U}(\mathbb{C}^n)$ of Λ_0 on $\mathbb{C}^n$, such that the following hold:*

(a) *V_0 and v_0 have opposing cocycles;*

(b) *$\mathrm{S}_0 = (\mathbb{C}^n \otimes \mathscr{H}_0, \epsilon_n \times u_0, v_0 \times V_0)$ is a CSR in $\mathcal{CSR}_{\Lambda_0}$, where ϵ_n is the trivial representation of $\mathbb{G}$ on $\mathbb{C}^n$;*

(c) *S_0 and S are isomorphic in $\mathcal{CSR}_{\Lambda_0}$.*

Proof. Uniqueness is almost clear once we finish the proof of existence, which we do now. Let U be a unitary intertwiner from u to $\epsilon_n \otimes u_0$. Noting that u is pure and replacing S with $U\mathrm{S}U^*$ if necessary, we may assume $\mathscr{H} = \mathbb{C}^n \otimes \mathscr{H}_0$ and $u = \epsilon_n \times u_0 = (u_0)_{23}$. For any $r_0 \in \Lambda_0$, we claim that there exists a unique $v_0(r_0) \in \mathcal{B}(\mathbb{C}^n)$ such that $w(r_0) = v_0(r_0) \otimes V_0(r_0)$. Admitting the claim for the moment, the unitarity of $v_0(r_0)$ follows from the unitarity of $w(r_0)$ and $V_0(r_0)$, and w being a representation and V_0 being a projective representation force v_0 to be a unitary projective representation with a cocycle opposing to the cocycle of V_0. Thus the proposition follows from the claim, which we now prove. Since $\mathcal{B}(\mathbb{C}^n \otimes \mathscr{H}_0) = \mathcal{B}(\mathbb{C}^n) \otimes \mathcal{B}(\mathscr{H}_0)$ by the usual identification, there exists an $m \in \mathbb{N}$, $A_1, \ldots, A_m \in \mathcal{B}(\mathbb{C}^n)$ and $B_1, \ldots, B_m \in \mathcal{B}(\mathscr{H}_0)$, such that

$$w(r_0) = \sum_{i=1}^{m} A_i \otimes B_i. \tag{II.9.1}$$

Furthermore, we can and do choose these operators so that $A_1, \ldots, A_m$ are linearly independent in $\mathcal{B}(\mathbb{C}^n)$. Since u and w are covariant, we have

$$\big(w(r_0) \otimes 1\big)u = \big[(\mathrm{id}_{\mathscr{H}} \otimes \alpha^*_{r_0})u\big]\big(w(r_0) \otimes 1\big). \tag{II.9.2}$$

Substituting $u = (u_0)_{23}$ and (II.9.1) in (II.9.2) yields

$$\begin{aligned} &\sum_{i=1}^{m} A_i \otimes [(B_i \otimes 1)u_0] \\ = &\sum_{i=1}^{m} A_i \otimes \Big(\big[(\mathrm{id}_{\mathscr{H}_0} \otimes \alpha^*_{r_0})u_0\big]\big(B_i \otimes 1\big)\Big) \in \mathcal{B}(\mathbb{C}^n) \otimes \mathcal{B}(\mathscr{H}_0) \otimes \mathrm{Pol}(\mathbb{G}). \end{aligned} \tag{II.9.3}$$

Since $A_1, \dots, A_m$ are linearly independent, there exists linear functionals $l_1, \dots, l_m$ on $\mathcal{B}(\mathbb{C}^n)$ such that $l_i(A_j) = \delta_{i,j}$. Applying $l_i \otimes \mathrm{id}_{\mathscr{H}_0} \otimes \mathrm{id}_{\mathrm{Pol}(\mathbb{G})}$ on (II.9.3) shows that for each $i = 1, \dots, m$,

$$(B_i \otimes 1)u_0 = \left[(\mathrm{id} \otimes \alpha^*_{r_0})u_0\right](B_i \otimes 1), \tag{II.9.4}$$

or equivalently

$$B_i \in \mathrm{Mor}_{\mathbb{G}}(r_0 \cdot u_0, u_0) = \mathbb{C}V_0(r_0). \tag{II.9.5}$$

Now the claim follows from (II.9.1) and (II.9.5). □

Conversely, we have

Proposition II.9.6. *Fix a $\Lambda_0 \in \mathcal{G}_{\mathrm{iso}}(\Lambda)$, $x \in \mathrm{Irr}(\mathbb{G})$ with $\Lambda_0 \subseteq \Lambda_x$. Take a $u \in x$ acting on some finite dimensional Hilbert space $\mathscr{H}$, and a covariant projective Λ_0-representation V of u, then for any unitary projective representation $v \colon \Lambda_0 \to \mathcal{U}(\mathscr{K})$ of Λ_0 with cocycle opposing the cocycle of V, the unitary representation $v \times V$ of Λ_0 is covariant with the unitary representation $\mathrm{id}_{\mathscr{K}} \otimes u = \epsilon_{\mathscr{K}} \times u$ of $\mathbb{G}$, where $\epsilon_{\mathscr{K}}$ is the trivial representation of $\mathbb{G}$ on $\mathscr{K}$, i.e. $(\mathscr{K} \otimes \mathscr{H}, \epsilon_{\mathscr{K}} \times u, v \times V)$ is a stably pure CSR in $\mathcal{CSR}_{\Lambda_0}$.*

Proof. Since V and u are covariant, for any $r_0 \in \Lambda_0$, we have

$$\big(V(r_0) \otimes 1\big)u = \left[(\mathrm{id} \otimes \alpha^*_{r_0})u\right]\big(V(r_0) \otimes 1\big). \tag{II.9.6}$$

The proposition follows by tensoring $v(r_0)$ on the left in (II.9.6). □

By Proposition II.9.5 and Proposition II.9.6, we now have a satisfactory description of stably pure CSRs in $\mathcal{CSR}_{\Lambda_0}$—from any irreducible representation u of $\mathbb{G}$ on $\mathscr{H}$ such that $\Lambda_0 \cdot [u] = [u]$, one choose a covariant projective Λ_0-representation V of u with some cocycle ω, then any unitary projective representation v of Λ_0 with cocycle $\omega^{-1} = \overline{\omega}$ gives rise to a stably pure CSR in $\mathcal{CSR}_{\Lambda_0}$, namely $\mathcal{S}(u, V, v) = (\mathscr{K} \otimes \mathscr{H}, \epsilon_{\mathscr{K}} \times u, v \times V)$; and all stably pure CSRs in $\mathcal{CSR}_{\Lambda_0}$ arise in this way up to isomorphism.

Remark II.9.7. Using the above notations, while it is true that V is determined by u to a great extent due to the restriction of Schur's lemma, it is still not completely determined (see Proposition II.8.7), and a choice of this V is vitally relevant as is demonstrated by Remark II.8.4 applied to v. This is why V can *not* be suppressed in our notation $\mathcal{S}(u, V, v)$.

Definition II.9.8. Let $\Lambda_0 \in \mathcal{G}_{\mathrm{iso}}(\Lambda)$. A triple (u, V, v) is called a **representation parameter** for $\mathbb{G} \rtimes \Lambda$ associated with Λ_0, if it the following hold:

- u is an irreducible unitary representation of $\mathbb{G}$ on some finite dimensional Hilbert space $\mathscr{H}$;
- V is a covariant projective Λ_0-representation of u;
- v is a unitary projective representation of Λ_0 (possibly on Hilbert spaces other than $\mathscr{H}$), such that v and V have opposing cocycles.

If (u, V, v) is a representation parameter, the stably pure CSR $\mathcal{S}(u, V, v)$ in $\mathcal{CSR}_{\Lambda_0}$ is called the CSR parametrized by the representation parameter (u, V, v). If furthermore the unitary projective representation v is irreducible, we say the representation parameter (u, V, v) is *irreducible*.

Thus Proposition II.9.5 immediately implies the following corollary.

Corollary II.9.9. *Fix a $\Lambda_0 \in \mathcal{G}_{\mathrm{iso}}(\Lambda)$, then every stably pure CSR associated with Λ_0 is parameterised by some representation parameter associated with Λ_0.* □

Definition II.9.10. Fix a $\Lambda_0 \in \mathcal{G}_{\mathrm{iso}}(\Lambda)$. Let u be an irreducible unitary representation of $\mathbb{G}$ such that $\Lambda_0 \cdot [u] = [u]$, V_1 and V_2 are two covariant projective Λ_0-representations of u, the unique mapping $\mathrm{b}\colon \Lambda_0 \to \mathbb{T}$ such that $V_2 = \mathrm{b}V_1$ is called the u-**transitional mapping** from V_1 to V_2 (note that we do *not* require V_1 and V_2 to have the same cocycle here).

Proposition II.9.11. *Fix a $\Lambda_0 \in \mathcal{G}_{\mathrm{iso}}(\Lambda)$. For $i = 1, 2$, let (u_i, V_i, v_i) be a representation parameter associated with Λ_0, U_i denote the unitary representation $\mathscr{R}_{\Lambda_0}\Big(\mathcal{S}(u_i, V_i, v_i)\Big)$ of $\mathbb{G} \rtimes \Lambda_0$, then the following holds:*

(a) *if $[u_1] \neq [u_2]$ in $\mathrm{Irr}(\mathbb{G})$, then $\dim \mathrm{Mor}_{\mathbb{G}\rtimes\Lambda_0}(U_1, U_2) = 0$;*

(b) *if $u_1 = u_2 = u$, and $\mathrm{b}\colon \Lambda_0 \to \mathbb{T}$ the u-transitional map from V_1 to V_2, then*

$$\dim \mathrm{Mor}_{\mathbb{G}\rtimes\Lambda_0}(U_1, U_2) = \dim \mathrm{Mor}_{\Lambda_0}(v_1, \mathrm{b}v_2)\,. \tag{II.9.7}$$

Proof. Let h be the Haar state of $\mathbb{G}$, by (II.1.7), the Haar state h_{Λ_0} of $\mathbb{G}\rtimes\Lambda_0$ is the linear functional on $A\otimes C(\Lambda_0)$ defined by $a\otimes\delta_{r_0} \mapsto |\Lambda_0|^{-1}h(a)$, where $a \in A$, $r_0 \in \Lambda_0$ (recall that $A = C(\mathbb{G})$).

Suppose $[u_1] \neq [u_2]$. For any $i = 1, 2$, by choosing a Hilbert space basis for the representation of u_i, one can write u_i as a square matrix $\left(u^{(i)}_{jk}\right)$ over $\mathrm{Pol}(\mathbb{G}) \subseteq A$, and V_i as a matrix $\left(V^{(i)}_{jk}\right)$ over $C(\Lambda_0)$ of the same size of $\left(u^{(i)}_{jk}\right)$. Then the character χ_i of U_i is given by

$$\chi_i = \sum_{r_0\in\Lambda_0}\sum_{j=1}^{n_i}\mathrm{Tr}(v_i)\left(\sum_{k=1}^{n_i}V^{(i)}_{kj}(r_0)u^{(i)}_{jk}\right)\otimes\delta_{r_0} \in \mathrm{Pol}(\mathbb{G})\otimes C(\Lambda_0). \tag{II.9.8}$$

The orthogonality relation for the nonequivalent irreducible representations u_1 and u_2 implies that

$$\forall j_1, k_1, j_2, k_2, \quad h\left(\left(u^{(1)}_{j_1k_1}\right)^* u^{(2)}_{j_2k_2}\right) = 0. \tag{II.9.9}$$

Hence, by (II.9.8) and (II.9.9),

$$\begin{aligned}
&\dim \mathrm{Mor}_{\mathbb{G}\rtimes\Lambda_0}(U_1, U_2) = h_{\Lambda_0}\left(\overline{\chi_1}\chi_2\right)\\
&= |\Lambda_0|^{-1}\sum_{r_0\in\Lambda_0}\sum_{j_1,k_1=1}^{n_1}\sum_{j_2,k_2=1}^{n_2}\overline{\mathrm{Tr}\big(v_1(r_0)\big)}\,\mathrm{Tr}\big(v_2(r_0)\big)\\
&\qquad \overline{V^{(1)}_{k_1j_1}(r_0)}V^{(2)}_{k_2j_2}(r_0)h\left(\left(u^{(1)}_{j_1k_1}\right)^* u^{(2)}_{j_2k_2}\right)\\
&= 0.
\end{aligned} \tag{II.9.10}$$

This proves (a).

Under the hypothesis of (b), using the same notations as in the previous paragraph, we have $n_1 = n_2 = \dim U$. We may assume that $e^{(1)}_j = e^{(2)}_j = e_j$, hence $u_{jk} := u^{(1)}_{jk} = u^{(2)}_{jk}$ for all possible j, k. Note that $V_2 = \mathrm{b}V_1$, and $\mathcal{S}(u, V_2, v_2) = \mathcal{S}(u, V_1, \mathrm{b}v_2)$

because $\mathrm{b}V_1 \times v_2 = V_1 \times \mathrm{b}v_2$, we may assume that $V_2 = V_1 = V$ and $\mathrm{b} = 1$, with $V_{jk} := V_{jk}^{(1)} = V_{jk}^{(2)} \in C(\Lambda_0)$ for all possible j,k. Let ρ be the unique invertible positive operator in $\mathrm{Mor}_{\mathbb{G}}(u, u^{cc})$ such that $\mathrm{Tr}(\cdot\,\rho) = \mathrm{Tr}(\cdot\,\rho^{-1})$ on $\mathrm{End}_{\mathbb{G}}(u)$. With these assumptions, by (II.9.10), the orthogonality relation takes the form

$$h(u_{ij}^* u_{kl}) = \frac{\delta_{j,l}(\rho^{-1})_{ki}}{\dim_q U} \tag{II.9.11}$$

where $\dim_q U = \mathrm{Tr}(\rho) = \mathrm{Tr}(\rho^{-1})$ is the quantum dimension of U (see (Neshveyev and Tuset, 2013, §1.4)). Since ρ is positive, we might choose the basis $e_1, \dots, e_n$ to diagonize ρ, so that $\rho_{ki} = (\rho^{-1})_{ki} = 0$ whenever $k \neq i$. Using this basis, (II.9.11) and (II.9.10), we have

$$\begin{aligned}
&\dim \mathrm{Mor}_{\mathbb{G}\rtimes\Lambda_0}(U_1, U_2)\\
&= |\Lambda_0|^{-1} \sum_{r_0\in\Lambda_0} \sum_{j_1,k_1=1}^{n} \sum_{j_2,k_2=1}^{n} \overline{\mathrm{Tr}\big(v_1(r_0)\big)}\, \mathrm{Tr}\big(v_2(r_0)\big)\\
&\qquad\qquad \cdot \overline{V_{k_1j_1}(r_0)} V_{k_2j_2}(r_0) h\left(\left(u_{j_1k_1}\right)^* u_{j_2k_2}\right)\\
&= |\Lambda_0|^{-1} \sum_{r_0\in\Lambda_0} \sum_{j_1,k_1=1}^{n} \sum_{j_2,k_2=1}^{n} \overline{\mathrm{Tr}\big(v_1(r_0)\big)}\, \mathrm{Tr}\big(v_2(r_0)\big) \overline{V_{k_1j_1}(r_0)} V_{k_2j_2}(r_0)\\
&\qquad\qquad \cdot \frac{\delta_{j_1,j_2}\delta_{k_1,k_2}(\rho^{-1})_{j_2j_1}}{\dim_q U} \qquad\qquad \text{(II.9.12)}\\
&= |\Lambda_0|^{-1} \overline{\mathrm{Tr}\big(v_1(r_0)\big)}\, \mathrm{Tr}\big(v_2(r_0)\big) \sum_{r_0\in\Lambda_0} \sum_{j=1}^{n} \left\{ \sum_{k=1}^{n} \overline{V_{kj}(r_0)} V_{kj}(r_0) \right\} \frac{(\rho^{-1})_{jj}}{\dim_q U}\\
&\text{(Note that } V(r_0) \text{ is unitary)}\\
&= |\Lambda_0|^{-1} \sum_{r_0\in\Lambda_0} \overline{\mathrm{Tr}\big(v_1(r_0)\big)}\, \mathrm{Tr}\big(v_2(r_0)\big) \frac{\sum_{j=1}^{n} (\rho^{-1})_{jj}}{\dim_q U}\\
&= |\Lambda_0|^{-1} \sum_{r_0\in\Lambda_0} \overline{\mathrm{Tr}\big(v_1(r_0)\big)}\, \mathrm{Tr}\big(v_2(r_0)\big) = \dim \mathrm{Mor}_{\Lambda_0}(v_1, v_2).
\end{aligned}$$

This proves (b). ☐

The following corollary is now clear.

Corollary II.9.12. *Fix a $\Lambda_0 \in \mathcal{G}_{\mathrm{iso}}(\Lambda)$. Let (u, V, v) be a representation parameter associated with Λ_0, then the representation $\mathscr{R}_{\Lambda_0}\big(\mathcal{S}(u, V, v)\big)$ of $\mathbb{G} \rtimes \Lambda_0$ is irreducible if and only if the representation parameter (u, V, v) is irreducible.* ☐

II.10 Distinguished representation parameters and distinguished representations

Fix a $\Lambda_0 \in \mathcal{G}_{\mathrm{iso}}(\Lambda)$. For any unitary projective representation $V\colon \Lambda_0 \to \mathcal{U}(\mathscr{H})$ of Λ_0, and any $r \in \Lambda$, define $r \cdot V$ to be the unitary projective representation of $r\Lambda_0 r^{-1}$ on $\mathscr{H}$ sending $s_0 = r r_0 r^{-1} \in r\Lambda_0 r^{-1}$ to $(V \circ \mathrm{Ad}_{r^{-1}})(s_0) = V(r_0)$. Then $(rs)\cdot V = r\cdot(s\cdot V)$ for all $r, s \in \Lambda$ with $1_\Lambda \cdot V = V$, in other words, this defines an action of the group Λ on the class of all unitary projective representations of general isotropy subgroups of Λ.

It is easy to see from Proposition II.3.9 that whenever $\mathrm{S} = (\mathscr{H}, u, w) \in \mathcal{CSR}_{\Lambda_0}$, the triple $r \cdot \mathrm{S} = (\mathscr{H}, r \cdot u, r \cdot w)$ is a CSR in $\mathcal{CSR}_{r\Lambda_0 r^{-1}}$. If $U = \mathscr{R}_{\Lambda_0}(\mathrm{S})$ is the unitary representation of $\mathbb{G} \rtimes \Lambda_0$, then it is easy to see by restriction that $\mathscr{R}_{r\Lambda_0 r^{-1}}(r \cdot \mathrm{S})$ is the unitary representation $r \cdot U = (\mathrm{id} \otimes \alpha^*_{r^{-1}} \otimes \mathrm{Ad}^*_{r^{-1}})(U)$ of $\mathbb{G} \rtimes r\Lambda_0 r^{-1}$, as described in Proposition II.5.1. Thus by Corollary II.4.4, we see that $\mathrm{Ind}(U)$ and $\mathrm{Ind}(r \cdot U)$ are equivalent representations of $\mathbb{G} \rtimes \Lambda$.

Similarly, for any representation parameter (u, V, v) associated with Λ_0 and any $r \in \Lambda$, the triple $(r \cdot u, r \cdot V, r \cdot v)$ is a representation parameter associated with $r\Lambda_0 r^{-1}$, which we denoted by $r \cdot (u, V, v)$. This clearly defines an Λ-action on the proper class of all representation parameters associated with any group in some conjugacy class of a general isotropy subgroup of Λ. A simple calculation shows that (recall $\mathcal{S}(u, V, v)$ is the CSR parameterized by (u, V, v))

$$\forall r \in \Lambda, \quad r \cdot \mathcal{S}(u, V, v) = \mathcal{S}\big(r \cdot (u, V, v)\big). \tag{II.10.1}$$

Definition II.10.1. Let (u, V, v) be a representation parameter associated with some $\Lambda_0 \in \mathcal{G}_{\mathrm{iso}}(\Lambda)$, the induced representation $\mathrm{Ind}\Big(\mathscr{R}_{\Lambda_0}\big(\mathcal{S}(u, V, v)\big)\Big)$ of $\mathbb{G} \rtimes \Lambda$ is called the representation of $\mathbb{G} \rtimes \Lambda$ parameterized by (u, V, v).

Proposition II.10.2. *Let (u, V, v) be a representation parameter associated with some $\Lambda_0 \in \mathcal{G}_{\mathrm{iso}}(\Lambda)$. Then for any $r \in \Lambda$, the representation parameters (u, V, v) and $r \cdot (u, V, v)$ parameterize equivalent representations of $\mathbb{G} \rtimes \Lambda$.*

Proof. Since $\mathscr{R}_{\Lambda_0}\big(\mathcal{S}(u, V, v)\big)$ and $\mathscr{R}_{r\Lambda_0 r^{-1}}\big(r \cdot \mathcal{S}(u, V, v)\big)$ induces equivalent representations of $\mathbb{G} \rtimes \Lambda$, the proposition now follows from equation (II.10.1) and Definition II.10.1. □

Proposition II.10.3. *Fix a $\Lambda_0 \in \mathcal{G}_{\mathrm{iso}}(\Lambda)$. Let (u, V, v) be an irreducible representation parameter associated with Λ_0, U denote the representation $\mathscr{R}_{\Lambda_0}\Big(\mathcal{S}(u, V, v)\Big)$. If $\Lambda_0 = \Lambda_{[u]}$, then the the induced representation* $\mathrm{Ind}(U)$ *of* $\mathbb{G} \rtimes \Lambda$ *is irreducible.*

Proof. By Corollary II.6.4, the proposition amounts to show that

$$\begin{aligned} &\forall r, s \in \Lambda, \\ &r^{-1}s \notin \Lambda_0 \implies \dim \mathrm{Mor}_{\mathbb{G} \rtimes \Lambda(r,s)}\Big((r \cdot U)|_{\mathbb{G} \rtimes \Lambda(r,s)}, (s \cdot U)|_{\mathbb{G} \rtimes \Lambda(r,s)}\Big) = 0, \end{aligned} \tag{II.10.2}$$

where $\Lambda(r, s) = r\Lambda_0 r^{-1} \cap s\Lambda_0 s^{-1}$. Since $\Lambda_0 = \Lambda_{[u]}$, by the definition of $\Lambda_{[u]}$, we have $[r \cdot u] \neq [s \cdot u]$ whenever $r^{-1}s \notin \Lambda_0$. Now condition (II.10.2) holds by Proposition II.9.11. □

Definition II.10.4. Fix a $\Lambda_0 \in \mathcal{G}_{\mathrm{iso}}(\Lambda)$, an *irreducible* representation parameter (u, V, v) associated with Λ_0 is called **distinguished** if $\Lambda_0 = \Lambda_{[u]}$. When this is the case, the irreducible unitary representation $\mathrm{Ind}(U)$ of $\mathbb{G} \rtimes \Lambda$ is called **distinguished**, where U is the unitary representation $\mathscr{R}_{\Lambda_0}\Big(\mathcal{S}(u, V, v)\Big)$ of $\mathbb{G} \rtimes \Lambda_0$.

Remark II.10.5. The associated group of a distinguished representation parameter must be an isotropy subgroup of Λ for the action $\Lambda \curvearrowright \mathrm{Irr}(\mathbb{G})$. More precisely, a representation parameter (u, V, v) is distinguished if and only if its associated group is exactly the isotropy subgroup of $[u] \in \mathrm{Irr}(\mathbb{G})$ under the action $\Lambda \curvearrowright \mathrm{Irr}(\mathbb{G})$. As we will see presently, in the formulation of our results on the classification of irreducible representations of $\mathbb{G} \rtimes \Lambda$ and the conjugation on $\mathrm{Irr}(\mathbb{G})$, only distinguished representation parameters are needed. This makes one wonder why we pose the family of general isotropy subgroup $\mathcal{G}_{\mathrm{iso}}(\Lambda)$ instead of only isotropy subgroups. The main reason we need general isotropy subgroups of Λ is that in proving these results, as well as the formulation and the proof of the fusion rules, we need to express the dimensions of various intertwiner spaces. The calculation of the dimensions of these intertwiner spaces will rely on Proposition II.6.3, which clearly requires us to consider the intersections of isotropy subgroups, i.e. general isotropy subgroups.

Definition II.10.6. Let Λ_0 be an isotropy subgroup of Λ for the action $\Lambda \curvearrowright \mathrm{Irr}(\mathbb{G})$. Suppose (u_1, V_1, v_1) and (u_2, V_2, v_2) are two distinguished representation parameters associated with Λ_0. If the CSRs $\mathcal{S}(u_1, V_1, v_1)$ and $\mathcal{S}(u_2, V_2, v_2)$ are isomorphic in $\mathcal{CSR}_{\Lambda_0}$, we say (u_1, V_1, v_1) and (u_2, V_2, v_2) are equivalent.

The following proposition serves to characterize equivalence of distinguished representation parameters in some more concrete ways.

Proposition II.10.7. *Let Λ_0 be an isotropy subgroup of Λ for the action $\Lambda \curvearrowright \mathrm{Irr}(\mathbb{G})$, (u_1, V_1, v_1) and (u_2, V_2, v_2) two distinguished representation parameters associated with Λ_0. The following are equivalent:*

(a) *(u_1, V_1, v_1) and (u_2, V_2, v_2) are equivalent;*

(b) *(u_1, V_1, v_1) and (u_2, V_2, v_2) parameterize equivalent representations of $\mathbb{G} \rtimes \Lambda_0$;*

(c) *there exists a mapping $\mathrm{b}\colon \Lambda_0 \to \mathbb{T}$ such that $\mathrm{b}V_1$ and V_2 share the same cocycle, and both $\mathrm{Mor}_{\mathbb{G}}(u_1, u_2) \cap \mathrm{Mor}_{\Lambda_0}(\mathrm{b}V_1, V_2)$ and $\mathrm{Mor}_{\Lambda_0}(v_1, \mathrm{b}v_2)$ are nonzero;*

(d) *there exists a mapping $\mathrm{b}\colon \Lambda_0 \to \mathbb{T}$ such that $\mathrm{b}V_1$ and V_2 share the same cocycle, and both $\mathrm{Mor}_{\mathbb{G}}(u_1, u_2) \cap \mathrm{Mor}_{\Lambda_0}(\mathrm{b}V_1, V_2)$ and $\mathrm{Mor}_{\Lambda_0}(v_1, \mathrm{b}v_2)$ contain unitary operators.*

Proof. The equivalence of (a) and (b) follows directly from the definitions. It is also clear that (d) implies (c). If (c) holds, and

$$\begin{aligned} 0 \neq S &\in \mathrm{Mor}_{\mathbb{G}}(u_1, u_2) \cap \mathrm{Mor}_{\Lambda_0}(\mathrm{b}V_1, V_2), \\ \text{and} \quad 0 \neq T &\in \mathrm{Mor}_{\Lambda_0}(v_1, \mathrm{b}v_2) = \mathrm{Mor}_{\Lambda_0}(\mathrm{b}^{-1}v_1, v_2), \end{aligned} \tag{II.10.3}$$

then both S and T are invertible by Schur's lemma as $u_1, u_2, \mathrm{b}^{-1}v_1, v_2$ are all irreducible. Since $u_1, u_2, \mathrm{b}V_1, V_2, v_1, \mathrm{b}v_2$ are all unitary, we have

$$0 \neq \Upsilon_S \in \mathrm{Mor}_{\mathbb{G}}(u_1, u_2) \cap \mathrm{Mor}_{\Lambda_0}(\mathrm{b}V_1, V_2), \quad \text{and} \quad 0 \neq \Upsilon_T \in \mathrm{Mor}_{\Lambda_0}(v_1, \mathrm{b}v_2), \tag{II.10.4}$$

where $S = \Upsilon_S|S|$ is the polar decomposition of S, and $T = \Upsilon_T|T|$ the polar decomposition of T. As S, T are invertible, Υ_S and Υ_T are unitary. This proves that (c) implies (d).

Let $\mathscr{K}_i$ be the representation space of v_i for $i = 1, 2$. By definition, $\mathcal{S}(u_i, V_i, v_i) = (\mathrm{id}_{\mathscr{K}_i} \otimes u, v_i \times V_i)$, and $\mathrm{b}^{-1}v_i \times \mathrm{b}V_i = v_i \times V_i$ for any mapping $\mathrm{b}\colon \Lambda_0 \to \mathbb{T}$. If (c) holds, let S, T be operators as in (II.10.3), then

$$T \otimes S \in \mathrm{Mor}_{\mathbb{G}}(\mathrm{id}_{\mathscr{K}_1} \otimes u_1, \mathrm{id}_{\mathscr{K}_2} \otimes u_2) \cap \mathrm{Mor}_{\Lambda_0}(v_1 \times V_1, v_2 \times V_2). \tag{II.10.5}$$

Now (a) follows from (II.10.5), Proposition II.7.4 and the fact that both S and T are invertible. Thus (c) implies (a).

We conclude the proof by showing (a) implies (d). By Schur's lemma, and the irreducibility of u_1 and u_2, it is easy to see that

$$\mathrm{Mor}_{\mathbb{G}}(\mathrm{id}_{\mathscr{K}_1} \otimes u_1, \mathrm{id}_{\mathscr{K}_2} \otimes u_2) = \mathcal{B}(\mathscr{K}_1, \mathscr{K}_2) \otimes \mathrm{Mor}_{\mathbb{G}}(u_1, u_2). \tag{II.10.6}$$

Suppose (a) holds. Then the intertwiner space given by the intersection in (II.10.5) is nonzero, and

$$\mathrm{Mor}_{\mathbb{G}}(u_1, u_2) = \mathbb{C}W_r \tag{II.10.7}$$

for some unitary operator W_r. By (II.10.6) and (a), there exists a unitary $W_l \in \mathcal{B}(\mathscr{K}_1, \mathscr{K}_2)$ such that

$$W_l \otimes W_r \in \mathrm{Mor}_{\Lambda_0}(v_1 \times V_1, v_2 \times V_2) = \mathrm{Mor}_{\Lambda_0}\big((\mathrm{b}^{-1}v_1) \times (\mathrm{b}V_1), v_2 \times V_2\big). \tag{II.10.8}$$

By (II.10.7), both $W_r V_1 W_r^*$ and V_2 are covariant projective Λ_0-representations of u_2. Thus we can take a u_2-transitional mapping b from $W_r V_1 W_r^*$ to V_2 (see Definition II.9.10), i.e. a mapping $\mathrm{b}\colon \Lambda_0 \to \mathbb{T}$ such that

$$W_r(\mathrm{b}V_1)W_r^* = \mathrm{b}(W_r V_1 W_r^*) = V_2, \tag{II.10.9}$$

which forces the cocycles of $\mathrm{b}V_1$ and V_2 coincide, and

$$W_r \in \mathrm{Mor}_{\Lambda_0}(\mathrm{b}V_1, V_2) \cap \mathrm{Mor}_{\mathbb{G}}(u_1, u_2). \tag{II.10.10}$$

Now (II.10.8) and (II.10.10) forces

$$W_l \in \mathrm{Mor}_{\Lambda_0}(\mathrm{b}^{-1}v_1, v_2) = \mathrm{Mor}_{\Lambda_0}(v_1, \mathrm{b}v_2). \tag{II.10.11}$$

Thus (d) holds by (II.10.10) and (II.10.11). □

II.11 Density of matrix coefficients of distinguished representations

The aim of this section is to show that the linear span of matrix coefficients of distinguished representations of $\mathbb{G} \rtimes \Lambda$ is exactly $\mathrm{Pol}(\mathbb{G}) \otimes C(\Lambda)$, hence is dense in $C(\mathbb{G} \rtimes \Lambda) = A \otimes C(\Lambda)$ in particular. As a consequence, any irreducible unitary representation of $\mathbb{G} \rtimes \Lambda$ is equivalent to a distinguished one.

The following lemma essentially establishes the density of the linear span of matrix coefficients of distinguished representations of $\mathbb{G} \rtimes \Lambda$ in $C(\mathbb{G} \rtimes \Lambda) = A \otimes C(\Lambda)$.

Lemma II.11.1. *Let u be an irreducible unitary representation of $\mathbb{G}$ on some finite dimensional Hilbert space $\mathscr{H}$, $x = [u] \in \mathrm{Irr}(\mathbb{G})$, V the covariant projective Λ_x-representation of u with cocycle ω. Let $M(u)$ denote the linear subspace of $\mathrm{Pol}(\mathbb{G}) \otimes C(\Lambda)$ spanned by matrix coefficients of distinguished representations of $\mathbb{G} \rtimes \Lambda$ parameterized by distinguished representation parameters of the form (u, V, v), where v runs through all irreducible unitary projective representations of Λ_x with cocycle $\omega^{-1} = \overline{\omega}$. For any $r \in \Lambda$, suppose $M_c(r \cdot u)$ is the linear subspace of $\mathrm{Pol}(\mathbb{G})$ spanned by matrix coefficients of $r \cdot u$, then*

$$M(u) = \sum_{r \in \Lambda} M_c(r \cdot u) \otimes C(\Lambda) = \left(\sum_{r \in \Lambda} M_c(r \cdot u)\right) \otimes C(\Lambda). \tag{II.11.1}$$

Proof. Take any irreducible unitary projective representation v of Λ_x on some finite dimensional Hilbert space $\mathscr{K}$ with cocycle $\overline{\omega}$, then (u, V, v) is a distinguished representation parameter. The distinguished CSR $\mathcal{S}(u, V, v)$ subordinate to Λ_x parameterized by (u, V, v) is given by

$$\mathcal{S}(u, V, v) = (\mathscr{K} \otimes \mathscr{H}, \mathrm{id}_{\mathscr{K}} \otimes u, v \times V) \tag{II.11.2}$$

by definition. Let $U = \mathscr{R}_{\Lambda_0}\big(\mathcal{S}(u, V, v)\big)$, then the distinguished representation $W = \mathrm{Ind}(U)$ of $\mathbb{G} \rtimes \Lambda$ parameterized by (u, V, v) is obtained as follows by the construction of induced representations presented in § II.4. First we define a unitary representation

$$\begin{aligned} \widetilde{W} &= \sum_{r,s\in\Lambda} e_{rs^{-1},r} \otimes \mathrm{id}_{\mathscr{K}} \otimes [(\mathrm{id}_{\mathscr{H}} \otimes \alpha^*_{rs^{-1}})(u)] \otimes \delta_s \\ &\in \mathcal{B}(\ell^2(\Lambda)) \otimes \mathcal{B}(\mathscr{K}) \otimes \mathcal{B}(\mathscr{H}) \otimes \mathrm{Pol}(\mathbb{G}) \otimes C(\Lambda) \end{aligned} \tag{II.11.3}$$

of $\mathbb{G} \rtimes \Lambda$ on $\ell^2(\Lambda) \otimes \mathscr{K} \otimes \mathscr{H}$. The subspace

$$\mathscr{H}_{(u,V,v)} = \left\{ \sum_{r\in\Lambda} \delta_r \otimes \zeta_r \ : \ \begin{array}{r} \zeta_r \in \mathscr{K} \otimes \mathscr{H}, \text{ and } \zeta_{r_0 r} = \big(v(r_0) \otimes V(r_0)\big)\zeta_r \\ \text{for all } r_0 \in \Lambda_0, r \in \Lambda \end{array} \right\} \tag{II.11.4}$$

of $\ell^2(\Lambda) \otimes \mathscr{K} \otimes \mathscr{H}$ is invariant under $\widetilde{W}$ and W is the subrepresentation $\mathscr{H}_{(u,V,v)}$ of $\widetilde{W}$. Recall (Lemma II.4.5) that the projection $\pi \in \mathcal{B}(\ell^2(\Lambda) \otimes \mathscr{K} \otimes \mathscr{H})$ with range $\mathscr{H}_{u,V,v}$ is given by

$$\pi = \frac{1}{|\Lambda_x|} \sum_{r_0\in\Lambda_x} \sum_{s\in\Lambda} e_{r_0 s,s} \otimes v(r_0) \otimes V(r_0). \tag{II.11.5}$$

Since vectors of the form $\delta_r \otimes \xi \otimes \eta$, $r \in \Lambda$, $\xi \in \mathscr{K}$, $\eta \in \mathscr{H}$ span $\ell^2(\Lambda) \otimes \mathscr{K} \otimes \mathscr{H}$, the matrix coefficients of W is spanned by elements of $\mathrm{Pol}(\mathbb{G}) \otimes C(\Lambda)$ of the form

$$\begin{aligned} &c(v; r, s, \xi_1, \xi_2, \eta_1, \eta_2) \\ &= (\omega_{\pi(\delta_r\otimes\xi_1\otimes\eta_1),\pi(\delta_s\otimes\xi_2\otimes\eta_2)} \otimes \mathrm{id}_{\mathrm{Pol}(\mathbb{G})} \otimes \mathrm{id}_{C(\Lambda)})(W) \\ &= (\omega_{\delta_r\otimes\xi_1\otimes\eta_1,\delta_s\otimes\xi_2\otimes\eta_2} \otimes \mathrm{id} \otimes \mathrm{id})\big((\pi \otimes 1 \otimes 1)\widetilde{W}(\pi \otimes 1 \otimes 1)\big) \\ &= (\omega_{\delta_r\otimes\xi_1\otimes\eta_1,\delta_s\otimes\xi_2\otimes\eta_2} \otimes \mathrm{id} \otimes \mathrm{id})\big(\widetilde{W}(\pi \otimes 1 \otimes 1)\big), \end{aligned} \tag{II.11.6}$$

where the last equality follows from Lemma II.4.5, and $\omega_{x,y}$ is the linear form $\langle \cdot\, x\, ,\ y\rangle$.

By (II.11.3) and (II.11.5), we see that

$$\begin{aligned} &|\Lambda_0| \cdot \big[\widetilde{W}(\pi \otimes 1 \otimes 1)\big](\delta_r \otimes \xi_1 \otimes \eta_1 \otimes 1 \otimes 1) \\ &= \sum_{\substack{r',s',t\in\Lambda,\\ r_0\in\Lambda_x}} \Big[e_{r's'^{-1},r'} e_{r_0 t,t} \otimes v(r_0) \otimes \Big(\big((s'r'^{-1}) \cdot u\big)\big(V(r_0) \otimes 1\big)\Big) \otimes \delta_{s'}\Big] \\ &\qquad\qquad \cdot (\delta_r \otimes \xi_1 \otimes \eta_1 \otimes 1 \otimes 1) \\ &\quad \text{(Only terms with } t = r, \text{ and } r' = r_0 t = r_0 r \text{ can be nonzero)} \\ &= \sum_{s'\in\Lambda} \sum_{r_0\in\Lambda_x} \delta_{r_0 r s'^{-1}} \otimes [v(r_0)\xi_1] \otimes \Big[\big((s' r^{-1} r_0^{-1}) \cdot u\big)\big(V(r_0)\eta_1 \otimes 1\big)\Big] \otimes \delta_{s'}. \end{aligned} \tag{II.11.7}$$

Note that $r_0 r s'^{-1} = s \iff s' = s^{-1} r_0 r \iff s' r^{-1} r_0^{-1} = s^{-1}$, by (II.11.6) and (II.11.7), we have

$$c(v; r, s, \xi_1, \xi_2, \eta_1, \eta_2) = \sum_{r_0\in\Lambda_x} \omega_{\xi_1,\xi_2}\big(v(r_0)\big)\big[(\omega_{V(r_0)\eta_1,\eta_2} \otimes \mathrm{id})(s^{-1} \cdot u)\big] \otimes \delta_{s^{-1} r_0 r}. \tag{II.11.8}$$

For any $r_0 \in \Lambda_x$, we have

$$\omega_{\xi_1,\xi_2}\big(v(r_0)\big) \in \mathbb{C} \quad \text{and} \quad \big[(\omega_{V(r_0)\eta_1,\eta_2} \otimes \mathrm{id})(s^{-1} \cdot u)\big] \in M_c(s^{-1} \cdot u). \tag{II.11.9}$$

By (II.11.8) and (II.11.9), we have

$$c(v; r, s, \xi_1, \xi_2, \eta_1, \eta_2) \in M_c(s^{-1} \cdot u) \otimes C(\Lambda), \tag{II.11.10}$$

which proves that

$$M(u) \subseteq \sum_{r' \in \Lambda} M_c(r' \cdot u) \otimes C(\Lambda) = \left(\sum_{r' \in \Lambda} M_c(r' \cdot u)\right) \otimes C(\Lambda). \tag{II.11.11}$$

It remains to establish the reverse inclusion, which is easily seen to be equivalent to show that for any $r_1, r_2 \in \Lambda$, we have

$$M(u) \supseteq M_c(r_1 \cdot u) \otimes \delta_{r_2}. \tag{II.11.12}$$

By the general theory of projective representations, there exists irreducible unitary projective representations $v_1, \ldots, v_m$ on $\mathscr{K}_1, \ldots, \mathscr{K}_m$ respectively, all with cocycle $\overline{\omega}$, and $\xi_1^{(i)}, \xi_2^{(i)} \in \mathscr{K}_i$, such that

$$\sum_{i=1}^{m} \left(\omega_{\xi_1^{(i)},\xi_2^{(i)}} \otimes \mathrm{id}\right)(v_i) = \delta_e \in C(\Lambda_x). \tag{II.11.13}$$

By (II.11.8) and (II.11.13), we see that for any $r, s \in \Lambda$, and any $\eta_1, \eta_2 \in \mathscr{H}$, $M(u)$ contains

$$\begin{aligned} &\sum_{i=1}^{n} c(v_i; r, s\xi_1^{(i)}, \xi_2^{(i)}, \eta_1, \eta_2) \\ = &\sum_{r_0 \in \Lambda_x} \delta_e(r_0)\big[(\omega_{V(r_0)\eta_1,\eta_2} \otimes \mathrm{id})(s^{-1} \cdot u)\big] \otimes \delta_{s^{-1}r_0 r} \\ &\text{(Only terms with } r_0 = e \text{ can be nonzero, and } V(e) = \mathrm{id}_{\mathscr{H}}) \\ = &\big[(\omega_{\eta_1,\eta_2} \otimes \mathrm{id})(s^{-1} \cdot u)\big] \otimes \delta_{s^{-1}r}. \end{aligned} \tag{II.11.14}$$

Taking $s = r_1^{-1}$ and $r = sr_2 = r_1^{-1} r_2$ in (II.11.14) proves (II.11.12) and finishes the proof of the lemma. □

Proposition II.11.2. *The linear span of matrix coefficients of distinguished representations of* $\mathbb{G} \rtimes \Lambda$ *in* $\mathrm{Pol}(\mathbb{G}) \otimes C(\Lambda)$ *is* $\mathrm{Pol}(\mathbb{G}) \otimes C(\Lambda)$ *itself. In particular, every unitary irreducible representation of* $\mathbb{G} \rtimes \Lambda$ *is unitarily equivalent to a distinguished one.*

Proof. The first assertion follows from Lemma II.11.1, and the second assertion follows from the first and the orthogonality relations of irreducible representations of $\mathbb{G} \rtimes \Lambda$. □

II.12 Classification of irreducible representations of $\mathbb{G} \rtimes \Lambda$

For each isotropy subgroup Λ_0 of Λ, let $\mathfrak{D}_{\Lambda_0}$ denotes the collection of equivalence classes of distinguished representation parameters associated with Λ_0. By Proposition II.10.7, the mapping

$$\begin{aligned} \Psi_{\Lambda_0} : \mathfrak{D}_{\Lambda_0} &\longrightarrow \mathrm{Irr}(\mathbb{G} \rtimes \Lambda) \\ [(u, V, v)] &\longmapsto \mathscr{R}_{\Lambda_0}\big(\mathcal{S}(u, V, v)\big) \end{aligned} \tag{II.12.1}$$

is well-defined and injective. In particular, $\mathfrak{D}_{\Lambda_0}$ is a set (instead of a proper class). Let $\mathfrak{D}$ be the collection of equivalence classes of distinguished representation parameters associated with any isotropy subgroup of Λ. By definition, $\mathfrak{D}$ is the disjoint union of $\mathfrak{D}_{\Lambda_0}$ as Λ_0 runs through all isotropy subgroups of Λ, hence $\mathfrak{D}$ is also a set. For any $[(u, V, v)] \in \mathfrak{D}_{\Lambda_0}$ and any $r \in \Lambda$, $r \cdot [(u, V, v)] = [r \cdot (u, V, v)]$ is a well-defined class in $\mathfrak{D}_{r\Lambda_0 r^{-1}}$. This defines an action of Λ on $\mathfrak{D}$. We are now ready to state and prove the classification of irreducible representations of $\mathbb{G} \rtimes \Lambda$.

Theorem II.12.1 (Classification of irreducible representations of $\mathbb{G} \rtimes \Lambda$)**.** *The mapping*

$$\begin{aligned} \Psi \colon \mathfrak{D} &\to \operatorname{Irr}(\mathbb{G} \rtimes \Lambda) \\ [(u, V, v)] \in \mathfrak{D}_{\Lambda_0} &\mapsto \Psi_{\Lambda_0}\big([(u, V, v)]\big) = \operatorname{Ind}\Big(\mathscr{R}_{\Lambda_0}\big(\mathcal{S}(u, V, v)\big)\Big) \end{aligned} \tag{II.12.2}$$

is surjective, and the fibers of Ψ *are exactly the* Λ*-orbits in* $\mathfrak{D}$*.*

Proof. By Proposition II.11.2, Ψ is surjective. By Corollary II.5.2 and (II.10.1), each Λ-orbits in $\mathfrak{D}$ maps to the same point under Ψ. It remains to show that if (u_i, V_i, v_i) is a distinguished representation parameter with associated subgroup Λ_i for $i = 1, 2$, and

$$\Psi\big([(u_1, V_1, v_1)]\big) = \Psi\big([(u_2, V_2, v_2)]\big), \tag{II.12.3}$$

then there exists an $r_0 \in \Lambda$, such that

$$r_0 \cdot [(u_1, V_1, v_1)] = [(u_2, V_2, v_2)] \in \mathfrak{D}_{\Lambda_2}. \tag{II.12.4}$$

Let $\mathrm{S}_i = \mathcal{S}(u_i, V_i, v_i)$, $U_i = \mathscr{R}_{\Lambda_i}(\mathrm{S}_i)$ for $i = 1, 2$. If $[u_2] \notin \Lambda \cdot [u_1]$, then by Proposition II.9.11, we have

$$\forall r, s \in \Lambda,\ \dim \operatorname{Mor}_{\mathbb{G} \rtimes \Lambda(r,s)}\Big((r \cdot U_1)|_{\mathbb{G} \rtimes \Lambda(r,s)}, (s \cdot U_2)|_{\mathbb{G} \rtimes \Lambda(r,s)}\Big) = 0, \tag{II.12.5}$$

where $\Lambda(r, s) = r\Lambda_1 r^{-1} \cap s\Lambda_2 s^{-1}$. This is because $(r \cdot U_1)|_{\mathbb{G} \rtimes \Lambda(r,s)}$ is parameterized by the representation parameter $(u_1, V_1|_{\Lambda(r,s)}, v_1|_{\Lambda(r,s)})$ associated with $\Lambda(r, s)$, and a similar assertion holds for $(s \cdot U_2)|_{\mathbb{G} \rtimes \Lambda(r,s)}$. Thus

$$\dim \operatorname{Mor}_{\mathbb{G} \rtimes \Lambda}\big(\operatorname{Ind}(U_1), \operatorname{Ind}(U_2)\big) = 0 \tag{II.12.6}$$

by Proposition II.6.3, which contradicts (II.12.3).

Thus $[u_2] \in \Lambda \cdot [u_1]$, by replacing $[(u_1, V_1, v_1)]$ with $r_0 \cdot [(u_1, V_1, v_1)]$ for some $r_0 \in \Lambda$ if necessary, we may assume without loss of generality that $[u_1] = [u_2] \in \operatorname{Irr}(\mathbb{G})$, and $\Lambda_1 = \Lambda_2$, which we now denote by Λ_0. It remains to prove that under this assumption, we have

$$[(u_1, V_1, v_1)] = [(u_2, V_2, v_2)] \in \mathfrak{D}_{\Lambda_0} \tag{II.12.7}$$

Since when $r^{-1}s \notin \Lambda_0$ if and only if $r \cdot [u_1] \neq s \cdot [u_2]$, we have

$$\forall r, s \in \Lambda, \quad r^{-1}s \notin \Lambda_0 \implies \dim \operatorname{Mor}_{\mathbb{G} \rtimes \Lambda(r,s)}\big((r \cdot U_1)|_{\mathbb{G} \rtimes \Lambda(r,s)}, (r \cdot U_2)|_{\mathbb{G} \rtimes \Lambda(r,s)}\big) = 0. \tag{II.12.8}$$

Note that when $r^{-1}s \in \Lambda_0$, we have $\Lambda(r, s) = r\Lambda_0 r^{-1} = s\Lambda_0 s^{-1}$, and $[\Lambda \colon \Lambda(r, s)] = [\Lambda \colon \Lambda_0]$. By (II.12.3), (II.12.8) and Proposition II.6.3, we have

$$1 = \frac{1}{|\Lambda_0|^2 [\Lambda \colon \Lambda_0]} \sum_{\substack{r,s \in \Lambda, \\ r^{-1}s \in \Lambda_0}} \dim \operatorname{Mor}_{\mathbb{G} \rtimes r\Lambda_0 r^{-1}}\big(r \cdot U_1, s \cdot U_2\big). \tag{II.12.9}$$

Since $r \cdot U_1$, $s \cdot U_2$ are both irreducible, we have

$$r^{-1}s \in \Lambda_0 \implies \dim \mathrm{Mor}_{\mathbb{G} \rtimes r\Lambda_0 r^{-1}}\big(r \cdot U_1, s \cdot U_2\big) = 0 \text{ or } 1. \tag{II.12.10}$$

Note that there are $|\Lambda_0|^2[\Lambda\colon \Lambda_0] = |\Lambda| \cdot |\Lambda_0|$ terms on the right side of (II.12.9), (II.12.10) forces

$$r^{-1}s \in \Lambda_0 \implies \dim \mathrm{Mor}_{\mathbb{G} \rtimes r\Lambda_0 r^{-1}}\big(r \cdot U_1, s \cdot U_2\big) = 1. \tag{II.12.11}$$

In particular, taking $r = s = 1_\Lambda$ in (II.12.11) shows that U_1 and U_2 are equivalent, hence (II.12.7) holds by Proposition II.10.7. This finishes the proof of the theorem. □

II.13 The conjugate representation of distinguished representations

We now study the conjugation of irreducible representations of $\mathbb{G} \rtimes \Lambda$ in terms of the classification presented in Theorem II.12.1. There is a small complication here in the non-Kac type case, where the contragredient of a unitary representation need not be unitary. Resolving this kind of question involves the modular operator, just as in Proposition II.7.7.

We begin with a simple lemma on linear operators.

Lemma II.13.1. *Let $\mathscr{H}$ be a Hilbert space, $U, P \in \mathcal{B}(\mathscr{H})$ such that U is unitary, P is invertible and positive, if PUP^{-1} is unitary, then $PUP^{-1} = U$, i.e. P commutes with U.*

Proof. Let $V = PUP^{-1}$. We have

$$PU^*P^{-1} = PU^{-1}P^{-1} = V^{-1} = V^* = P^{-1}U^*P. \tag{II.13.1}$$

Thus U^* commutes with the positive operator P^2. Hence U^* commutes with $(P^2)^{1/2} = P$, i.e. $U^*P = PU^*$. Taking adjoints of this proves $PU = UP$. □

Proposition II.13.2. *Let u be an irreducible unitary representation of $\mathbb{G}$, Λ_0 a subgroup of the isotropy subgroup $\Lambda_{[u]}$, V a covariant projective Λ_0-representation of u. Then any operator $\rho \in \mathrm{Mor}_{\mathbb{G}}(u, u^{cc})$ commutes with V (i.e. $\rho V(r_0) = V(r_0)\rho$ for all $r_0 \in \Lambda_0$).*

Proof. Since u is irreducible, $\mathrm{Mor}_{\mathbb{G}}(u, u^{cc})$ is a one dimensional space spanned by an invertible positive operator ((Neshveyev and Tuset, 2013, Lemma 1.3.12)). By definition (see (Neshveyev and Tuset, 2013, Proposition 1.4.4 and Definition 1.4.5)), the conjugation $\overline{u}$ of u is given by

$$\overline{u} = \big(j(\rho_u)^{1/2} \otimes 1\big)u^c\big(j(\rho_u)^{-1/2} \otimes 1\big), \tag{II.13.2}$$

where ρ_u is the unique *positive* operator in $\mathrm{Mor}_{\mathbb{G}}(u, u^{cc})$ with $\mathrm{Tr}(\rho_u) = \mathrm{Tr}(\rho_u^{-1})$. Since $\mathrm{Mor}_{\mathbb{G}}(u, u^{cc}) = \mathbb{C}\rho_u$, it suffices to show that ρ_u commutes with V.

Since u, V are covariant, we have

$$\forall r_0 \in \Lambda_0,\ \big(V(r_0) \otimes 1\big)(r_0 \cdot u) = u\big(V(r_0) \otimes 1\big). \tag{II.13.3}$$

Taking the adjoint of both sides of (II.13.3) then applying $j \otimes \mathrm{id}$, we get

$$\forall r_0 \in \Lambda_0,\ \big(V^c(r_0) \otimes 1\big)(r_0 \cdot u^c) = u^c\big(V^c(r_0) \otimes 1\big), \tag{II.13.4}$$

where

$$V^c = (j \otimes \mathrm{id})(V^{-1}) = (j \otimes \mathrm{id})(V^*) \tag{II.13.5}$$

is the contragredient of V, and

$$u^c = (j \otimes \mathrm{id})(u^{-1}) = (j \otimes \mathrm{id})(u^*) \tag{II.13.6}$$

the contragredient of u. We pose

$$\overline{V} = \big(j(\rho_u)^{1/2} \otimes 1\big)V^c\big(j(\rho_u)^{-1/2} \otimes 1\big), \tag{II.13.7}$$

then by (II.13.4) and (II.13.2), we have

$$\forall r_0 \in \Lambda_0,\ \big(\overline{V}(r_0) \otimes 1\big)(r_0 \cdot \overline{u}) = \overline{u}\big(\overline{V}(r_0) \otimes 1\big). \tag{II.13.8}$$

Thus for any $r_0 \in \Lambda_0$, $\overline{V}(r_0) \in \mathrm{Mor}_{\mathbb{G}}(r_0 \cdot \overline{u}, \overline{u})$, which is a one dimensional space spanned by a unitary operator since both $r_0 \cdot \overline{u}$ and $\overline{u}$ are irreducible unitary representations of $\mathbb{G}$. Note that $V^c(r_0) = j\big(V(r_0)^*\big)$ is unitary, by (II.13.7), we have

$$\det\big(\overline{V}(r_0)\big) = \det\big(j(\rho_u)^{1/2}V^c(r_0)j(\rho_u)^{-1/2}\big) = \det\big(V^c(r_0)\big) \in \mathbb{T}. \tag{II.13.9}$$

This forces $\overline{V}(r_0)$ to be unitary since it is a scalar multiple of a unitary operator. Applying Lemma II.13.1 to (II.13.7) (evaluated on each $r_0 \in \Lambda_0$), we see that

$$V^c = \overline{V} = \big(j(\rho_u)^{1/2} \otimes 1\big)V^c\big(j(\rho_u)^{-1/2} \otimes 1\big). \tag{II.13.10}$$

Applying $j \otimes \mathrm{id}$ to the inverse of both sides of (II.13.10) and note that $V^{cc} = V$, we see that

$$V = V^{cc} = (\rho_u^{1/2} \otimes 1)V^{cc}(\rho_u^{-1/2} \otimes 1) = (\rho_u^{1/2} \otimes 1)V(\rho_u^{-1/2} \otimes 1), \tag{II.13.11}$$

i.e. $\rho_u^{1/2}$ (hence ρ_u) commutes with V. □

Proposition II.13.3. *Let (u, V, v) be a representation parameter associated with some $\Lambda_0 \in \mathcal{G}_{\mathrm{iso}}(\Lambda)$, U is the unitary representation of $\mathbb{G} \rtimes \Lambda_0$ parameterized by (u, V, v), then the following hold:*

(a) *$(\overline{u}, V^c, v^c)$ is also a representation parameter;*

(b) *$\rho_U = \mathrm{id}_{\mathscr{H}_v} \otimes \rho_u$, where ρ_U (resp. ρ_u) is the modular operator for the representation U (resp. u);*

(c) *$\overline{U}$ is parameterized by $(\overline{u}, V^c, v^c)$.*

Proof. As we've seen in Proposition II.13.2 and its proof, we have $\overline{V}(r_0) \in \mathrm{Mor}_{\mathbb{G}}(r_0 \cdot u, u)$ for all $r_0 \in \Lambda_0$, thus $V^c = \overline{V}$ is covariant with $\overline{u}$. Since

$$\begin{aligned} \forall r_0 \in \Lambda_0, \qquad (v^c \times V^c)(r_0) &= j\big([v(r_0)]^{-1} \otimes [V(r_0)]^{-1}\big) \\ &= j\big(\{[(v \times V)](r_0)\}^{-1}\big) \end{aligned} \tag{II.13.12}$$

$v^c \times V^c$ is the contragredient of the unitary representation $v \times V$ of Λ_0, hence is a unitary representation itself. Thus v^c and V^c are unitary projective representations with opposing cocycles. This proves (a).

To prove (b), by the characterizing property of ρ_U, it suffices to show that the invertible positive operator $\mathrm{id}\otimes\rho_u$ satisfies

$$\mathrm{id}\otimes\rho_u \in \mathrm{Mor}_{\mathbb{G}}(U, U^{cc}) \tag{II.13.13}$$

and (by Proposition II.7.4 and Schur's lemma applied to the irreducible representation u)

$$\begin{aligned}&\mathrm{Tr}((\cdot)(\mathrm{id}\otimes\rho_u)) = \mathrm{Tr}((\cdot)(\mathrm{id}\otimes\rho_u^{-1}))\\ \in\ &\mathrm{End}_{\mathbb{G}\rtimes\Lambda_0}(U) = \mathrm{End}_{\mathbb{G}}(\mathrm{id}\otimes u)\cap\mathrm{End}_{\Lambda_0}(v\times V) \subseteq \mathcal{B}(\mathscr{H}_v)\otimes\mathbb{C}\,\mathrm{id}\,.\end{aligned} \tag{II.13.14}$$

Since $\mathrm{Tr}(\rho_u) = \mathrm{Tr}(\rho_u^{-1})$, (II.13.14) holds. We now prove (II.13.13). As is seen in the proof of Proposition II.13.2, condition (II.13.3) holds, and a similar calculation by applying $j\otimes\mathrm{id}$ to the inverse of both sides of (II.13.3) yields (note that $V^{cc} = V$),

$$\forall r_0\in\Lambda_0,\ \big(V(r_0)\otimes 1\big)(r_0\cdot u^{cc}) = u^{cc}\big(V(r_0)\otimes 1\big). \tag{II.13.15}$$

By definition, we have

$$\begin{aligned}U &= (\mathrm{id}\otimes u)_{123}(v\times V)_{124} = (\mathrm{id}\otimes u\otimes 1)v_{14}V_{24}\\ &\in \mathcal{B}(\mathscr{H}_v)\otimes\mathcal{B}(\mathscr{H}_u)\otimes\mathrm{Pol}(\mathbb{G})\otimes C(\Lambda_0).\end{aligned} \tag{II.13.16}$$

Thus

$$U^c = (j\otimes j\otimes\mathrm{id}\otimes\mathrm{id})(U^{-1}) = (\mathrm{id}\otimes u^c\otimes 1)v_{14}^c V_{24}^c, \tag{II.13.17}$$

and

$$U^{cc} = (\mathrm{id}\otimes u^{cc}\otimes 1)v_{14}^{cc}V_{24}^{cc} = (\mathrm{id}\otimes u^{cc}\otimes 1)v_{14}V_{24}. \tag{II.13.18}$$

By (II.13.16), (II.13.18) and Proposition II.13.2, we have

$$\begin{aligned}&(\mathrm{id}\otimes\rho_u\otimes 1\otimes 1)U = (\mathrm{id}\otimes\rho_u\otimes 1\otimes 1)(\mathrm{id}\otimes u\otimes 1)v_{14}V_{24}\\ &= (\mathrm{id}\otimes u^{cc}\otimes 1)v_{14}[(\mathrm{id}\otimes\rho_u\otimes 1\otimes 1)V_{24}]\\ &= (\mathrm{id}\otimes u^{cc}\otimes 1)v_{14}V_{24}(\mathrm{id}\otimes\rho_u\otimes 1\otimes 1)\\ &= U^{cc}(\mathrm{id}\otimes\rho_u\otimes 1\otimes 1).\end{aligned} \tag{II.13.19}$$

This proves (II.13.13) and finishes the proof of (b).

By Proposition II.7.7 and (b), $\overline{U}$ corresponds to the CSR $(\overline{\mathscr{H}_u}, u', w')$ in $\mathcal{CSR}_{\Lambda_0}$, where $\mathscr{H}_u$ is the underlying finite dimensional Hilbert space of u,

$$u' = (\mathrm{id}\otimes j(\rho)^{1/2}\otimes 1)(\mathrm{id}\otimes u^c)(\mathrm{id}\otimes j(\rho)^{-1/2}\otimes 1) = \mathrm{id}\otimes\overline{u}, \tag{II.13.20}$$

and

$$\begin{aligned}w' &= (\mathrm{id}\otimes j(\rho)^{1/2}\otimes 1)(v_{13}^c V_{23}^c)(\mathrm{id}\otimes j(\rho)^{-1/2}\otimes 1)\\ &= v_{13}^c[(\mathrm{id}\otimes j(\rho)^{1/2}\otimes 1)V_{23}^c(\mathrm{id}\otimes j(\rho)^{1/2}\otimes 1)]\\ &= v_{13}^c V_{23}^c = v^c\times V^c.\end{aligned} \tag{II.13.21}$$

Thus the CSR $(\overline{\mathscr{H}_u}, u', w')$, and consequently $\overline{U}$, is indeed parameterized by $(\overline{u}, v^c, V^c)$, which proves (c). ◻

Proposition II.13.2 motivates the following definition.

Definition II.13.4. Let (u, V, v) be a representation parameter associated with some $\Lambda_0 \in \mathcal{G}_{\text{iso}}(\Lambda)$, the representation parameter $(\overline{u}, V^c, v^c)$ is called the **conjugate** of (u, V, v).

By Proposition II.10.7 and Corollary II.9.12, it is clear that the conjugation of an irreducible representation parameter is irreducible, and $\overline{[(u, V, v)]} = [(\overline{u}, V^c, v^c)]$ gives a well-defined mapping $\overline{(\cdot)} : \mathfrak{D} \to \mathfrak{D}$. The following theorem describes how the conjugate representation of irreducible (unitary) representation of $\mathbb{G} \rtimes \Lambda$ looks like in terms of the classification given in Theorem II.12.1.

Theorem II.13.5. *Let* $[(u, V, v)] \in \mathfrak{D}$, $x = \Psi([(u, V, v)]) \in \operatorname{Irr}(\mathbb{G} \rtimes \Lambda)$, *then*

$$\overline{x} = \Psi(\overline{[(u, V, v)]}) = \Psi([(\overline{u}, V^c, v^c)]). \tag{II.13.22}$$

Proof. This follows immediately from Proposition II.13.3 and the character formula (II.5.3) for representations induced from representations of principal subgroups of $\mathbb{G} \rtimes \Lambda$. □

II.14 The incidence numbers

We now turn our attention to the fusion rules of $\mathbb{G} \rtimes \Lambda$. We define and study incidence numbers in this section, and use these numbers to express the fusion rules in § II.15.

Definition II.14.1. For $i = 1, 2, 3$, let $\Lambda_i \in \mathcal{G}_{\text{iso}}(\Lambda)$. Suppose U_i is a unitary representation of $\mathbb{G} \rtimes \Lambda_i$, and $r_i \in \Lambda$, then the **incidence number** of (r_1, r_2, r_3) relative to (U_1, U_2, U_3), denoted by $m_{U_1,U_2,U_3}(r_1, r_2, r_3)$, is defined by

$$\begin{aligned} &m_{U_1,U_2,U_3}(r_1, r_2, r_3) \\ &= \dim \operatorname{Mor}_{\mathbb{G}\rtimes\Lambda_0} \left((r_1 \cdot U_1)|_{\mathbb{G}\rtimes\Lambda_0}, (r_2 \cdot U_2)|_{\mathbb{G}\rtimes\Lambda_0} \times (r_3 \cdot U_3)|_{\mathbb{G}\rtimes\Lambda_0} \right), \end{aligned} \tag{II.14.1}$$

where $\Lambda_0 = \cap_{i=1}^{3} r_i \Lambda_i r_i^{-1}$.

We now aim to express the incidence numbers in terms of characters. Let Θ, Ξ be two subgroups of Λ with $\Theta \subseteq \Xi$. Recall that $C(\mathbb{G}) = A$. Suppose $F = \sum_{r\in\Xi} a_r \otimes \delta_r$, $a_r \in A$ is an element of $C(\mathbb{G}) \otimes C(\Xi) = A \otimes C(\Xi)$. We use $F|_{\mathbb{G}\rtimes\Theta}$ to denote the element $\sum_{r\in\Theta} a_r \otimes \delta_r$ in $\mathbb{G} \rtimes \Theta$, and call it the restriction of F to $\mathbb{G} \rtimes \Theta$. A simple calculation shows that this restriction operation gives a surjective unital morphism of C^*-algebras from $C(\mathbb{G} \rtimes \Xi) = A \otimes C(\Xi)$ to $C(\mathbb{G} \rtimes \Theta) = A \otimes C(\Theta)$ that also preserves comultiplication, thus allows us to view $\mathbb{G} \rtimes \Theta$ as a closed subgroup of $\mathbb{G} \rtimes \Xi$ in the sense of Definition II.3.1. Recall that we also have the extension morphism $E_{\Lambda_0} : C(\Lambda_0) \to C(\Lambda)$, $\delta_{r_0} \mapsto \delta_{r_0}$ for every subgroup Λ_0 of Λ, which simply sends each function in $C(\Lambda_0)$ to its unique extension in $C(\Lambda)$ that vanishes outside Λ_0. Finally, we use h^{Λ_0} to denote the Haar state on $\mathbb{G}\rtimes\Lambda_0$. For $i = 1, 2, 3$, let $\Lambda_i \in \mathcal{G}_{\text{iso}}(\Lambda)$. Suppose U_i is a unitary representation of $\mathbb{G}\rtimes\Lambda_i$, χ_i is the character of U_i. Let $\Lambda_0 = \cap_{i=1}^{3} r_i \Lambda_i r_i^{-1}$. Then we have the following formula to calculate the incidence numbers in terms of characters.

$$\begin{aligned} \forall r_1, r_2, r_3 \in \Lambda, \qquad &m_{U_1,U_2,U_3}(r_1, r_2, r_3) \\ &= h^{\Lambda_0} \left(\overline{(r_1 \cdot \chi_1)|_{\mathbb{G}\rtimes\Lambda_0}} (r_2 \cdot \chi_2)|_{\mathbb{G}\rtimes\Lambda_0} (r_3 \cdot \chi_3)|_{\mathbb{G}\rtimes\Lambda_0} \right). \end{aligned} \tag{II.14.2}$$

Proposition II.14.2. *Using the above notations, the incidence number $m_{U_1,U_2,U_3}(s_1,s_2,s_3)$ depends only on the classes $[U_1]$, $[U_2]$, $[U_3]$ of equivalent unitary representations and the left cosets $r_1\Lambda_1$, $r_2\Lambda_2$, $r_3\Lambda_3$.*

Proof. Note that for any $i = 1,2,3$, $s_i\Lambda_i s_i^{-1} = r_i\Lambda_i r_i^{-1}$ whenever $r_i^{-1}s_i \in \Lambda_i$. The proposition follows from (II.14.2) and Lemma II.5.3 (b). □

By Proposition II.14.2, we see immediately that the following definition is well-defined.

Definition II.14.3. For $i = 1,2,3$, let $\Lambda_i \in \mathcal{G}_{\mathrm{iso}}(\Lambda)$. Suppose x_i is a class of equivalent unitary representations of $\mathbb{G}\rtimes\Lambda_i$, and $z_i \in \Lambda/\Lambda_i$ is a left coset of Λ_i in Λ, then the **incidence number** of (z_1,z_2,z_3) relative to (x_1,x_2,x_3), denoted by $m_{x_1,x_2,x_3}(z_1,z_2,z_3)$, is defined by

$$m_{x_1,x_2,x_3}(z_1,z_2,z_3) = m_{U_1,U_2,U_3}(r_1,r_2,r_3) \tag{II.14.3}$$

where $U_i \in x_i$, $r_i \in z_i$ for $i = 1,2,3$.

The rest of this section is devoted to the calculation of the incidence number (II.14.3) in terms of more basic ingredients when $x_i = \Phi_{\Lambda_i}(\mathfrak{p}_i)$ for some $\mathfrak{p}_i \in \mathfrak{D}_{\Lambda_i}$ (see § II.12), as this will be the case we need in the calculation of fusion rules for $\mathbb{G}\rtimes\Lambda$ in § II.15. We begin with a result on the structure of unitary projective representations of some $\Lambda_0 \in \mathcal{G}_{\mathrm{iso}}(\Lambda)$ that are covariant with some unitary representation of $\mathbb{G}$.

Lemma II.14.4. *Fix a $\Lambda_0 \in \mathcal{G}_{\mathrm{iso}}(\Lambda)$. Let u_0 be an irreducible unitary representation of $\mathbb{G}$, $[u_0] \in \mathrm{Irr}(\mathbb{G})$ the class of u_0, such that $\Lambda_0 \subseteq \Lambda_{[u_0]}$. Suppose u is a unitary representation of $\mathbb{G}$, $V\colon \Lambda_0 \to \mathcal{U}(\mathscr{H}_u)$ is a unitary projective representation covariant with u, p is the minimal central projection in $\mathrm{End}_{\mathbb{G}}(u)$ corresponding to the maximal pure subrepresentation of u supported by $[u_0] \in \mathrm{Irr}(\mathbb{G})$. Let $q = 1-p$, then V is diagonalizable along p in the sense that*

$$\begin{aligned}&(p\otimes 1)V = V(p\otimes 1), \quad (q\otimes 1)V = V(q\otimes 1),\\ &\textit{and } (p\otimes 1)V(q\otimes 1) = (q\otimes 1)V(p\otimes 1) = 0.\end{aligned} \tag{II.14.4}$$

Proof. Since V and u are covariant, we have

$$\forall r_0 \in \Lambda_0,\ \big(V(r_0)\otimes 1\big)(r_0\cdot u) = u\big(V(r_0)\otimes 1\big). \tag{II.14.5}$$

Note that $p \in \mathrm{End}_{\mathbb{G}}(u) = \mathrm{End}_{\mathbb{G}}(r_0\cdot u)$ (see (II.8.1)), then for every $r_0 \in \Lambda_0$, it follows that

$$\begin{aligned}&\big([pV(r_0)q]\otimes 1\big)[(q\otimes 1)(r_0\cdot u)] = (p\otimes 1)\big(V(r_0)\otimes 1\big)(q\otimes 1)(r_0\cdot u)\\ =\ &(p\otimes 1)\big(V(r_0)\otimes 1\big)(r_0\cdot u)(q\otimes 1) = (p\otimes 1)u\big(V(r_0)\otimes 1\big)(q\otimes 1)\\ =\ &[(p\otimes 1)u](p\otimes 1)\big(V(r_0)\otimes 1\big)(q\otimes 1) = [(p\otimes 1)u]\big([pV(r_0)q]\otimes 1\big).\end{aligned} \tag{II.14.6}$$

Let u_p (resp. u_q) be the subrepresentation of u corresponding to p (resp. q), then $r_0^{-1}\cdot u_p$ is equivalent to u_p for all $r_0 \in \Lambda_0$ since $\Lambda_0 \subseteq \Lambda_{[u]}$, and

$$\mathrm{Mor}_{\mathbb{G}}(r_0\cdot u_q, u_p) = \mathrm{Mor}_{\mathbb{G}}(u_q, r_0^{-1}\cdot u_p) = \mathrm{Mor}_{\mathbb{G}}(u_q, u_p) = 0. \tag{II.14.7}$$

By (II.14.6), the operator $pV(r_0)q$, when viewed as an operator from $p(\mathscr{H}_u)$ to $q(\mathscr{H}_u)$, intertwines $r_0\cdot u_q$ and u_p. Thus by (II.14.7),

$$\forall r_0 \in \Lambda_0, \quad pV(r_0)q = 0. \tag{II.14.8}$$

Similarly,

$$\forall r_0 \in \Lambda_0, \quad qV(r_0)p = 0. \tag{II.14.9}$$

Hence

$$pV(r_0) = pV(r_0)(p+q) = pV(r_0)p = (p+q)V(r_0)p = V(r_0)p, \tag{II.14.10}$$

and similarly,

$$qV(r_0) = qV(r_0)(p+q) = qV(r_0)q = (p+q)V(r_0)q = V(r_0)q. \tag{II.14.11}$$

Now (II.14.4) follows from equations (II.14.8), (II.14.9), (II.14.10), and (II.14.11). □

We also need to generalize the notion of representation parameter a little, as the natural candidate of the "tensor product" of two representation parameters need not be a representation parameter, but it still possesses the same covariant property.

Definition II.14.5. Let $\Lambda_0 \in \mathcal{G}_{\mathrm{iso}}(\Lambda)$, we call a triple (u, V, v) a **generalized representation parameter ((GRP)** for short) associated with Λ_0, if the following hold:

(a) V is a unitary projective representation of Λ_0 on $\mathscr{H}_u$, such that

$$\forall r_0 \in \Lambda_0, \quad V(r_0) \in \mathrm{Mor}_{\mathbb{G}}(r_0 \cdot u, u); \tag{II.14.12}$$

(b) v is a unitary projective representation (on some other finite dimensional Hilbert space $\mathscr{H}_v$) of Λ_0, such that the cocycles of v and V are opposite to each other.

Proposition II.14.6. *If (u, V, v) is a GRP associated with some $\Lambda_0 \in \mathcal{G}_{\mathrm{iso}}(\Lambda)$, then $(\mathscr{H}_v \otimes \mathscr{H}_u, \mathrm{id} \otimes u, v \times V) \in \mathcal{CSR}_{\Lambda_0}$.*

Proof. The proof of Proposition II.9.6 applies almost verbatim here. □

Definition II.14.7. If (u, V, v) is a GRP associated with $\Lambda_0 \in \mathcal{G}_{\mathrm{iso}}(\Lambda)$, then the CSR $\mathrm{S} := (\mathscr{H}_v \otimes \mathscr{H}_u, \mathrm{id} \otimes u, v \times V)$ associated with Λ_0 and the unitary representation $\mathscr{R}_{\Lambda_0}(\mathrm{S})$ of $\mathbb{G} \rtimes \Lambda_0$ are said to be parameterized by (u, V, v).

We now describe a reduction process for generalized representation parameters, which leads to our desired calculation of the incidence numbers using more basic ingredients—the dimension of a certain intertwiner space of two projective representations of some generalized isotropy subgroup of Λ.

Proposition II.14.8. *Fix a $\Lambda_0 \in \mathcal{G}_{\mathrm{iso}}(\Lambda)$. Let (u, V, v) be a GRP associated with Λ_0, $x \in \mathrm{Irr}(\mathbb{G})$ such that $\Lambda_0 \subseteq \Lambda_x$, and $u_0 \in x$. Suppose p is the minimal central projection of $\mathrm{End}_{\mathbb{G}}(u)$ corresponding to the maximal pure subrepresentation of u supported by x. The following holds:*

(a) *(u_p, V_p, v) is a GRP, where u_p (resp. V_p) is the subrepresentation of u (resp. V) on $p(\mathscr{H}_u)$;*

(b) *let $n \in \mathbb{N}$ be the multiplicity of x in u, V_0 a covariant projective Λ_0-representation of u_0, then* up to unitary equivalence*, there exists a unique unitary projective representation V_1 of Λ_0 on $\mathbb{C}^n$, such that V_p is unitarily equivalent to $V_1 \times V_0$;*

(c) *$(u_0, V_0, v\times V_1)$ is representation parameter, and the CSR $(\mathscr{H}_v \otimes p(\mathscr{H}_u), \mathrm{id}\otimes u_p, v\times V_p)$ parameterized by (u_p, V_p, v) is isomorphic to the CSR $\mathcal{S}(u_0, V_0, v\times V_1)$ parameterized by $(u_0, V_0, v\times V_1)$ in the category $\mathcal{CSR}_{\Lambda_0}$. In particular, the representation parameter $(u_0, V_0, v\times V_1)$ and the GRP $(u_p, V_p, v\times V_p)$ parameterize equivalent unitary representations of $\mathbb{G}\rtimes\Lambda_0$.*

Proof. By Lemma II.14.4, u_p and V_p are covariant. Since V_p is a subrepresentation of V, it has the same cocycle as V, hence V_p and v have opposing cocycles. This proves (a).

The proof of (b) parallels that of Proposition II.9.5. Since u_p is equivalent to a direct sum of n copies of u_0, thus there exists a unitary operator $U \in \mathrm{Mor}_{\mathbb{G}}(\mathrm{id}_{\mathbb{C}^n}\otimes u_0, u_p)$. Replace (u_p, V_p, v) with $(U^* u_p U, U^* V_p U, v)$ if necessary, we may assume $u_p = \mathbb{C}^n \otimes u_0$. Repeat the proof of Proposition II.9.5 with the small modification of replacing the unitary representation w there with the unitary projective representation V_p, we see that there exists a unique unitary projective representation $V_1\colon \Lambda_0 \to \mathcal{U}(\mathbb{C}^n)$, such that $V_p = V_1\times V_0$. This proves (b).

By (b) and its proof, we may suppose $u_p = \mathrm{id}_{\mathbb{C}^n}\otimes u_0$. Note that the CSR parameterized by (u_p, V_p, v) is exactly $(\mathrm{id}_{\mathscr{H}_v}\otimes \mathrm{id}_{\mathbb{C}^n}\otimes u_0, v\times V_p)$, which coincides exactly with the CSR parameterized by $(\mathrm{id}_{\mathbb{C}^n\otimes\mathscr{H}_v}\otimes u_0, V_0, v\times V_1)$ since $v\times V_p = v\times V_1\times V_0$. This proves (c). □

Definition II.14.9. Using the notation of Proposition II.14.8, the representation parameter $(u_0, V_0, v\times V_1)$ is called a **reduction** of the GRP (u, V, v) along (u_0, V_0).

Remark II.14.10. Since V_1 is determined up to unitary equivalence, so is the reduction $(u_0, V_0, v\times V_1)$.

The following result describes the incidence numbers $m_{[U_1],[U_2],[U_3]}(z_1, z_2, z_3)$ in terms of the dimension of the intertwiner space of some projective representations of Λ_0.

Proposition II.14.11. *Suppose we are given the following data for each $i = 1, 2, 3$:*

- *a $\Lambda_i \in \mathcal{G}_{\mathrm{iso}}(\Lambda)$, a left coset z_i in Λ/Λ_i and a $r_i \in z_i$;*
- *a representation parameter (u_i, V_i, v_i) associated with Λ_i;*
- *the unitary representation U_i of $\mathbb{G}\rtimes\Lambda_i$ parameterized by (u_i, V_i, v_i).*

Let $\Lambda_0 = \cap_{i=1}^3 r_i\Lambda_i r_i^{-1} = \cap_{i=1}^3 z_i\Lambda_i z_i^{-1}$. Suppose

$$\big(r_1\cdot u_1, (r_1\cdot V_1)|_{\Lambda_0}, (r_2\cdot v_2)|_{\Lambda_0}\times(r_3\cdot v_3)|_{\Lambda_0}\times V\big)$$

is the reduction of the GRP

$$\big((r_2\cdot u_2)\times(r_3\cdot u_3), (r_2\cdot V_2)|_{\Lambda_0}\times(r_3\cdot V_3)|_{\Lambda_0}, (r_2\cdot v_2)|_{\Lambda_0}\times(r_3\cdot v_3)|_{\Lambda_0}\big)$$

along $\big(r_1\cdot u_1, (r_1\cdot V_1)|_{\Lambda_0}\big)$. Then the unitary projective representations $(r_1\cdot v_1)|_{\Lambda_0}$ and

$$(r_2\cdot v_2)|_{\Lambda_0}\times(r_3\cdot v_3)|_{\Lambda_0}\times V$$

of Λ_0 have the same cocycle, and

$$\begin{aligned} &m_{[U_1],[U_2],[U_3]}(z_1, z_2, z_3) \\ &= \dim \mathrm{Mor}_{\Lambda_0}\big((r_1\cdot v_1)|_{\Lambda_0}, (r_2\cdot v_2)|_{\Lambda_0}\times(r_3\cdot v_3)|_{\Lambda_0}\times V\big). \end{aligned} \tag{II.14.13}$$

Proof. It is easy to check that $\big((r_2\cdot u_2)\times(r_3\cdot u_3), (r_2\cdot V_2)|_{\Lambda_0}\times(r_3\cdot V_3)|_{\Lambda_0}, (r_2\cdot v_2)|_{\Lambda_0}\times (r_3\cdot v_3)|_{\Lambda_0}\big)$ is indeed a generalized representation parameter. Take the minimal central projection p of $\mathrm{End}_{\mathbb{G}}\big((r_2\cdot u_2)\times(r_3\cdot u_3)\big)$ corresponding to the maximal pure subrepresentation u_p of $(r_2\cdot u_2)\times(r_3\cdot u_3)$ that is supported by $[r_1\cdot u_1]\in\mathrm{Irr}(\mathbb{G})$. Suppose $q=1-p$. By Lemma II.14.4, q also corresponds to a subrepresentation u_q of $(r_2\cdot u_2)\times(r_3\cdot u_3)$ on $q(\mathscr{H}_{u_2}\otimes\mathscr{H}_{u_3})$. Similarly, let V_p (resp. V_q) be the subrepresentation of the unitary projective representation $(r_2\cdot v_2)|_{\Lambda_0}\times(r_3\cdot v_3)|_{\Lambda_0}$ on $p(\mathscr{H}_{u_2}\otimes\mathscr{H}_{u_3})$ (resp. $q(\mathscr{H}_{u_2}\otimes\mathscr{H}_3)$). Let U_p (resp. U_q) be the representation of $\mathbb{G}\rtimes\Lambda_0$ parameterized by the GRP $(u_p, V_p, (r_2\cdot v_2)|_{\Lambda_0}\times(r_3\cdot v_3)|_{\Lambda_0})$ (resp. $(u_q, V_q, (r_2\cdot v_2)|_{\Lambda_0}\times(r_3\cdot v_3)|_{\Lambda_0})$). By construction, the unitary representation U of $\mathbb{G}\rtimes\Lambda_0$ parameterized by $\big((r_2\cdot u_2)\times(r_3\cdot u_3), (r_2\cdot V_2)|_{\Lambda_0}\times(r_3\cdot V_3)|_{\Lambda_0}, (r_2\cdot v_2)|_{\Lambda_0}\times(r_3\cdot v_3)|_{\Lambda_0}\big)$ is the direct sum of U_p and U_q. By definition,

$$\begin{aligned} m_{[U_1],[U_2],[U_3]}(z_1,z_2,z_3) &= \dim\mathrm{Mor}_{\mathbb{G}\rtimes\Lambda_0}(U_1,U) \\ &= \dim\mathrm{Mor}_{\mathbb{G}\rtimes\Lambda_0}(U_1,U_p)+\dim\mathrm{Mor}_{\mathbb{G}\rtimes\Lambda_0}(U_1,U_q). \end{aligned} \tag{II.14.14}$$

From our construction, the matrix coefficients of u_p and u_q are orthogonal with respect to the Haar state h of $\mathbb{G}$. Thus the proof of Proposition II.9.11 (a) applies almost verbatim, and shows that

$$\dim\mathrm{Mor}_{\mathbb{G}\rtimes\Lambda_0}(U_1,U_q)=0. \tag{II.14.15}$$

On the other hand, the cocycles of both $(r_1\cdot v_1)|_{\Lambda_0}$ and $(r_2\cdot v_2)|_{\Lambda_0}\times(r_3\cdot v_3)|_{\Lambda_0}\times V$ are both opposite to that of $(r_1\cdot V_1)|_{\Lambda_0}$ by the reduction process described above, hence these cocycles coincide. By Proposition II.9.11 (b) and Proposition II.14.8 (c), we have

$$\dim\mathrm{Mor}_{\mathbb{G}\rtimes\Lambda_0}(U_1,U_p)=\dim\mathrm{Mor}_{\Lambda_0}\big((r_1\cdot v_1)|_{\Lambda_0},(r_2\cdot v_2)|_{\Lambda_0}\times(r_3\cdot v_3)|_{\Lambda_0}\times V\big). \tag{II.14.16}$$

Now (II.14.13) follows from (II.14.15) and (II.14.16). ◻

II.15 Fusion rules

We now calculate the fusion rules of $\mathbb{G}\rtimes\Lambda$. From the classification theorem (Theorem II.12.1), up to unitary equivalence, all unitary irreducible representations of $\mathbb{G}\rtimes\Lambda$ are distinguished. Thus the task falls to the calculation of

$$\dim\mathrm{Mor}_{\mathbb{G}}\big(\mathrm{Ind}(U_1),\mathrm{Ind}(U_2)\times\mathrm{Ind}(U_3)\big), \tag{II.15.1}$$

where, for $i=1,2,3$, U_i is the irreducible unitary representation of $\mathbb{G}\rtimes\Lambda_i$ parameterized (see Definition II.10.1 and Definition II.10.4) by some *distinguished* representation parameter (u_i,V_i,v_i) associated with Λ_i (recall that $\Lambda_i=\Lambda_{[u_i]}$ since (u_i,V_i,v_i) is distinguished). Let h be the Haar state on $C(\mathbb{G})=A$. For any subgroup Λ_0 of Λ, we use h^{Λ_0} to denote the Haar state on $C(\mathbb{G}\rtimes\Lambda_0)=A\otimes C(\Lambda_0)$, and $E_{\Lambda_0}\colon C(\Lambda_0)\to C(\Lambda)$ denotes the linear embedding such that $\delta_{r_0}\in C(\Lambda_0)\mapsto\delta_{r_0}\in C(\Lambda)$(the extension of functions in $C(\Lambda_0)$ to functions in $C(\Lambda)$ that vanishes outside Λ_0). In particular, h^{Λ} is the Haar state on $C(\mathbb{G}\rtimes\Lambda)=A\otimes C(\Lambda)$. For $i=1,2,3$, let $\chi_i=(\mathrm{Tr}\otimes\mathrm{id})(U_i)\in A\otimes C(\Lambda_i)$ be the character of U_i, and $r\cdot\chi_i$ is defined to be the character of the representation $r\cdot U_i$ of $\mathbb{G}\rtimes r\Lambda_i r^{-1}$.

Using these notations, by Proposition II.5.1, we have the following formula for the character of $\mathrm{Ind}(U_i)$,

$$\forall i=1,2,3,\quad \chi(\mathrm{Ind}(U_i))=|\Lambda_i|^{-1}\sum_{r_i\in\Lambda}(\mathrm{id}\otimes E_{r_i\Lambda_i r_i^{-1}})(r_i\cdot\chi_i). \tag{II.15.2}$$

Thus

$$\begin{aligned}&\dim \mathrm{Mor}_{\mathbb{G}\rtimes\Lambda}\big(\mathrm{Ind}(U_1),\mathrm{Ind}(U_2)\times\mathrm{Ind}(U_3)\big)\\&=h^{\Lambda}\big(\overline{\chi(\mathrm{Ind}(U_1))}[\chi(\mathrm{Ind}(U_2))][\chi(\mathrm{Ind}(U_3))]\big)\\&=\sum_{r_1,r_2,r_3}\frac{h^{\Lambda}\big(\chi(r_1,r_2,r_3)\big)}{|\Lambda_1|\cdot|\Lambda_2|\cdot|\Lambda_3|},\end{aligned}\tag{II.15.3}$$

where

$$\begin{aligned}&\chi(r_1,r_2,r_3)\\&=\overline{(\mathrm{id}\otimes E_{r_1\Lambda_1 r_1^{-1}})(r_1\cdot\chi_1)}[(\mathrm{id}\otimes E_{r_2\Lambda_2 r_2^{-1}})(r_2\cdot\chi_2)][(\mathrm{id}\otimes E_{r_3\Lambda_3 r_3^{-1}})(r_3\cdot\chi_3)].\end{aligned}\tag{II.15.4}$$

If Θ,Ξ are subgroups of Λ with $\Theta\subseteq\Xi$, and $\sum_{r\in\Xi}a_r\otimes\delta_r$ is an arbitrary element of $A\otimes C(\Xi)$ with all $a_r\in A$, we call the element $\sum_{r\in\Theta}a_r\otimes\delta_r$ of $A\otimes C(\Theta)$ the restriction of $\sum_{r\in\Xi}a_r\otimes\delta_r$ and denote it by $(\sum_{r\in\Xi}a_r\otimes\delta_r)|_{\mathbb{G}\rtimes\Theta}$. Recall that

$$h^{\Lambda_0}=[\Lambda\colon\Lambda_0]\cdot h^{\Lambda}\circ(\mathrm{id}\otimes E_{\Lambda_0})\tag{II.15.5}$$

for any subgroup Λ_0 of Λ, posing

$$\Lambda(r_1,r_2,r_3)=\bigcap_{i=1}^{3}r_i\Lambda_i r_i^{-1},\tag{II.15.6}$$

we have

$$\begin{aligned}&h^{\Lambda}\big(\chi(r_1,r_2,r_3)\big)=h^{\Lambda}\left(\chi(r_1,r_2,r_3)|_{\mathbb{G}\rtimes\Lambda(r_1,r_2,r_3)}\right)\\&=\frac{h^{\Lambda(r_1,r_2,r_3)}\left(\overline{(r_1\cdot\chi_1)|_{\mathbb{G}\rtimes\Lambda(r_1,r_2,r_3)}}(r_2\cdot\chi_2)|_{\mathbb{G}\rtimes\Lambda(r_1,r_2,r_3)}(r_3\cdot\chi_3)|_{\mathbb{G}\rtimes\Lambda(r_1,r_2,r_3)}\right)}{[\Lambda\colon\Lambda(r_1,r_2,r_3)]}\\&=[\Lambda\colon\Lambda(r_1,r_2,r_3)]^{-1}m_{U_1,U_2,U_3}(r_1,r_2,r_3),\end{aligned}\tag{II.15.7}$$

where $m_{U_1,U_2,U_3}(r_1,r_2,r_3)$ is the incidence number of (r_1,r_2,r_3) relative to (U_1,U_2,U_3).

By (II.15.3) and (II.15.7), we have

$$\begin{aligned}&\dim \mathrm{Mor}_{\mathbb{G}\rtimes\Lambda}\big(\mathrm{Ind}(U_1),\mathrm{Ind}(U_2)\times\mathrm{Ind}(U_3)\big)\\&=\sum_{r_1,r_2,r_3\in\Lambda}\frac{m_{U_1,U_2,U_3}(r_1,r_2,r_3)}{|\Lambda_1|\cdot|\Lambda_2|\cdot|\Lambda_3|\cdot[\Lambda\colon\Lambda(r_1,r_2,r_3)]}.\end{aligned}\tag{II.15.8}$$

As we've seen in Definition II.14.3 and the discussion before it, we have

$$\begin{aligned}&(\forall i=1,2,3,\ r_i\in z_i\in\Lambda/\Lambda_i)\\&\implies m_{[U_1],[U_2],[U_3]}(z_1,z_2,z_3)=m_{U_1,U_2,U_3}(r_1,r_2,r_3),\end{aligned}\tag{II.15.9}$$

where $\Lambda(z_1,z_2,z_3):=\cap_{i=1}^{3}r_i\Lambda_i r_i^{-1}$ does not depend on the choices for $r_i\in z_i$, $i=1,2,3$. Thus (II.15.8) can be written more succinctly as

$$\begin{aligned}&\dim \mathrm{Mor}_{\mathbb{G}\rtimes\Lambda}\big(\mathrm{Ind}(U_1),\mathrm{Ind}(U_2)\times\mathrm{Ind}(U_3)\big)\\&=\sum_{z_1\in\Lambda/\Lambda_1}\sum_{z_2\in\Lambda/\Lambda_2}\sum_{z_3\in\Lambda/\Lambda_3}\frac{m_{[U_1],[U_2],[U_3]}(z_1,z_2,z_3)}{[\Lambda\colon\Lambda(z_1,z_2,z_3)]}.\end{aligned}\tag{II.15.10}$$

We formalize the above calculation as the following theorem, which describes the fusion rules of $\mathbb{G}\rtimes\Lambda$ in terms of the more basic ingredients of incidence numbers, which in turn is completely determined by the representation theory of $\mathbb{G}$, the action of Λ on $\mathrm{Irr}(\mathbb{G})$, and various unitary projective representations of some naturally appeared subgroups in $\mathcal{G}_{\mathrm{iso}}(\Lambda)$.

Theorem II.15.1. *The fusion rules for $\mathbb{G} \rtimes \Lambda$ is given as the following. For $i = 1, 2, 3$, let W_i be an irreducible representation of $\mathbb{G} \rtimes \Lambda$. Suppose U_i is the distinguished representation parameterized by some distinguished representation parameter (u_i, V_i, v_i) associated with some isotropy subgroup Λ_i of Λ, such that W_i is equivalent to $\operatorname{Ind}(U_i)$, then*

$$\begin{aligned} &\dim \operatorname{Mor}_{\mathbb{G}\rtimes\Lambda}(W_1, W_2 \times W_3) \\ &= \sum_{z_1 \in \Lambda/\Lambda_1} \sum_{z_2 \in \Lambda/\Lambda_2} \sum_{z_3 \in \Lambda/\Lambda_3} \frac{m_{[U_1],[U_2],[U_3]}(z_1, z_2, z_3)}{[\Lambda \colon \Lambda(z_1, z_2, z_3)]}. \end{aligned} \tag{II.15.11}$$

Here the incidence numbers

$$\begin{aligned} &m_{[U_1],[U_2],[U_3]}(z_1, z_2, z_3) = m_{U_1,U_2,U_3}(r_1, r_2, r_3) \\ &= \dim \operatorname{Mor}_{\Lambda(z_1,z_2,z_3)}\big((r_1 \cdot v_1)|_{\Lambda_0}, (r_2 \cdot v_2)|_{\Lambda_0} \times (r_3 \cdot v_3)|_{\Lambda_0} \times V\big), \end{aligned} \tag{II.15.12}$$

where $r_i \in z_i$ for $i = 1, 2, 3$, and the unitary projective representation V of $\Lambda(z_1, z_2, z_3)$ is taken from the reduction

$$\big(r_1 \cdot u_1, (r_1 \cdot V_1)|_{\Lambda_0}, (r_2 \cdot v_2)|_{\Lambda_0} \times (r_3 \cdot v_3)|_{\Lambda_0} \times V\big)$$

of the generalized representation parameter

$$\big((r_2 \cdot u_2) \times (r_3 \cdot u_3), (r_2 \cdot V_2)|_{\Lambda_0} \times (r_3 \cdot V_3)|_{\Lambda_0}, (r_2 \cdot v_2)|_{\Lambda_0} \times (r_3 \cdot v_3)|_{\Lambda_0}\big)$$

along $\big(r_1 \cdot u_1, (r_1 \cdot V_1)|_{\Lambda_0}\big)$.

Proof. The above calculation proves (II.15.11), and (II.15.12) follows from Proposition II.14.11. □

Chapter III

Some examples of bicrossed product with property (RD)

Introduction

This chapter focuses on producing explicit examples of bicrossed products whose dual has property (RD). Of course, the more interesting ones are those without polynomial growth. The main idea is to first twist semidirect products by a finite subgroup to obtain nontrivial bicrossed products, then utilise the theories of Chapter I and Chapter II to treat the technical issues that appear in this process. Of course, the most central results are Theorem I.7.3 and Theorem I.8.4, whose application has one major difficulty, namely, how does one show the length functions on the discrete group and the dual of the compact group are actually matched in the sense of Definition I.6.9. This is essentially addressed by a careful analysis of the representation theory of a classical compact group, which is in fact the classical case of the more general theory presented in Chapter II. However, due the twisting process, some technical hypothesis on the invariance of certain length functions arises. This technical issue is only partially resolved, which is not totally satisfactory in the author's opinion. On the one hand, under a seemingly mild condition, we have a completely satisfactory characterization result which gives us a powerful process of producing interesting bicrossed products (see Theorem III.5.1), which is explained in § III.5. On the other hand, there does exist many interesting examples that violates this seemingly mild condition, which is described in § III.6. In any case, many interesting concrete examples of bicrossed products are constructed here, and it is the hope of the author that the dichotomy mentioned above will pique the reader's interests and perhaps stimulate further investigation.

III.1 Nontrivial bicrossed product from semidirect product

Let G be a compact group, Γ a discrete group acting on G via topological automorphisms given by a group morphism $\tau : \Gamma \longrightarrow \mathrm{Aut}(G)$. Using these data, one can form the semidirect product $G \rtimes_\tau \Gamma$, which is a locally compact group whose underlying topological space is the topological product $G \times \Gamma$, and whose group law is given by

$$(g_1, \gamma_1)(g_2, \gamma_2) = \left(g_1 \tau_{\gamma_1}(g_2), \gamma_1\gamma_2\right) . \tag{III.1.1}$$

It is easy to see that the insertion $\iota_\Gamma : \Gamma \to G \rtimes_\tau \Gamma$, $\gamma \mapsto (e_G, \gamma)$ is a group morphism. In particular, the mapping

$$\begin{aligned} \theta : \Gamma &\to \operatorname{Aut}(G \rtimes_\tau \Gamma) \\ \gamma &\mapsto (\operatorname{Ad} \circ \iota_\Gamma)(\gamma) = \operatorname{Ad}_{(e_G, \gamma)} \end{aligned} \tag{III.1.2}$$

is a group morphism. For all $(g, r) \in G \rtimes_\tau \Gamma$ and $\gamma \in \Gamma$, we have

$$(e_G, \gamma)(g, r)(e_G, \gamma)^{-1} = \big(\tau_\gamma(g), \gamma r\big)\,\big(e_G, \gamma^{-1}\big) = \big(\tau_\gamma(g), \gamma r \gamma^{-1}\big)\,. \tag{III.1.3}$$

Thus as a map from the set $G \times \Gamma$ to itself, we have

$$\theta_\gamma := \theta(\gamma) = \tau_\gamma \times \operatorname{Ad}_\gamma : G \times \Gamma \to G \times \Gamma. \tag{III.1.4}$$

Now consider any *finite* subgroup Λ of Γ. The group morphism θ defined in (III.1.2) restricts to the subgroup Λ to give an action $\Lambda \curvearrowright G \rtimes_\tau \Gamma$ by topological automorphisms. This allows us to form yet another semidirect product $(G \rtimes_a \Gamma) \rtimes_\theta \Lambda$, whose underlying topological space is $G \times \Gamma \times \Lambda$. It is clear that the group law on $(G \rtimes_\tau \Gamma) \rtimes_\theta \Lambda$ is given by

$$\begin{aligned} &(g_1, \gamma_1, r_1)(g_2, \gamma_2, r_2) = \big((g_1, \gamma_1)\theta_{r_1}(g_2, \gamma_2), r_1 r_2\big) \\ &= \Big((g_1, \gamma_1)\big(\tau_{r_1}(g_2), r_1 \gamma_2 r_1^{-1}\big), r_1 r_2\Big) \\ &= \big(g_1 \tau_{\gamma_1 r_1}(g_2), \gamma_1 r_1 \gamma_2 r_1^{-1}, r_1 r_2\big). \end{aligned} \tag{III.1.5}$$

By (III.1.5), both the mapping

$$\begin{aligned} \iota_{1,3} : G \rtimes_\tau \Lambda &\to (G \rtimes_\tau \Gamma) \rtimes_\theta \Lambda \\ (g, r) &\mapsto (g, e, r), \end{aligned} \tag{III.1.6}$$

and

$$\begin{aligned} \iota_2 : \Gamma &\to (G \rtimes_\tau \Gamma) \rtimes_\theta \Lambda \\ \gamma &\mapsto (e_G, \gamma, e) \end{aligned} \tag{III.1.7}$$

are injective group morphisms, such that for all $\gamma \in \Gamma$, $(g, r) \in G \rtimes_\tau \Lambda$, we have

$$\forall g \in G,\ \gamma, r \in \Gamma, \quad \iota_{1,3}(g, r)\iota_2(r^{-1}\gamma r) = (g, e, r)(e_G, r^{-1}\gamma r, e) = (g, \gamma, r), \tag{III.1.8}$$

which implies that

$$\iota_{1,3}(G \rtimes_\tau \Lambda)\iota_2(\Gamma) = (G \rtimes_\tau \Gamma) \rtimes_\theta \Lambda. \tag{III.1.9}$$

It is clear that

$$\iota_{1,3}(G \rtimes_\tau \Lambda) \cap \iota_2(\Gamma) = \{(e_G, e, e)\}. \tag{III.1.10}$$

Moreover,

$$\begin{aligned} \forall g \in G,\ \gamma, r \in \Gamma, \quad &\iota_2(\gamma)\iota_{1,3}(g, r) = (e_G, \gamma, e)(g, e, r) \\ &= \big(\tau_\gamma(g), \gamma, r\big) = \big(\tau_\gamma(g), e, r\big)(e_G, r^{-1}\gamma r, e) \\ &= \iota_{1,3}(g, r)\iota_2(r^{-1}\gamma r). \end{aligned} \tag{III.1.11}$$

The following result already appeared as special case in (Fima et al., 2017, §7.1.1).

Proposition III.1.1. *Let Γ be a discrete group, G a compact group, and $\tau : \Gamma \to \mathrm{Aut}(G)$ a left action of Γ on G by topological automorphisms. If Λ is a* finite *subgroup of Γ, then $(\Gamma\,,\, G \rtimes_\tau \Lambda)$ is a matched pair of groups with left action*

$$\begin{aligned}\alpha^\Lambda : \Gamma \times (G \rtimes_\tau \Lambda) &\to G \rtimes_\tau \Lambda \\ \big(\gamma, (g, r)\big) &\mapsto \big(\tau_\gamma(g), r\big),\end{aligned} \tag{III.1.12}$$

and right action

$$\begin{aligned}\beta^\Lambda : \Gamma \times (G \rtimes_\tau \Lambda) &\to \Gamma \\ \big(\gamma, (g, r)\big) &\mapsto r^{-1}\gamma r.\end{aligned} \tag{III.1.13}$$

Moreover, the following hold.

(a) *The action α^Λ is trivial if and only if τ is trivial;*

(b) *The action β^Λ is trivial if and only if $\Lambda \subseteq Z(\Gamma)$, where $Z(\Gamma)$ is the centre of Γ.*

Proof. That $(\Gamma\,,\, G \rtimes_\tau \Lambda)$ is a matched pair with the actions α^Λ and β^Λ follows from (III.1.9), (III.1.10) and (III.1.11). (a) and (b) are direct consequences of the definition of α^Λ and β^Λ. □

III.2 Some notations

For the convenience of our discussion, we now introduce and fix some notations related to the bicrossed product of the matched pair $(\Gamma\,,\, G \rtimes_\tau \Lambda)$ with the actions α^Λ and β^Λ.

The bicrossed product of the matched pair $(\Gamma\,,\, G \rtimes_\tau \Lambda)$ is denoted by $\Gamma\,{}_{\alpha^\Lambda}\!\bowtie_{\beta^\Lambda} (G \rtimes_\tau \Lambda)$. When there is no risk of confusion, we often omit the actions and simply write $G \rtimes \Lambda$ and $\Gamma \bowtie (G \rtimes \Lambda)$. Moreover, $\mathrm{Aut}(G)$ denotes the group of topological automorphism of G.

The isotropy subgroup of $G \rtimes \Lambda$ fixing $\gamma \in \Gamma$ with respect to the action β^Λ is easily seen to be $G \rtimes \Lambda_\gamma$, where

$$\Lambda_\gamma := \{r \in \Lambda \ : \ \gamma r = r\gamma\}. \tag{III.2.1}$$

For $x \in \mathrm{Irr}(G)$, $\gamma \in \Gamma$, we denote the isotropy subgroup of Λ_γ fixing x with respect to the action $\Lambda_\gamma \curvearrowright \mathrm{Irr}(G)$, $(r, [u]) \mapsto [u \circ \tau_r]$ by $\Lambda_{\gamma,x}$, i.e.

$$\Lambda_{\gamma,x} := \big\{r \in \Lambda_\gamma \ : \ r \cdot x = x\big\}. \tag{III.2.2}$$

We also need to fix some notations concerning from the representation theory of $\Gamma \bowtie (G \rtimes \Lambda)$. We assume familiarity with § I.4 and § II.10–§ II.12.

Let $\gamma \in \Gamma$. Suppose Λ_0 is an isotropy subgroup of Λ_γ with respect to the action $\Lambda_\gamma \curvearrowright \mathrm{Irr}(G)$. Let $\mathfrak{D}_{\gamma,\Lambda_0}$ denotes the set of equivalent distinguished representation parameters (see § II.10, Definition II.10.4) associated with Λ_0, and

$$\Psi_{\gamma,\Lambda_0} : \mathfrak{D}_{\gamma,\Lambda_0} \to \mathrm{Irr}(G \rtimes \Lambda_\gamma) \tag{III.2.3}$$

is the injection used to classify irreducible unitary representations of $G \rtimes \Lambda_\gamma$ as in § II.12. Let $\mathfrak{D}_\gamma$ be the set of equivalency classes of all distinguished representation parameters for the semidirect product $G \rtimes \Lambda_\gamma$, we then have an action of Λ_γ on the class of all distinguished representation parameters for $G \rtimes \Lambda_\gamma$, which passes to the quotient and yields an action $\Lambda_\gamma \curvearrowright \mathfrak{D}_\gamma$ as presented in § II.12. We thus have the classification *surjection*

$$\begin{aligned} \Psi_\gamma : \mathfrak{D}_\gamma &\to \operatorname{Irr}(G \rtimes \Lambda_\gamma) \\ [(u, V, v)] \in \mathfrak{D}_{\gamma,\Lambda_0} &\mapsto \Psi_{\gamma,\Lambda_0}\big([(u, V, v)]\big), \end{aligned} \tag{III.2.4}$$

whose fibers are exactly the orbits for the action $\Lambda_\gamma \curvearrowright \mathfrak{D}_\gamma$.

When $\gamma = e_\Gamma$, we then have $\Lambda_\gamma = \Lambda$ and we write Ψ_{e_Γ} simply as Ψ.

We use $\operatorname{Orb}_{\beta^\Lambda}$ to denote the set of β^Λ-orbits. For each $\mathscr{O} \in \operatorname{Orb}_{\beta^\Lambda}$, let $\mathfrak{R}_\mathscr{O}$ be the mapping from the class of $\mathscr{O}$-representations of $G \rtimes \Lambda$ to the class of finite dimensional unitary representations of the bicrossed product $\Gamma \bowtie (G \rtimes \Lambda)$ as in (I.4.6) of Lemma I.4.4, and let $\operatorname{Irr}_\mathscr{O}(G \rtimes \Lambda)$ denote the set of equivalency classes of $\mathscr{O}$-irreducible $\mathscr{O}$-representations. We thus have the classification bijection

$$\begin{aligned} \mathfrak{R} : \coprod_{\mathscr{O} \in \operatorname{Orb}_{\beta^\Lambda}} \operatorname{Irr}_\mathscr{O}(G \rtimes \Lambda) &\to \operatorname{Irr}\big(\Gamma \bowtie (G \rtimes \Lambda)\big) \\ [U] \in \operatorname{Irr}_\mathscr{O}(G \rtimes \Lambda) &\mapsto [\mathfrak{R}_\mathscr{O}(U)]. \end{aligned} \tag{III.2.5}$$

III.3 A sufficient condition

We first establish the following technical result.

Lemma III.3.1. *Suppose $l_{\widehat{G}} : \operatorname{Irr}(G) \to \mathbb{R}_{\geq 0}$ is a Γ-invariant length function on $\widehat{G}$, i.e. $l_{\widehat{G}}([u^x \circ \tau_\gamma]) = l_{\widehat{G}}(x)$ whenever $\gamma \in \Gamma$, $u^x \in x \in \operatorname{Irr}(G)$, and l_Γ is a β^Λ-invariant length function on Γ. Then*

$$\begin{aligned} l_{\widehat{G \rtimes \Lambda}} : \operatorname{Irr}(G \rtimes \Lambda) &\to \mathbb{R}_{\geq 0} \\ \Psi([(u, V, v)]) &\mapsto l_{\widehat{G}}([u]) \end{aligned}$$

is a well-defined length function on $\widehat{G \rtimes \Lambda}$ such that the pair $\big(l_\Gamma\,, l_{\widehat{G \rtimes \Lambda}}\big)$ is matched.

Proof. The fact that $l_{\widehat{G \rtimes \Lambda}}$ is well-defined (does not depend on the choice of the distinguished representation parameter (u, V, v)) follows from Theorem II.12.1 and the Λ-invariance of $l_{\widehat{G}}$. We now show that $\big(l_\Gamma\,, l_{\widehat{G \rtimes \Lambda}}\big)$ is matched.

For all $\mathscr{O} \in \operatorname{Orb}_{\beta^\Lambda}$, define $l_\mathscr{O} : \operatorname{Irr}_\mathscr{O}(G \rtimes \Lambda) \to \mathbb{R}_{\geq 0}$ via the following procedure. Take any $\gamma \in \mathscr{O}$, and let $\Phi_\gamma : \operatorname{Irr}(G \rtimes \Lambda_\gamma) \to \operatorname{Irr}_\mathscr{O}(G \rtimes \Lambda)$ be the canonical bijection as in Notations I.4.7. To avoid over-complication of our notations, we often implicitly identify $\operatorname{Irr}(G \rtimes \Lambda_\gamma)$ with $\operatorname{Irr}_\mathscr{O}(G \rtimes \Lambda)$ via the bijection Φ_γ, when doing so won't cause a risk of confusion. For all distinguished representation parameter (u, V, v) of $G \rtimes \Lambda_\gamma$, let

$$l_\mathscr{O}\left(\Psi_\gamma\big([(u, V, v)]\big)\right) := l_{\widehat{G}}([u]) + l_\Gamma(\gamma). \tag{III.3.1}$$

By Theorem II.12.1 again, we see that (III.3.1) yields a well-defined mapping $l_\mathscr{O} : \operatorname{Irr}_\mathscr{O}(G \rtimes \Lambda) \to \mathbb{R}_{\geq 0}$. It is clear that $l_{\{e_\Gamma\}} = l_{\widehat{G}}$ via the identification of $\operatorname{Irr}_{\{e_\Gamma\}}(G \rtimes \Lambda)$ with $\operatorname{Irr}(G \rtimes \Lambda)$ by Φ_{e_Γ}. Moreover, it is clear that $[\varepsilon_\mathscr{O}] = \Psi_\gamma([\varepsilon_G, \varepsilon_{\Lambda_\gamma}, \varepsilon_{\Lambda_\gamma}])$, so that

$$l_\Gamma(\gamma) = l_{\widehat{G}}([\varepsilon_G]) + l_\Gamma(\gamma) = l_\mathscr{O}([\varepsilon_\mathscr{O}]).$$

Therefore, to finish the proof, it remains to show that $(l_\mathscr{O})_{\mathscr{O} \in \operatorname{Orb}_{\beta^\Lambda}}$ is an affording family in the sense of Definition I.6.2. By definition, it is clear that

$$l_{\{e_\Gamma\}}([\varepsilon_{G \rtimes \Lambda}]) = l_{\widehat{G}}([\varepsilon_G]) + l_\Gamma(e_\Gamma) = 0.$$

The condition $l_{\mathscr{O}}([U]) = l_{\mathscr{O}^{-1}}([U^\dagger])$ can also be easily checked. Indeed, if $[U]$ is given by $\Psi_\gamma\big([(u,V,v)]\big)$, then $[U^\dagger]$ is given by $\Psi_{\gamma^{-1}}\big([(\overline{u},V^c,v^c)]\big)$ (see Proposition II.13.3). By (III.3.1), we now have

$$l_{\mathscr{O}}([U]) = l_{\widehat{G}}([u]) + l_\Gamma(\gamma) = l_{\widehat{G}}([\overline{u}]) + l_\Gamma(\gamma^{-1}) = l_{\mathscr{O}^{-1}}([U^\dagger]). \tag{III.3.2}$$

By Definition I.6.2, it remains only to establish the following claim.
<u>**Claim.**</u> For $i = 1,2,3$, let $\mathscr{O}_i \in \mathrm{Orb}_{\beta^\Lambda}$, and $[U_i] \in \mathrm{Irr}_{\mathscr{O}_i}(G \rtimes \Lambda)$, with

$$U_i = \sum_{r,s\in\mathscr{O}_i} e_{r,s} \otimes u^{(i)}_{r,s}$$

being an $\mathscr{O}_i$-irreducible $\mathscr{O}_i$-representation of $G \rtimes \Lambda$ on $\ell^2(\mathscr{O}_i) \otimes \mathscr{H}_i$. If

$$\dim \mathrm{Mor}_{G\rtimes\Lambda_\gamma}\left(u^{(3)}_{\gamma,\gamma}|_{G\rtimes\Lambda_\gamma}, U_1 \times_\gamma U_2\right) \neq 0 \tag{III.3.3}$$

for some (hence for all, by Lemma I.4.18) $\gamma \in \mathscr{O}_3$, then

$$l_{\mathscr{O}_3}([U_3]) \leq l_{\mathscr{O}_1}([U_1]) + l_{\mathscr{O}_2}([U_2]). \tag{III.3.4}$$

Before proving the claim, we remark that until now, only the Λ-invariance of $l_{\widehat{G}}$ is needed. The hypothesis that $l_{\widehat{G}}$ is Γ-invariant will play an important role in the proof of the claim as we will presently see.

We now prove the claim. Suppose $[U_i]$ is given by some $\Psi_{\gamma_i}\big([(u_i,V_i,v_i)]\big)$ and let $\Lambda_i := \Lambda_{\gamma_i,[u_i]}$ for each $i = 1,2,3$. Define $\mu \cdot u := u \circ \tau_\mu$ to be the left action of Γ on the class of finite dimensional unitary representation of G, and let $M(u)$ denote the vector space of matrix coefficients of u. Using the character formulae for $U_1 \times_\gamma U_2$ and for $\Psi_{\gamma_3}\big([(u_3,V_3,v_3)]\big)$, as well as the construction of Ψ_{γ_3}, we see that as elements in $\mathrm{Pol}(G) \otimes C(\Lambda_\gamma)$, we have

$$\chi\left(u^{(3)}_{\gamma,\gamma}|_{G\rtimes\Lambda_\gamma}\right) \in \mathrm{Vect}\left(\bigcup_{r\in\Lambda} M(r\cdot u_3)\right) \otimes C(\Lambda_\gamma) \subseteq \mathrm{Pol}(G) \otimes C(\Lambda_\gamma), \tag{III.3.5}$$

and

$$\chi\left(U_1 \times_\gamma U_2\right) \in \mathrm{Vect}\Big(\big[\Gamma\cdot M(u_1)\big]\big[\Gamma\cdot M(u_2)\big]\Big) \otimes C(\Lambda_\gamma), \tag{III.3.6}$$

where

$$\forall i = 1,2, \qquad \Gamma\cdot M(u_i) := \bigcup_{r_i\in\Gamma} M(r_i\cdot u_i)$$

and $\big[\Gamma\cdot M(u_1)\big]\big[\Gamma\cdot M(u_2)\big]$ denotes product of form $\varphi_1\varphi_2 \in \mathrm{Pol}(G)$ where $\varphi_i \in \Gamma\cdot M(u_i)$ for $i = 1,2$. By (III.3.5), (III.3.6) and a simple calculation using the Haar state on $C(G)\otimes C(\Lambda_\gamma) = C(G\rtimes\Lambda_\gamma)$, it is clear that (III.3.3) implies the existence of $r \in \Lambda$, $r_1, r_2 \in \Gamma$, such that $M(r\cdot u_3)$ and $M(r_1\cdot u_1)\cdot M(r_2\cdot u_2)$ are *not* orthogonal with respect to the Haar measure on G. Since the representation $r\cdot u_3$ of G is irreducible, this forces that

$$\dim \mathrm{Mor}_G\big(r\cdot u_3, (r_1\cdot u_1)\times(r_2\cdot u_2)\big) \neq 0.$$

Hence

$$r\cdot[u_3] \subseteq (r_1\cdot[u_1]) \otimes (r_2\cdot[u_2]). \tag{III.3.7}$$

Since $l_{\widehat{G}}$ is a Γ-invariant length function, by (III.3.7), we have

$$l_{\widehat{G}}([u_3]) = l_{\widehat{G}}(r\cdot[u_3]) \leq l_{\widehat{G}}(r_1\cdot[u_1]) + l_{\widehat{G}}(r_2\cdot[u_2]) = l_{\widehat{G}}([u_1]) + l_{\widehat{G}}([u_2]). \tag{III.3.8}$$

On the other hand, (III.3.3) also implies that $\gamma_3 \in \mathscr{O}_3 \subseteq \mathscr{O}_1\mathscr{O}_2$, so there is $s_i \in \mathscr{O}_i$, $i = 1, 2$, such that $s_1 s_2 = \gamma_3$. Using the fact that l_Γ is a β^Λ-invariant length function, we have

$$l_\Gamma(\gamma_3) = l_\Gamma(s_1 s_2) \leq l_\Gamma(s_1) + l_\Gamma(s_2) = l_\Gamma(\gamma_1) + l_\Gamma(\gamma_2). \tag{III.3.9}$$

By (III.3.8), (III.3.9) and (III.3.1) again, we have

$$\begin{aligned} l_{\mathscr{O}_3}([U_3]) &= l_{\widehat{G}}([u_3]) + l_\Gamma(\gamma_3) \\ &\leq l_{\widehat{G}}([u_1]) + l_{\widehat{G}}([u_2]) + l_\Gamma(\gamma_1) + l_\Gamma(\gamma_2) \\ &= l_{\mathscr{O}_1}([U_1]) + l_{\mathscr{O}_2}([U_2]). \end{aligned}$$

This finishes the proof of the claim, and hence the lemma. □

We can now give the following sufficient condition for $\widehat{\Gamma \bowtie (G \rtimes \Lambda)}$ to have property (RD) (resp. polynomial growth).

Theorem III.3.2. *In the above settings. If there is a Γ-invariant length function $l_{\widehat{G}}$ on $\widehat{G}$, and a β^Λ-invariant length function l_Γ on Γ, such that both $\left(\widehat{G}, l_{\widehat{G}}\right)$ and (Γ, l_Γ) have polynomial growth (resp. (RD)), then the dual of the bicrossed product, namely $\widehat{\Gamma \bowtie (G \rtimes \Lambda)}$ also has polynomial growth (resp. (RD)).*

Proof. This follows from Lemma III.3.1, Theorem I.7.3 and Theorem I.8.4. □

Remark III.3.3. It is however, unknown to the author that whether the polynomial growth (resp. (RD)) of the dual $\widehat{\Gamma \bowtie (G \rtimes \Lambda)}$ implies the existence of a Γ-*invariant* length function $l_{\widehat{G}}$ on $\widehat{G}$ witnessing the polynomial growth (resp. (RD)) of $\widehat{G}$. Later we will show that if the composition $\Gamma \xrightarrow{\tau} \mathrm{Aut}(G) \to \mathrm{Out}(G)$ has finite image, then the converse of Theorem III.3.2 also holds (see Theorem III.5.1).

III.4 Invariance of length functions

In this section, we partially treat the difficulty of the technical assumption on the Γ-invariance of the length function $l_{\widehat{G}}$ on $\widehat{\Gamma}$ that witnesses the polynomial growth or (RD) of $\widehat{G}$, as presented in Theorem III.3.2. The results here will be used in § III.5 in which we give some concrete examples of bicrossed products whose dual has (RD) but does *not* have polynomial growth. We also point out here that the examples given in § III.6 do *not* fit into this framework, thus we only have a partial understanding of the situation.

We begin by considering a technical lemma on the Fourier transform and the Sobolev-0-norm compact quantum groups of Kac type.

Lemma III.4.1. *Let $\mathbb{H}$ be a compact quantum group of Kac type. Suppose $\theta : C(\mathbb{H}) \to C(\mathbb{H})$ is an automorphism of C^*-algebras that intertwines the comultiplication Δ of $\mathbb{H}$ (i.e. θ is an automorphism of the quantum group $\mathbb{H}$). Then there exists an automorphism $\widehat{\theta}$ of the involutive algebra $c_c(\widehat{\mathbb{H}})$, such that*

$$\forall a \in c_c(\widehat{\mathbb{H}}), \quad \mathscr{F}_{\mathbb{H}}\left(\widehat{\theta}(a)\right) = \theta\big(\mathscr{F}_{\mathbb{H}}(a)\big) \quad \textit{and} \quad \left\|\widehat{\theta}(a)\right\|_{\mathbb{H},0} = \|a\|_{\mathbb{H},0}. \tag{III.4.1}$$

Proof. Choose a complete set of representatives $\{u^x : x \in \mathrm{Irr}(\mathbb{H})\}$ for $\mathrm{Irr}(\mathbb{H})$, and denote the finite dimensional Hilbert space underlying the unitary representation u^x by $\mathscr{H}_x$, so that

$$c_c(\widehat{\mathbb{H}}) = \bigoplus_{x\in\mathrm{Irr}(\mathbb{H})}^{\mathrm{alg}} \mathcal{B}(\mathscr{H}_x).$$

For each finite dimensional unitary representation $u \in \mathcal{B}(\mathscr{H}) \otimes \mathrm{Pol}(\mathbb{H})$ of $\mathbb{H}$ on $\mathscr{H}$, since θ is an automorphism of $\mathbb{H}$, the unitary operator

$$\theta_*(u) := (\mathrm{id} \otimes \theta)(u) \in \mathcal{B}(\mathscr{H}) \otimes \mathrm{Pol}(\mathbb{H}) \tag{III.4.2}$$

remains a unitary representation of $\mathbb{H}$ on the same space $\mathscr{H}$. It is clear that θ_* also passes to a bijection of the set $\mathrm{Irr}(\mathbb{H})$ to itself, which we still denote by θ_* by abuse of notation, via $\theta_*([u]) = [\theta_*(u)]$. In particular, for each $x \in \mathrm{Irr}(\mathbb{H})$, we have $\left[u^{\theta_*(x)}\right] = \theta_*(x) = [\theta_*(u^x)]$, thus there exists a *unitary*

$$T_x \in \mathrm{Mor}_{\mathbb{H}}\left(u^{\theta_*(x)}, \theta_*(u^x)\right) \subseteq \mathcal{B}\left(\mathscr{H}_{\theta_*(x)}, \mathscr{H}_x\right),$$

which is uniquely determined up to a multiple of a scalar in $\mathbb{T}$.

Take any

$$a = (a_x)_{x\in\mathrm{Irr}(\mathbb{H})} = \sum_{x\in\mathrm{Irr}(\mathbb{H})} a_x \in c_c(\widehat{\mathbb{H}}), \tag{III.4.3}$$

where the sum is finite (meaning all but finitely many $a_x \in \mathcal{B}(\mathscr{H}_x)$ is 0). For each $x \in \mathrm{Irr}(\mathbb{H})$, we pose

$$b_{\theta_*(x)} := T_x^* a_x T_x \in \mathcal{B}\left(\mathscr{H}_{\theta_*(x)}\right). \tag{III.4.4}$$

Then

$$\dim\bigl(\theta_*(x)\bigr) = \dim x. \tag{III.4.5}$$

By the choice of T_x, we have

$$\begin{aligned}
&\left(\mathrm{Tr}_{\mathscr{H}_{\theta_*(x)}} \otimes \mathrm{id}\right)\left(u^{\theta_*(x)}(b_{\theta_*(x)} \otimes 1)\right)\\
&= \left(\mathrm{Tr}_{\mathscr{H}_{\theta_*(x)}} \otimes \mathrm{id}\right)\left((T_x^* \otimes 1)\bigl[[\theta_*(u^x)](a_x \otimes 1)\bigr](T_x \otimes 1)\right)\\
&= (\mathrm{Tr}_{\mathscr{H}_x} \otimes \mathrm{id})\left([\theta_*(u^x)](a_x \otimes 1)\right)\\
&= \theta\left[(\mathrm{Tr}_{\mathscr{H}_x} \otimes \mathrm{id})\left([u^x(a_x \otimes 1)\right)\right],
\end{aligned} \tag{III.4.6}$$

and

$$\begin{aligned}
&\left(\mathrm{Tr}_{\mathscr{H}_{\theta_*(x)}} \otimes \mathrm{id}\right)(b_{\theta_*(x)}^* b_{\theta_*(x)})\\
&= \left(\mathrm{Tr}_{\mathscr{H}_{\theta_*(x)}} \otimes \mathrm{id}\right)(T_x^* a_x^* a_x T_x)\\
&= \left(\mathrm{Tr}_{\mathscr{H}_{\theta_*(x)}} \otimes \mathrm{id}\right)(a_x^* a_x).
\end{aligned} \tag{III.4.7}$$

We now define

$$\widehat{\theta}(a) := \sum_{x\in\mathrm{Irr}(\mathbb{H})} b_{\theta_*(x)} \tag{III.4.8}$$

Since $\theta_* : \mathrm{Irr}(\mathbb{H}) \longrightarrow \mathrm{Irr}(\mathbb{H})$ is a bijection, it is clear that (III.4.8) defines an automorphism $\widehat{\theta}$ of the involutive algebra $c_c(\widehat{\mathbb{H}})$. Finally, (III.4.1) follows from (III.4.6) and (III.4.7). □

Remark III.4.2. Lemma III.4.1 also applies to non-Kac type $\mathbb{H}$ with almost the same proof, with the caveats that the Fourier transform and the Sobolev norm needs to be adjusted using quantum dimensions of representations, which is not needed hence not introduced here (see (Vergnioux, 2007) or (Bhowmick et al., 2015) the discussion of non-Kac type Fourier transforms and Sobolev norms).

Recall the notations in Notations I.5.9, we have the following result.

Proposition III.4.3. *Let* $\mathbb{H} = \big(C(\mathbb{H}), \Delta\big)$ *be a compact quantum group of Kac type. Suppose* Θ *is a* finite *subgroup of* $\mathrm{Aut}\big(C(\mathbb{H}), \Delta\big)$. *The following are equivalent.*

(a) *There exists a length function* l *on* $\widehat{\mathbb{H}}$ *and* $P(X) \in \mathbb{R}[X]$, *such that*

$$\forall k \in \mathbb{N}, \quad a \in Q_{l,k}c_c(\widehat{\mathbb{H}}) \implies \|\mathcal{F}_{\mathbb{H}}(a)\| \leq P(k)\|a\|_{\mathbb{H},0}. \tag{III.4.9}$$

(b) *There exists a* Θ-invariant *length function* l_Θ *on* $\widehat{\mathbb{H}}$ *and* $Q(X) \in \mathbb{R}[X]$, *such that*

$$\forall k \in \mathbb{N}, \quad a \in Q_{l_\Theta,k}c_c(\widehat{\mathbb{H}}) \implies \|\mathcal{F}_{\mathbb{H}}(a)\| \leq Q(k)\|a\|_{\mathbb{H},0}. \tag{III.4.10}$$

Proof. Obviously (b) implies (a).

Now suppose (a) holds and let's prove (b). Let $n = |\Theta|$ and suppose $\theta_1, \dots, \theta_n$ form an enumeration of all elements of Θ. Let l_i denote the length function $l \circ (\theta_i)_*$ on $\widehat{\mathbb{H}}$ (see the discussion after (III.4.2) in the proof of Lemma III.4.1). Put

$$l_\Theta := \frac{1}{|\Theta|}\sum_{i=1}^{n} l_i, \tag{III.4.11}$$

then it is clear that l_Θ is a Θ-invariant length function on $\widehat{\mathbb{H}}$. For each $k \in \mathbb{N}$, define

$$F_{\Theta,k} := \{x \in \mathrm{Irr}(\mathbb{H}) \ : \ l_\Theta(x) < k+1\}, \tag{III.4.12}$$

and for $i = 1, \dots, n$, put

$$F_{i,k} := \{x \in \mathrm{Irr}(\mathbb{H}) \ : \ l_i(x) < k+1\}, \tag{III.4.13}$$

By (III.4.9), (III.4.13) and (III.4.11), we have

$$F_{\Theta,k} \subseteq \bigcup_{i=1}^{n} F_{i,k}. \tag{III.4.14}$$

Define

$$\begin{aligned} \xi : F_{\Theta,k} &\to \{1, \dots, n\} \\ x &\mapsto \inf\big\{i \ : \ x \in F_{i,k}\big\}. \end{aligned} \tag{III.4.15}$$

Note that (III.4.14) guarantees that ξ is well-defined.

We now prove (III.4.10) holds for some suitable polynomial $Q(X) \in \mathbb{R}[X]$, which will finish the proof. Since $a \in Q_{l_\Theta,k}c_c(\widehat{\mathbb{H}})$, there exists a finite subset F of $F_{\Theta,k}$, such that

$$a = \sum_{x \in F} a_x = \sum_{i=1}^{n} a_i, \tag{III.4.16}$$

where for each i,

$$a_i := \sum_{x \in F \cap \xi^{-1}(i)} a_x \in Q_{l_i,k}. \tag{III.4.17}$$

By Lemma III.4.1 and (a), we have

$$\forall i = 1, \ldots, n, \quad \|\mathcal{F}_{\mathbb{H}}(a_i)\| \leq P(k)\|a_i\|_{\mathbb{H},0}, \tag{III.4.18}$$

hence

$$\begin{aligned} \|\mathcal{F}_{\mathbb{H}}(a)\|^2 &\leq \left(\sum_{i=1}^{n} \|\mathcal{F}_{\mathbb{H}}(a_i)\|\right)^2 \leq n\left(\sum_{i=1}^{n} \|\mathcal{F}_{\mathbb{H}}(a_i)\|^2\right) \\ &\leq n[P(k)]^2 \left(\sum_{i=1}^{n} \|a_i\|^2_{\mathbb{H},0}\right) = |\Theta|[P(k)]^2\|a\|^2_{\mathbb{H},0}. \end{aligned} \tag{III.4.19}$$

Thus posing $Q(X) = \sqrt{|\Theta|}P(X) \in \mathbb{R}[X]$, we have (III.4.10). □

Corollary III.4.4. *The following are equivalent:*

(a) Γ *has polynomial growth (resp.* (RD)*);*

(b) *there exists a* β^Λ*-invariant length function* l_Γ *on* Γ*, such that* (Γ, l_Γ) *has polynomial growth (resp.* (RD)*).*

Proof. This follows from Proposition III.4.3 by posing $\Theta = \{\mathrm{Ad}_r \in \mathrm{Aut}(\Gamma) : r \in \Lambda\}$ and $\mathbb{H} = \widehat{\Gamma}$. □

III.5 Examples of bicrossed products with rapid decay but not polynomial growth–part I

We begin by observing more closely the action $\Gamma \curvearrowright \mathrm{Irr}(G)$. It is clear that this action is actually given by $\mathrm{Aut}(G)$ acting on G, and the group morphism $\tau : \Gamma \to \mathrm{Aut}(G)$ with respect to which we form the semidirect product (see the beginning of § III.1). More precisely, there is a natural action $\mathrm{Aut}(G) \curvearrowright \mathrm{Irr}(G)$ by letting $(\theta, [u]) \mapsto [\theta_*(u)]$, and the action $\Gamma \curvearrowright \mathrm{Irr}(G)$ is given by $(\gamma, x) \mapsto \tau(\gamma) \cdot x$. By definition, one has

$$\mathrm{Inn}(G) \subseteq \bigcap_{x \in \mathrm{Irr}(G)} [\mathrm{Aut}(G)]_x, \tag{III.5.1}$$

where

$$[\mathrm{Aut}(G)]_x := \{\theta \in \mathrm{Aut}(G) : \theta \cdot x = x\}.$$

Thus passing to the quotient, it is in fact $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$ that acts on $\mathrm{Irr}(G)$. Thus to talk about the Γ invariance of a given length function l on $\widehat{G}$, it suffices to consider the invariance of l under the image of the composition of group morphisms $\tau\colon \Gamma \to \mathrm{Aut}(G)$ and the canonical projection $\mathrm{Aut}(G) \to \mathrm{Out}(G)$.

With the above considerations in mind, we can now establish the following theoretical result.

Theorem III.5.1. *Let* $\widetilde{\tau} : \Gamma \to \mathrm{Out}(G)$ *be the composition of* $\tau : \Gamma \to \mathrm{Aut}(G)$ *with the canonical projection* $\mathrm{Aut}(G) \to \mathrm{Out}(G)$*. If* $\mathrm{Image}(\widetilde{\tau})$ *is finite, then the following are equivalent:*

(a) $\Gamma \bowtie \widehat{(G \rtimes \Lambda)}$ *has polynomial growth (resp.* (RD)*);*

(b) *both* Γ *and* $\widehat{G}$ *have polynomial growth (resp.* (RD)*).*

Proof. This follows from Theorem III.3.2, the proof of Proposition III.4.3 (posing $\mathbb{H} = G$ and $\Theta = \mathrm{Image}(\widetilde{\tau})$, and noting that inner automorphisms of G acts trivially on $\mathrm{Irr}(G)$ and hence on length functions) and Corollary III.4.4. □

We will also frequently use Jolissaint's theorem on rapid decay of amalgamated product of groups, which we record here for convenience of the reader.

Theorem III.5.2 (Jolissaint)**.** *Suppose* Γ_1, Γ_2 *are two discrete groups with property* (RD), A *is a finite group,* $j_i : A \hookrightarrow \Gamma_i$ *is an injective group morphism for* $i = 1, 2$, *then the amalgamated product* $\Gamma_1 *_A \Gamma_2$ *with respect to* j_1, j_2 *also has property* (RD).

Proof. This is part of (Jolissaint, 1990, Theorem 2.2.2). □

We will refer Theorem III.5.2 as Jolissaint's theorem hereafter.

Example III.5.3. Take $\Gamma = \mathrm{PSL}_2(\mathbb{Z}) \simeq (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/3\mathbb{Z})$, with the isomorphism determined by identifying $\mathbb{Z}/2\mathbb{Z}$ with the cyclic group generated by $s \in \Gamma$, and $\mathbb{Z}/3\mathbb{Z}$ with the cyclic group generated by $t \in \Gamma$, where

$$s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad t = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

(see e.g. (Brown and Ozawa, 2008, Example E.10 on page 476) for a discussion of this amalgamated product decomposition of $\mathrm{SL}_2(\mathbb{Z})$). Let G be any compact connected real Lie group that admits an element $x \in G$ of order 2, and an element y of order 3, such that $\{x, y\} \not\subseteq Z(G)$ (e.g. $G = \mathrm{SO}(3, \mathbb{R})$, x is any rotation by π, y any rotation by $2\pi/3$), where $Z(G)$ is the center of G. Now the mapping $s \mapsto \mathrm{Ad}_x$, $t \mapsto \mathrm{Ad}_y$ determines a unique group morphism

$$\tau : \Gamma \to \mathrm{Inn}(G) \subseteq \mathrm{Aut}(G)$$

so $\widetilde{\tau} : \Gamma \to \mathrm{Out}(G)$ is trivial (hence of finite image). Put $\Lambda < \Gamma$ to be $< s >$ or $< t >$. Since $\Lambda \not\subseteq Z(\Gamma)$, it follows from the choice of x and y that the resulted bicrossed product $\mathbb{G} := \Gamma_{\alpha^\Lambda} \bowtie_{\beta^\Lambda} (G \rtimes_\tau \Lambda)$ is nontrivial (Proposition III.1.1).

By Jolissaint's theorem, $\mathrm{PSL}_2(Z)$ has (RD), but $\mathrm{PSL}_2(\mathbb{Z})$ does not have polynomial growth since it is not virtually nilpotent (Gromov's theorem, see (Gromov, 1981)), and (Vergnioux, 2007) showed that $\widehat{G}$ has polynomial growth, thus Theorem III.5.1 applies and we see that $\widehat{\mathbb{G}}$ has (RD) but not polynomial growth.

Example III.5.4. Let G be any compact group with $\widehat{G}$ having polynomial growth (e.g. all connected compact real Lie group), and Λ a finite subgroup of $\mathrm{Aut}(G)$. Take Γ to be a nontrivial semidirect product of the free group $\mathbb{F}_2$ on two generators (here $\mathbb{F}_2$ can be replaced by any discrete group with (RD) but without polynomial growth) with Λ (in particular, Λ is nontrivial). Then the obvious action of Λ on G and the canonical projection $\mathbb{F}_2 \rtimes \Lambda \to \Lambda$ together yield a nontrivial left action τ of Γ on G by topological automorphisms. The same reasoning as in the above Example shows that $\Gamma \bowtie (G \rtimes \Lambda)$ is also a bicrossed product whose dual has (RD) but not polynomial growth.

Many more examples can be constructed in the same spirit as in the above examples, showing that Theorem III.5.1 is an applicable procedure to produce bicrossed products whose dual has (RD) but not polynomial growth.

III.6 Examples of bicrossed products with rapid decay but not polynomial growth–part II

Despite of the fact that Theorem III.5.1 yields many interesting concrete examples of bicrossed products with property (RD) as shown in § III.5, it is worth pointing out that the restriction the finiteness of the image $\mathrm{Image}(\widetilde{\tau})$ is too strong to include many interesting examples, which we will now show in this section. To make the contrast even more dramatic, we show how to construct examples of nontrivial bicrossed product of the form $\Gamma \bowtie (G \rtimes \Lambda)$ whose dual has (RD) but *not* polynomial growth, while $\mathrm{Image}(\widetilde{\tau})$ as in Theorem III.5.1 is infinite (hence Theorem III.5.1 no longer applies).

We begin with a simple result in finite group theory.

Lemma III.6.1. *If A is a finite abelian group, then there exists infinitely many finite abelian group B, such that A is isomorphic to a subgroup of $\mathrm{Aut}(B)$.*

Proof. Since A is a direct sum of finite cyclic groups, without loss of generality, we may assume A is cyclic of order n, with a as a generator. Pose B to be the n-fold direct sum of any nontrivial finite abelian group C, and define $\sigma(a) \in \mathrm{Aut}(B)$ to be the permutation

$$(c_1, \ldots, c_n) \mapsto (c_2, \ldots, c_n, c_1).$$

Then it is clear that

$$\begin{aligned} \sigma : A &\longrightarrow \mathrm{Aut}(B) \\ a^m &\mapsto [\sigma(a)]^m \end{aligned}$$

is a well-defined injective group morphism. □

As we will see later, Theorem III.5.1 no longer applies for the examples constructed in this section due to the violation of the hypothesis of the finiteness of $\mathrm{Image}(\widetilde{\tau})$. This we will have to resort to Theorem III.3.2 to prove the rapid decay of the dual of the bicrossed product $\Gamma \bowtie (G \rtimes \Lambda)$. Here, the β^Λ-invariance of the length function on Γ poses no problem thanks to Corollary III.4.4. But the Γ-invariance of the length function on $\widehat{G}$ requires a little more work.

Lemma III.6.2. *Suppose $\Xi_1, \Xi_2, \ldots$ is a sequence of finite discrete (hence compact) groups. The product group $\prod_{i=1}^\infty \mathrm{Aut}(\Xi_i)$ naturally acts pointwise on the direct sum $\oplus_{i=1}^\infty \Xi_i$, hence we have a canonical inclusion $\prod_{i=1}^\infty \mathrm{Aut}(\Xi_i) \subseteq \mathrm{Aut}(\oplus_{i=1}^\infty \Xi_i)$. With these settings, there exists a $\prod_{i=1}^\infty \mathrm{Aut}(\Xi_i)$-invariant length function l on the discrete group $\oplus_{i=1}^\infty \Xi_i$, such that the pair $\left(\oplus_{i=1}^\infty \Xi_i\, ,\, l\right)$ has polynomial growth.*

Proof. Let $N_i = |\Xi_i|$ for all $i \in \mathbb{N}_{>0}$ and pose $M_k = \prod_{i=1}^k N_i$ for all $k \in \mathbb{N}$ (we make the convention that $M_0 = 1$). Let e_i be the identity of the group Ξ_i, and denote the characteristic function of $\Xi_i \setminus \{e_i\}$ by χ_i. Define

$$\begin{aligned} l : \oplus_{i=1}^\infty \Xi_i &\longrightarrow \mathbb{R}_{\geq 0} \\ (\xi_i) &\mapsto \sum_{i=1}^\infty \chi_i(\xi_i) M_i. \end{aligned}$$

Then it is clear that l is a $\prod_{i=1}^{\infty}\mathrm{Aut}(G_i)$-invariant length function on $\oplus_{i=1}^{\infty}\Xi_i$. Moreover, for all $n \in \mathbb{N}_{>0}$, there exists a unique $k \geq 1$, such that $M_{k-1} \leq n < M_k$. Then, by the definition of l, we have

$$\left\{\xi = (\xi_i) \in \oplus_{i=1}^{\infty}\Xi_i \ : \ l(\xi) < n\right\} \subseteq \left\{(\xi_i) \in \oplus_{i=1}^{\infty}\Xi_i \ : \ \forall i \geq k,\ \xi_i = e_i\right\}.$$

Thus

$$\left|\left\{\xi = (\xi_i) \in \oplus_{i=1}^{\infty}\Xi_i \ : \ l(\xi) < n\right\}\right| \leq \prod_{i=1}^{k-1} N_i = M_{k-1} \leq n.$$

In particular, $\left(\oplus_{i=1}^{\infty}\Xi_i\,,\ l\right)$ has polynomial growth. □

We are now prepared to give the construction of new examples of bicrossed product of the form $\Gamma \bowtie (G \rtimes \Lambda)$ that don't fit into the framework of Theorem III.5.1.

Example III.6.3. Let Λ be any nontrivial finite abelian group. By Lemma III.6.1, one can take a sequence of finite abelian groups $(G_i)_{i=1}^{\infty}$, such that Λ is isomorphic to a subgroup of $\mathrm{Aut}(G_i)$ for each $i = 1, 2, \ldots$ via an injective group morphism $j_i : \Lambda \hookrightarrow \mathrm{Aut}(G_i)$. Equip each G_i with the discrete topology, and $G := \prod_{i=1}^{\infty} G_i$ the product topology. Then G is a compact abelian group. In particular, the character group $\chi(G)$ of G is a complete set of representatives of $\mathrm{Irr}(G)$. By Pontryagin's duality, we have $\chi(G) \simeq \oplus_{i=1}^{\infty}\chi(G_i)$, and it is clear that length functions on $\widehat{G}$ become exactly length functions on the discrete group $\chi(G)$ of continuous characters of G. But as finite abelian groups, each G_i is isomorphic to $\chi(G_i)$ (albeit the isomorphism is not natural in the categorical sense). Thus Lemma III.6.2 shows that there exists a $\prod_{i=1}^{\infty}\mathrm{Aut}(G_i)$-invariant length function l_G on $\widehat{G}$, such that $(G\,,\ l_G)$ has polynomial growth, where we've used the canonical inclusion $\prod_{i=1}^{\infty}\mathrm{Aut}(G_i) \subseteq \mathrm{Aut}(G)$.

The construction of Γ takes some more work which we now explain. First we take Λ' to be any nontrivial finite group and pose Γ_1 to be the free product $\Lambda * \Lambda'$. It follows from Jolissaint's theorem and Gromov's theorem that Γ_1 has (RD) but not polynomial growth. Define $j : \Lambda \hookrightarrow \prod_{i=1}^{\infty}\mathrm{Aut}(G_i)$ to be the mapping $\lambda \mapsto \left(j_1(\lambda), j_2(\lambda), \cdots\right)$. Take any infinite discrete subgroup Γ_2' of $\oplus_{i=1}^{\infty}\mathrm{Aut}(G_i) \subseteq \prod_{i=1}^{\infty}\mathrm{Aut}(G_i)$ such that $j(\Lambda)$ is contained in the normalizer of Γ_2' in $\prod_{i=1}^{\infty}\mathrm{Aut}(G_i)$. Obviously $j(\Lambda)$ and Γ_2' intersect trivially, thus the subgroup of $\prod_{i=1}^{\infty}\mathrm{Aut}(G_i)$ generated by $j(\Lambda)$ and Γ_2' is the (internal) semidirect product of Γ_2' with $j(\Lambda)$, which we denote by Γ_2. Since $\oplus_{i=1}^{\infty}\mathrm{Aut}(G_i)$ has polynomial growth by Lemma III.6.2, it follows that Γ_2', hence Γ_2 (note that $[\Gamma_2 : \Gamma_2'] = |\Lambda|$ is finite) has polynomial growth. In particular, Γ_2 has (RD), and $j : \Lambda \hookrightarrow \prod_{i=1}^{\infty}\mathrm{Aut}(G_i)$ restricts an injective group morphism, which we still denote by j, from Λ into Γ_2. To facilitate our discussion, we identify Λ with its copy in $\Gamma_1 = \Lambda * \Lambda'$ and in Γ via j. This allows us to form the amalgamated product of Γ_1 and Γ_2 over A, which we denote by Γ. Jolissaint's theorem applies again and proves that Γ has (RD). Moreover, Γ does not have polynomial growth since its subgroup Γ_1 does not. We also make the obvious identification of Λ with $j(\Lambda)$ in Γ. By Corollary III.4.4, there exists a Λ-invariant length function l_Γ on Γ, meaning $l_\Gamma = l_\Gamma \circ \mathrm{Ad}_r$ for all $r \in \Lambda$, such that $(\Gamma\,,\ l_\Gamma)$ has (RD).

Finally, let's explain how the action, which is a group morphism $\tau : \Gamma \longrightarrow \mathrm{Aut}(G)$, is defined. The trivial group morphism $\Lambda' \longrightarrow \mathrm{Aut}(G)$, together with $j : \Lambda \longrightarrow \prod_{i=1}^{\infty}\mathrm{Aut}(G_i) \subseteq \mathrm{Aut}(G)$ and the universal property of free products, yields a group morphism $\tau_1 : \Gamma_1 \longrightarrow \mathrm{Aut}(G)$. Let τ_2 be the simple inclusion $\Gamma_2 \hookrightarrow \prod_{i=1}^{\infty}\mathrm{Aut}(G_i) \subseteq \mathrm{Aut}(G)$. It is clear that τ_1 and τ_2 agree on Λ, thus the universal property of $\Gamma_1 *_\Lambda \Gamma_2$ applies and determines a unique group morphism $\tau : \Gamma \longrightarrow \mathrm{Aut}(G)$. We can finally construct the bicrossed product $\Gamma \bowtie (G \rtimes \Lambda)$, and we conclude by Theorem III.3.2 that the dual of $\Gamma \bowtie (G \rtimes \Lambda)$ has (RD) (it does not have polynomial growth because of Theorem I.7.3 and the fact that Γ does not have polynomial growth).

It is clear by our construction that $\mathrm{Image}(\tau) = \Gamma_2 \subseteq \mathrm{Aut}(G) = \mathrm{Out}(G)$ is infinite, thus Theorem III.5.1 does not apply.

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