

# Commented bibliography on two-neighbour bootstrap percolation on the grid

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## Abstract

This document aims to provide an exhaustive bibliography on two-neighbour bootstrap percolation on the square lattice  $\mathbb{Z}^2$ . The scope is so restrictive, because the bootstrap percolation literature is too vast. We seek depth rather than breadth.

This is the best understood bootstrap percolation model, so the list may serve to recap what the typical questions of interest are and what techniques and ideas are available for tackling them. These questions, ideas and techniques may then be adapted to the reader's favourite setting.

In case of omissions or inaccuracies, please contact me.

## 1 Mathematical surveys

- [Mor17, Section 2]
- [Har22, Section 1.4.3]

## 2 Critical scaling

- [van87] Critical probability. First rigorous work on bootstrap percolation.
- [BP98] (posterior to [AL88], but independent), [AL88], [Hol03], [GH08], [GHM12], [HM19], [HT24] Progressively sharper critical scaling results.

## 3 The critical window and inside it

- [BB03] Probability-window.
- [GH08, Theorem 2] Size-window.
- [Har22, Proposition 1.4.3] Time-window.

- [BP15a] Noise sensitivity.
- [HL25] Hitting time result.

## 4 Large deviations

- [Sch92] Exponential decay.
- [AMS95] Large deviation rate.

## 5 Extremal questions

- The fact that infecting an  $n \times m$  rectangle requires  $\lceil (n + m)/2 \rceil$  infections is folklore.
- [Mor09] Size of (inclusion-wise) minimal percolating sets in a square.
- [BP13] Maximum percolation time for initial condition of minimal size in a square.
- [BP15b] Maximum percolation time in a square (any percolating initial condition).

## 6 Non-rigorous works

- [KL81] First work on the model (the name is from [CLR79]).
- [Bdd84, NT86, AA88, ASA89, vAD90, Adl91, AL03, dLBD04, dLBD05, dLD09, TS14, HT25] Results of numerics or simulations and comments on the bootstrap percolation paradox.

## 7 Mathematical works on slight variants of the model

The following works concern slight variants of the two-neighbour model on  $\mathbb{Z}^2$ . These include local two-neighbour, modified two-neighbour, Froböse,  $k$ -cross, triangular lattice three-neighbour bootstrap percolation models, etc. still in two dimensions.

- [Sch90] Critical probability.
- [HLR03, GH09, BM12, CG14, Har23] Critical scaling.
- [And93, Mou92] Large deviation rate.
- [SS91] Enumeration of percolating configurations with minimal size.

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