Homework n° 10

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due for April 5th

All the following questions are taken from past examinations of the prestigious University of Cambridge, UK. So you can't blame your math teacher's poor English if you don't understand them or can't answer to them. Beware, some of the mathematical conventions used in UK are not the same as the ones used in our country, for example 'positive' means « **strictement** positif » and 'non negative' is used for « supérieur ou égal à 0 ». You may try to give tour answers in English, but it's not mandatory (and not recommanded by the teacher).

Question 1.

Find the coefficient of the x^4 term in the expansion of $x^2\left(2x+\dfrac{1}{x}\right)^6$.

Question 2.

$(2x+1)$ and $(x-2)$ are factors of $2x^3+px^2+q$. What is the value of $2p+q$?

Question 3.

A large circular table has 40 chairs round it. What is the smallest number of people who can be sitting at the table already such that the next person to sit down **must** sit next to someone ?

Question 4.

The numbers a, b and c are each greater than 1. The following logarithms are all to the same base : $\log(ab^2c)=7$, $\log(a^2bc^2)=11$ and $\log(a^2b^2c^3)=15$. What is this base (five possible answers : 'a', 'b', 'c', 'none of a, b or c', 'there is insufficient information given to determine the base') ?

Question 5.

Consider the following inequality : $a|x|+1 \leqslant |x-2|$, where a is a real constant. Determine the complete set of values of a such that the inequality is true for all real x.

Question 6.

Find the value of the expression $\sqrt{8-4\sqrt{2}+1}+\sqrt{9-12\sqrt{2}+8}$ (possible answers : $\sqrt{26-16\sqrt{2}}$, $4\sqrt{2}-4$, -2, $4-4\sqrt{2}$, 2, $\sqrt{26}-4\sqrt{2}$, 1).

Question 7.

p is a positive constant. Find the area enclosed between the curves $y=p\sqrt{x}$ and $x=p\sqrt{y}$.

Question 8.

x satisfies the simultaneous equations $\sin 2x + \sqrt{3}\cos 2x = -1$ and $\sqrt{3}\sin 2x?\cos 2x = \sqrt{3}$, where $x \in [0, 2\pi]$. Find the sum of the possible values of x.

Question 9.

Given that $2\int_0^1 f(x)\,dx + 5\int_1^2 f(x)\,dx = 14$ and $\int_0^1 f(x+1)\,dx = 6$, find the value of $\int_0^2 f(x)\,dx$.

Question 10.

Find the shortest distance between the curve $y = x^2 + 4$ and the line $y = 2x - 2$.

Problem 1.

Find the two fixed points of the Möbius transformation $z \mapsto \omega = \dfrac{3z+1}{z+3}$, that is, find the two values of z for which $\omega = z$.

Given that $c \neq 0$ and $(a-d)^2 + 4bc \neq 0$, show that a general Möbius transformation $z \mapsto \omega = \dfrac{az+b}{cz+d}$, $ad - bc \neq 0$, has two fixed points α, β given by $\alpha = \dfrac{a-d+m}{2c}$ and $\beta = \dfrac{a-d-m}{2c}$, where $\pm m$ are the square roots of $(a-d)^2 + 4bc$.

Show that such a transformation can be expressed in the form $\dfrac{\omega-\alpha}{\omega-\beta} = k\dfrac{z-\alpha}{z-\beta}$, where k is a constant that you should determine.

Problem 2.

Solve the initial value problem $\dfrac{dx}{dt} = x(1-x)$, $x(0) = x_0$, and sketch the phase portrait. Describe the behaviour as $t \to +\infty$ and as $t \to -\infty$ of solutions with initial value satisfying $0 < x_0 < 1$.

Problem 3.

Let f be a continuous function from $[0,1]$ to $[0,1]$ such that $f(x) < x$ for every $0 < x < 1$. We write f^n for the n-fold composition of f with itself (so for example $f^2(x) = f(f(x))$).

1. Prove that for every $0 < x < 1$ we have $f^n(x) \to 0$ as $n \to \infty$.
2. Must it be the case that for every $\varepsilon > 0$ there exists n with the property that $f^n(x) < \varepsilon$ for all $0 < x < 1$? Justify your answer.
3. Now suppose that we remove the condition that f be continuous.
   Give an example to show that it need not be the case that for every $0 < x < 1$ we have $f^n(x) \to 0$ as $n \to \infty$.
4. Must it be the case that for **some** $0 < x < 1$ we have $f^n(x) \to 0$ as $n \to \infty$? Justify your answer.