Packaging Mathematical Structures

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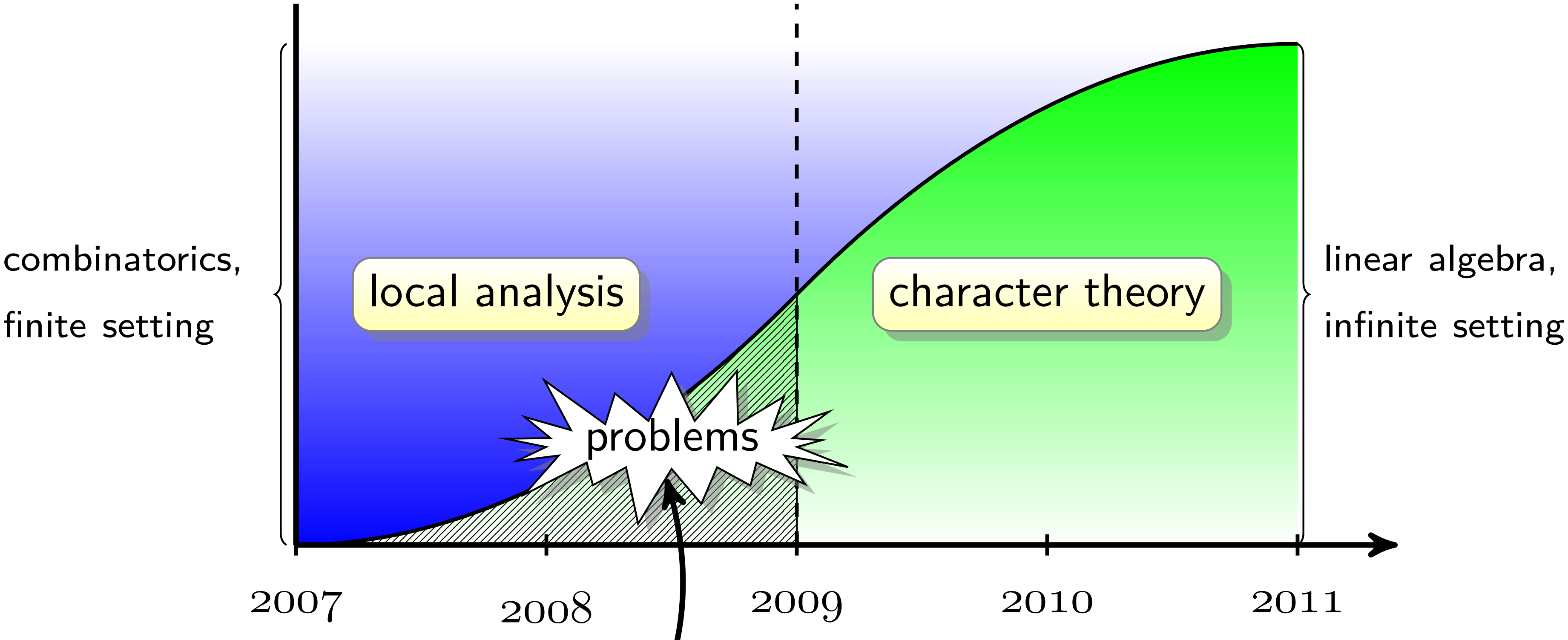
Formalizing Finite Group Algebra

the Feit-Thompson proof



This paper presents our solutions using Coq's dependent records, coercions, type inference.

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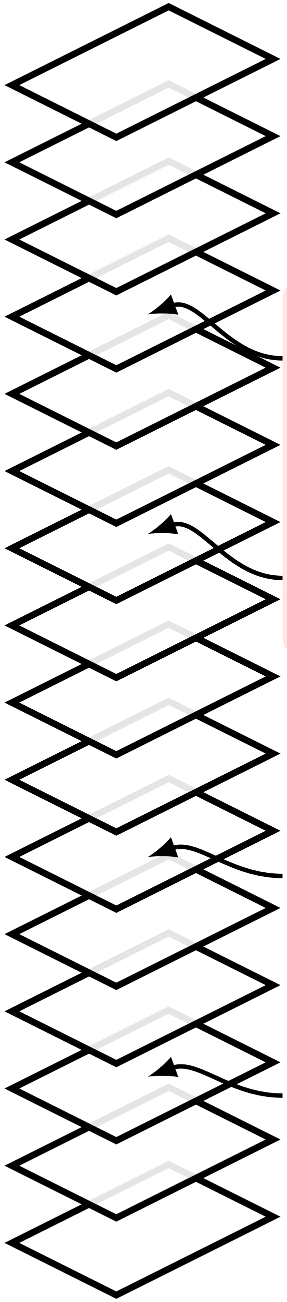


Composable mathematical structures

Our algebraic hierarchy

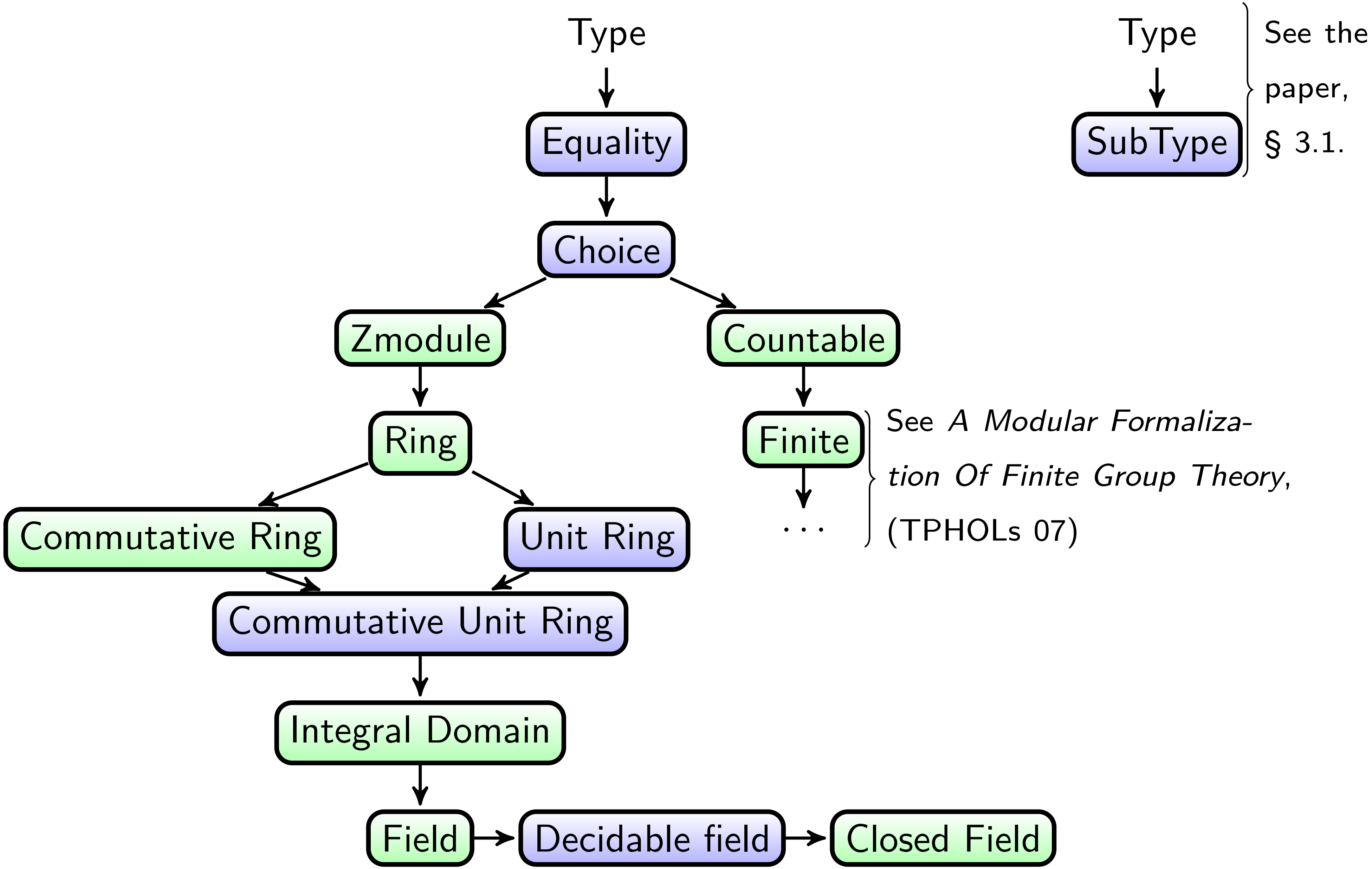
This talk

Hierarchy details,
a nice alternative to Σ-types

Two proof examples, in depth

Our paper

Our Algebraic Hierarchy



Type

Equality

Choice

Zmodule

Countable

Ring

Finite

…

Commutative Ring

Unit Ring

Commutative Unit Ring

Integral Domain

Field

Decidable field

Closed Field

Type

SubType

See the paper, § 3.1.

See *A Modular Formalization Of Finite Group Theory*, (TPHOLs 07)

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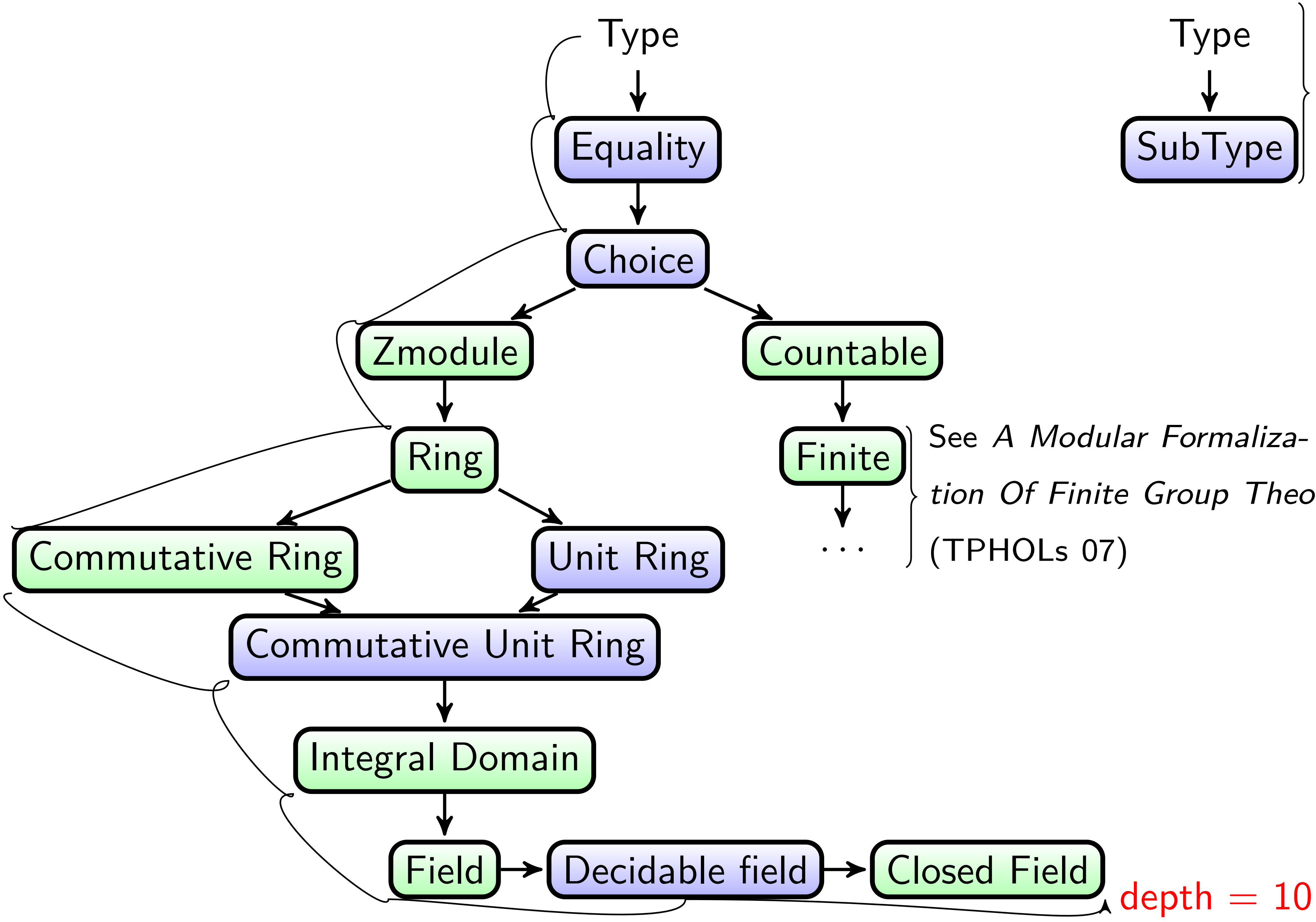

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depth = 10

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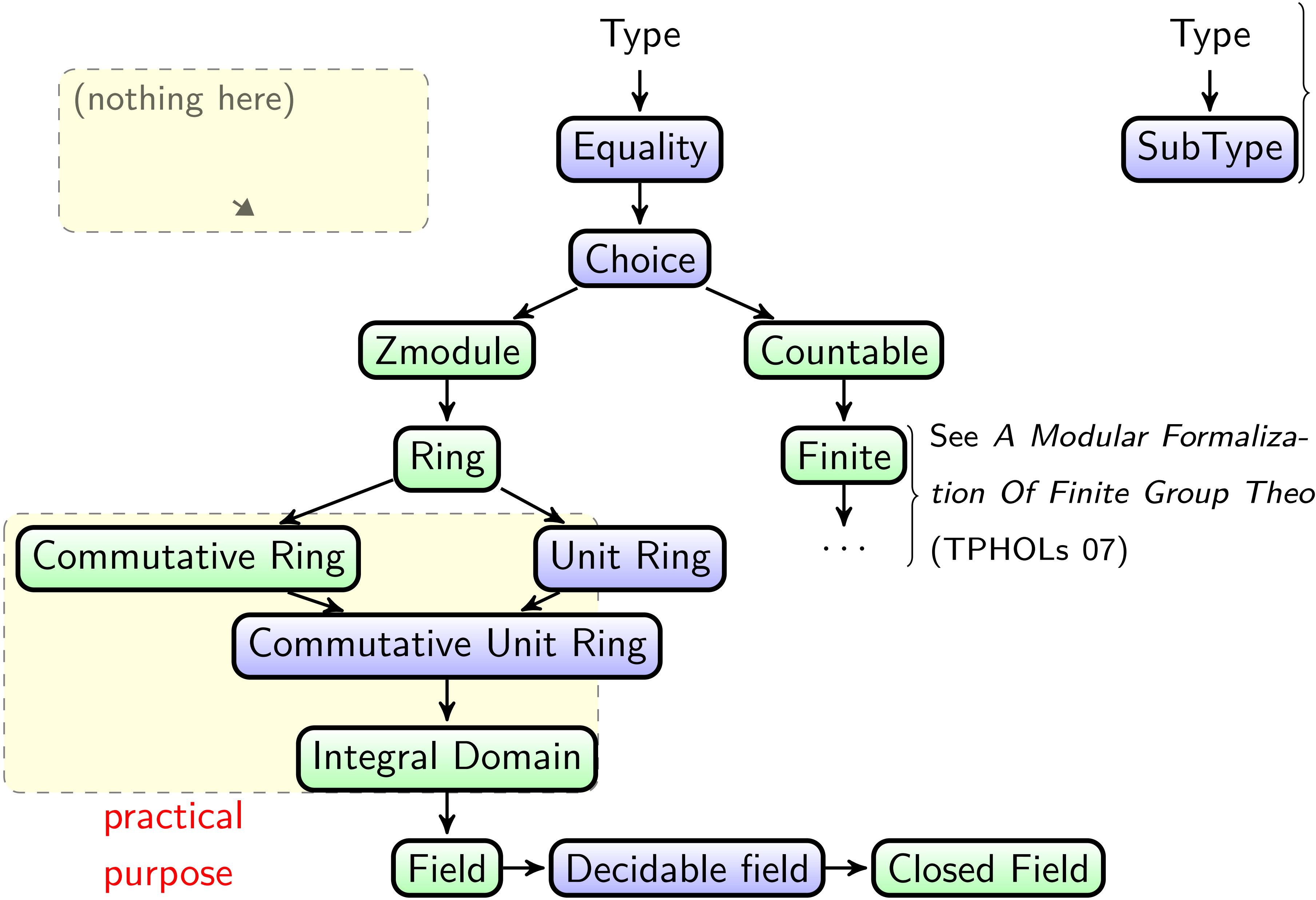

(nothing here)

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practical purpose

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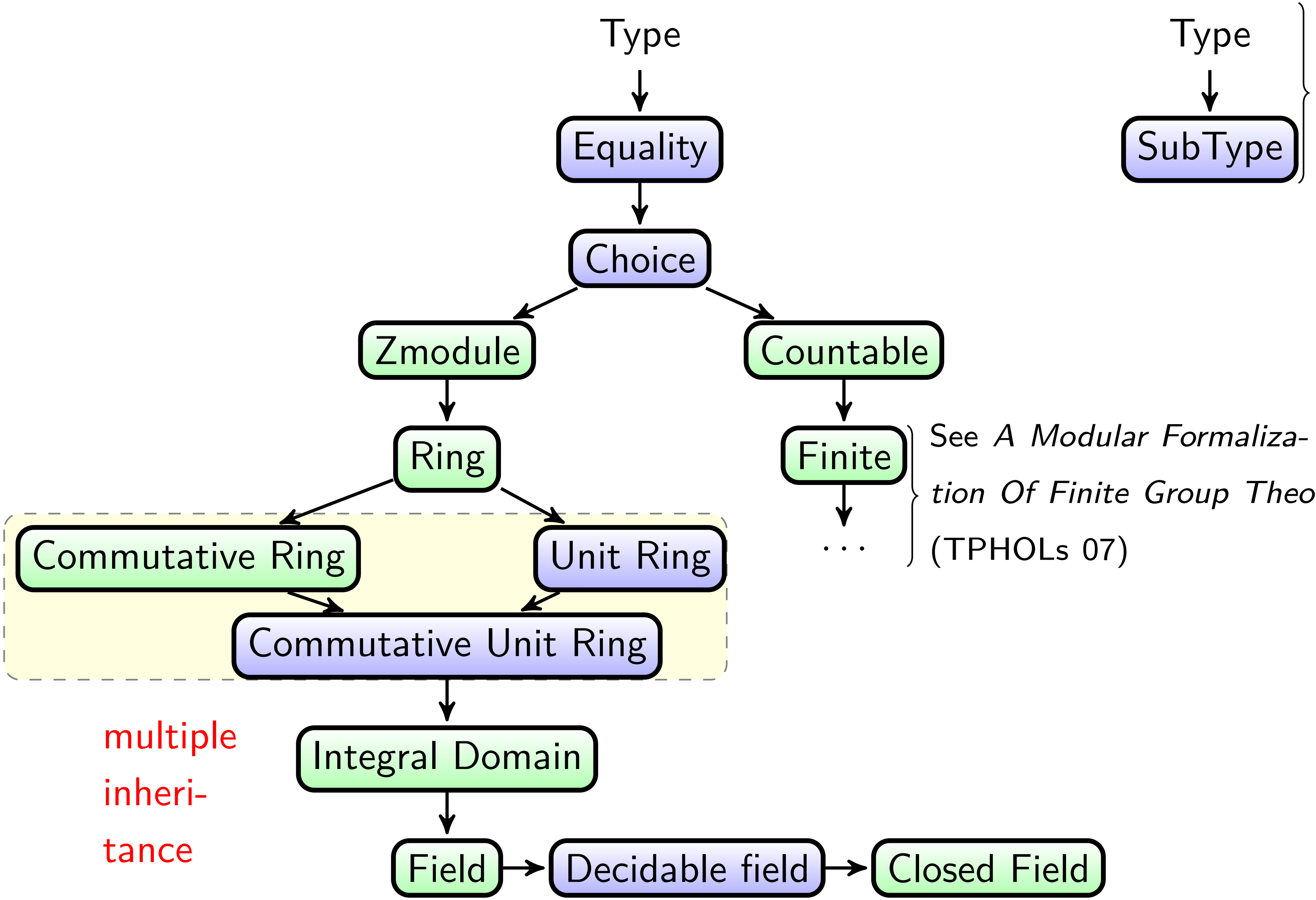

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multiple inheritance

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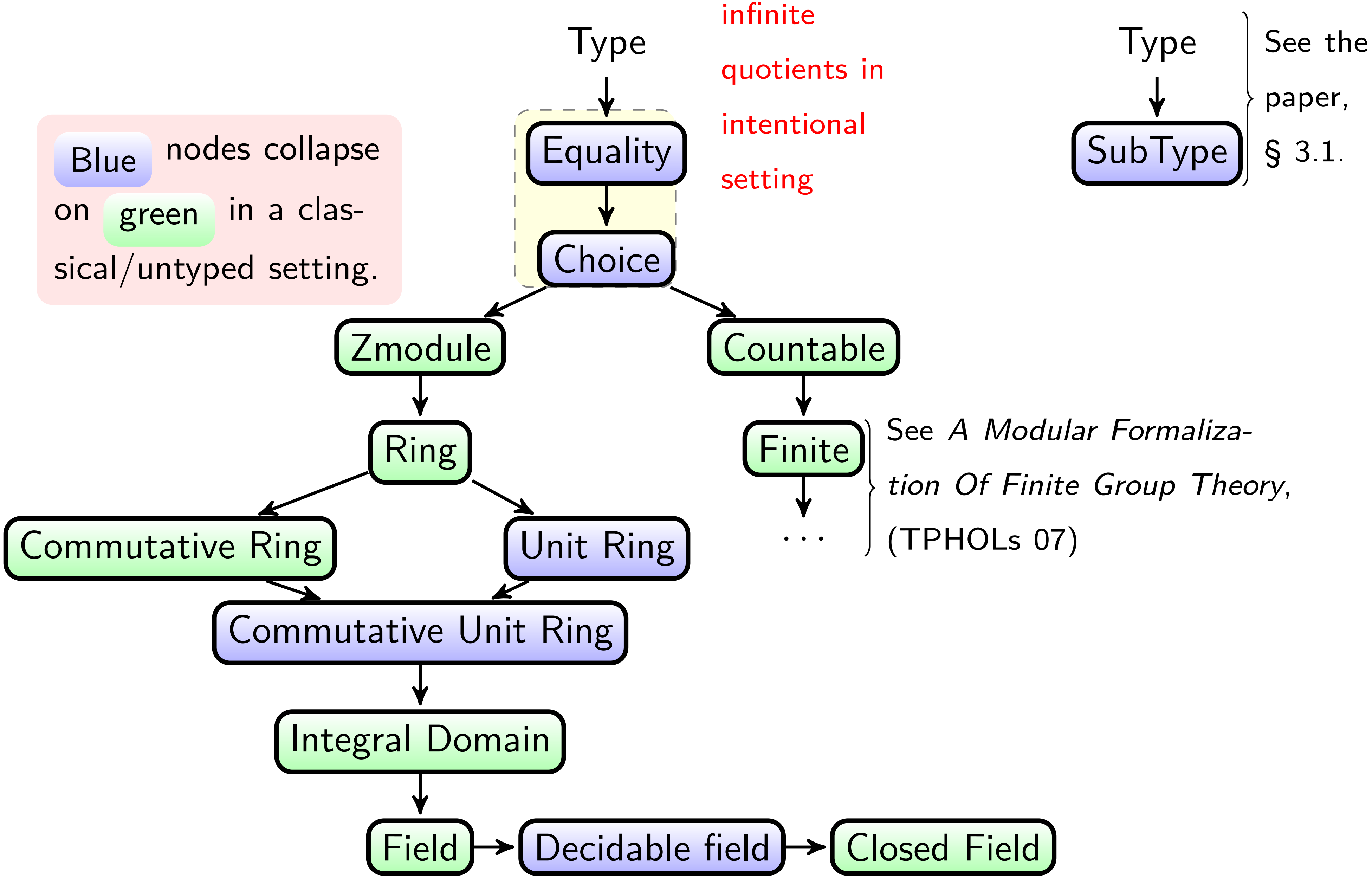

Blue nodes collapse on green in a classical/untyped setting.

infinite quotients in intentional setting

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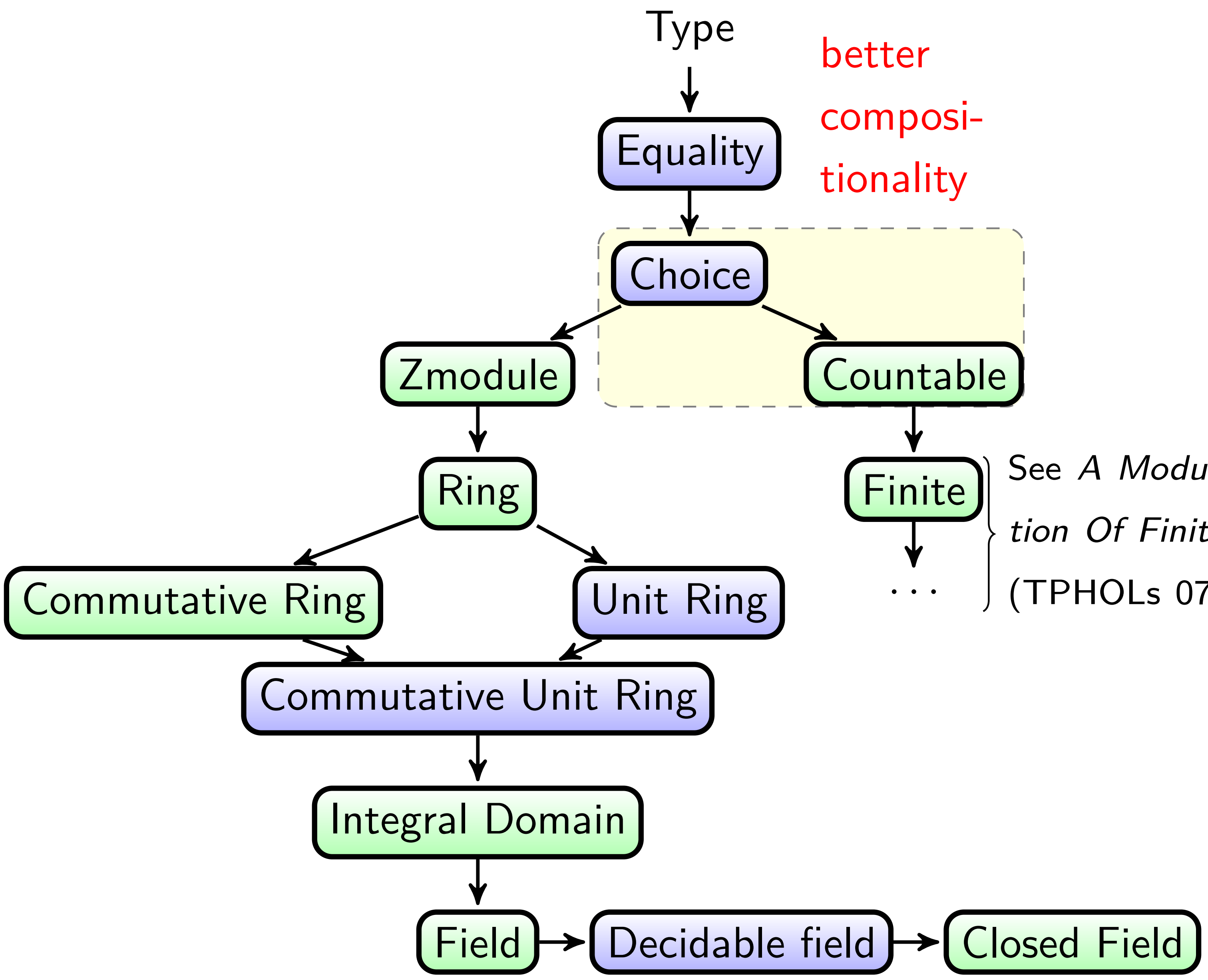

better compositionality

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(6) algebraic properties $\triangleq$ interface programming

Representing objects and structures

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(Cayley-Hamilton)

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(Cayley-Hamilton)

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But we do not want to specify the corresponding structure:

$$\left[\begin{array}{l} \text{The } \mathtt{Ring} \text{ of polynomials over} \\ \text{the } \mathtt{Ring} \text{ of matrices} \\ \text{over a general } \mathtt{Commutative\ Ring}. \end{array}\right]$$

A Type class

`ZModule (M:Type)`

- a constant `zero : M`
- an operation : `add : M-> M -> M`
- axiom(s) verified by `add` on `M`

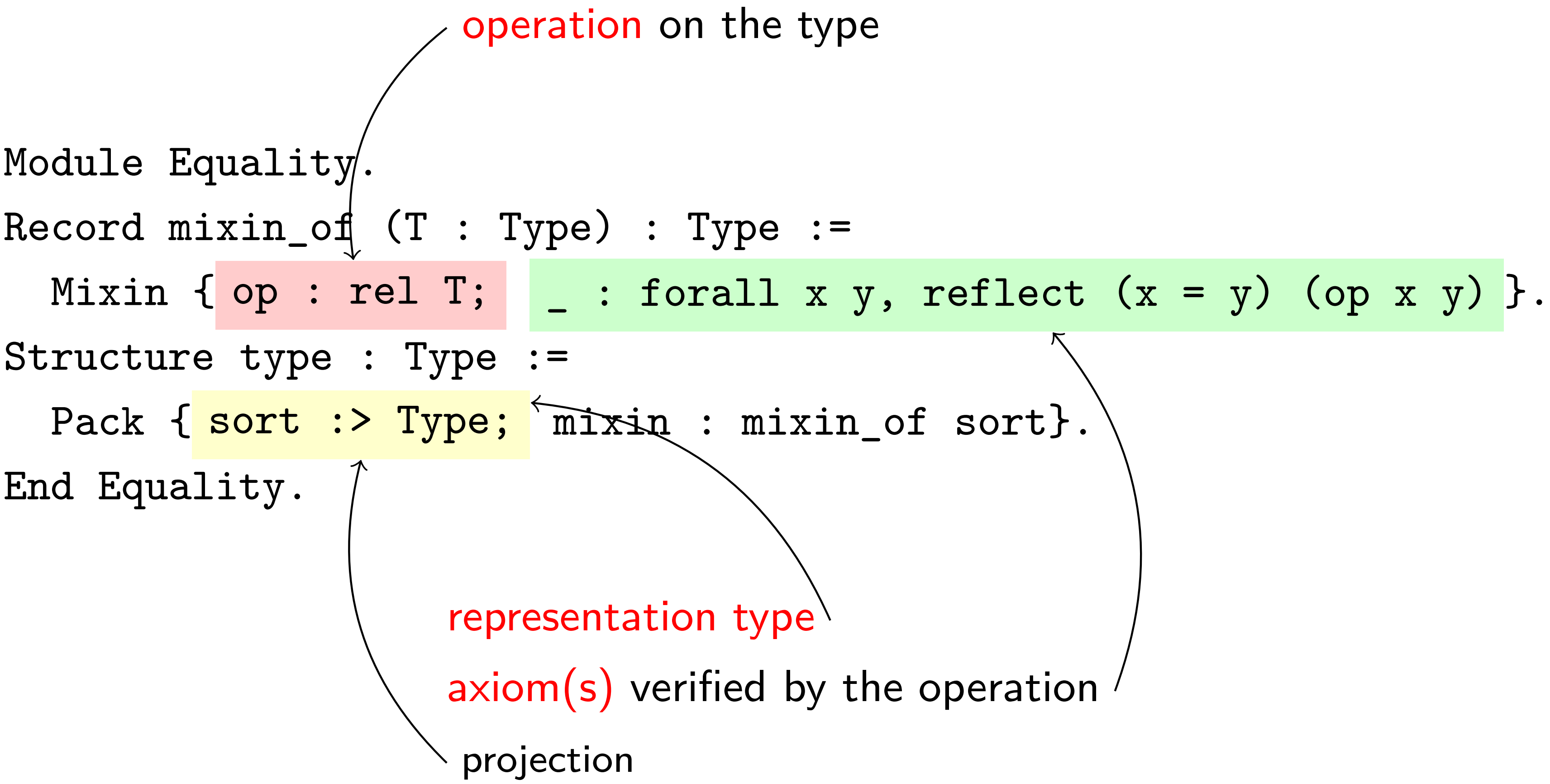
  associative add;
- . . .

C elements per structure, n nested (parametric) structures in which the parameter occurs at every element: term size in C^n

- How do we pass structures when enunciating lemmas ?
- Proofs introduce structures.

A Typical structure

We fill the blanks using `Canonical Structures`.



operation on the type

```
Module Equality.
Record mixin_of (T : Type) : Type :=
  Mixin { op : rel T;  _ : forall x y, reflect (x = y) (op x y) }.
Structure type : Type :=
  Pack { sort :> Type; mixin : mixin_of sort}.
End Equality.
```

representation type

axiom(s) verified by the operation

projection

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  Mixin {    : _    ; _ : _                   }.
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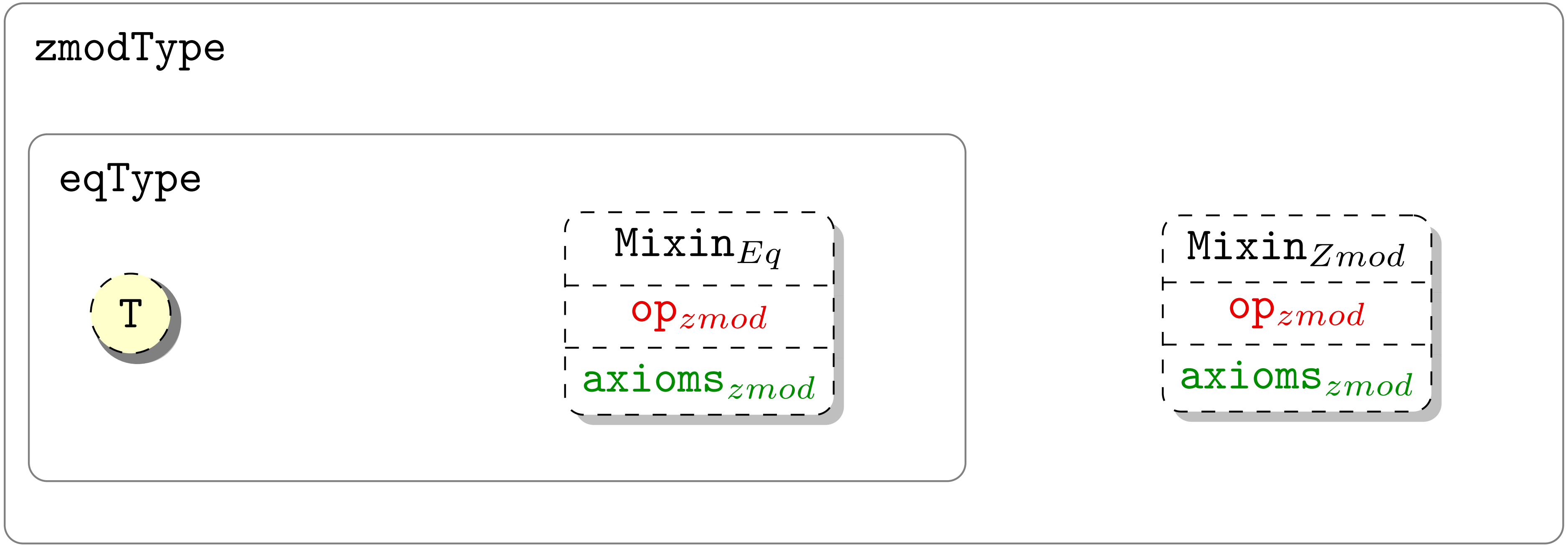
End Equality.
```

This is generic: we use modules as namespaces only:

```
Notation eqType := Equality.type.
```

Canonical Structure Composition

- Most widely used: Telescopes



Telescopes : Canonical Structure inference

$2 + 2$

Notation

$\texttt{Zmodule.add}\ \alpha\ \texttt{2}\ \texttt{2}$

type inference

$\texttt{Equality.sort}(\texttt{Zmodule.sort}\alpha) \equiv_{\beta\iota\delta} \texttt{int}$

lookup $(\texttt{Equality.sort},\texttt{int}) \rightsquigarrow \texttt{int}_{eqType}$

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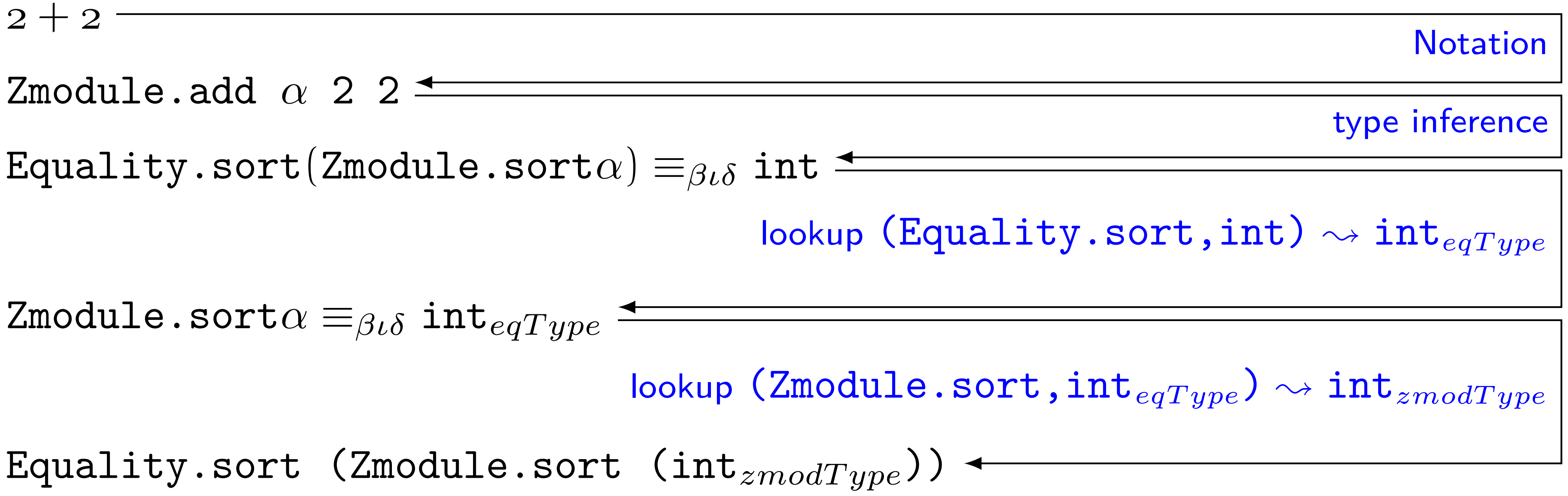

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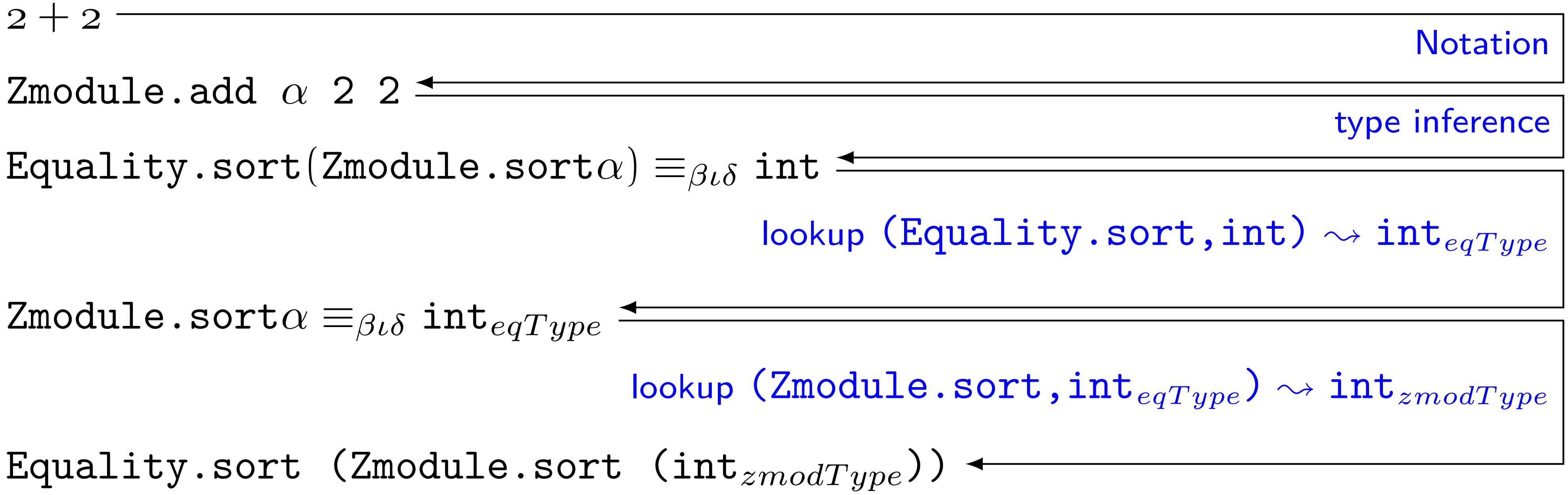

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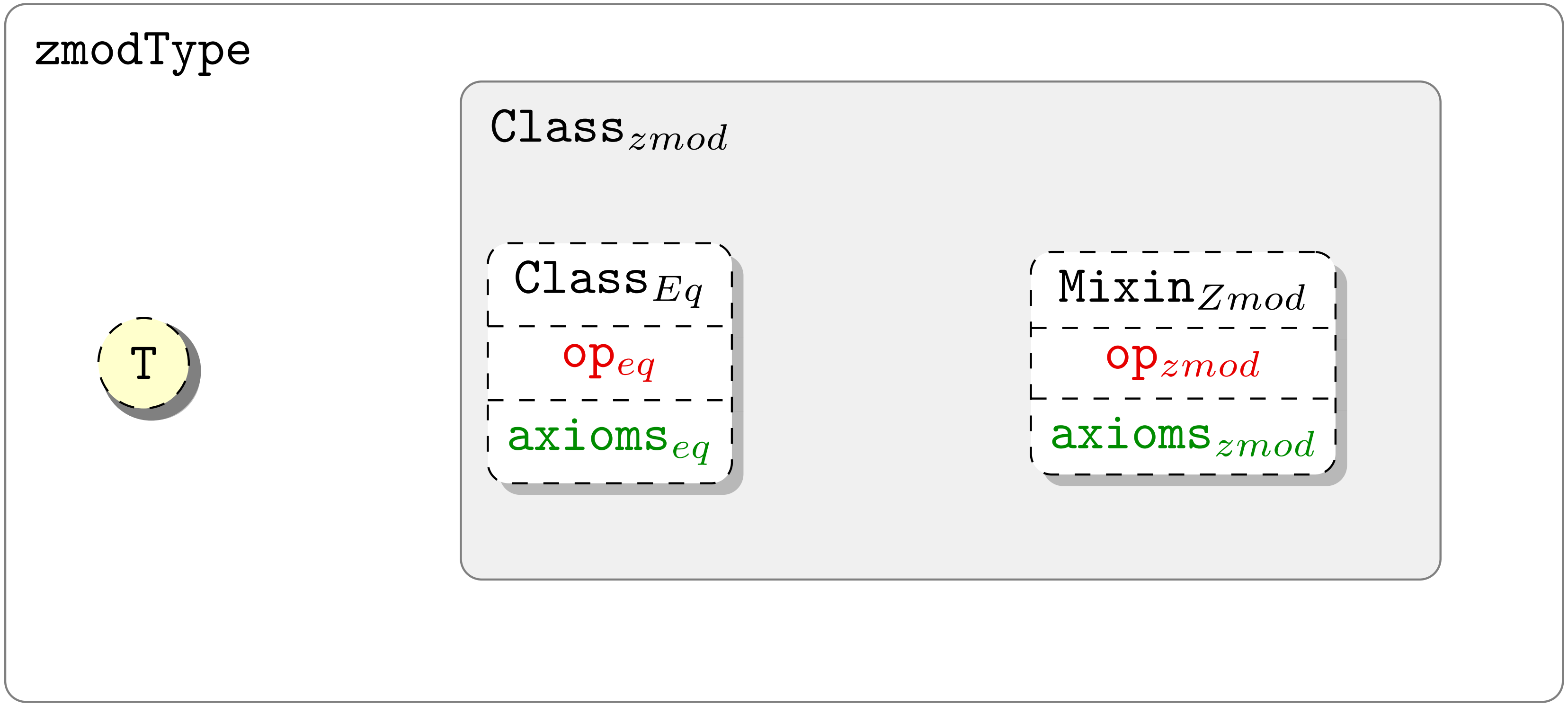
- as is, no multiple inheritance

Packed Classes



zmodType

$\texttt{Class}_{zmod}$

T

$\texttt{Class}_{Eq}$

$\texttt{op}_{eq}$

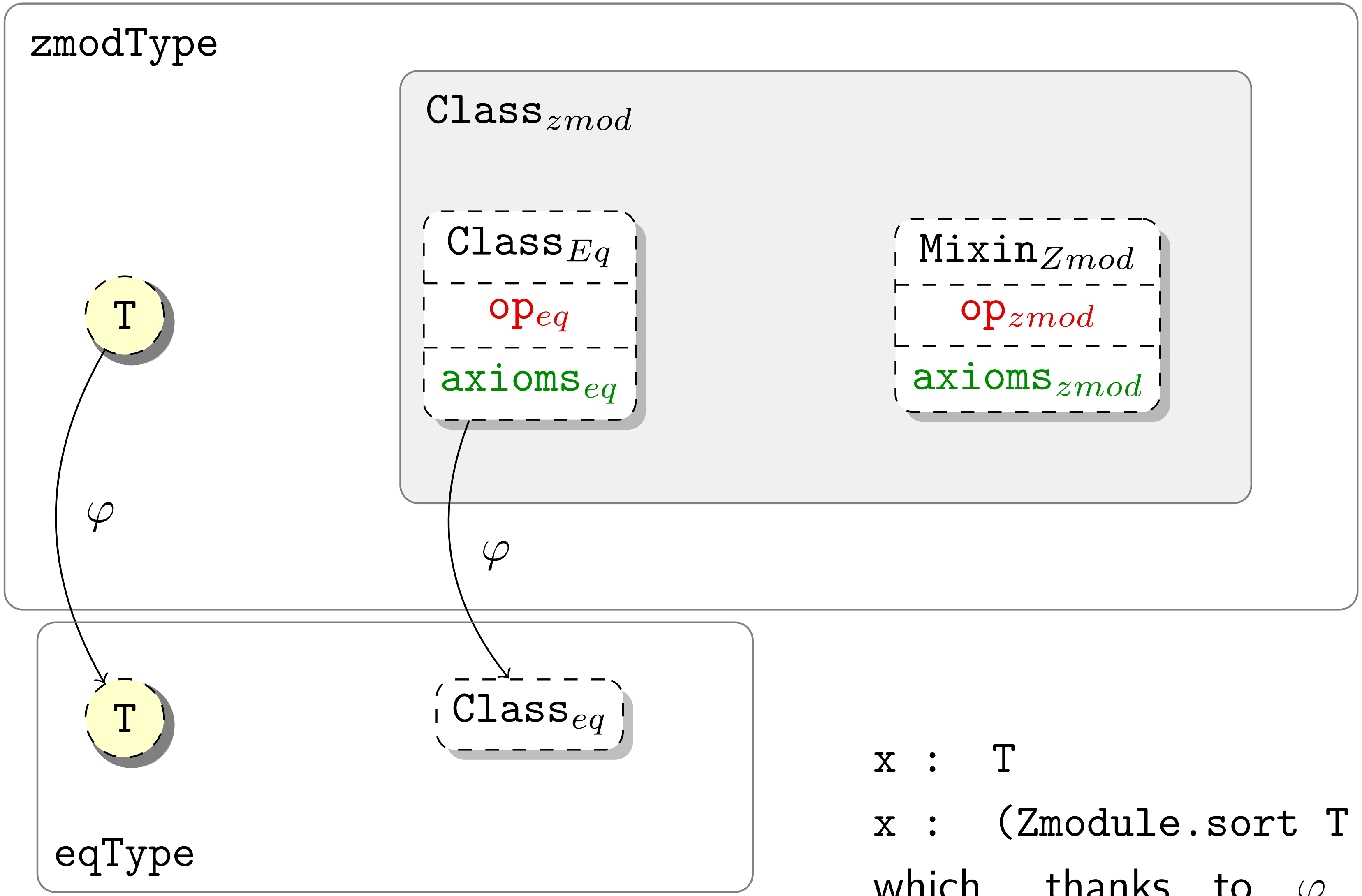
$\texttt{axioms}_{eq}$

$\texttt{Mixin}_{Zmod}$

$\texttt{op}_{zmod}$

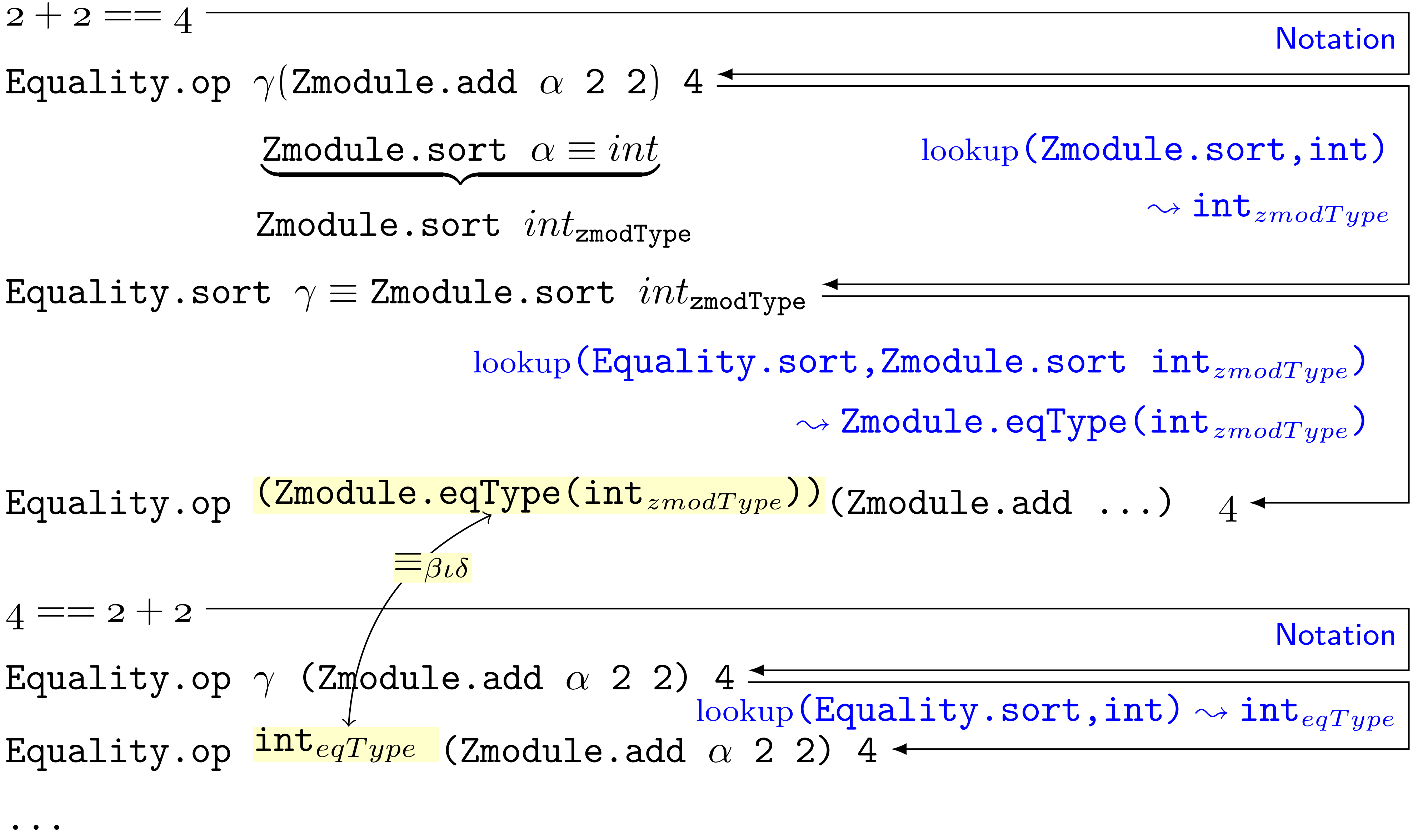
$\texttt{axioms}_{zmod}$

Packed Classes



`x : T` denotes `x : (Zmodule.sort T` α`)`, which, thanks to φ, has a canonical eqType structure.

Packed Classes



$2 + 2 == 4$

Notation

$\texttt{Equality.op}\ \gamma(\texttt{Zmodule.add}\ \alpha\ 2\ 2)\ 4$

$$\underbrace{\texttt{Zmodule.sort}\ \alpha \equiv int}_{\texttt{Zmodule.sort}\ int_{\texttt{zmodType}}}$$

lookup(Zmodule.sort,int) $\rightsquigarrow$ $\texttt{int}_{zmodType}$

$\texttt{Equality.sort}\ \gamma \equiv \texttt{Zmodule.sort}\ int_{\texttt{zmodType}}$

lookup(Equality.sort,Zmodule.sort $\texttt{int}_{zmodType}$)

$\rightsquigarrow$ Zmodule.eqType($\texttt{int}_{zmodType}$)

$\texttt{Equality.op}\ (\texttt{Zmodule.eqType}(\texttt{int}_{zmodType}))(\texttt{Zmodule.add}\ \ldots)\ 4$

$\equiv_{\beta\iota\delta}$

$4 == 2 + 2$

Notation

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lookup(Equality.sort,int) $\rightsquigarrow$ $\texttt{int}_{eqType}$

$\texttt{Equality.op}\ \texttt{int}_{eqType}\ (\texttt{Zmodule.add}\ \alpha\ 2\ 2)\ 4$

...

Packed Classes

- Coherent coercions,
- `x:T` interpreted as `Zmodule.sort T`.
- Multiple inheritance:

```
Module ComUnitRing.
Record class_of (R : Type) : Type := Class {
  base1 :> ComRing.class_of R;
  ext :> UnitRing.mixin_of (Ring.Pack base1 R)}.
Coercion base2 R m := UnitRing.Class (@ext R m).
...
End ComUnitRing.
```

Conclusion

- algebraic structure composition is interface programming
- representing a large hierarchy implies packaging
- know the work ahead,
- and take the time to build the tools you will need

- `A + x`

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- `add (matrix T) (zeromx T) (addmx T)`

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- `... (addmx (poly T) (zeropx T) (addpx T) )`

◆ `A + x`

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◆ `... (addmx (poly T) (zeropx T) (addpx T) )`

◆ and `zeromx` is itself `(zeromx (poly T) (zeropx T) (addpx T) )` ...