Working effectively with generic subgroups

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work in progress

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Classifying (some) simple finite groups

$\forall n \neq 1$, there exists a sequence

$$n = n_0 \geq n_1 \geq n_2 \geq \ldots \geq \ldots n_{r-1} \geq n_r = 1$$

such that n_i/n_{i+1} is **prime**, and that sequence of primes and their multiplicity are unique for each n.

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(Jordan-Hölder)
Given a finite group G, there exists a sequence

$$G = G_0 \supset G_1 \supset G_2 \supset \ldots \supset G_{r-2} \supset G_{r-1} \supset G_r = \{1\}$$

such that $G_{i+1} \lhd G_i$, G_i/G_{i+1} is **simple**, and that *composition series* is unique up to isomorphism and permutation.

The Feit-Thompson theorem

Finite groups of odd order are solvable.

Abel prize in 2008

Feit, Walter; Thompson, John G. , *Solvability of groups of odd order* ,

Pacific Journal of Mathematics 13: 775-1029 , 1963

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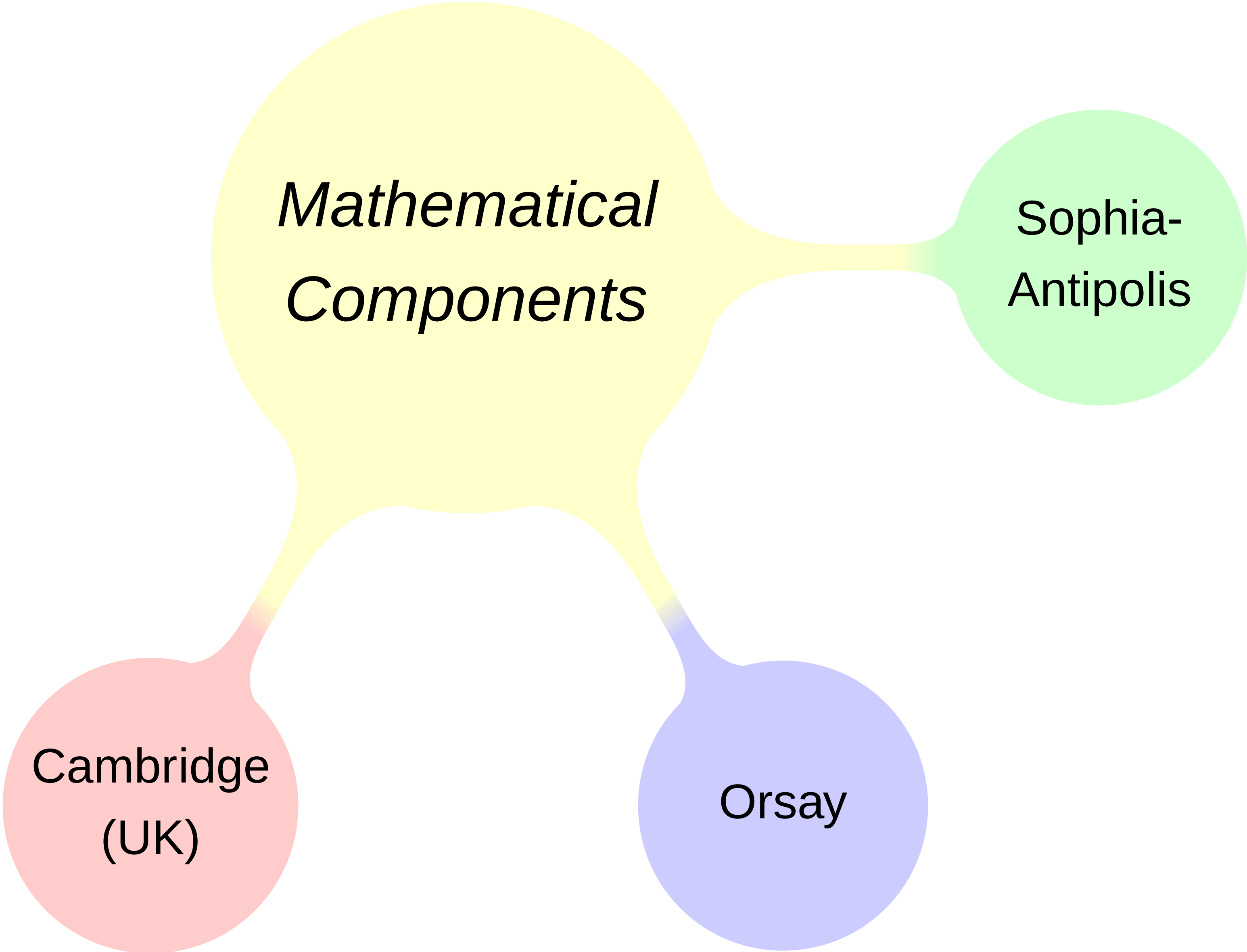
254 pages !

One team, several locations



Characteristic groups

the set G, $\bullet : G \times G \to G$

+ *associativity, identity, inverse*

A **subgroup** H of a group G is a **characteristic subgroup** of G if $H^{\varphi} = H$ for all $\varphi \in \mathrm{Aut}(G)$.

Bijective morphisms of $G \to G$

Subgroups defined by functorials

Let F be a function from groups to groups that returns a specific subgroup,

$$F : G \mapsto H$$

We want

$$\forall \varphi \in \mathrm{Aut}(G), F(G)^{\varphi} = F(G \quad)$$

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Formalized Groups

finGroupType

- `T :` Type, with decidable equality and finite enumeration
- `* : T -> T -> T`
- `1 : T`
- `_^-1 : T -> T`
- associativity, unit, inverse properties

Details : TPHOLS 2009!

Formalized Groups

{group gT}

- a **set** S : an indicator function on the enumeration of gT.
- closure of $*$ w.r.t. S
- $1 \in$ S

finGroupType

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Subgroups defined by functorials

$$F : \forall gT, \{group\, gT\} \to \{group\, gT\}$$

Example:

```
Definition Frattini (A:set gT): set gT :=
  \bigcap_(G : group gT | maximal_eq G A) G.

Canonical Structure Frattini_group A : group gT :=
  Eval hnf in [group of Frattini A].
```

We require

$$\forall \varphi \in \mathrm{Aut}(G), F(G)^{\varphi} \subset F(G\quad)$$

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Subfunctors defined by functorials

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d, "F"] & H \arrow[d, "F"] \\ F(G) & F(H) \end{tikzcd}$$

gFunc

- $F : \forall gT, \{groupgT\} \rightarrow \{groupgT\}$
- $\forall G, F(G) \subset G$
- $\forall \varphi \in Aut(G)$, $F(G)^{\varphi} \subset F(G^{\varphi})$

Subfunctors defined by functorials

In **Grp** with arrows restricted to isomorphisms, F is the obj. mapping of a subfunctor $\mathcal{F}$ of I iff

$$F(G)^{\varphi} \subset F(H)$$

Then, $\mathcal{F}\varphi = \varphi|_{F(G)}$

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d, "F"] & H \arrow[d, "F"] \\ F(G) \arrow[r, "\varphi|_{F(G)}"] & F(H) \end{tikzcd}$$

gFunc

- $F \ : \ \forall gT, \{groupgT\} \to \{groupgT\}$
- $\forall G, F(G) \subset G$
- $\forall \varphi \in \mathrm{Aut}(G)$, $F(G)^{\varphi} \subset F(G^{\varphi})$

Instances

L	Φ	$Z(G)$	O_p
J	G'	$F(G)$	O_π
$\mho_p$	$L_{(n)}$	$F^*(G)$	
Ω_p	$G^{(n)}$	$U_{(n)}$	

(plus some deprived of a nice notation)

How are (some of) these cases handled ?

1.23. Let x and y in F have m and n conjugates, respectively, in G. For each $u \in G$, $u^{-1}x^{-1}u = (u^{-1}xu)^{-1}$ and $u^{-1}(xy)u = (u^{-1}xu)(u^{-1}yu)$. Thus x^{-1} and xy have at most m and mn conj… F is a subgroup of G. Similarly, for … the same number of conjugates as x. … group (see [22]).

Theorem 5.21. *For every group G, the higher commutator subgroups are characteristic, hence normal subgroups.*

Proof. The proof is by induction on $i \geq 1$. Recall that the commutator subgroup $G' = G^{(1)}$ is generated by all commutators; that is, by all elements of the form $aba^{-1}b^{-1}$. If φ is an automorphism of G, then $\varphi(aba^{-1}b^{-1}) =$ $\varphi(a)\varphi(b)\varphi(a)^{-1}\varphi(b)^{-1}$ is also a commutator, and so $\varphi(G') \leq G'$. For the in… …at $G^{(i+1)}$ char $G^{(i)}$; since $G^{(i)}$ char G, by … $G^{(i+1)}$ is characteristic in G. ∎

characteristic subgroups of G. Another example of a characteristic subgroup is $Z(G)$. Indeed, for $x \in Z(G)$, $g \in G$, $\alpha \in \operatorname{Aut} G$,

$$x^\alpha g^\alpha = (xg)^\alpha = (gx)^\alpha = g^\alpha x^\alpha,$$

and since $G = \{g^\alpha \mid g \in G\}$ we have $x^\alpha \in Z(G)$.

a Hierarchy of subfunctors

I

$$\forall \varphi \in \mathrm{Aut}(G), F(G)^{\varphi} \subset F(G^{\varphi})$$

II

$$\forall \varphi, F(G)^{\varphi} \subset F(G^{\varphi})$$

III $\forall \varphi, F(G)^{\varphi} \subset F(G^{\varphi})$

$H \leq G \rightarrow F(H) \leq F(G)$

IV $\forall \varphi, F(G)^{\varphi} \subset F(G^{\varphi})$

$H \leq G \rightarrow F(G) \cap H \leq F(H)$

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9

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Subfunctors for finite group theory

- Simpler characteristicity proofs for everyone.
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Subfunctors for finite group theory

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- Simple additional algebraic properties. *e.g.* :

$$\forall F \in \mathsf{IV}, F \circ F = F$$

- Better compositionality than characteristic subgroups. *e.g.*:If φ surjective,

$$H \,\mathrm{char}\, G \not\Rightarrow H^{\varphi} \,\mathrm{char}\, G^{\varphi}$$

$$F(G)^{\varphi} = F(G^{\varphi}) \,\mathrm{char}\, G^{\varphi}$$

Expressing parametricity ?

- still have to prove $\forall \varphi \in \mathrm{Aut}(G'), F(G)^{\varphi} = F(G^{\varphi})$
- Cayley (regular) representation of groups:

$$a \in G \rightsquigarrow (x \mapsto a \bullet x) \in \mathrm{perm}(G)$$

$$G \rightleftharpoons^{\psi}_{\psi^{-1}} G_p : \{\mathrm{group}(\mathrm{permI}_{|G|})\}$$

- Define functors as monomorphic on the Cayley representation. Define a uniform mapping of elements of gT to the Cayley representation.