Triangular plaquettes

Ivailo Hartarsky
joint with Laurent Bartholdi and Ivan Mitrofanov

Colloquium Mathematical Physics
Bielefeld, 26 June 2026

Model definition 1: a light game

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Magic square game released in 1978
'best selling toy and game item in America in 1980'

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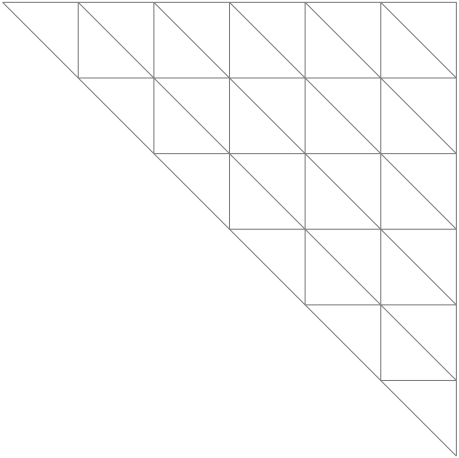

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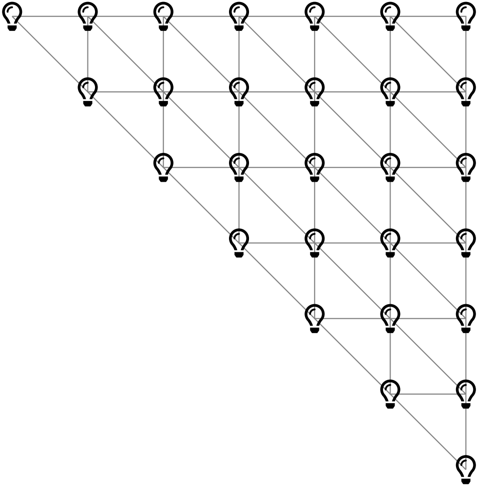

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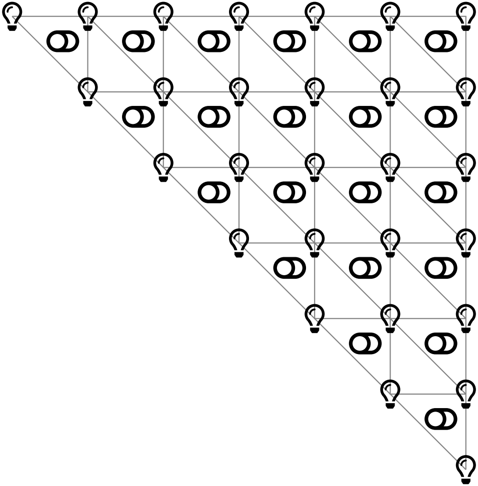

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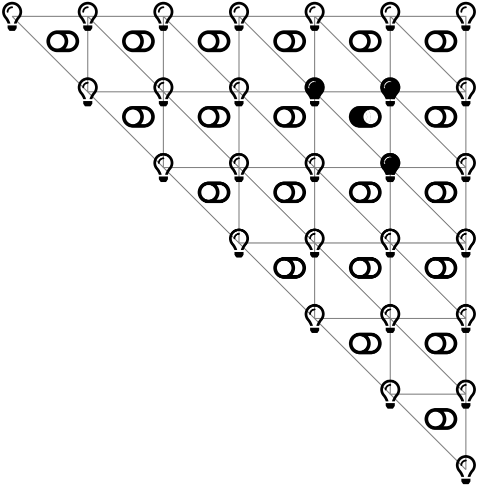

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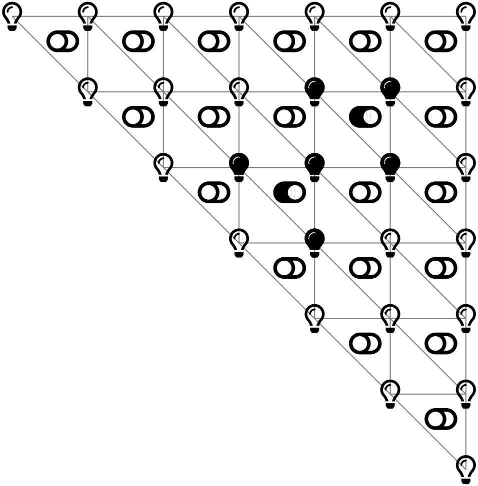

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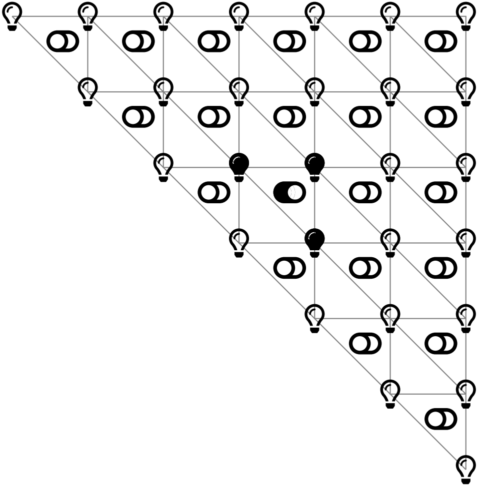

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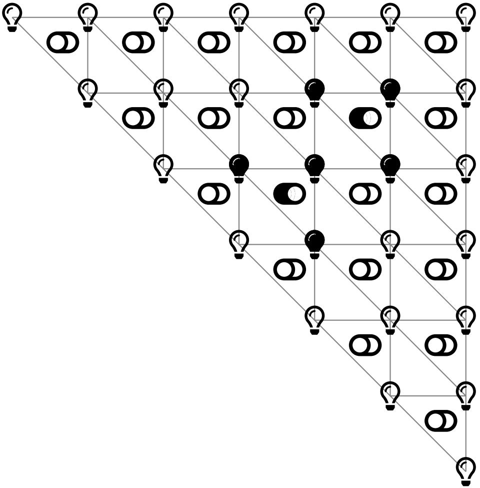

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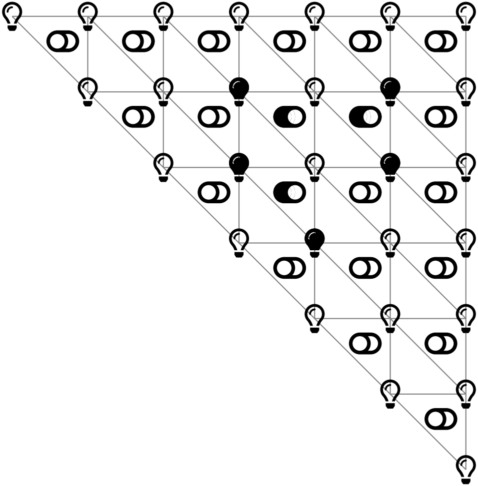

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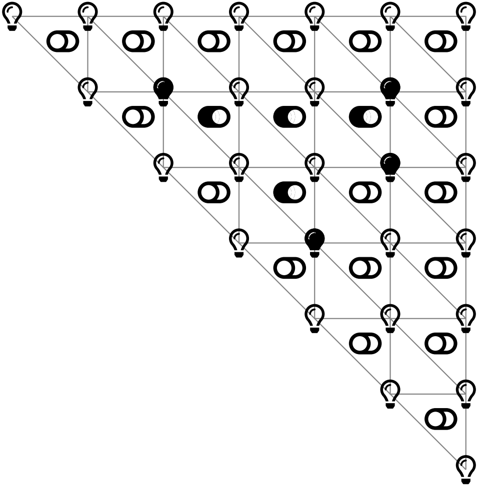

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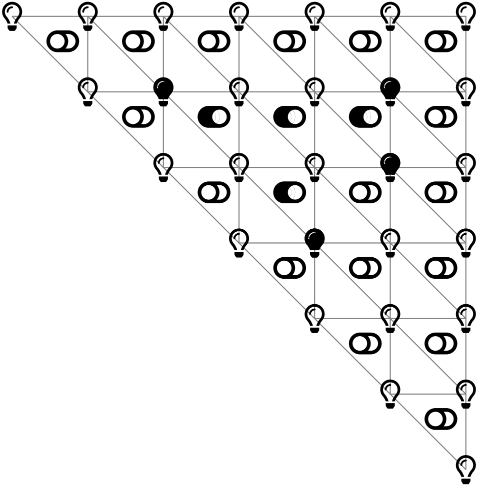

Energy consumption: 1W/💡.

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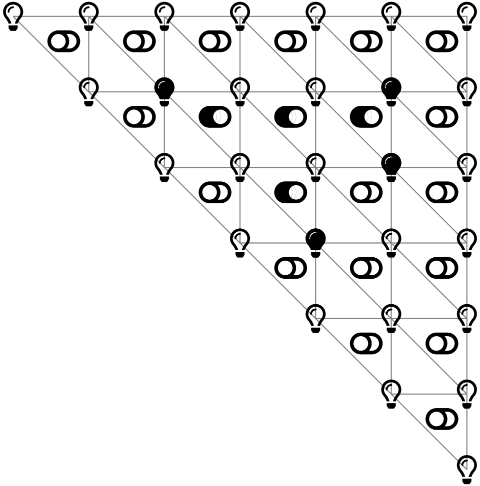

Energy consumption: 1W/ . Glauber dynamics on $\{\ ,\ \}^{\mathbb{Z}^2}$.

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[3] Rate of correlation decay.

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In even dimension, $-1^{\mathbb{Z}^d}$ and $1^{\mathbb{Z}^d}$ have highest and lowest energy. In odd dimension, they both have lowest energy.
In terms of lamps, we have the product Rademacher measure with parameter

$$p = \frac{1}{1 + e^{2\beta}} \approx e^{-2\beta},$$

but a $(d+1)$-site dynamics.

Model definition 3: group and ergodic theories

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Groundstates (switch configurations with no 💡) are known as the Ledrappier subshift from 1978.

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Theorem (Bartholdi, H., Mitrofanov'26+)

In dimension $d = 2$,

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Conjectured in [Newmann, Moore'99].

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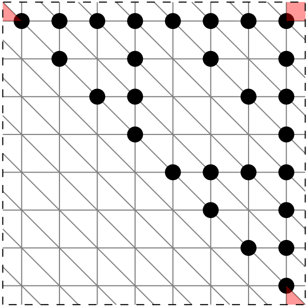

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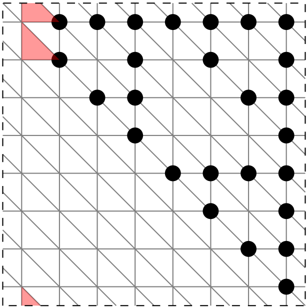

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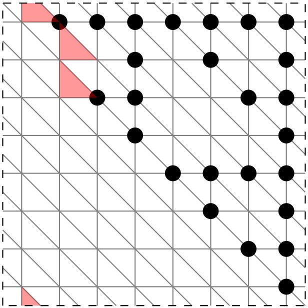

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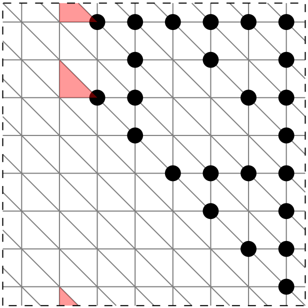

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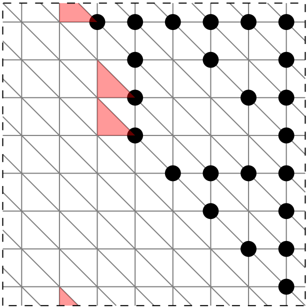

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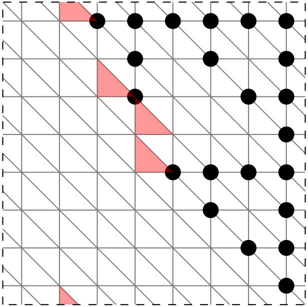

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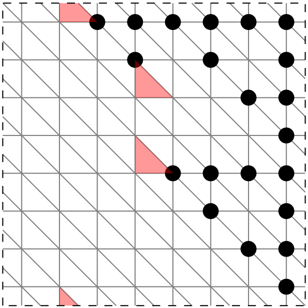

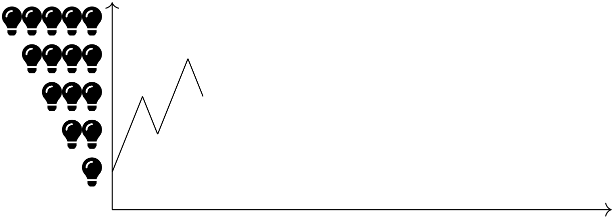

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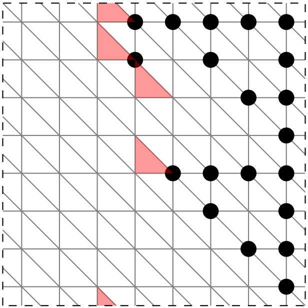

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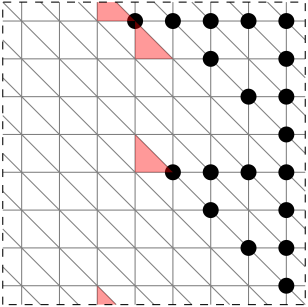

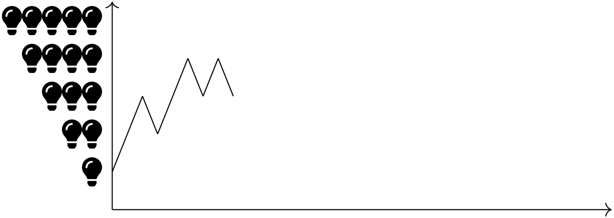

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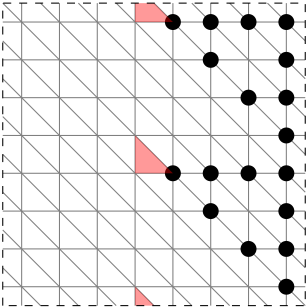

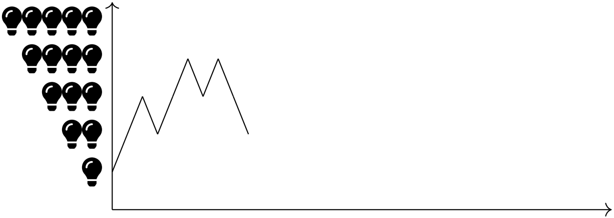

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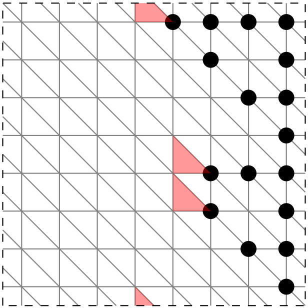

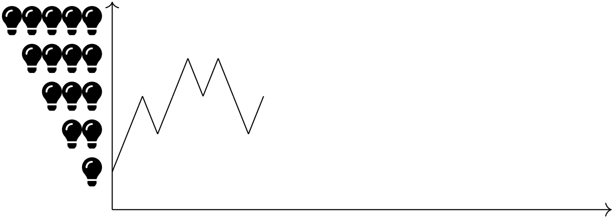

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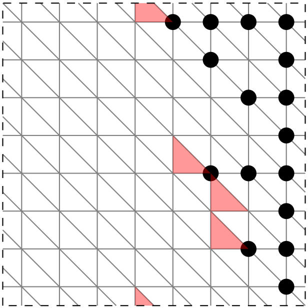

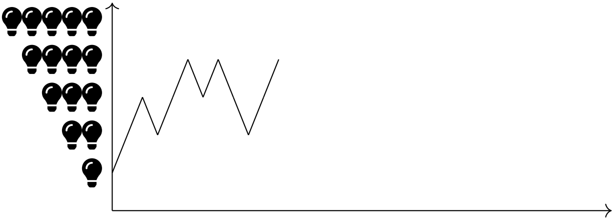

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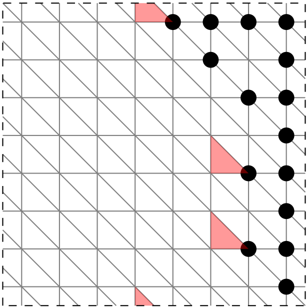

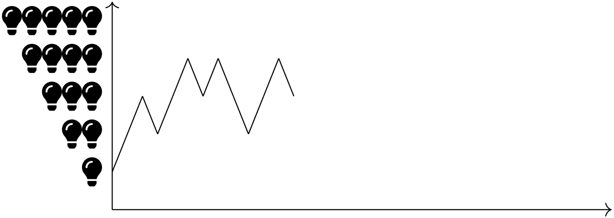

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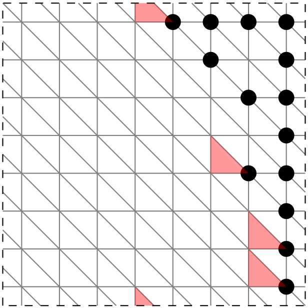

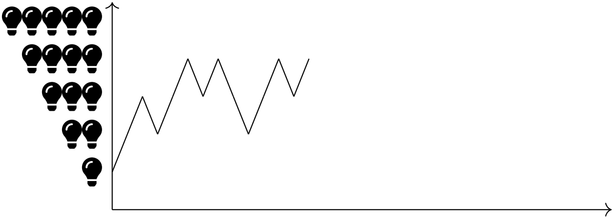

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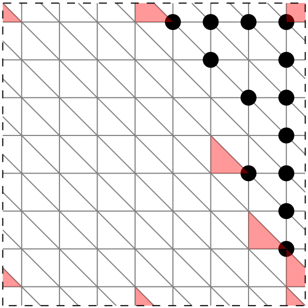

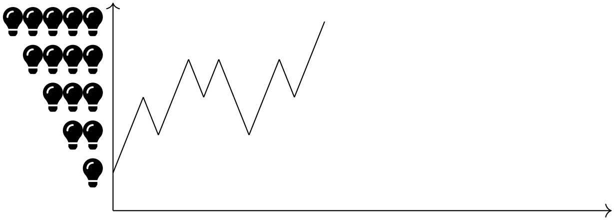

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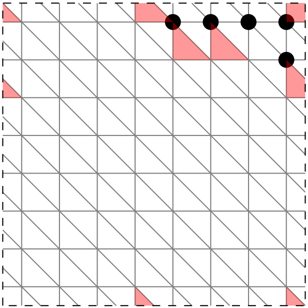

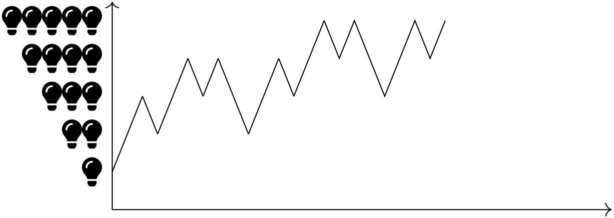

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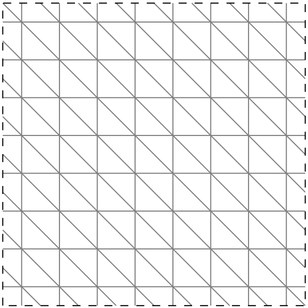

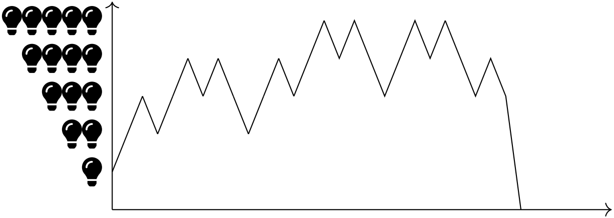

Canonical paths

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Bisection

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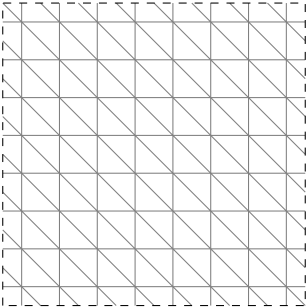

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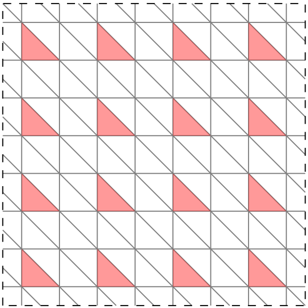

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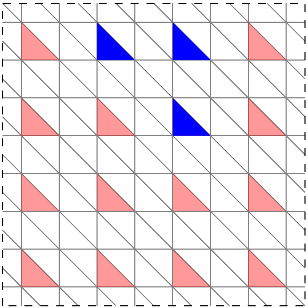

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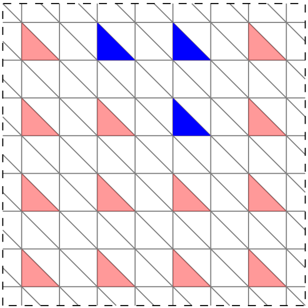

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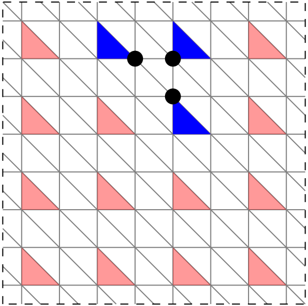

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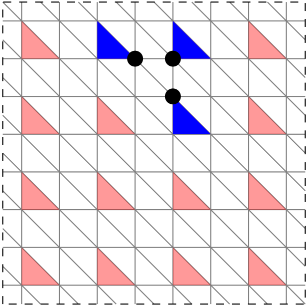

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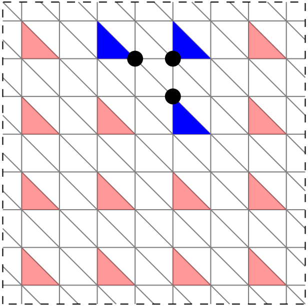

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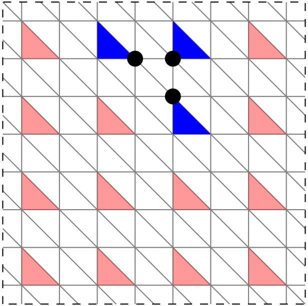

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Combinatorial bottleneck

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Proposition (Bartholdi, H., Mitrofanov'26+)

Let $k \in \mathbb{N}$, and set $B_k = B_\infty(O, 6 \cdot 2^k)$. Consider a sequence of configurations $W_0, \ldots, W_n$ (sets of  *differing by one*  */* *). If $W_0 = \{0\}$ and $W_n \cap B_k = \varnothing$, then for some index i we have*

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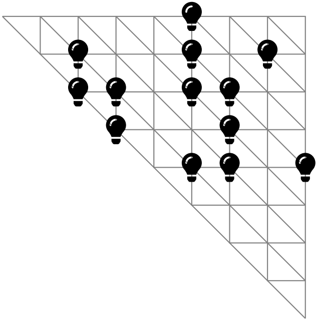

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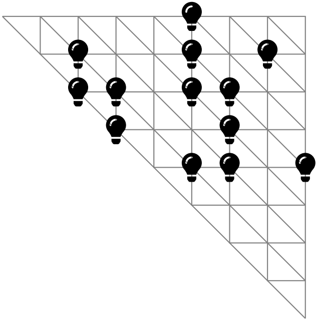

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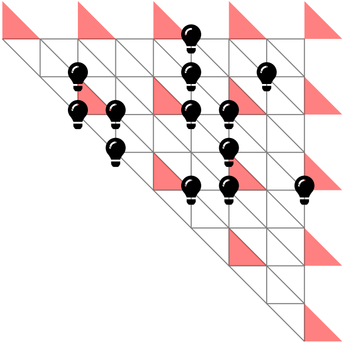

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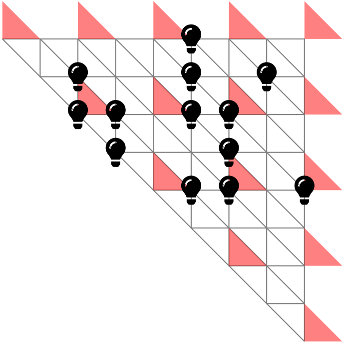

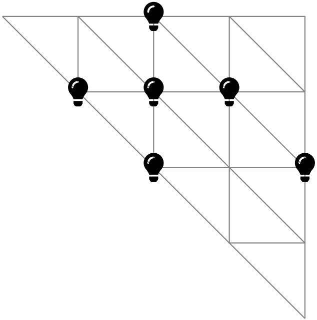

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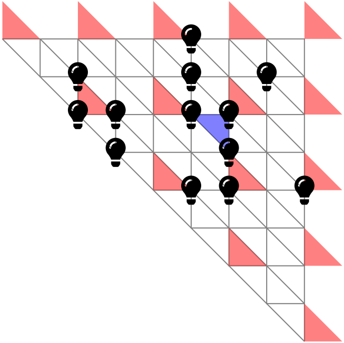

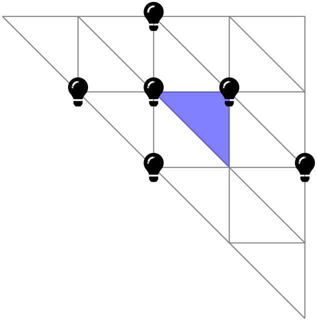

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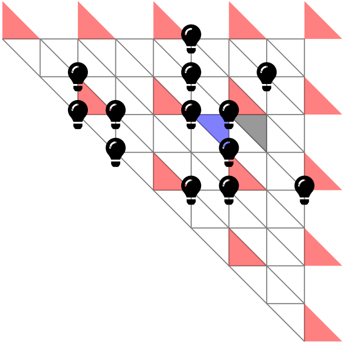

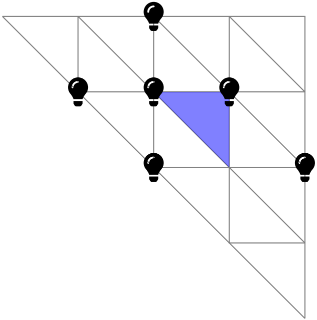

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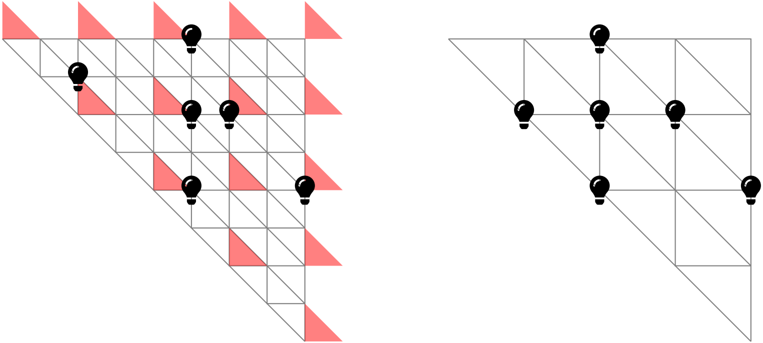

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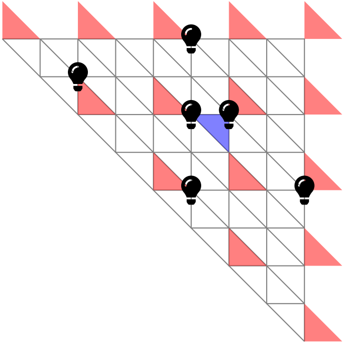

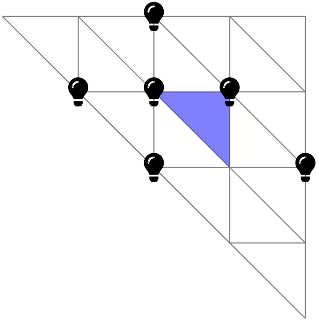

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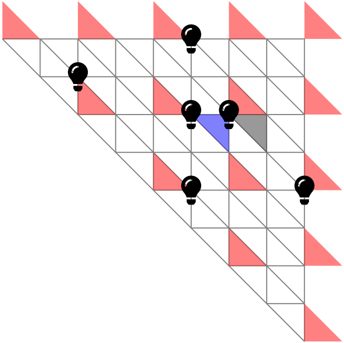

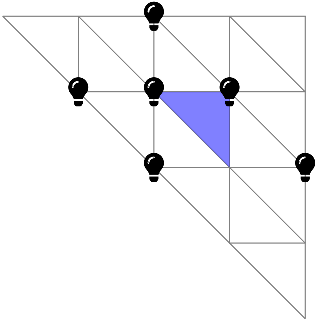

Infinite volume

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For any $d \geqslant 2$ there exists $C = C(d) \geqslant 1$ such that, for all inverse temperature $\beta > 0$, we have

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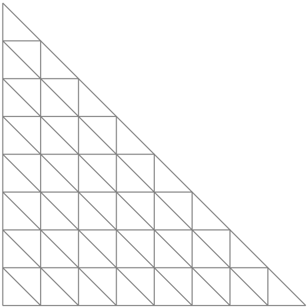
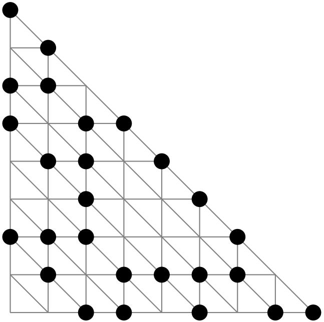
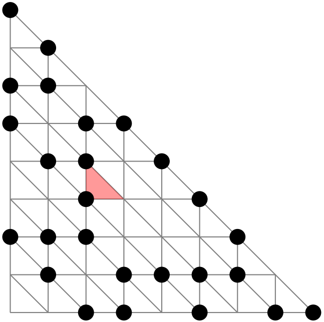
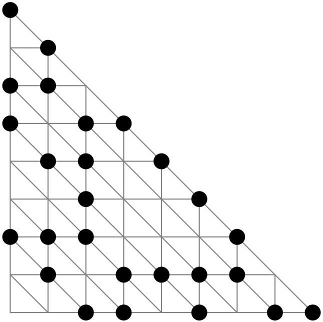
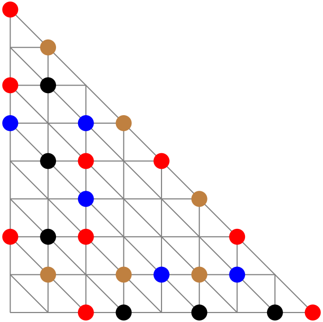
If $t < 1$ and n is suitably large, then $G_{d,n}(t) \approx 1$.

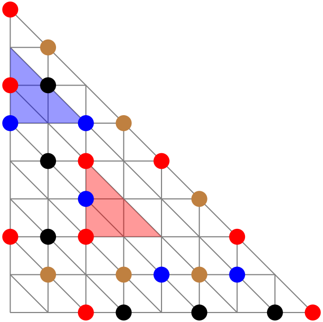


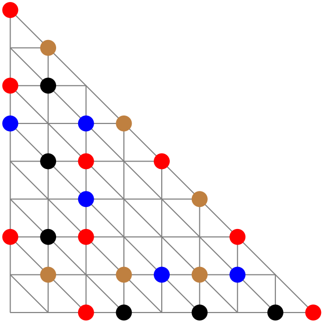


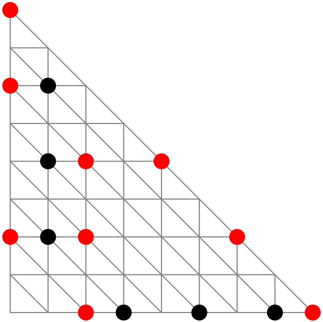


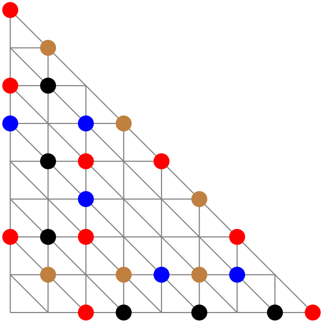


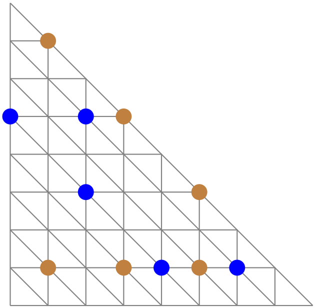


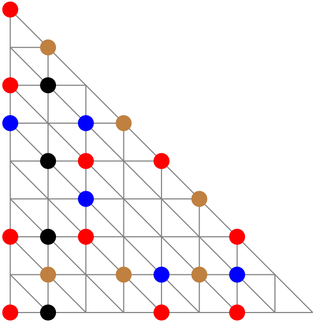


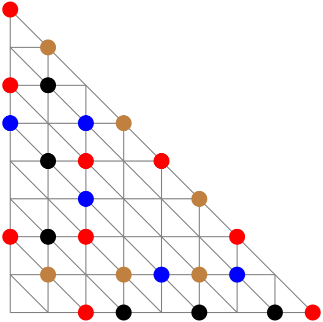


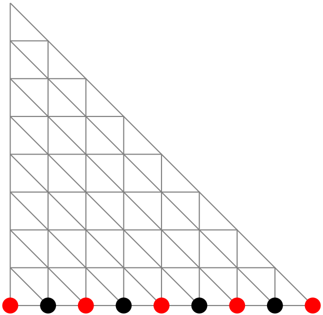












Finite volume

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Theorem (Bartholdi, H., Mitrofanov'26+)

For any $d \geqslant 2$ there exists $C = C(d) \geqslant 1$ such that the following holds. If $n \leqslant e^{\beta/C}$, then

$$e^{\beta n^{1/C}}/C \leqslant T_{\mathrm{rel}}^{\Lambda_{d,n},\mathbf{1}} \leqslant Ce^{Cn^d(\beta+1)}.$$

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size	torus	simplex
$e^{\beta/C}$	e^{β^2}	$e^{e^{\beta}}$
$e^{\beta\cdot C}$	e^{β^2}	?

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The strong mixing length scale is $e^{2\beta}$.

Question

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Is T_{rel} of order e^{β^2} or e^{e^β}?