Coalescing and branching simple exclusion and Fredrickson-Andersen models\textsuperscript{1}

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2 June 2020

Analysis-Probability seminar CEREMADE, Paris

\textsuperscript{1}Supported by ERC Starting Grant 680275 MALIG
Coalescing Random Walks with Neighbour Births

\[ G = (V, E) \] is a connected graph.
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**CRWNB representation**

Random walk jumping along each edge at rate 1.
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*Independent* random walks jumping along each edge at rate 1.
# Coalescing Random Walks with Neighbour Births

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\[ G = (V, E) \] is a connected graph.

**CRWNB representation**

Coalescing independent random walks jumping along each edge at rate 1 and giving birth to a particle at each neighbour independently at rate \( \beta \).
History
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- Biased voter model
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- Williams-Bjerknes tumour growth model [WB’72]
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- On $\mathbb{Z}^d$ for $\beta > 0$ – limit shape, cutoff [Bramson,Griffeath’80,81; Durrett,Griffeath’82]
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- On $\mathbb{Z}^d$ for $\beta > 0$ – limit shape, cutoff [Bramson,Griffeath’80,81; Durrett,Griffeath’82]
- On $\mathbb{Z}$ for $\beta \to 0$ Brownian net [Sun,Swart’08]
Coalescing and Branching 
Simple Exclusion Process

$G = (V, E), \Omega = \{0, 1\}^V, 0 < p < 1, \pi = \text{Ber}(p)^\otimes V$
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\text{CBSEP representation}

Each edge \( e \) containing a particle resamples at rate 1 from \( \pi_e \) conditioned to still contain a particle.
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**CBSEP representation**

Each edge \( e \) containing a particle resamples at rate 1 from \( \pi_e \)
conditioned to still contain a particle. In other words along \( e \):

- (SEP) a particle swaps with a hole with rate \( (1 - p)/(2 - p) \);
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- (SEP) a particle swaps with a hole with rate \( (1 - p)/(2 - p) \);
- (B) a particle fills the adjacent hole with rate \( p/(2 - p) \);
- (C) two particles coalesce at uniformly chosen of the two positions at rate \( 2(1 - p)/(2 - p) \).
What is so nice about CBSEP?

\[ \mu := \pi \left( \cdot | \Omega^+ \right) \]

is reversible, where \( \Omega^+ = \{ \text{at least one particle} \} \).

CBSEP is the same as CRWNB with \( \beta = \frac{p}{1-p} \) slowed down by a factor \( \frac{1-p}{2-p} \).

Nice dual model (in two distinct ways).

Lots of embedded random walks (even more than those in the CRWNB representation).
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- Nice dual model (in two distinct ways).
- Lots of embedded random walks (even more than those in the CRWNB representation).
Mixing times

Let $h^t_\omega(\cdot) = P^t_\omega(\cdot)/\mu(\cdot)$ be the density of the law of CBSEP started at $\omega$ w.r.t. the reversible measure $\mu$. 
Mixing times

\[ h^t_\omega(\cdot) = p^t_\omega(\cdot)/\mu(\cdot) \]

Let \[ \|f\|_q = \left( \int f^q \, d\mu \right)^{1/q} = (\mu(f^q))^{1/q} \]
for \( q \in [1, \infty] \).
Mixing times

\[ h^t_\omega(\cdot) = P^t_\omega(\cdot)/\mu(\cdot) \]
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\[ \|h^t_\omega - 1\|_1 = 2d_{TV}(P^t_\omega, \mu) \]
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\[ T_q = \inf \{ t > 0, \max_\omega \| h_t^\omega - 1 \|_q \leq 1/e \} \]
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\[ \forall q \in [1, \infty], \quad T_q \leq O\left(\log \log \frac{1}{\mu_*}\right) T_{Sob}, \]

\[ \mu_* = \min_\omega \mu(\omega); \quad T_{Sob} \text{ is ‘the inverse rate of decay of entropy’} \]
Commuting and meeting

- The commute time $T_{\text{com}}^{x,y}$ of a RW between $x, y \in V$ is $\mathbb{E}_x[\tau_y] + \mathbb{E}_y[\tau_x]$. 
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- $T_{\text{meet}}^{x,y}$ is the expected meeting time of $x$ and $y$. 
Commuting and meeting

- The commute time $T_{\text{com}}^{x,y}$ of a RW between $x, y \in V$ is $E_x[\tau_y] + E_y[\tau_x]$.
- It's also $2|V|R_{x,y}$, where $R_{x,y}$ is the resistance between $x, y$.
- $T_{\text{meet}}^{x,y}$ is the expected meeting time of $x$ and $y$.
- In all examples we will encounter (and many others) we have

$$T_{\text{meet}} := \frac{1}{|V|^2} \sum_{x,y} T_{\text{meet}}^{x,y} \preceq \frac{1}{|V|^2} \sum_{x,y} T_{\text{com}}^{x,y} \preceq \max_{x,y} T_{\text{meet}}^{x,y} \preceq \max_{x,y} T_{\text{com}}^{x,y} =: T_{\text{com}}$$

and these are known up to a constant factor (or better).
Setting

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T_{\text{com}} \approx n \times \begin{cases} 
n & d = 1 \\
\log n & d = 2 \\
1 & d \geq 3
\end{cases}
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\[ T_{\text{com}} \simeq n \times \begin{cases} n & d = 1 \\ \log n & d = 2 \\ 1 & d \geq 3 \end{cases} \]

- uniform random regular graph $G(n, d)$. 
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- complete binary tree.
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\[
T_{\text{com}} \propto n \times \begin{cases} 
  n & d = 1 \\
  \log n & d = 2 \\
  1 & d \geq 3
\end{cases}
\]

- uniform random regular graph \( G(n, d) \). \( T_{\text{com}} \propto n \)
- complete binary tree. \( T_{\text{com}} \propto n \log n \)
- hypercube of dimension \( \log_2 n \).
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- complete binary tree. $T_{com} \asymp n \log n$
- hypercube of dimension $\log_2 n$. $T_{com} \asymp n / \log n$
Theorem (Martinelli, Toninelli, H.’20)

Let $p_n = \Theta(1/n)$ and $G_n = (V_n, E_n)$ be a sequence of ‘nice’ graphs with $|V_n| = n$. Then

$$\Omega(T_{\text{meet}}) \leq T_{\text{CBSEP}} \leq O(T_{\text{com}} \log n).$$

\(^a\)E.g. with bounded degree or rapidly mixing with degree at most $n^{1/5}$. This is only needed for the upper bound.
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Corollary

If \( G_n \) is the \( d \)-dimensional torus, then

\[
\Omega(n^2) \leq T_{\text{Sob}}^{\text{CBSEP}} \leq O(n^2 \log n) \quad d = 1
\]

\[
\Omega(n \log n) \leq T_{\text{Sob}}^{\text{CBSEP}} \leq O(n \log^2 n) \quad d = 2
\]

\[
\Omega(n) \leq T_{\text{Sob}}^{\text{CBSEP}} \leq O(n \log n) \quad d \geq 3
\]
FA1f

\[ G = (V, E), \Omega = \{0, 1\}^V, \quad 0 < p < 1, \quad \pi = Ber(p)^\otimes V \]
### Definition (FA1f)

Each vertex $v \in V$ such that there is a neighbouring particle (i.e. $\{u, v\} \in E$ with $\omega_u = 1$) resamples at rate 1 from $\pi_v$. 

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\[ \mu = \pi(.|\Omega_+) \text{ is reversible.} \]
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$-$ Not attractive (and does not have a dual).
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- Not attractive (and does not have a dual).
- No other (known) nice representations.
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+ \( \mu = \pi(\cdot | \Omega_+) \) is reversible.

- Not attractive (and does not have a dual).
- No other (known) nice representations.
- No (known) embedded random walks.
- Not well understood even for \( p = 1/10 \) on \( \mathbb{Z} \).
Observation

A particle can perform a SEP move by creating a second one which kills the initial one.
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\[ D_{\text{CBSEP}} \leq O(d_{\text{max}}/p)D_{\text{FA1f}}. \]
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\[ \mathcal{D}^{\text{CBSEP}} \leq O(d_{\text{max}}/p) \mathcal{D}^{\text{FA1f}}. \]

Definition \((T_{\text{Sob}})\)

\(T_{\text{Sob}}\) is the smallest constant such that

\[ \text{Ent}_\mu(f^2) := \mu(f^2 \log(f^2/\mu(f^2))) \leq T_{\text{Sob}} \mathcal{D}(f). \]
Observation

A particle can perform a SEP move by creating a second one which kills the initial one. In terms of Dirichlet forms this reads

\[ D_{CBSEP} \leq O(d_{max}/p)D_{FA1f}. \]

Definition \( (T_{Sob}) \)

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\[ \text{Ent}_\mu(f^2) := \mu(f^2 \log(f^2/\mu(f^2))) \leq T_{Sob}D(f). \]

Corollary

\[ T_{Sob}^{FA1f} \leq O(d_{max}/p)T_{Sob}^{CBSEP} \]
Corollary

With $p = \Theta(1/n)$ on the torus of dimension $d$, for all $q \geq 1$

$$T^*_q \leq O(\log n) T^*_{\text{Sob}} \leq O(n \log n) T^*_{\text{CBSEP}} \leq \begin{cases} O(n^3 \log^2 n) & d = 1 \\ O(n^2 \log^3 n) & d = 2 \\ O(n^2 \log^2 n) & d \geq 3 \end{cases}$$
**Corollary**

*With $p = \Theta(1/n)$ on the torus of dimension $d$, for all $q \geq 1$*

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T_q^{FA} \leq O(\log n) T_{Sob}^{FA} \leq O(n \log n) T_{Sob}^{CBSEP} \leq \begin{cases} 
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**Theorem (Pillai,Smith’17; Pillai,Smith’19)**

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\Omega(n^2) \leq T_{mix}^{FA} \leq \begin{cases} 
O(n^2 \log^{14} n) & d = 2 \\
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Corollary

With \( p = \Theta(1/n) \) on the torus of dimension \( d \), for all \( q \geq 1 \)

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T^\text{FA}_q \leq O(\log n) T^\text{FA}_{\text{Sob}} \leq O(n \log n) T^\text{CBSEP}_{\text{Sob}} \leq \begin{cases} 
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- Simpler proof.
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- Stronger mixing notion.
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- Simpler proof.
- Stronger mixing notion.
- General graphs and choices of \( p \).
Definition

\( G = (V, E), \Omega = S^V, S = S_1 \sqcup S_0 \) is finite, \( \rho \) is a product probability measure on \( \Omega \).

We say there is a particle at \( v \in V \) if \( \omega_v \in S_1 \).
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We say there is a particle at \( v \in V \) if \( \omega_v \in S_1 \).

Definition (g-CBSEP)

Each edge \( e \) containing a particle resamples at rate 1 from \( \rho_e \) conditioned to still contain a particle.
Definition

\[ G = (V, E), \quad \Omega = S^V, \quad S = S_1 \sqcup S_0 \text{ is finite}, \quad \rho \text{ is a product probability measure on } \Omega. \]
We say there is a particle at \( v \in V \) if \( \omega_v \in S_1 \).

Definition (g-CBSEP)

Each edge \( e \) containing a particle resamples at rate 1 from \( \rho_e \) conditioned to still contain a particle.

Remark

The projection on \( \{0, 1\}^V \) of g-CBSEP is CBSEP with \( p = \rho(S_1) \).
Theorem (Martinelli, Toninelli, H.’20)

\[ T_{\text{mix}}^{\text{CBSEP}} \leq T_{\text{mix}}^{g-CBSEP} \leq O(T_{\text{mix}}^{\text{CBSEP}} + T_{\text{cov}}^{rw}). \]
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Proof idea.
Once CBSEP couples, wait for one of the random walks to cover \( G \).
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**Remark**

Not true for \( T_{\text{mix}}^{g-\text{CBSEP}} \).
### Theorem (Martinelli, Toninelli, H.’20)

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### Remark

It is known that \( T_{\text{com}} \leq T_{\text{cov}}^{\text{rw}} \leq O(T_{\text{com}} \log |V|) \), so on ‘nice’ graphs with \( p_n = \Theta(1/n) \) we get

\[
\Omega(T_{\text{meet}}) \leq T_{\text{mix}}^{g-\text{CBSEP}} \leq O(T_{\text{com}} \log^2 |V|).
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**Theorem (Martinelli, Toninelli, H.’20)**

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\[ \Omega(T_{\text{meet}}) \leq T_{\text{mix}}^{g-\text{CBSEP}} \leq O(T_{\text{com}} \log^2 |V|). \]

**Corollary**

On \( \{1, \ldots, L\}^d, d \geq 2 \) with \( p = \Theta(1/L^d) \) we have

\[ T_{\text{mix}}^{g-\text{CBSEP}} = L^d (\log L)^{O(1)}. \]
\[ d \geq j \geq 2, \quad \Omega = \{0, 1\}^{\mathbb{Z}^d}, \quad 0 < p < 1, \quad \pi = \text{Ber}(p) \otimes \mathbb{Z}^d \]
\( \text{FA}_{jf} \)

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**Definition (FA\( jf \))**

Each vertex \( v \in \mathbb{Z}^d \) such that there are at least \( j \) neighbouring particles resamples at rate 1 from \( \pi_v \).
\[ d \geq j \geq 2, \quad \Omega = \{0, 1\}^{\mathbb{Z}_d}, \quad 0 < p < 1, \quad \pi = \text{Ber}(p)^{\otimes \mathbb{Z}_d} \]

**Definition (FA}_j^n f \)**

Each vertex \( v \in \mathbb{Z}_d \) such that there are at least \( j \) neighbouring particles resamples at rate 1 from \( \pi_v \).

**Definition (j-neighbour bootstrap percolation)**

Each vertex \( v \in \mathbb{Z}_d \) such that there are at least \( j \) neighbouring particles becomes filled at rate 1.
Bootstrap percolation

Theorem (Gravner, Holroyd’08 + Morris, H.’19; first term: Holroyd’03)

For $d = j = 2$ w.h.p. the origin becomes filled at time

$$\exp \left( \frac{\pi^2}{18p} - \frac{\Theta(1)}{\sqrt{p}} \right).$$
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$$

Morally: a particle reaches the origin starting from a small anomalously occupied region called droplet, which occurs with probability

$$
q = \exp \left( -\frac{\pi^2}{9p} + \frac{\Theta(1)}{\sqrt{p}} \right)
$$

and invades space at linear speed.
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Theorem (Balogh, Bollobás, Duminil-Copin, Morris’12+Uzzell’19)

For \( d \geq j \geq 2 \) there exists an explicit constant\(^a\) \( \lambda(d, j) > 0 \) such that w.h.p. the filling time \( \tau \) of the origin satisfies

\[
\exp^{j-1} \left( \frac{\lambda(d, j) - o(1)}{p^{1/(d-j+1)}} \right) \leq \tau \leq \exp^{j-1} \left( \frac{\lambda(d, j)}{p^{1/(d-j+1)}} - \frac{\Omega(1)}{p^{1/(2(d-j+1))}} \right).
\]

\(^a\)This notation is not the standard one in bootstrap percolation.
FA2f in 2d

**Conjecture (Toninelli’03)**

\[ T_{\text{rel}} = \exp \left( \frac{\pi^2 + o(1)}{9p} \right) \]
FA2f in 2d

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\[ \exp \left( \frac{\pi^2 - o(1)}{18p} \right) \leq T_{\text{rel}} \leq \exp \left( \frac{O(1)}{p^5} \right). \]
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\[ T_{\text{rel}} = \exp \left( \frac{\pi^2}{9p} + \frac{O(\log(1/q))}{\sqrt{p}} \right). \]
FAjf, $d \geq j \geq 3$

**Theorem (Cancrini, Martinelli, Roberto, Toninelli’08)**

\[
\exp^{j-1} \left( \frac{\lambda(d, j) - o(1)}{p^{1/(d-j+1)}} \right) \leq T_{\text{rel}} \leq \exp^{d-1} \left( \frac{O(1)}{p} \right).
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Generalisations:

- Sharp thresholds for most other models available in bootstrap percolation transfer to KCM.
- The proof extends to all models for which similar results hold (called isotropic). [H.’20+]
- The techniques allow proving tight upper bounds completing universality for critical KCM. [Marêché, H.20+; H.’20+]
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Ivailo Hartarsky
CBSEP and FA
Some questions that are not crazy (any more)

- Close the logarithmic gap for $T_{Sob}$. Also between $T_{Sob}$ and $T_{mix}$.
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Thank you.
Theorem

There exists $c > 0$ s.t. for any $p_n \to 0$

$$T_{Sob} \leq c \max \left( \frac{d_{\text{avg}} d_{\text{max}}}{d_{\text{min}}^2} T_{\text{mix}}^{\text{rw}} \log(n), \left( \max_y \bar{R}_y \right) n|\log(p_n)| \right),$$

where $T_{\text{mix}}^{\text{rw}}$ is the mixing time of the lazy simple random walk on $G$.

[Alon-Kozma’18+Lee-Yau’98]