

Actions of higher rank Lie groups

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Around the Zimmer program

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Introduction

This text is devoted to the study of lattices in higher rank Lie groups. All these concepts will be defined in the words to come, but in the meantime the reader should keep the following picture in mind. G stands for a connected semi-simple real Lie group with no compact factors. Γ will stand for a lattice in G , which is in some sense an abstraction of $\mathbb{Z}^n$ lying inside $\mathbb{R}^n$.

The main point of this dissertation is to shed light on the rigidity properties of some of these lattices in the case when G has "higher rank", that is real rank greater than or equal to 2. In particular, we will focus on actions of these lattices on measured spaces. One common theme will be that of extending such actions to G itself, or that is bootstrapping Γ -spaces to G -spaces.

Towards that general goal, we will encounter several classical theorems in that field. The exposition of these theorems will be the core of the text, and will occupy most if not all of the third section. The first of these will be the Margulis factor theorem. This theorem classifies the Γ -spaces which are in some sense quotients of G/P , where P is a parabolic subgroup of G . In particular, all these Γ -spaces can be given a natural structure of G -space. As a side product of this factor theorem, we will get the very celebrated Margulis normal subgroup theorem which states that normal subgroups of Γ are either of finite index or are finite and central.

As a generalization of the factor theorem, we will then classify the G -spaces Y which lie in some exact sequence $G/P \times X \rightarrow Y \rightarrow X$, where X is an ergodic probability measure preserving G -space. Upon setting $X = G/\Gamma$ we will then be able to recover the normal subgroup theorem.

Finally, we will turn our attention to the Stuck-Zimmer theorem which classifies the possible structures for stabilizers of G -actions. From this very striking result, we will once again deduce the Margulis normal subgroup theorem.

The first section is mainly preliminaries on groups and group actions. The second one is way more involved, and contains classical material about amenability and property (T) . The third section is the core of this text and contains the exposition of all of the aforementioned theorems.

I do apologize in advance for all the mistakes and misprints which definitely do remain in this text. I warmly thank Cyril Houdayer for introducing me to this beautiful topic and guiding me through it. I also thank Soham Chakraborty for taking the time to evaluate this work and being part of the examination committee.

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1 Locally compact groups

1.1 Measures and lattices

1.1.1 Generalities

Let X be a topological space. Recall that X is said to be locally compact if and only if it is Hausdorff (every two distinct points have disjoint neighborhoods) and every point in X has a compact neighborhood in X . A topological space X is said to be second countable if and only if there is a countable basis of open sets for the topology of X . It is said to be first countable if and only if every point in X has a countable basis of neighborhood. It is clear that second countable implies first countable. Finally, X is said to be σ -compact if and only if it is the countable union of compact subsets of itself. The word compact will always stand for compact Hausdorff. Now we turn our attention to topological groups.

Definition 1.1. Let G be a group that is also a topological space.

We say that G is a topological group if and only if the multiplication map $G \times G \rightarrow G$ and the inversion map $G \rightarrow G$ are continuous.

If moreover G is locally compact as a topological space, one says that G is a locally compact group.

The following lemma will turn out to be useful in some occasions.

Lemma 1.2. *Let G be a locally compact group. Then G is second countable if and only if it is first countable and σ -compact.*

Proof. Let G be a locally compact group.

Suppose first that G is second countable. Then it is obviously first countable. Let us prove it is σ -compact. To do so, start by considering a countable basis $\mathcal{B}$ for the topology of G , and for every $x \in G$ consider a compact neighborhood U_x of x . Write :

$$\mathcal{B}' = \{\overline{B} \in \mathcal{B} \mid \exists x \in G, x \in B \subseteq U_x\}$$

Now all sets of $\mathcal{B}'$ are compact, there are a countable number of them, and $\mathcal{B}$ being a basis ensures that they cover G . Hence G is σ -compact.

Conversely, suppose that G is first-countable and σ -compact. Consider a countable basis $\mathcal{B}_0$ of open sets for the topology at the point $e \in G$. Write $G = \bigcup_{n \in \mathbb{N}} K_n$ where the K_n are compact sets. Now for every $n \in \mathbb{N}$, every $B \in \mathcal{B}_0$ there is a finite set $X_{n,B} \subseteq K_n$ such that :

$$K_n \subseteq \bigcup_{x \in X_{n,B}} xB$$

Take $\mathcal{B}$ to be the set of xB for $B \in \mathcal{B}_0$ and x in some $X_{n,B}$. $\mathcal{B}$ is countable, and it is a basis for the topology of G . Indeed, let $y \in G$ and U an open set containing y . One has $y \in K_n$ for some n , hence for every $B \in \mathcal{B}_0$ there is some $x_B \in X_{n,B}$ such that $y \in x_B B$ i.e. $x_B \in yB^{-1}$. Taking B small enough so that $B^{-1} \cdot B \subseteq y^{-1}U$ which is a neighborhood of e , we then have hence :

$$y \in x_B B \subseteq yB^{-1} \cdot B \subseteq yy^{-1}U = U$$

and we are done. □

Locally compact second countable groups will sometimes be called lcsc groups for convenience.

1.1.2 The Haar measure

Before defining the Haar measure on a locally compact group, let us recall a few facts from measure theory.

Definition 1.3. Let X be a locally compact Hausdorff space. A Borel measure μ on X is called a regular measure if and only if the three following conditions hold.

- (i) For every compact subset $K \subseteq X$, one has $\mu(K) < +\infty$ (local finiteness).
- (ii) For every Borel subset $E \subseteq X$, one has $\mu(E) = \inf\{\mu(U) \mid U \text{ open in } X, E \subseteq U\}$ (outer regularity).
- (iii) For every Borel subset $E \subseteq X$, one has $\mu(E) = \sup\{\mu(K) \mid K \text{ compact}, K \subseteq E\}$ (inner regularity).

We will need the following measure-theoretic result at some point in the future. We record it here for convenience.

Theorem 1.4 (Riesz-Markov-Kakutani theorem on linear forms and measures). *Let X be a compact Hausdorff space, and $\lambda : \mathcal{C}(X) \rightarrow \mathbb{R}$ a linear form that is non-negative, i.e. if $f \geq 0$ then $\lambda(f) \geq 0$.*

Then there exists a unique positive Radon measure ν on X such that :

$$\forall f \in \mathcal{C}(X), \lambda(f) = \int_X f d\nu$$

Proof. This is extremely classical, and covered in most of measure theory textbooks. See for example [2]. □

Consider a locally compact group G . Haar's theorem states that there is in some way a canonical way to integrate functions on G . Here is a more precise statement.

Theorem 1.5 (Haar). *Let G be a locally compact group. There is a unique (up to scaling) non-zero regular (see definition 1.3) Borel measure μ_G on G that satisfies the two following equivalent conditions :*

- (i) *For every Borel set $U \subseteq G$, every $g \in G$ one has $\mu_G(gU) = \mu_G(U)$.*
- (ii) *For every function $f \in \mathcal{C}_c(G)$, one has :*

$$\int_G f(gx) d\mu_G(x) = \int_G f(x) d\mu_G(x)$$

Such a measure will be called a Haar measure on G . It will be said to be "left-invariant" under the action of G .

Proof. See [5]. □

If G is a locally compact group, μ_G will always denote any Haar measure on G . Note that some information about the topology of G can be recovered from the measure μ_G .

Lemma 1.6. *Let G be a locally compact group. Then G is compact if and only if $\mu_G(G) < +\infty$.*

Proof. If G is compact then local finiteness of μ_G ensures that $\mu_G(G) < +\infty$. Conversely, suppose that G is not compact. Then one can find $g_1 \in G \setminus KK^{-1}$, i.e. such that $g_1K \cap K = \emptyset$. By induction, using the fact that G is not compact, one defines a sequence $(g_n)_{n \in \mathbb{N}^*} \in G^{\mathbb{N}^*}$ such that $g_{n+1} \notin (K \cup g_1K \cup \dots \cup g_nK)K^{-1}$ for every $n \in \mathbb{N}$. Then the sequence of compact sets $g_nK \subseteq G$ are all disjoint and have the same non-zero measure, which ensures that $\mu_G(G) = +\infty$. $\square$

Of course for every $g \in G$, we may define a new left-invariant measure μ_g on G by letting $\mu_g(B) = \mu_G(Bg)$ for every Borel set B . By the uniqueness part of the above theorem there must be some $\Delta_G(g) \in \mathbb{R}_+^*$ such that $\mu_g = \Delta_G(g)\mu_G$, and one can show that this defines a continuous morphism of groups $\Delta_G : G \rightarrow \mathbb{R}_+^*$ (see for example [1]) which we call the modular function of G . The group G is called unimodular if and only if $\Delta_G = 1$.

The following examples are left as an exercise for the reader.

Example 1.7. *Let G be a locally compact group. If G is abelian, compact, or discrete then it is unimodular.*

After having read section 1.3, the reader may also want to check that semi-simple Lie groups are unimodular.

Now consider a locally compact group G and a closed subgroup H of G . If H is normal in G , then the quotient G/H is endowed with its Haar measure, which is of finite total mass if and only if G/H is compact. G/H is acted on by G on the left, and this measure $\mu_{G/H}$ is of course invariant under this action of G . It turns out that in general, we may not need H to be normal in G for the quotient to have a measure which is left-invariant.

Proposition 1.8. *Let G be a locally compact group, and H a closed subgroup of G . The quotient G/H space carries a non-zero left-invariant Borel regular measure if and only if $\Delta_{G|H} = \Delta_H$. Moreover, in that case such a measure is unique up to scaling.*

Proof. See again [1]. $\square$

We will denote such a measure by $\mu_{G/H}$. This does not conflict the notation for Haar measures, since when H is normal in G the Haar measure on G/H satisfies the requirements of proposition 1.8 above.

1.1.3 Lattices in locally compact groups

We then define lattices in terms of finite invariant measures the following way.

Definition 1.9. Let Γ be a discrete subgroup in a locally compact group G . One says that Γ is a lattice in G if and only if G/Γ carries a left-invariant regular Borel measure of finite total mass.

Note that G/Γ carries a finite left-invariant Borel measure if and only if the left quotient $\Gamma \backslash G$ does. Thus the notion of lattice does not depend on the side on which we take the quotient.

Not all locally compact groups contain lattices. Here is a strong restriction against it.

Proposition 1.10. *Let G be a locally compact group which contains a lattice Γ . Then G is unimodular.*

Proof. Let G be a locally compact group which contains a lattice Γ . let us prove that G is unimodular. First of all, since there exists a left-invariant Borel measure on the quotient G/Γ , it must be that $\Delta_{G/\Gamma} = \Delta_\Gamma$. But since Γ is discrete, the Haar measure on Γ is the counting measure, which is both left and right invariant, hence $\Delta_\Gamma = 1$. In particular, it follows that $\Delta_{G/\Gamma} = 1$.

Now consider the left-invariant measure $\mu_{G/\Gamma}$ which we normalize to be a probability measure. Since Δ_G is trivial on Γ , we may factor it through G/Γ and get a continuous mapping $\Delta_{G/\Gamma} : G/\Gamma \rightarrow \mathbb{R}_+^*$. Now consider the push-forward measure $\eta = \Delta_{G/\Gamma*} \mu_{G/\Gamma}$ defined on $\mathbb{R}_+^*$ by $\eta(B) = \mu_{G/\Gamma}(\Delta_{G/\Gamma}^{-1}(B))$ for every Borel set B . η is a probability measure on $\mathbb{R}_+^*$. Moreover, it is left-invariant under the action of the subgroup $\overline{\Delta}(G/\Gamma) = \Delta(G)$ of $\mathbb{R}_+^*$. If this subgroup is non trivial, then it is clear that there is no such probability measure, hence $\Delta(G)$ must be $\{1\}$. This concludes the proof. $\square$

One important example of lattices is the following.

Proposition 1.11. *Let Γ be a discrete subgroup of a locally compact unimodular group G . If the quotient G/Γ is compact, then Γ is a lattice in G .*

Proof. First we have to prove that $\Delta_{G/\Gamma} = \Delta_\Gamma$ for a left-invariant regular Borel measure to exist on the quotient G/Γ . Note that since Γ is discrete, μ_Γ is clearly the counting measure, thus $\Delta_\Gamma = 1$ which is indeed equal to $\Delta_{G/\Gamma}$ by unimodularity of G . Now by local finiteness in the definition of regular measures (see definition 1.3) and since G/Γ is compact one has $\mu_{G/\Gamma}(G/\Gamma) < +\infty$, thus Γ is a lattice in G . $\square$

Such a lattice will be called a uniform lattice. Not all lattices are uniform : it could be that the quotient G/Γ is non-compact but still carries a left-invariant regular measure. The following gives an example of this fact.

Example 1.12. *For every $n \geq 2$, $SL_n(\mathbb{Z})$ is a non-uniform lattice in $SL_n(\mathbb{R})$.*

We first prove that it is indeed a lattice. Some calculations in the proof are skipped for convenience. These can all be found in [7].

Proposition 1.13. *For every $d \geq 2$, $SL_d(\mathbb{Z})$ is a lattice in $SL_d(\mathbb{R})$.*

Proof. Write $K = SO_d(\mathbb{R})$, A the subgroup of $SL_d(\mathbb{R})$ consisting of diagonal matrices with positive entries, and N the subgroup of $SL_d(\mathbb{R})$ of upper triangular matrices with ones on the diagonal. We claim that :

$$SL_d(\mathbb{R}) = K \cdot A \cdot N$$

Indeed, by Gram-Schmidt orthogonalization one get for every $m \in \mathrm{SL}_d(\mathbb{R})$ two matrices t and $s \in \mathrm{O}_d(\mathbb{R})$ with t upper triangular such that $m = st$. Now upon extracting the diagonal, write $t = a_0 u$ with $u \in N$ and a_0 diagonal. We now have $m = sa_0 t$. Upon switching the signs of some columns of s , which does not change the fact that $s \in \mathrm{O}_d(\mathbb{R})$, we may as well suppose that the coefficients of a are positive, hence $a \in A$ and $\det(a)\det(s) = 1$ so $\det(s) > 0$ and $s \in \mathrm{SO}_d(\mathbb{R})$, hence the wanted decomposition.

Now endow K, A and N with their respective Haar measures dk, da and dn . Write $A = \mathrm{Diag}(\lambda_1, \dots, \lambda_d)$. A simple calculation then shows that the push forward through the map $K \times A \times N \rightarrow \mathrm{SL}_d(\mathbb{R})$ of the measure :

$$\prod_{1 \leq i < j \leq d} \frac{\lambda_i}{\lambda_j} dk da dn$$

is a Haar measure on $\mathrm{SL}_d(\mathbb{R})$.

Now consider for every $t, u > 0$:

$$\begin{aligned} A_t &= \{\mathrm{Diag}(\lambda_1, \dots, \lambda_d) \in A \mid \forall i \in \llbracket 1, n-1 \rrbracket, \lambda_i < t\lambda_{i+1}\} \\ N_u &= \{n = (n_{i,j})_{1 \leq i, j \leq n} \in N \mid \forall i, j \in \llbracket 1, n \rrbracket, |n_{i,j}| \leq u\} \\ \Sigma_{t,u} &= KA_t N_u \end{aligned}$$

We will now prove that for every $t, u > 0$, $\Sigma_{t,u}$ has finite volume in $\mathrm{SL}_d(\mathbb{R})$. Since K and N_u are compact, it thus suffices to prove that :

$$\int_{A_t} \prod_{1 \leq i < j \leq n} \frac{\lambda_i}{\lambda_j} da < +\infty$$

and a fully explicit calculation shows that indeed :

$$\int_{A_t} \prod_{1 \leq i < j \leq n} \frac{\lambda_i}{\lambda_j} da = t^{\frac{d(d+1)(d+2)}{6}} \frac{1}{d!^2}$$

Now since we know that $\Sigma_{u,t}$ has finite measure, to conclude it suffices to show that for some choice of $t, u > 0$ one has $\mathrm{SL}_d(\mathbb{R}) = \Sigma_{u,t} \cdot \mathrm{SL}_d(\mathbb{Z})$. To that extent, we first prove by induction on d that $N = N_{\frac{1}{2}}(N \cap \mathrm{SL}_d(\mathbb{Z}))$. Geometric considerations that we will not develop here then show that for $t = \frac{2}{\sqrt{3}}$ and $u = \frac{1}{2}$, we have the wanted decomposition. $\square$

Let us now explain why $\mathrm{SL}_n(\mathbb{Z})$ is in fact a non-uniform lattice, that is the quotient $\mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})$ is not compact. To that extent, consider $\gamma_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ and consider $\gamma \in \mathrm{SL}_d(\mathbb{R})$ made of γ_0 on the top left corner, ones on the diagonal and zeroes everywhere else. Write $g_n = \mathrm{Diag}(n^{-1}, n, 1, \dots, 1) \in \mathrm{SL}_d(\mathbb{R})$ for every $n \in \mathbb{N}^*$. A simple calculation shows that $g_n \gamma g_n^{-1}$ goes to Id_d as n goes to $+\infty$, thus the following lemma proves that the quotient $\mathrm{SL}_d(\mathbb{R})/\mathrm{SL}_d(\mathbb{Z})$ cannot be compact.

Lemma 1.14. *Let G be a locally compact group and Γ a uniform lattice in G . Then there is some neighborhood U of the identity element $e \in G$ such that :*

$$\forall g \in G, g\Gamma g^{-1} \cap U = \{e\}$$

Proof. Keep the above notations. Take two symmetric neighborhoods W and V of e such that $\Gamma \cap W = \{e\}$ and $VVV \subseteq W$. By compactness of G/Γ , one has $g_1, \dots, g_k \in G$ such that $G/\Gamma = g_1p(V) \cup \dots \cup g_kp(V)$, where $p : G \rightarrow G/\Gamma$ denotes the canonical projection. Write $U = g_1Vg_1^{-1} \cap \dots \cap g_kVg_k^{-1}$.

We will show that U satisfies the requirements of the lemma. Take $g \in G$ and $\gamma \in \Gamma$ such that $u = g\gamma g^{-1} \in U$. One can write $g\Gamma = g_i h \Gamma$ for some $i \in \llbracket 1, k \rrbracket$ and some $h \in V$, thus $g^{-1}g_i \in \Gamma h \subseteq \Gamma V$. Then one has $g\gamma g^{-1} = g_i h' g_i^{-1}$ for some $h' \in V$, hence $\gamma = (g^{-1}g_i)h'(g^{-1}g_i)^{-1} \subseteq (\Gamma h)h'(\Gamma h)^{-1} \subseteq \Gamma h h' h^{-1} \Gamma$, thus $h h' h^{-1} \in \Gamma$ and since $h h' h^{-1} \in VVV \subseteq W$ it follows that $h h' h^{-1} = e$, that is $h' = e$ and therefore $\gamma = e$ and $u = e$, which concludes the proof. $\square$

Let us now consider a semi-simple real Lie group $G = G_1 \cdots G_k$ where the G_i are simple, and let Γ be a lattice in G . It might be that Γ decomposes into a lattice in each G_i . In that case we may say that the lattice Γ is reducible. The following notion of irreducibility aims to encapsulate the opposite way a lattice can sit inside G .

Definition 1.15. The lattice $\Gamma \subseteq G$ is said to be irreducible if and only if for every $i \in \llbracket 1, k \rrbracket$, the projection of Γ into G/G_i is dense.

Let us give one example. Consider the group $G = \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$. Then $\Gamma = \mathrm{SL}_2(\mathbb{Z}[\sqrt{2}])$ embeds in G in the following way : for every $g \in \mathrm{SL}_2(\mathbb{Z}[\sqrt{2}])$, consider $(g, \sigma(g)) \in G$ where σ is the only non-trivial field automorphism of $\mathbb{Q}[\sqrt{2}]$. This gives an embedding $\Gamma \rightarrow G$. We still denote by Γ the image of that embedding.

It can be checked that Γ is indeed a lattice in G . Moreover, since $\mathrm{SL}_2(\mathbb{Z}[\sqrt{2}])$ is dense in $\mathrm{SL}_2(\mathbb{R})$ (this is easily checked using the fact that transvections generate $\mathrm{SL}_2(\mathbb{R})$), this lattice is in fact irreducible.

Of course, if G is simple then every lattice in G is irreducible.

Now suppose that G is a semi-simple real Lie group and that Γ is a non-irreducible lattice in G .

Writing $G = G_1 \cdots G_k$ with the G_i being simple, we may thus suppose that the image of Γ in $\widehat{G}_1 = G/G_1$ is discrete. Denote it by Λ . Now $\widehat{G}_1/\Lambda = \frac{G}{\langle \Gamma, G_1 \rangle} = \frac{G/\Gamma}{G_1}$ as a metric space acted on by G . In particular, it is a quotient of the homogenous space G/Γ which carries a left-invariant probability, hence it also carries a left-invariant probability measure. Thus, Λ is a lattice in $\widehat{G}_1$.

1.2 Actions of locally compact groups

1.2.1 G -spaces

Definition 1.16. A measurable space $(X, \mathcal{T})$ is said to be a standard Borel space if there exists a metric on X that makes it complete separable, in such a way that $\mathcal{T}$ is then the Borel σ -algebra.

On a standard Borel space, there can be many different measures. The notion of measure class allows to distinguish those in a slightly less rigid way than equality.

Definition 1.17. Let $(X, \mathcal{T})$ be a standard Borel space. Two measures μ, ν on $(X, \mathcal{T})$ are said to be in the same measure class if and only if they have the same null sets. This defines an equivalence relation on the set of all measures on $(X, \mathcal{T})$.

Definition 1.18. Let G be a locally compact group.

A G -space is a measured standard Borel space $(X, \mathcal{T})$ endowed with a measure μ that is σ -finite, and with a measurable action of G that is measure class preserving.

The G -space $(X, \mathcal{T}, \mu)$ is said to be measure preserving if and only if $\mu(gA) = \mu(A)$ for every $A \in \mathcal{T}$ and every $g \in G$.

Here, measurable action means that the map $G \times X \rightarrow X$ is a measurable map, where $G \times X$ is endowed with the product σ -algebra.

The action being measure class preserving means that for every $g \in G$, the measure on $(X, \mathcal{T})$ given by $A \mapsto \mu(gA)$ is in the same measure class as μ .

When defining a G -space, we will generally omit the mention of the σ -algebra $\mathcal{T}$ and of the measure μ which will be implied by the context.

Definition 1.19. Let G be a locally compact group and X be a G -space.

- The G -space is said to be ergodic if and only if every G -invariant measurable subset of X is either null or conull.
- We say it is properly ergodic if and only if it is ergodic and every G -orbit has measure 0.
- It is said to be irreducible if and only if every normal non-central subgroup acts ergodically.
- It is said to be faithful if and only if for every $g \in G$, the set $\{x \in X \mid gx \neq x\}$ has non-zero measure.
- It is said to be essentially transitive if and only if there is a conull G -orbit.

Note that checking all of these definitions only require the knowledge of the measure class of μ , not of μ itself. We may therefore slightly abuse notation by considering a G -space to be defined only by the data of a measure class. Note that, if the G -space is G/H for some closed subgroup $H \subseteq G$, then there is a canonical measure class on G/H which is the one we will be using without referring to it explicitly.

1.2.2 Unitary representations

One of the main recurrent themes in this text will be that of actions of locally compact groups on certain sets. Probably one of the easiest and widely used example of such actions is the notion of unitary representation.

Definition 1.20. Let G be a group. A unitary representation of G is a morphism $G \rightarrow \mathcal{U}(\mathcal{H})$, where $\mathcal{H}$ is a separable Hilbert space.

The spaces on which we will let G act are most generally Hilbert spaces. Of course, we want to keep track of the topology of G in that action, that is ask our action to be "continuous" in some way. To that extent, we will need to choose a topology on $\mathcal{U}(\mathcal{H})$ for every Hilbert space $\mathcal{H}$. It turns out that the right choice is that of the strong operator topology.

Definition 1.21. Let $\mathcal{H}$ be a Hilbert space, $\mathcal{L}(\mathcal{H})$ the space of bounded operators on $\mathcal{H}$. The strong operator topology on $\mathcal{L}(\mathcal{H})$ is the coarsest topology on $\mathcal{L}(\mathcal{H})$ which makes all the maps $\mathcal{L}(\mathcal{H}) \rightarrow \mathcal{H} : f \mapsto f(\xi)$ (for $\xi \in \mathcal{H}$) continuous. It can easily be checked that a basis of open neighborhoods of $g \in \mathcal{L}(\mathcal{H})$ in this topology is given by the $U_{\Xi, \varepsilon} = \{f \in \mathcal{L} \mid \forall \xi \in \Xi, \|f(\xi) - g(\xi)\| < \varepsilon\}$ where Ξ is any finite subset of $\mathcal{H}$ and $\varepsilon > 0$.

We can now define the right notion of continuous representations of locally compact groups.

Definition 1.22. Let G be a locally compact group.

A continuous unitary representation ("cur" for short) of G is a continuous morphism of groups $\pi : G \rightarrow \mathcal{U}(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space and $\mathcal{U}(\mathcal{H}) \subseteq \mathcal{L}(\mathcal{H})$ is the group of unitary operators on $\mathcal{H}$ endowed with the strong operator topology.

The following proposition, of independent interest, can turn out to be extremely useful when checking that a morphism is indeed a cur.

Lemma 1.23. *Let G be a locally compact group and $\mathcal{H}$ be a separable Hilbert space. Consider a morphism of groups $\pi : G \rightarrow \mathcal{U}(\mathcal{H})$.*

Suppose that for every $\eta, \xi \in \mathcal{H}$, the function $g \mapsto \langle \pi(g)(\eta), \xi \rangle$ from G to $\mathbb{R}$ is Borel measurable. Then π is a cur.

Proof. We refer to [7]. □

Let us now give a few examples of such representations.

First of all, if one considers the Hilbert space $L^2(G)$ of real-valued functions from G to $\mathbb{R}$, one can note that $L^2(G)$ is acted on by G : for every $f \in L^2(G)$ and every $g \in G$, one can define $\lambda_g(f) : G \rightarrow \mathbb{R}$ to be the function $x \mapsto f(g^{-1}x)$. This makes λ_g a unitary operator on $L^2(G)$, and makes the application $\lambda : g \mapsto \lambda_g$ a continuous unitary representation of G .

More generally, consider a G -space X . Then $L^2(X)$ is a Hilbert space which is acted on by G . If the G -space is measure preserving, then the action of G on $L^2(X)$ is given by unitaries, and is therefore a continuous unitary representation of G .

One of the most important features of representations is invariant vectors. We now give the following definitions which will turn out to be useful when defining property (T).

Definition 1.24. Let G be a locally compact group, and $\pi : G \rightarrow \mathcal{U}(\mathcal{H})$ a continuous unitary representation of G . We define the following two properties of π :

- (i) One says that π has invariant vectors if and only if there is some non-zero $\xi \in \mathcal{H}$ such that for every $g \in G$, $\pi(g)(\xi) = \xi$.
- (ii) One says that π has almost invariant vectors if and only if, for every $\varepsilon > 0$ and every compact subset $K \subseteq G$, there is $\xi \in \mathcal{H} \setminus \{0\}$ such that :

$$\forall g \in K, \|\pi(g)(\xi) - \xi\| \leq \varepsilon \|\xi\|$$

To give a first example of invariant vectors, let us investigate when the representations of G discussed above do have invariant vectors.

Proposition 1.25. *Let G be a locally compact group σ -compact group, and let H be a closed subgroup of G such that the quotient G/H carries a left-invariant regular measure. Then the representation of G on $L^2(G/H)$ has invariant vectors if and only if $\mu_{G/H}(G/H) < +\infty$.*

Proof. It is clear that if G/H has finite total mass then the constant function equal to 1 is a non-zero invariant vector in $L^2(G/H)$. Let us now consider the converse implication. Suppose there is some $f \in L^2(G/H)$ that is invariant under the action of G on the left. Then, for every $g \in G$ one has :

$$f(gx) = f(x) \quad \text{for almost every } x \in G/H$$

Hence one has by Fubini's theorem :

$$0 = \int_{g \in G} \int_{x \in G} (f(gx) - f(x))^2 d\mu_G(x) d\mu_G(g) = \int_{x \in G} \int_{g \in G} (f(gx) - f(x))^2 d\mu_G(x) d\mu_G(g)$$

Therefore, there is some $x \in G$ such that for almost every $g \in G$ one has $f(gx) = f(x)$ and f is thus constant almost everywhere. Since it is in $L^2(G/H)$ and non-zero, this implies that G/H must have finite total mass, and we are done. $\square$

1.2.3 Induction

Let us now explain one of the most important tools of representation theory of locally compact groups : induced representations.

We consider a locally compact second countable group G , and a lattice Γ in G . We know there exists a left-invariant probability measure $\mu_{G/\Gamma}$ on the homogeneous space G/Γ . Let $\pi : \Gamma \rightarrow \mathcal{U}(\mathcal{H})$ be a continuous unitary representation, where $\mathcal{H}$ is a Hilbert space.

Now consider the space $\mathcal{A}$ of measurable functions $f : G \rightarrow \mathcal{H}$ such that the two following conditions hold :

- (i) For almost every $g \in G$ and every $\gamma \in \Gamma$ one has : $f(g\gamma) = \pi(\gamma^{-1})f(g)$.
- (ii) The integral $\int_{G/\Gamma} \|f(x)\|^2 d\mu_{G/\Gamma}(\bar{x})$ is finite. Note that this integral makes sense since $\|f(g)\|$ only depends on the class of $G \bmod \Gamma$ by (i).

We endow $\mathcal{A}$ with a positive bilinear form by setting for every $f, g \in \mathcal{A}$:

$$\langle f, g \rangle = \int_{G/H} \langle f(x), g(x) \rangle d\mu_{G/\Gamma}(\bar{x})$$

Once again, this is well defined since $\langle f(x), g(x) \rangle$ only depends on $\bar{x} = x\Gamma$, the class of x in the quotient G/Γ . Taking the Hilbert closure of $\mathcal{A}$ relative to this positive bilinear form, we get a Hilbert space $\tilde{\mathcal{H}}$. Now G acts on $\mathcal{A}$ on the left by the formula :

$$\forall g \in G, \forall x \in G, (g \cdot f)(x) = f(g^{-1}x)$$

And therefore it acts on $\tilde{\mathcal{H}}$ as well. This provides a representation $\text{Ind}_\Gamma^G \pi : G \rightarrow \mathcal{U}(\tilde{\mathcal{H}})$ that is indeed a continuous unitary representation, which we call the "induced representation" of π to G .

The following lemma will turn out to be very useful when relating certain properties of lattices to those of the ambient locally compact group.

Lemma 1.26. *Let G be a locally compact group, and Γ a lattice in G , and let $\pi : \Gamma \rightarrow \mathcal{U}(\mathcal{H})$ be a cur of Γ . Then :*

- (i) *π has invariant vectors if and only if $\text{Ind}_\Gamma^G \pi$ has invariant vectors.*
- (ii) *If π almost has invariant vectors then $\text{Ind}_\Gamma^G \pi$ also almost has invariant vectors.*

Proof. Let us first prove (i). Suppose first that π has invariant vectors. Let us consider some $\xi \in \mathcal{H}$ such that $\pi(\gamma)(\xi) = \xi$ for every $\gamma \in \Gamma$. It is then rather easy to convince oneself that the constant function $f : G \rightarrow \mathcal{H}$ equal to ξ is indeed in $\tilde{\mathcal{H}}$ and is invariant under the action of G .

Conversely, if $\text{Ind}_\Gamma^G \pi$ had invariant vectors then there is some non-zero function $f \in \tilde{\mathcal{H}}$ that is invariant under the action of G and thus constant (it can be proven in exactly the same way as in the proof of proposition 1.25) equal to some non-zero vector $\xi \in \mathcal{H}$. Using the fact that $f(\gamma) = \pi(\gamma^{-1})f(1_\Gamma)$ for every $\gamma \in \Gamma$, it follows that ξ is an invariant vector for π , hence the result.

For the proof of (ii) we refer to [7]. □

1.3 Semi-simple Lie groups

1.3.1 Lie groups, Lie algebras and root systems

In all that follows, we will assume some familiarity with the notion of real Lie group, more specifically semi-simple Lie groups. Since this material has been treated in numerous monographs, we will not give any of the proofs of the results mentioned below, which are only stated here for the sake of completeness.

Definition 1.27. Let G be a real connected Lie group which is non-compact. One says that G is simple if and only if all its proper normal subgroups are finite and central.

One says that G is semi-simple if and only if G is a central quotient of $G_1 \times \cdots \times G_k$ for some G_i which are all simple real Lie groups.

Here, by central quotient we mean a quotient by a subgroup of the center (which will of course always be normal). Note that if G is a real semi-simple Lie group, one can always write $G = G_1 \cdots G_k$ for some simple real Lie groups $G_1, \dots, G_k$ of G such that every element of G_i commute with every element of G_j , whenever $i \neq j$. This decomposition is in fact unique, and is referred to as the decomposition of G as a product of simple subgroups. The G_i are the simple Lie subgroups of G .

Definition 1.28. Let G be a real Lie group. The tangent space at the origin of G is called the Lie algebra of G , and is denoted by $\mathfrak{g}$. It is endowed with a Lie bracket operation denoted by $[\cdot, \cdot]$ and inherits the action of G on itself by conjugation, called the adjoint action of G

on $\mathfrak{g}$.

The exponential of an element $g \in \mathfrak{g}$ is denoted by $\exp(g)$ or e^g and is defined to be equal to $f(1)$, where f is the unique solution to the differential equation $f'(t) = g \cdot f(t)$ with initial condition $f(0) = e$.

Let us now explain how the bracket operation as well as the adjoint action of G are defined on the Lie algebra.

We start with the adjoint action. Consider some $x \in \mathfrak{g}$ and some $g \in G$. We then have the one parameter subgroup $t \mapsto e^{tx}$. We can conjugate it by g to get the arc $f : t \mapsto ge^{tx}g^{-1}$. We then define $\text{ad}_g(x)$ to be equal to $f'(0)$. This gives an action of G on $\mathfrak{g}$ that we call the adjoint action of G on its Lie algebra.

Let us now move to the definition of the bracket operation. Consider two vectors $x, y \in \mathfrak{g}$. Then the exponential gives one parameter subgroups e^{tx} and e^{ty} . We can now form the arc $f(t) = [e^{tx}, e^{ty}] = e^{tx}e^{ty}e^{-tx}e^{-ty}$. The goal to define $[x, y]$ will be to look at its derivative at $t = 0$. However, it turns out that $f'(0) = 0$ So we need to look at $f''(0)$. To do so, note that for every $t \in \mathbb{R}$, $f'(t) \in T_{f(t)}G$ but since the tangent bundle TG is trivial (multiplication by elements of G allow us to go from one tangent space to another), $T_{f(t)}G$ is naturally identified with $\mathfrak{g}$. Thus, we can see f' as a function $\mathbb{R} \rightarrow \mathfrak{g}$. This function has a derivative that we denote by f'' which takes values in $\mathfrak{g}$, hence $f''(0) \in \mathfrak{g}$. This $f''(0)$ is what we define to be the bracket of x and y , which is denoted by $[x, y]$.

Before moving to the theory of root systems, we give a few important definitions regarding Lie algebras which will be useful in the following.

Definition 1.29. Let $\mathfrak{g}$ be a Lie algebra.

For every subsets A, B of $\mathfrak{g}$, denote by $[A, B]$ the Lie sub-algebra of $\mathfrak{g}$ generated by the elements of the form $[a, b]$ for $a \in A$ and $b \in B$.

Denote $D^0(\mathfrak{g}) = \mathfrak{g}$ and $D^{n+1}(\mathfrak{g}) = [D^n(\mathfrak{g}), D^n(\mathfrak{g})]$ for every $n \in \mathbb{N}$.

Write $C^0(\mathfrak{g}) = \mathfrak{g}$ and $C^{n+1}(\mathfrak{g}) = [C^n(\mathfrak{g}), \mathfrak{g}]$ for every $n \in \mathbb{N}$.

The Lie algebra $\mathfrak{g}$ is said to be solvable if $D^n(\mathfrak{g}) = \{0\}$ for some $n \in \mathbb{N}$. It is said to be nilpotent if $C^n(\mathfrak{g}) = \{0\}$ for some $n \in \mathbb{N}$.

Let us move to the definition of a Cartan sub-algebra, which is the first step towards the theory of roots systems.

Definition 1.30. A sub-algebra $\mathfrak{h}$ of $\mathfrak{g}$ (that is, a real subspace stable by the Lie bracket) is called abelian if and only if the Lie bracket is zero on $\mathfrak{h}$.

A maximal abelian Lie sub-algebra of $\mathfrak{g}$ is called a Cartan sub-algebra of $\mathfrak{g}$.

In what follows, let $\mathfrak{h}$ be a Cartan sub-algebra of $\mathfrak{g}$. Now $\mathfrak{h}$ acts linearly on $\mathfrak{g}$ by the Lie bracket operation. This actions vanishes on $\mathfrak{h}$ since it is an abelian Lie algebra, hence $\mathfrak{h}$ actually acts on $\mathfrak{g}/\mathfrak{h}$, and this action is commutative. We thus may co-diagonalize it if the eigen-values are real. In everything that follows, we assume that this is the case. This will always be true when dealing with the Lie algebra of a real semi-simple Lie group without compact factors.

More specifically, there is a finite set of linear forms $\Phi \subseteq \mathfrak{h}^*$ and a decomposition of $\mathfrak{g}$ of the form :

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_{\alpha} \quad (1)$$

such that for every $\alpha \in \Phi$, every $h \in \mathfrak{h}$ acts on $\mathfrak{g}_{\alpha}$ by multiplication by $\alpha(h)$. Such a decomposition will be called a Cartan decomposition of the Lie algebra $\mathfrak{g}$. It consists in the data of Φ , $\mathfrak{h}$ and the subspaces $\mathfrak{g}_{\alpha}$.

This set Φ is called the set of roots of $\mathfrak{g}$, or of G . It turns out that this set has very specific geometric properties. We summarize those that we will use in the following.

Proposition 1.31. *Let G be a semi-simple Lie group, $\Phi \subseteq \mathfrak{h}^*$ the set of its roots.*

There exists two subsets $\Delta \subset \Phi$ and $\Phi^+ \subset \Phi$ which are respectively called a basis of Φ and a set of positive roots of Φ such that the following propositions hold.

1. $\alpha \mapsto -\alpha$ is a bijection between Φ^+ and $\Phi \setminus \Phi^+$.
2. Every root in Φ^+ can be written uniquely as a sum of elements of Δ .
3. Δ is a linear basis of $\mathfrak{h}^*$.

1.3.2 Zoology of subgroups

In this section, we fix G a real semi-simple Lie group with a Cartan decomposition :

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_{\alpha}$$

where Φ is the set of roots. We also consider Φ^+ a set of positive roots, and Δ a basis for Φ . We will define a few groups associated to this decomposition.

Upon setting $\mathfrak{n}^+ = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_{\alpha}$ and $\mathfrak{n}^- = \bigoplus_{\alpha \in \Phi \setminus \Phi^+} \mathfrak{g}_{\alpha}$, one has the decomposition :

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}^+ \oplus \mathfrak{n}^-$$

Tori and their subgroups.

First of all, we need the notion of maximal split torus.

Definition 1.32. Let G be a real semi-simple Lie group. A split torus in G is a closed subgroup of G isomorphic to $\mathbb{R}^n$ for some integer $n \in \mathbb{N}$. A maximal split torus is a split torus that is maximal for inclusion among all split tori.

Proposition 1.33. *Consider a Cartan decomposition of G . Then the exponential of the Cartan sub-algebra $\mathfrak{h}$ is a maximal split torus. In particular, it is contained in all parabolic subgroups associated to this Cartan decomposition.*

This particular torus will be called the maximal split torus associated to the given Cartan decomposition. We will generally denote it by S . We will not use it in the following, but the reader should be aware of the fact that proposition 1.36 is also valid for split tori :

Proposition 1.34. *Let G be a real semi-simple Lie group, and S a maximal split torus in G . Then every split torus in G is conjugated to some closed subgroup of S . Moreover, every maximal split torus in G is conjugated to S itself.*

In particular, the dimension of a maximal split torus of G is independent of the chosen torus. It is also equal to the dimension of a Cartan sub-algebra $\mathfrak{h}$. This integer will be called the rank of G , and will be denoted by $\text{rank}(G)$. The group G will be said to be of higher rank if and only if $\text{rank}(G) \geq 2$.

Now one can also define closed connected subgroups of the maximal split torus $S = \exp(\mathfrak{h})$, which correspond to subspaces of $\mathfrak{h}$, the following way. If θ is a subset of Δ , define S_θ to be the exponential of the intersection of kernels of elements of θ . It is then clear from the fact that Δ is a basis of $\mathfrak{h}^*$ that $S_\emptyset = S$, $S_\Delta = \{e\}$ and that S_θ is always isomorphic to $\mathbb{R}^{\text{rank}_{\mathbb{R}}(G) - |\theta|}$.

Now fix some subset $\theta \subseteq \Delta$ and define the following two subgroups :

$$\begin{aligned} R_\theta &= \exp(\{x \in \mathfrak{h} \mid \forall \alpha \in \Delta, \alpha(x) \leq 0, \forall x \in \Delta \setminus \theta, \alpha(x) < 0\}) \\ D_\theta &= S_\theta \cap R_\theta = \exp(\{x \in \mathfrak{h} \mid \forall \alpha \in \theta, \alpha(x) = 0, \forall \alpha \in \Delta \setminus \theta, \alpha(x) < 0\}) \end{aligned}$$

Parabolic subgroups.

Now one can note that $\mathfrak{p} = \mathfrak{h} \oplus \mathfrak{n}^+$ is a solvable Lie sub-algebra of $\mathfrak{g}$. Moreover, it is a maximal such algebra. This gives rise to the following definitions :

Definition 1.35. Let G be a real semi-simple Lie group.

A Borel sub-algebra of $\mathfrak{g}$ is a maximal solvable Lie sub-algebra of $\mathfrak{g}$.

A Borel subgroup of G is a maximal connected solvable Lie subgroup of G .

Proposition 1.36. *Let G be a real semi-simple Lie group.*

All Borel sub-algebras of $\mathfrak{g}$ are conjugated one to another by the adjoint action of G . They are all of the form discussed in the above paragraph.

Borel subgroups of G are exactly exponentials of Borel sub-algebras of $\mathfrak{g}$. They are all conjugated in G .

We typically denote by P a Borel subgroup of G . The above proposition shows that P is the exponential of a Borel Lie sub-algebra $\mathfrak{p}$ in $\mathfrak{g}$. This Lie algebra $\mathfrak{p}$ is itself associated to some Cartan decomposition of $\mathfrak{g}$ by the construction given in the above paragraph. Such a Cartan decomposition will naturally be called "associated to P ".

All of the closed subgroups of G containing P are called parabolic subgroups. To classify them, let us introduce some more notation.

For every subset $A \subseteq \Phi$, define G_A to be the subgroup of G generated by the exponential of $\mathfrak{h} \oplus \bigoplus_{\alpha \in A} \mathfrak{g}_\alpha$, and U_A the subgroup of G generated by the exponential of $\bigoplus_{\alpha \in A} \mathfrak{g}_\alpha$. For every $\theta \subseteq \Delta$, define $[\theta] \subseteq \Phi$ to be the set of roots expressible as linear combinations with integer coefficients of elements of θ .

For every subset $\theta \subseteq \Delta$, define :

$$P_\theta = G_{\Phi^+ \cup [\theta]}$$

It is clear that $P = P_\emptyset$ is the Borel subgroup associated to the given Cartan decomposition, and that $G = P_\Delta$. The following proposition shows that there are a finite number of parabolic subgroups associated to this Cartan decomposition, that is only a finite number of closed subgroups of G containing P .

Proposition 1.37. *The connected close subgroups of G containing P are exactly the P_θ for $\theta \subseteq \Delta$. They are all distinct.*

These groups are called parabolic subgroups of G (associated to the given Cartan decomposition). Of course, the smallest of them is the Borel subgroup P .

We may now give the definition of a few more subgroups which will be used throughout the rest of this text :

$$P_\theta^- = G_{\Phi \setminus \Phi^+ \cup [\theta]} \quad V_\theta = U_{\Phi^+ \setminus [\theta]} \quad V_\theta^- = U_{(\Phi \setminus \Phi^+) \setminus [\theta]}$$

Finally, and this will be the last ad-hoc definition of this section, we set for every $\theta \subseteq \Delta$:

$$L_\theta^- = Z_G(S_\theta) \cap V^- \quad L_\theta = Z_G(S_\theta) \cap V$$

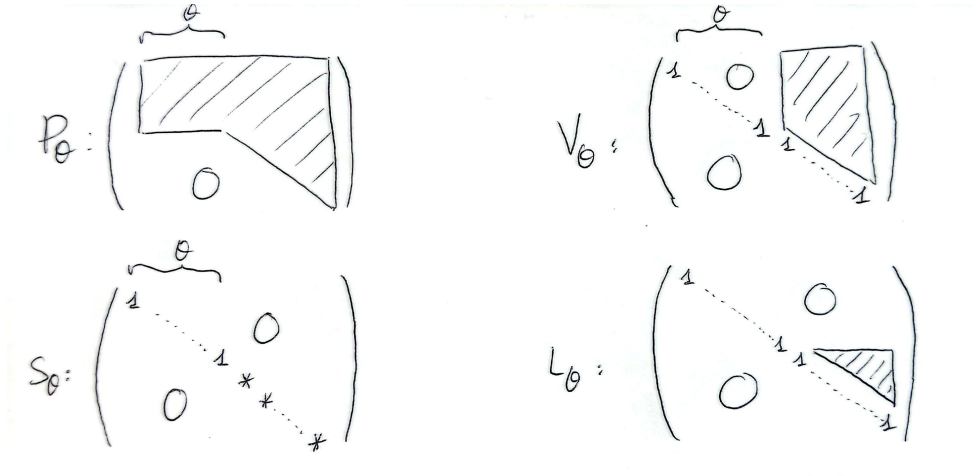
Where $Z_G(S_\theta)$ is of course the centralizer of S_θ in G .

Here are a few relations between all of these groups that the reader will find in [8].

Lemma 1.38. *Let $\theta \subseteq \Delta$. One has :*

$$Z_G(S_\theta) = P_\theta \cap P_\theta^- \quad P_\theta^- = Z_G(S_\theta) \rtimes V_\theta^- \quad L_\theta^- = P_\theta \cap V^- \quad V^- = L_\theta^- \rtimes V_\theta^-$$

Here is a drawing in the case of $\mathrm{GL}_n(\mathbb{R})$ which gives an idea of why this should be true. This is of course not to be taken too seriously, since $\mathrm{GL}_n(\mathbb{R})$ is not semi-simple.



Now the following lemma, which essentially boils down to some rather unappealing calculations, will be used later on. We state it here for convenience.

Lemma 1.39. *For every $\theta \subseteq \Delta$ and every $s \in R_\theta$, the automorphism $p \mapsto sps^{-1}$ of V_θ as well as the automorphism $p \mapsto s^{-1}ps$ of V_θ^- are contracting.*

For every $s \in R_\Delta$, the automorphism $p \mapsto sps^{-1}$ of P is non-expanding.

Here, by contracting automorphism we mean that upon applying it enough times, every compact goes into every fixed neighborhood of the identity.

$\varphi : H \rightarrow H$ being non-expanding means the following : for every $U \subseteq H$ open neighborhood of e , there is some V open neighborhood of e such that $\varphi^n(V) \subseteq U$ for every $n \geq 0$.

Proof. See [8], Lemma II.3.1. for a rigorous proof.

We will still give a general idea of why this should be true, without getting into the technical details of the proof. Consider a set $\theta \subseteq \Delta$, and some $s \in R_\theta$. Take $t \in \mathfrak{h}$ (the Cartan algebra) which is sent to s by the exponential map. Denote once again by $[\theta]$ the subset of Δ made of the elements of Ψ which are linear combinations of elements of θ with non-negative integer coefficients.

By definition of R_θ , the eigenvalues of the action of t on all $\mathfrak{g}_\alpha$, for $\alpha \in \theta$, are negative, thus also for $\alpha \in [\theta]$. Since $s = \exp(t)$, it follows that s acts on $\mathfrak{g}_\alpha$ with eigenvalues of modulus < 1 , for every $\alpha \in [\theta]$, hence acts contractively. Since the sum of these spaces $\mathfrak{g}_\alpha$ is the Lie algebra of V_θ , s acts as a contracting automorphism of the Lie algebra of V_θ . It turns out that the identification between V_θ and its Lie algebra preserves this, essentially because taking the exponential commutes with conjugation. $\square$

The reader should definitely convince themselves of this lemma by working out by hand the case of $\mathrm{SL}_2(\mathbb{R})$.

1.3.3 The case of $\mathrm{SL}_n(\mathbb{R})$

In this section, we will explicitly compute all of the invariants defined in the above paragraphs in the case of the group $\mathrm{SL}_n(\mathbb{R})$. Let us therefore consider $n \geq 2$, fixed in all of this section.

The Lie algebra $\mathfrak{sl}_n(\mathbb{R})$ of $\mathrm{SL}_n(\mathbb{R})$ is the subspace of $\mathcal{M}_n(\mathbb{R})$ of traceless matrices, which has dimension $n^2 - 1$. Now let us define $\mathfrak{h}$ to be the Lie sub-algebra of diagonal matrices in $\mathfrak{sl}_n(\mathbb{R})$:

$$\mathfrak{h} = \{m \in \mathcal{M}_n(\mathbb{R}) \mid \mathrm{Tr}(m) = 0, m \text{ is diagonal.}\}$$

it is clear that $\mathfrak{h}$ is abelian, has dimension $n - 1$, and is in fact a Cartan sub-algebra : any element of $\mathfrak{sl}_n(\mathbb{R})$ commuting with all of $\mathfrak{h}$ leaves fixed the eigenspaces of elements of $\mathfrak{h}$, and this is enough to show that it stabilizes all of the lines directed by an element of the canonical basis of $\mathbb{R}^n$, and is hence diagonal.

It follows that a maximal split torus of $\mathrm{SL}_n(\mathbb{R})$ is given by :

$$S = \exp(\mathfrak{h}) = \{m \in \mathcal{M}_n(\mathbb{R}) \mid \det(m) = 1, m \text{ is diagonal with positive diagonal entries.}\}$$

and that the rank of $\mathrm{SL}_n(\mathbb{R})$ is $n - 1$.

Let us now move to the explicit determination of the roots of $\mathrm{SL}_n(\mathbb{R})$, or (which is the same) of $\mathfrak{sl}_n(\mathbb{R})$. To do that, we define $e = (e_{i,j})_{1 \leq i,j \leq n}$ to be the canonical basis of $\mathcal{M}_n(\mathbb{R})$, and write $\varepsilon_i = e_{i,i} - e_{i+1,i+1}$ for every $i \in \llbracket 1, n-1 \rrbracket$. It follows that $\varepsilon = (\varepsilon_i)_{1 \leq i \leq n-1}$ is a basis of $\mathfrak{h}$.

Now, for every $i \in \llbracket 1, n-1 \rrbracket$, one has by a simple calculation, using the Kronecker delta notation :

$$[\varepsilon_i, e_{j,k}] = \delta_{i,j}e_{i,k} - \delta_{i+1,j}e_{i+1,k} - \delta_{k,i}e_{j,i} + \delta_{k,i+1}e_{j,i+1} = (\delta_{i,j} + \delta_{i,k-1} - \delta_{i,k} - \delta_{i,j-1})e_{j,k}$$

Hence for every $j, k \in \llbracket 1, n \rrbracket$ we may define $\alpha_{j,k}(\varepsilon_i) = \delta_{i,j} + \delta_{i,k-1} - \delta_{i,k} - \delta_{i,j-1}$ and get by linear extension a unique element $\alpha_{j,k} \in \mathfrak{h}^*$ that satisfies :

$$\forall h \in \mathfrak{h}, [h, e_{j,k}] = \alpha_{j,k}(h)e_{j,k}$$

Note moreover that if $\varepsilon^* = (\varepsilon_i^*)_{1 \leq i \leq n-1}$ is the basis of $\mathfrak{h}^*$ dual to ε , then one has for every $j, k \in \llbracket 1, n \rrbracket$:

$$\alpha_{j,k} = \varepsilon_j^* + \varepsilon_{k-1}^* - \varepsilon_k^* - \varepsilon_{j-1}^* = \beta_j - \beta_k$$

with $\beta_k = \varepsilon_{k-1}^* - \varepsilon_k^*$ and the natural convention $\varepsilon_0^* = \varepsilon_n^* = 0$, and with $\beta_k(h) = h_k$ for every $h = (h_1, \dots, h_n) \in \mathfrak{h}$.

Now all of the $\alpha_{j,k}$ live in $\mathfrak{h}^*$, and the only zero ones occur when $j = k$. Therefore, upon setting :

$$\Phi = (\alpha_{j,k})_{1 \leq j \neq k \leq n}$$

one has for every $j, k \in \llbracket 1, n \rrbracket$ distinct $\mathfrak{g}_{\alpha_{j,k}} = \mathrm{Vect}(e_{j,k})$, hence :

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_{\alpha}$$

and Φ is thus a set of roots for $\mathfrak{sl}_n(\mathbb{R})$. Note that it has cardinal $n^2 - n$. It remains to find a basis Δ for Φ which should have cardinal $n - 1$, and this comes in the form of the family :

$$\Delta = (\alpha_{j,j+1})_{1 \leq j \leq n-1}$$

Indeed, it can be checked that every element of Φ is a linear combination of elements of Δ using the fact that for $\alpha_{j,k} = \beta_k - \beta_j$. For this basis, the positive elements Φ^+ are exactly the $\alpha_{j,k}$ with $k > j$, as one can write $\alpha_{j,k} = \sum_{l=j}^{k-1} \alpha_{l,l+1}$ whenever $j < k$. There are $\binom{n}{2}$ of them.

Hence , it follows that :

$$\mathfrak{p} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_{\alpha} = \mathfrak{h} \oplus \bigoplus_{j < k} \mathrm{Vect}(e_{j,k})$$

which is nothing but the space of traceless upper triangular matrices. Therefore, the Borel subgroup P which is the exponential of $\mathfrak{p}$ is actually the subgroup of upper triangular matrices

with positive eigenvalues and determinant one. It is, without any surprise, solvable. In a similar fashion, one can give a description of P_θ for some subset $\theta \subseteq \Delta$. For example, if $\theta = \{\alpha_{1,2}, \dots, \alpha_{k,k+1}\}$ for some interger $k \in \llbracket 1, n \rrbracket$, then P_θ is the set of matrices of $\mathrm{SL}_n(\mathbb{R})$ of the form :

$$\begin{pmatrix} * & \cdots & * & \cdots & \cdots & * \\ \vdots & & \vdots & & & \vdots \\ * & \cdots & * & \cdots & \cdots & * \\ 0 & \cdots & 0 & \ddots & & \vdots \\ \vdots & & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & * \end{pmatrix}$$

Where the top left square block has size $(k+1) \times (k+1)$, and the diagonal coefficients from the $k+2$ -th to the n -th are positive.

Let us now quickly turn our attention to R_θ and S_θ , which are subgroups of the diagonal group S . More precisely, consider θ a subset of Δ , and write $\Delta \setminus \theta = \{\alpha_{i_1, i_1+1}, \dots, \alpha_{i_k, i_k+1}\} \subseteq \Delta$ where $1 \leq i_1 < \dots < i_k \leq n-1$.

Now to calculate R_θ , note that an element $h = (h_1, \dots, h_n) \in \mathfrak{h}$ satisfies $\alpha_{j,j+1} \leq 0$ for every $j \in \llbracket 1, n \rrbracket$ if and only if $h_1 \leq \dots \leq h_n$. Moreover, the condition $\alpha_{i_t, i_t+1} < 0$ for every $t \in \llbracket 1, k \rrbracket$ translates to $h_{i_t} < h_{i_t+1}$. Therefore, upon taking the exponential, we get that R_θ is the set of diagonal matrices with determinant 1 and positive diagonal entries $0 < \lambda_1 \leq \dots \leq \lambda_n$ satisfying $\lambda_{i_t} < \lambda_{i_t+1}$ for every $t \in \llbracket 1, k \rrbracket$.

In a similar fashion, we conclude that D_θ is the set of diagonal matrices of the form :

$$\begin{pmatrix} \mu_1 I_{i_1} & 0 & \cdots & 0 \\ 0 & \mu_2 I_{i_2-i_1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \mu_k I_{i_k-i_{k-1}} \end{pmatrix}$$

where $0 < \mu_1 < \dots < \mu_k$ and $\mu_1^{i_1} \mu_2^{i_2-i_1} \dots \mu_k^{i_k-i_{k-1}} = 1$.

1.3.4 Classification of real semi-simple Lie groups

Consider a real semi-simple Lie group $G = G_1 \cdots G_k$ where the G_i are its simple components. Then, G is a quotient of $G_1 \times \dots \times G_k$ by a finite subgroup, hence the Lie algebra of G and of $G_1 \times \dots \times G_k$ are the same. This phenomenon is what we call a local isomorphism.

Definition 1.40. Let G, H be two semi-simple Lie groups. One says that they are locally isomorphic if and only if their Lie algebras are isomorphic.

The main result which will enable us to classify real semi-simple Lie groups is the following. We will not prove it as it will not be crucial in the following developments.

Proposition 1.41. *Let G, H be connected real semi-simple Lie groups. Then the following two proposition hold :*

$\pi_1(G)$ is a finite group.

If G and H are simply-connected and locally isomorphic then they are isomorphic.

We easily deduce the following corollary.

Corollary 1.42. *Let G_1, G_2 be two connected real semi-simple Lie groups. Suppose that G_1 and G_2 are locally isomorphic. Then there exists a unique real semi-simple simply connected Lie group G such that G_1 and G_2 are finite quotients of G .*

Proof. Consider $\tilde{G}_1$ and $\tilde{G}_2$ the universal cover of G_1 and G_2 . These are also semi-simple real Lie groups, which are simply connected. Moreover, G_1 is a quotient of $\tilde{G}_1$ by $\pi_1(G_1)$ which is finite, thus G_1 and $\tilde{G}_1$ are locally isomorphic, and similarly for G_2 . Since $\tilde{G}_2$ and $\tilde{G}_1$ are locally isomorphic and simply connected, it follows from the above lemma that they are isomorphic, hence the desired result.

The uniqueness follows from the uniqueness of the universal cover. □

2 Amenability and property (T)

2.1 Amenable groups

2.1.1 Invariant means

Definition 2.1. Let G be a locally compact group, and W a left-invariant subspace of $L^\infty(G)$.

A mean on W is a linear form Λ on W which has norm 1 and is non-negative, that is for every $f \in W$ such that $f \geq 0$ almost everywhere, one has $\Lambda(f) \geq 0$.

Definition 2.2. G is said to be amenable if and only if there is a left invariant mean on $L^\infty(G)$, that is a mean $\Lambda : L^\infty(G) \rightarrow \mathbb{R}$ such that for every $f \in L^\infty(G)$, $g \in G$ one has $\Lambda(g \cdot f) = \Lambda(f)$.

Now it may happen that one gets a mean on some subset of $L^\infty(G)$, such as the set of bounded continuous functions or compactly supported continuous functions. It turns out that in such a case, it is always possible to extend such a mean to the whole of G , and thus G is amenable. This will be proved in proposition 2.4 below. Before stating it, we need to define several subspaces of $L^\infty(G)$.

Definition 2.3. Let X be a locally compact group. Define the following subsets of $L^\infty(G)$.

- $\mathcal{C}_b(G)$ is the space of continuous bounded functions on G .
- $LUC(G)$ is the space of bounded left-uniformly continuous functions of G , that is bounded functions $f : G \rightarrow \mathbb{R}$ such that for every $\varepsilon > 0$ there is some open neighborhood U of e such that for every $x \in G$ and every $y \in xU$, $|f(y) - f(x)| < \varepsilon$.
- $RUC(G)$ is the space of bounded right-uniformly continuous functions of G , that is bounded functions $f : G \rightarrow \mathbb{R}$ such that for every $\varepsilon > 0$ there is some open neighborhood U of e such that for every $x \in G$ and every $y \in Ux$, $|f(y) - f(x)| < \varepsilon$.
- $UC(G)$ is the space of bounded uniformly continuous functions, that is $UC(G) = LUC(G) \cap RUC(G)$.

Note that, defining the usual way the notion of convolution of a function with a measure, a function $f \in \mathcal{C}_b(G)$ is in $LUC(G)$ if and only if :

$$\begin{aligned} G &\rightarrow L^\infty(G) \\ g &\mapsto \delta_g * f \end{aligned}$$

is continuous. Similarly, it is in $RUC(G)$ if and only if the following map is continuous.

$$\begin{aligned} G &\rightarrow L^\infty(G) \\ g &\mapsto f * \delta_g \end{aligned}$$

Proposition 2.4. *Let G be a locally compact group. The following propositions are equivalent.*

- (i) G is amenable, that is there is a left-invariant mean on $L^\infty(G)$.
- (ii) There is a left-invariant mean on $\mathcal{C}_b(G)$.
- (iii) There is a left-invariant mean on $LUC(G)$.
- (iv) There is a left-invariant mean on $RUC(G)$.
- (v) There is a left-invariant mean on $UC(G)$.

Proof. It is clear that we only need to prove that $(v) \Rightarrow (i)$. This is done in [9], see Theorem 1.1.9. $\square$

2.1.2 Folner's condition

Amenability can be characterized in many way, generally involving the axiom of choice. One of the most important characterizations of amenability is the following, due to Folner.

Theorem 2.5 (Folner). *Let G be a locally compact group.*

Suppose that, for every finite subset A of G , every $\varepsilon > 0$, there exists U a measurable subset of G of non-zero measure such that $\forall g \in G$, $\mu_G(U \Delta gU) \leq \varepsilon \mu_G(U)$. Then G is amenable. Moreover, the converse is also true under the additional assumption that G is second countable.

The condition in the statement of the above theorem is generally referred to as "Folner's condition". The set F will be called a "Folner set".

Proof. One direction is rather easy, and the other one is what Folner is mostly remembered for.

First suppose that Folner's condition is satisfied, that is for every compact $K \subseteq G$ and every $\varepsilon > 0$ there exists $U \subseteq G$ such that $\mu_G(U) > 0$ and $\mu_G(U \Delta gU) \leq \varepsilon \mu_G(U)$ for every $g \in G$. We will construct a left-invariant mean on $L^\infty(G)$.

Let H be the linear subspace of $L^\infty(G)$ generated by the $f - g \cdot f$ for $f \in L^\infty(G)$ and $g \in G$ and $\overline{H}$ its closure in $L^\infty(G)$. Let $h \in H$. We will prove that $\text{ess-sup}(h) \geq 0$. Suppose by way of contradiction that is false. Then one gets some $\varepsilon > 0$ such that $h \leq -\varepsilon$ almost everywhere. Writing $h = \sum_i f_i - g_i \cdot f_i$ for i in some finite index set I , $g_i \in G$ and $f_i \in L^\infty(G)$, it follows that :

$$\forall x \in G, \sum_{i \in I} f_i(x) - f_i(g_i^{-1}x) \leq -\varepsilon$$

Now using Folner's condition, take a measurable set $U \subseteq G$ of non-zero measure such that $\mu_G(U \Delta g_i^{-1}U) \leq \varepsilon' \cdot \mu_G(U)$ for every $i \in I$, for some $\varepsilon' > 0$ to be chosen later. Integrating the above inequality over U , one then gets :

$$\begin{aligned} -\varepsilon \mu_G(U) &\geq \sum_{i \in I} \int_U f_i(x) - f_i(g_i^{-1}x) d\mu_G(x) \\ &\geq \sum_{i \in I} \int_U f_i(x) d\mu_G(x) - \int_{g_i^{-1}U} f_i(x) d\mu_G(x) \quad (\text{using left-invariance of the Haar measure}) \\ &\geq \sum_{i \in I} -\mu_G(U \Delta g_i^{-1}U) \cdot \|f_i\| \geq -\varepsilon' \cdot \mu_G(U) \cdot \left(\sum_{i \in I} \|f_i\| \right) \end{aligned}$$

So upon taking $\varepsilon' > 0$ such that $\varepsilon' \cdot (\sum_{i \in I} \|f_i\|) < \varepsilon$, we get the desired contradiction. Thus we have proven that $\text{ess-sup}(h) \geq 0$ for every $h \in H$. Now since H is dense in $\overline{H}$ and ess-sup is a continuous function, this stays true on $\overline{H}$:

$$\forall h \in \overline{H}, \text{ess-sup}(h) \geq 0$$

In particular, the constant function -1_G equal to -1 is not in $\overline{H}$, and since $\overline{H}$ is a linear subspace the same is true for all constant functions apart from 0. Therefore one has the direct sum $E = \overline{H} \oplus \mathbb{R}1_G$, with $\overline{H}$ and $\mathbb{R}1_G$ being closed sub-spaces. Thus we can define a linear functional Λ_0 on E by setting $\Lambda_0(H) = \{0\}$ and $\Lambda_0(1_G) = 1$, and by the closed graph theorem it will be continuous. Moreover, one has for every $h \in \overline{H}$ and every $\lambda \in \mathbb{R}$:

$$\|h + \lambda 1_G\| \geq \text{ess-sup}(h + \lambda 1_G) = \text{ess-sup}(h) + \lambda \geq \lambda = \Lambda_0(h + \lambda 1_G)$$

So Λ_0 has operator norm at most 1, and in fact exactly 1 since $\Lambda_0(1_G) = 1$. By the Hahn-Banach theorem, one can extend Λ_0 to a continuous linear map $\Lambda : L^\infty(G) \rightarrow \mathbb{R}$ of norm 1. Let us now prove that Λ is indeed a left-invariant mean. It is clearly left invariant since for every $f \in L^\infty(G)$ and every $g \in G$ one has $f - g \cdot f \in H$ so $\Lambda(f - g \cdot f) = 0$. Λ has norm one, so it only remains to show that Λ is non-negative. Let $f \in L^\infty(G)$ be such that $f \geq 0$ almost everywhere. Then, one has $\left\| \frac{\|f\|}{2} - f \right\| = \frac{\|f\|}{2}$ and since Λ has operator norm 1 one has :

$$\frac{\|f\|}{2} \geq \Lambda \left(\frac{\|f\|}{2} - f \right) = \frac{\|f\|}{2} - \Lambda(f) \quad \text{hence} \quad \Lambda(f) \geq 0$$

Therefore, Λ is a left-invariant mean, which proves that G is indeed amenable.

Conversely, let us now suppose that G is second countable and amenable, and let us prove Folner's condition. Let $\varepsilon > 0$ and A be a non-empty finite subset of G . Write $\varepsilon' = \frac{\varepsilon}{4|A|}$.

We will first prove that there is some $f \in L^1(G)$ which is almost everywhere non-negative and of norm 1 such that $\forall a \in A, \|f - a \cdot f\|_1 \leq \varepsilon'$. Suppose by way of contradiction that this is not the case, and write $\Phi = \{f \in L^1(G) \mid f \geq 0 \text{ with compact support, } \|f\|_1 = 1\}$. Then, the hypothesis ensures that the set $\Psi = \{(f - a \cdot f)_{a \in A} \mid f \in \Phi\} \subseteq (L^1(G))^A$ is separated away from zero : every element in this set has norm at least ε' . It is moreover clearly convex since Φ is. Now $\overline{\Psi}$, the closure of Ψ , is a closed convex set separated away from 0 : for every $x \in \overline{\Psi}$, one has $\|x\| \geq \varepsilon'$. Therefore by a geometric form of Hahn-Banach theorem, there must be a continuous linear form $\lambda : (L^1(G))^A \rightarrow \mathbb{R}$ such that $\lambda(\overline{\Psi}) \subseteq [1, +\infty[$. Since the dual of $L^1(G)$ is naturally isomorphic to $L^\infty(G)$ (here we use the fact that G be second countable), this gives $h_a \in L^\infty(G)$ for every $a \in A$ such that :

$$\forall f \in \Phi, \sum_{a \in A} \int_G (f - a \cdot f) h_a d\mu_G(x) \geq 1$$

Using the fact that simple functions are dense in $L^\infty(G)$, one may replace the h_a by simple functions (that is, linear combinations of indicators of Borel sets) if we allow the 1 on the right to decrease. It follows that for every $a \in A$, there are Borel sets $A_{a,1}, \dots, A_{a,p_a} \subseteq G$ and $b_{a,1}, \dots, b_{a,p_a} \in \mathbb{R}$ such that :

$$\forall f \in \Phi, \sum_{a \in A} \sum_{i=1}^{p_a} b_{a,i} \left(\int_{A_{a,i}} f - \int_{aA_{a,i}} f \right) \geq \frac{1}{2}$$

Now let us fix a function f of the form $f = \frac{1_U}{\|1_U\|_1} \in \Phi$ for some compact $U \subseteq G$ with non-zero measure. The above then becomes :

$$\frac{1}{2} \mu_G(U) \leq \sum_{a \in A} \sum_{i=1}^{p_a} b_{a,i} (\mu_G(U \cap A_{a,i}) - \mu_G(U \cap aA_{a,i})) = \sum_{a \in A} \sum_{i=1}^{p_a} b_{a,i} (\mu_G(U \cap A_{a,i}) - \mu_G(a^{-1}U \cap A_{a,i}))$$

Since this is true for U , it is also true for gU for every $g \in G$ and therefore upon setting $h_a(g) = \mu_G(gU \cap A_{a,i})$ for all $g \in G$, it follows that :

$$\frac{\mu_G(U)}{2} = \frac{\mu_G(gU)}{2} \leq \sum_{a \in A} \sum_{i=1}^{p_a} b_{a,i}(h_a - a \cdot h_a)$$

and by applying the left-invariant mean Λ to this inequality in $L^\infty(G)$, we get $\frac{\mu_G(U)}{2} \leq 0$, which gives the desired contradiction. Therefore, we have proven by contradiction that there is some function $f \in L^1(G)$ of norm 1 such that $f \geq 0$ almost everywhere, and such that for every $a \in A$, one has $\|f - a \cdot f\|_1 \leq \varepsilon'$.

For every $x \in \mathbb{R}$, define $U_x = f^{-1}([x, +\infty[)$. Then by usual integration techniques, one has : $\int_0^{+\infty} \mu(U_x) dx = \int_G f d\mu_G = 1$. Similarly, one computes for every $a \in A$:

$$\begin{aligned} \int_0^{+\infty} \mu_G(U_x \Delta aU_x) dx &= \int_0^{+\infty} \int_G 1_{U_x \Delta aU_x}(g) d\mu_G(g) dx = \int_0^{+\infty} \int_G 1_{U_x} + 1_{aU_x} - 2 \cdot 1_{U_x \cap aU_x} d\mu_G(g) dx \\ &= \int_G \int_0^{+\infty} 1_{f(g) \geq x} + 1_{a \cdot f(g) \geq x} - 2 \cdot 1_{\min(f, a \cdot f)(g) \geq x} dx d\mu_G(g) \\ &= \int_G f + a \cdot f - 2 \min(f, a \cdot f) d\mu_G \\ &\leq 2 \int_G |f - a \cdot f| = 2\|f - a \cdot f\|_1 \leq 2\varepsilon' = \int_0^{+\infty} 2\varepsilon' \mu_G(U_x) dx \end{aligned}$$

Thus, upon summing over $a \in A$ and using $\varepsilon = 4|A|\varepsilon'$ it follows that :

$$\int_0^{+\infty} \sum_{a \in A} \mu_G(U_x \Delta aU_x) dx \leq \int_0^{+\infty} \frac{\varepsilon}{2} \mu_G(U_x) dx < \int_0^{+\infty} \varepsilon \mu_G(U_x) dx$$

Therefore, there must be some $x \in \mathbb{R}_+$ such that $\sum_{a \in A} \mu_G(U_x \Delta aU_x) < \varepsilon \mu_G(U_x)$. Taking $F = U_x$, we clearly get that $\mu_G(F \Delta aF) \leq \varepsilon \mu_G(F)$ for every $a \in A$ and that $\mu_G(F) > 0$, which is the desired condition and thus concludes the proof. $\square$

2.1.3 Hereditary properties

Using the two characterizations (the first one involving invariant means, and the second one Folner's condition) of amenability we have developed in the above, we will be able to prove that amenability is in some sense a very stable property : it passes to closed subgroups and to quotient by closed subgroups. The quotient part is rather easy.

Proposition 2.6. *Suppose G is a locally compact amenable group and H is a closed normal subgroup of G . Then G/H is also amenable.*

Proof. Consider a left-invariant mean $\Lambda : L^\infty(G) \rightarrow \mathbb{R}$. Then if H is a closed normal subgroup of G and $p : G \rightarrow G/H$ is the canonical projection, then $\mu_{G/H} = p_* \mu_G$ defined by $\mu_{G/H}(U) = \mu_G(p^{-1}(U))$ for every Borel set U of G/H is indeed a Haar measure on G/H . Now write $\Lambda'(f) = \Lambda(f \circ p)$ for every $f \in L^\infty(G/H)$. Λ' is easily seen to be left-invariant mean on G/H . $\square$

The rest of this section will be devoted to the proof of this surprisingly hard theorem.

Theorem 2.7. *Let G be a locally compact group that is amenable, and H a closed subgroup of G . Then H is also amenable.*

In order to prove this fact, we will have to take a detour through the theory of Bruhat functions.

Definition 2.8. Let G be a locally compact group, H a subgroup of G . A Bruhat function for H is a continuous non-negative function $\beta : G \rightarrow \mathbb{R}$ such that the two following conditions hold :

- (i) For every compact subset K of G , the support of $\beta|_{KH}$ (the restriction of β to KH) is compact.
- (ii) For every $g \in G$, one has :

$$\int_H \beta(gh) d\mu_H(h) = 1$$

We will see that Bruhat functions always exist, and they will allow us to descend a left-invariant mean from any locally compact amenable group G to its closed subgroups.

Lemma 2.9. *Let G be a locally compact group, H a closed subgroup of G . Then there exists a Bruhat function $\beta : G \rightarrow \mathbb{R}$ for H .*

Proof. Start with a compactly supported continuous non-negative function $\varphi : G \rightarrow \mathbb{R}$ of integral 1 satisfying $\varphi(g) = \varphi(g^{-1})$ for every $g \in G$ and $\varphi(1) = 1$. Such a function can be constructed by Uryshon's lemma. Write $U = \varphi^{-1}(\mathbb{R}_+^*)$, whose closure is the support of φ . It is an open, symmetric, relatively compact neighborhood of the identity.

By Zorn's lemma, we may construct a subset S of G with the property that $s \notin UtH$ for every distinct $s, t \in S$ that is maximal with respect to set inclusion.

Let us now prove that for every compact set $K \subseteq G$, the set $\{s \in S \mid KH \cap \overline{U}s \neq \emptyset\}$ is finite. Let $K \subseteq G$ be compact. In order to prove this, suppose by way of contradiction that the above set is infinite. We may therefore find sequences $(s_n)_{n \in \mathbb{N}} \in S^{\mathbb{N}}$ and $(h_n)_{n \in \mathbb{N}} \in H^{\mathbb{N}}$ such that $s_n \neq s_m$ whenever $n \neq m$, and $s_n h_n \in K\overline{U}$. Upon extracting, we may suppose that $(s_n h_n)_{n \in \mathbb{N}}$ converges to some $g \in K\overline{U}$. Upon taking a neighborhood of the identity V such that $VV \subseteq U$, we may find distinct $n, m \in \mathbb{N}$ such that $s_n h_n, s_m h_m \in Vg$. Therefore $(s_n h_n)(s_m h_m)^{-1} \in Vg g^{-1} V^{-1} \subseteq U$ so that $s_n h_n \in U s_m h_m$ and finally $s_n \in U s_m H$ which contradicts the definition of S since $n \neq m$. Therefore, we have proven by contradiction that for every compact subset K of G the set $\{s \in S \mid KH \cap \overline{U}s \neq \emptyset\}$ is finite. This fact will be referred in what follows as the "finiteness property of S ".

It follows that we can set, for every $g \in G$:

$$f(g) = \sum_{s \in S} \varphi(g s^{-1})$$

Indeed, for every compact subset $K \subseteq G$ there is only a finite number of $s \in S$ such that $KH \cap \overline{U}s \neq \emptyset$ hence such that $Ks^{-1} \cap \overline{U} \neq \emptyset$ hence such that $\varphi(Ks^{-1}) \neq \{0\}$. Therefore, for every compact subset K of G the sum stays finite as g ranges over K . So f is well-defined

and continuous on the whole of G .

Now f already has the property (i) in the statement of the theorem. Indeed, let K be a compact subset of G . Then one has :

$$\{g \in KH \mid f(g) > 0\} = \bigcup_{s \in S} (Us \cap KH)$$

but the union is actually finite by the finiteness property of S and therefore since Us is relatively compact, $\{g \in KH \mid f(g) > 0\}$ also is, which proves that f has the property (i) in the definition of Bruhat functions.

The issue is now that f does not have the property (ii) in the definition of Bruhat functions. To solve this issue, define :

$$\begin{aligned} \alpha : G &\rightarrow \mathbb{R} \\ g &\mapsto \int_H f(gh) d\mu_H(h) \end{aligned}$$

α is well defined since $f|_{\{g\}H}$ has compact support (here we use the fact that f has the property (i) of the definition of Bruhat functions). Moreover, it is continuous. Indeed, still by the fact f has property (i), for every compact K there is a compact $C \subseteq H$ such that :

$$\forall g \in K, \alpha(g) = \int_C f(gh) d\mu_H(h)$$

hence α is continuous on K by the dominated convergence theorem and continuity of f . This being true for every compact K , α is therefore continuous on the whole of G .

We want to set $\beta = \frac{f}{\alpha}$, thus we have to check that α is always non-zero. Suppose there is some $g \in G$ such that $\alpha(g) = 0$. Then since f is continuous and non-negative, one must have $f|_{gH} = 0$, that is $\forall h \in H, \forall s \in S, ghs^{-1} \notin U$ i.e. $\forall s \in S, g \notin UsH$. Recalling the definition of S , this goes against its maximality, hence a contradiction. So α is everywhere non-zero, and we can finally set $\beta = \frac{f}{\alpha}$.

β is continuous and non-negative because f and α are. Since f has property (i) in the definition of Bruhat functions, β clearly satisfies it as well. Finally, one has for every $g \in G$, using the fact that μ_H is left-invariant :

$$\begin{aligned} \int_G \beta(gh) d\mu_H(h) &= \int_G \frac{f(gh)}{\alpha(gh)} d\mu_H(h) = \int_G f(gh) \left(\int_H f(ghh') d\mu_H(h') \right)^{-1} d\mu_H(h) \\ &= \int_G f(gh) \left(\int_H f(gh') d\mu_H(h') \right)^{-1} d\mu_H(h) = 1 \end{aligned}$$

Therefore β is a Bruhat function, which concludes the proof. $\square$

Using these Bruhat functions, we can now prove that amenability passes to closed subgroups.

Proof of Theorem 2. Let G be a locally compact amenable group, and H be a closed subgroup of G . By the above lemma, we can start with a Bruhat function $\beta : G \rightarrow \mathbb{R}$ for the subgroup H . Define an operator $T : L^\infty(H) \rightarrow L^\infty(G)$ by setting, for every $f \in L^\infty(H)$:

$$T(f) : g \mapsto \int_H f(h) \beta(g^{-1}h) d\mu_H$$

This gives a well defined linear operator $T : L^\infty(H) \rightarrow L^\infty(G)$ that has operator norm 1, sends the constant function 1_H to 1_G , sends non-negative functions to non-negative functions, and satisfies for every $f \in L^\infty(H)$ and every $h \in H : T(h \cdot f) = h \cdot T(f)$.

Now given a left-invariant mean $\Lambda : L^\infty(G) \rightarrow \mathbb{R}$, define $\Lambda' = \Lambda \circ T : L^\infty(H) \rightarrow \mathbb{R}$. It is clearly a non-negative linear operator since Λ is and T respects non-negativity. Moreover, it sends 1_H to 1, has norm 1 since it is the case of T and Λ , and is left-invariant under the action of H since Λ is and T is H -equivariant. Therefore Λ' is a left invariant mean on H , which proves that H is amenable. $\square$

We have proven that amenability passes to closed subgroups and quotients by closed normal subgroups. In fact, the implication also goes the other way round, as shown in the following.

Proposition 2.10. *Let G be a locally compact group, and H a closed normal subgroup of G such that both H and G/H are amenable. Then G is amenable.*

Proof. We keep the notations and assumptions of the statement above. Write Λ_H for a left-invariant mean on H and $\Lambda_{G/H}$ for a left-invariant mean on G/H . Consider $UC(G) \subseteq L^\infty(G)$ the space of uniformly continuous functions on G . For every $f \in UC(G)$, consider :

$$\begin{aligned} T(f) : G &\rightarrow \mathbb{R} \\ g &\mapsto \Lambda_H((g^{-1} \cdot f)|_H) \end{aligned}$$

$T(f)$ is a function defined on G . Moreover, it is continuous : indeed, Λ_H is and $g \mapsto (g \cdot f)|_H$ is also continuous because f is uniformly continuous. Therefore, we get a linear mapping $T : UC(G) \rightarrow \mathcal{C}(G)$ that clearly satisfies $\|T(f)\| \leq \|f\|$ for every $f \in \mathcal{C}_c(G)$. Note that we also have $T(g \cdot f) = g \cdot T(f)$ for every $g \in G$ and every $f \in UC(G)$.

Now consider $f \in \mathcal{C}_c(G)$. For every $h \in H$, every $x \in G$ one has $T(f)(xh) = \Lambda_H(h^{-1}x^{-1} \cdot f) = \Lambda_H(x^{-1} \cdot f) = T(f)(x)$ since Λ_H is invariant under the action of H on the left. Thus $T(f)$ is constant on every class modulo H and therefore it factors uniquely through G/H : there is a unique $T'(f) : G/H \rightarrow \mathbb{R}$ such that $T'(f) \circ p = T(f)$ where p is the natural projection from G to G/H . Clearly $T'(f)$ is bounded since $T(f)$ is, and it is measurable.

Thus we get a linear operator $T' : E \rightarrow L^\infty(G/H)$. It is moreover clear that T' sends 1_G to $1_{G/H}$, sends non-negative functions to non-negative functions, has norm 1 and is G -equivariant since it is the case for T . Now define for every $f \in UC(G) : \Lambda_0(f) = \Lambda_{G/H} \circ T'(f)$. Due to the above properties of T' , Λ_0 is a linear operator of norm 1 from $UC(G)$ to $\mathbb{R}$ which sends 1_G to 1, sends non-negative functions to non-negative functions and is left-invariant : indeed, for every $g \in G$ and every $f \in UC(G)$ one has :

$$\Lambda_0(g \cdot f) = \Lambda_{G/H}(T'(g \cdot f)) = \Lambda_{G/H}(\bar{g} \cdot T'(f)) = \Lambda_{G/H}(T'(f)) = \Lambda_0(f)$$

Thus we have a left-invariant mean on $UC(G)$, and proposition 2.4 concludes the proof. $\square$

2.1.4 Examples of amenable and non-amenable groups

In this section, we give a few examples of amenable and non-amenable groups. In short, we prove that compact groups, abelian groups and groups with sub-exponential growth are amenable whereas groups containing a non abelian free group are not.

Amenable groups.

Probably the easiest example of amenable groups is the following.

Proposition 2.11. *Let G be a compact group. Then G is amenable.*

Proof. The application :

$$L^\infty(G) \rightarrow L^\infty(G) \\ f \mapsto \frac{1}{\mu_G(G)} \int_G f(g) d\mu_G(g)$$

is already a left-invariant mean. □

Let us now give a somewhat subtler example of amenable groups.

Proposition 2.12. *Let G be a locally compact group that is abelian. Then G is amenable.*

Proof. Consider the set $K \subseteq L^\infty(G)^*$ of means on G (linear functionals of operator norm 1 that send 1_G to 1 and are non-negative). Now K is clearly convex. Moreover, it is a subset of the closed unit ball in $L^\infty(G)^*$ that is closed for the weak-* topology on $L^\infty(G)^*$. Therefore by Banach-Alaoglu's theorem, K is a convex subset of $L^\infty(G)^*$ that is compact for the weak-* topology.

Now for every $g \in G$, define $T_g : L^\infty(G)^* \rightarrow L^\infty(G)^*$ by setting $T_g(\Lambda)(f) = \Lambda(g^{-1} \cdot f)$ for every $\Lambda \in L^\infty(G)^*$, $f \in L^\infty(G)$. It is rather easy to see that for every $g \in G$, $T_g(K) \subseteq K$. It is also easily checked that the T_g for $g \in G$ are weak-* continuous. Finally, the T_g all commute since G is abelian. Therefore, Markov-Kakutani's fixed point theorem (theorem 2.13 below) applies and gives $\Lambda \in K$ that is a fixed point for all of the T_g . The said Λ is a left-invariant mean, thus proving amenability of G . □

Theorem 2.13 (Kakutani-Markov fixed point theorem). *Let V be a locally convex vector space, K a compact convex non-empty subset of V and let S be a set of continuous affine functions from V to V which stabilize K .*

Suppose all the elements of s commute. Then there is some $a \in K$ such that $s(a) = a$ for every $s \in S$.

Proof. For every $R \subseteq S$, write $K^R = \{x \in K \mid \forall s \in R, s(x) = x\}$. Clearly one has :

$$K^S = \bigcap_{s \in S} K^{\{s\}}$$

and the $K^{\{s\}}$ are closed in the compact space K . Therefore, to prove that $K^S \neq \emptyset$ it suffices to prove that for every finite subset $R \subseteq S$ one has $K^R \neq \emptyset$. So we may suppose without loss of generality that S is finite, and write $S = \{s_1, \dots, s_n\}$ for some $n \in \mathbb{N}$. Note that clearly one has $K^S = (K^{S'})^{\{s_n\}}$ with $S' = \{s_1, \dots, s_{n-1}\}$ and $K^{S'}$ being convex compact stable under the action of s_n (since s_n commutes with all elements of S'). Thus, by recurrence, it suffices to take care of the case where S is a singleton. Let us now do so.

Let s be an affine continuous map that leaves K invariant. Let us prove that K contains a fixed point of s . To that extent, consider $x \in K$ and define for every $n \in \mathbb{N}$:

$$u_n = \frac{1}{n+1} \sum_{i=0}^n s^i(x)$$

Since K is convex and stable under the action of s , $(u_n)_{n \in \mathbb{N}}$ is a sequence of elements of K . Since K is compact there is a convergent sub-sequence $(u_{n_i})_{i \in \mathbb{N}}$ of $(u_n)_{n \in \mathbb{N}}$ whose limit a is in K . Now we want to prove that a is a fixed point of s . Since V is locally convex, Hahn-Banach theorem shows that it suffices to prove that for every $f \in V^*$ one has $f(s(a)) = f(a)$. Consider some $f \in V^*$, and write :

$$|f(s(u_n)) - f(u_n)| = \frac{1}{n+1} |f(s^{n+1}(x) - f(x))| \leq \frac{2M}{n+1}$$

where $M = \sup_{y \in K} |f(y)|$ which is finite since K is compact. Upon letting $n = n_i$ and $i \rightarrow +\infty$ in the above, and using the fact that both s and f are continuous, it follows that $|f(s(a)) - f(a)| = 0$, which concludes the proof. $\square$

We deduce the following, using the fact that extension of amenable groups by amenable groups are amenable.

Corollary 2.14. *Every solvable locally compact group is amenable.*

Proof. This follows from the combination of proposition 2.12 and proposition 2.10. $\square$

An other interesting class of amenable groups is given by the following : a locally compact group needs to have exponential growth for it to even have a chance to be non-amenable.

Definition 2.15. Let G be a locally compact group. One says that G is of sub-exponential growth if and only if there is a Borel subset S of G of positive finite measure which generates the group G , satisfies $1 \in S = S^{-1}$ and such that :

$$(\mu_G(S^n))^{\frac{1}{n}} \xrightarrow{n \rightarrow +\infty} 1$$

Proposition 2.16. *Any locally compact group of sub-exponential growth is amenable.*

Proof. We will use theorem 2.5 above, and use the S^n to construct such Folner sets. Suppose by way of contradiction that there is some $\varepsilon > 0$ and some finite set $A \subseteq G$ such that for every measurable subset $F \subseteq G$ of non-zero measure, one has $\mu_G(F \Delta aF) \geq \varepsilon \mu_G(F)$ for some $a \in A$.

Since S generates G , upon replacing S by S^N for some large N (this will not change the fact that $\mu_G(S^n)^{\frac{1}{n}} \xrightarrow{n \rightarrow +\infty} 1$), we may suppose without loss of generality that $A \subseteq S$. We make this assumption in what follows.

Now for every $n \in \mathbb{N}$, one has $aS^n \subseteq S^{n+1}$ for every $a \in A$, thus for some $a \in A$ one has :

$$\varepsilon \mu_G(S^n) \leq \mu_G(S^n \Delta aS^n) = \mu_G(S^n \setminus aS^n) + \mu_G(aS^n \setminus S^n) = 2\mu_G(aS^n \setminus S^n) \leq \mu_G(S^{n+1} \setminus S^n)$$

That is : $\forall n \in \mathbb{N}$, $\mu_G(S^{n+1}) \geq (1 + \varepsilon) \mu_G(S^n)$. From this and the fact that $\mu_G(S) \neq 0$, it follows that $\liminf_{n \rightarrow +\infty} \mu_G(S^n)^{\frac{1}{n}} \geq 1 + \varepsilon$, hence the desired contradiction. $\square$

Remark 2.17. *This proof boils down to the fact that some sub-sequence of $(\mu_G(S^{n+1} \setminus S^n))$ goes to zero as n goes to infinity. It is still open whether or not the whole sequence goes to zero.*

Non-amenable groups.

Let us now provide a few example of non amenable groups.

Proposition 2.18. *The free group on two generators, denoted F_2 and endowed with the discrete topology, is not amenable.*

Proof. Let a, b be the two generators of the free group F_2 . Write every element of F_2 as a word on the letters a, b, a^{-1}, b^{-1} that does not contain any consecutive a and a^{-1} , nor any consecutive b and b^{-1} . A word in this form will be called "reduced". Every element of F_2 can be represented by a unique reduced word, which will be called its "reduced form". Denote by e the empty word, which is the neutral element of the group F_2 .

Let A^+ be the set of elements of F_2 whose reduced form starts with an a , A^- the set of those whose reduced form starts with a a^{-1} , and similarly for B^+ and B^- . It is then clear that :

$$\begin{aligned} F_2 &= \{e\} \cup A^+ \cup A^- \cup B^+ \cup B^- \\ F_2 &= aA^- \cup A^+ = bB^- \cup B^+ \end{aligned}$$

all the union being disjoint. These equalities transfer to characteristic functions in the following way :

$$1_{F_2} = 1_{\{e\}} + 1_{A^+} + 1_{A^-} + 1_{B^+} + 1_{B^-} = 1_{aA^-} + 1_{A^+} = 1_{bB^-} + 1_{B^+}$$

Now if we suppose that F_2 is amenable, we get a left-invariant mean Λ on F_2 . Upon applying it to the above equalities, it follows that :

$$1 = \Lambda(1_{\{e\}}) + \Lambda(1_{A^+}) + \Lambda(1_{A^-}) + \Lambda(1_{B^+}) + \Lambda(1_{B^-}) = \Lambda(1_{aA^-}) + \Lambda(1_{A^+}) = \Lambda(1_{bB^-}) + \Lambda(1_{B^+})$$

Now one has $\Lambda(1_{aA^-}) = \Lambda(a \cdot 1_{A^-}) = \Lambda(1_{A^-})$ since Λ is left-invariant, and similarly for b . Upon injecting that in the above inequalities, and using the fact that all the terms are positive, we easily end up with :

$$1 = \Lambda(1_{F_2}) = \Lambda(1_{\{e\}}) + \Lambda(1_{A^+}) + \Lambda(1_{A^-}) + \Lambda(1_{B^+}) + \Lambda(1_{B^-}) = 0$$

hence the desired contradiction. □

Corollary 2.19. *Every locally compact group which contains a discrete free group on two generators is not amenable.*

Proof. Use proposition 2.18 and theorem 2.7. □

Remark 2.20. *One can show that $SL_2(\mathbb{Z})$ contains a free subgroup on two generators. Hence by the above corollary it is not amenable, and neither are $SL_n(\mathbb{Z})$, $SL_n(\mathbb{R})$ or $GL_n(\mathbb{R})$ for every $n \geq 2$, with their usual topology. One specific pair of generators for a free group in $SL_2(\mathbb{Z})$ is given by :*

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

2.1.5 The ergodic point of view on amenability

In this section, we will provide an other characterization of amenability which is the one that will be the most useful throughout this text. It is essentially a fixed point theorem for actions of amenable groups on measured spaces.

Theorem 2.21. *Let G be a locally compact second countable group. The following two conditions are equivalent.*

(i) G is amenable.

(ii) *For every compact Hausdorff metrizable space X and every continuous group morphism $\varphi : G \rightarrow \text{Homeo}(X)$, there is a Borel probability measure ν on X such that $\varphi_g^* \nu = \nu$ for every $g \in G$. (Here, $\text{Homeo}(X)$ is endowed with the topology of simple convergence.)*

Proof. There are two implications to prove.

Let us first prove that (i) implies (ii). Let us therefore consider a locally compact group G , and a compact Hausdorff space (X, d) . Consider a continuous morphism $\varphi : G \rightarrow \text{Homeo}(X)$, where $\text{Homeo}(X)$ is endowed with the uniform convergence topology, and let us fix a left-invariant mean $\Lambda \in L^\infty(G)^*$ on G . Fix some Borel probability measure ν_0 on X .

Now for every $f \in \mathcal{C}(X)$, define :

$$T_f(g) = \int_X f \circ \varphi_g d\nu_0 \quad \text{and} \quad \lambda(f) = \Lambda(T_f)$$

Clearly for every $f \in \mathcal{C}(X)$, $T_f : G \rightarrow \mathbb{R}$ is a well defined function from G to $\mathbb{R}$ which is continuous by dominated convergence (here we use the fact that φ is continuous). One notes that $\|T_f\| \leq \|f\|$ since ν_0 is a probability measure. Moreover one has $T_f \geq 0$ almost everywhere if $f \geq 0$, and clearly $T_{1_X} = 1_G$. From this it follows that $\lambda : \mathcal{C}(X) \rightarrow \mathbb{R}$ is a well defined linear operator of operator norm 1, that sends 1_X to 1 and sends non-negative functions to $\mathbb{R}_+$. By the Riesz-Markov-Kakutani representation theorem (see theorem 1.4), there is a unique Radon probability measure ν on X such that :

$$\forall f \in \mathcal{C}(X), \lambda(f) = \int_X f d\nu$$

Then for every $f \in \mathcal{C}(X)$ and every $g \in G$ one has, using left-invariance of Λ :

$$\int_X f d(\varphi_g^* \nu) = \int_X f \circ \varphi_g d\nu = \lambda(f \circ \varphi_g) = \Lambda(T_{f \circ \varphi_g}) = \Lambda(g^{-1} \cdot T_f) = \Lambda(T_f) = \lambda(f) = \int_X f d\nu$$

$\varphi_g^* \nu$ is also a Radon measure since $\varphi_g \in \text{Homeo}(X)$, thus by the unicity part of the Riesz-Markov-Kakutani theorem this forces $\nu = \varphi_g^* \nu$, which concludes the proof.

Let us now suppose (ii) and prove (i). Consider the set $K \subseteq L^\infty(G)^*$ of means on G . This set is clearly closed for the weak-* topology on $L^\infty(G)^*$ and contained in the unit ball of $L^\infty(G)^*$. By the Banach-Alaoglu theorem, it is therefore a compact subset of the topological vector space $L^\infty(G)^*$ endowed with the weak-* topology. It is moreover metrizable as it is

Now for every $g \in G$, and every $\Lambda \in K$, consider $\varphi_g(\Lambda) \in L^\infty(G)^*$ defined by $\varphi_g(\Lambda)(f) =$

$\Lambda(g^{-1} \cdot f)$. This provides a continuous morphism of groups $\varphi : G \rightarrow \text{Homeo}(K)$ and since K is compact Hausdorff metrizable this gives a Borel probability measure ν on K such that $\varphi_g^* \nu = \nu$ for every $g \in G$. Now for every $f \in L^\infty(G)$, let us define :

$$\Lambda(f) = \int_K \lambda(f) d\nu(\lambda)$$

This gives a linear form $\Lambda_0 : L^\infty(G) \rightarrow \mathbb{R}$ that sends 1_G to 1 (since ν is a probability measure), sends non-negative functions to non-negative functions and has operator norm 1. Therefore Λ is a mean on G , and it is left-invariant. Indeed, one has for every $g \in G$ and every $f \in L^\infty(G)$:

$$\Lambda(g^{-1} \cdot f) = \int_K \lambda(g^{-1} \cdot f) d\nu = \int_K \varphi_g(\lambda)(f) d\nu = \int_K \lambda(f) d(\varphi_g^* \nu) = \int_K \lambda(f) d\nu = \Lambda(f)$$

Thus G is amenable. □

It has the following rather useful corollary.

Corollary 2.22. *Let us consider a locally compact second countable amenable group G acting on a compact convex non-empty subset C of a separable Banach space E by homeomorphisms. Then the action has a fixed point.*

Proof. The above theorem gives a G -invariant probability measure ν on C . Now consider the mean of C for this measure, that is :

$$c = \int_{x \in C} x d\nu \in C$$

It lies in C since C is closed and convex, and it is invariant under the action of G since ν is. □

2.2 Property (T) of Kazhdan

2.2.1 Definition and examples of property (T)

Definition 2.23. Let us start with the definition of property (T). Let G be a locally compact second countable group.

G is said to have property (T) if and only if every continuous unitary representation of G which almost has invariant vectors actually has invariant vectors.

Essentially, it states that G has property (T) if and only if for every continuous representation π of G which does not have invariant vectors, there exist a compact set $K \subseteq G$ and $\varepsilon > 0$ such that for every $\xi \in \mathcal{H}$ there exists $g \in K$ such that $\|\pi(g)(\xi) - \xi\| \geq \varepsilon \|\xi\|$. However it does not follow directly from the definition that such K, ε can be taken not depending on π . This is however quite surprisingly the case, as the following lemma shows.

Lemma 2.24. *Let G be a locally compact second countable group which has property (T). Then there is a compact subset K and $\varepsilon > 0$ such that every continuous unitary representation $\pi : G \rightarrow \mathcal{U}(\mathcal{H})$ without invariant vectors satisfies :*

$$\forall \xi \in \mathcal{H}, \exists g \in K, \|\pi(g)(\xi) - \xi\| \geq \varepsilon \|\xi\|$$

Proof. Take G a locally compact second countable group, and suppose that the conclusion of the lemma is false. Consider an increasing sequence of open relatively compact sets $(V_n)_{n \geq 0}$ whose increasing union is G , $K_n = \overline{V_n}$, and write for every $n \in \mathbb{N}$, $\varepsilon_n = 2^{-n}$. Then, for every $n \in \mathbb{N}$, there is a continuous unitary representation $\pi_n : G \rightarrow \mathcal{U}(\mathcal{H}_n)$ without invariant vectors such that :

$$\exists \xi_n \in \mathcal{H}_n, \forall g \in K_n, \|\pi_n(g)(\xi_n) - \xi_n\| < \varepsilon_n \|\xi_n\| \quad (*)$$

From this and the fact that every compact set is contained in some V_n for some $n \in \mathbb{N}$, it follows that the representation $\pi : \bigoplus_{n \in \mathbb{N}} \pi_n : G \rightarrow \mathcal{U}(\bigoplus_{n \in \mathbb{N}} \mathcal{H}_n)$ on the Hilbert sum of all $\mathcal{H}_n$ almost has invariant vectors. Thus since G has property (T) , it must have non-zero invariant vectors and upon projecting those on one of the $\mathcal{H}_n$, we get non-zero invariant vectors for some π_n , $n \in \mathbb{N}$. This is a contradiction. $\square$

One starts by noting that property (T) passes to quotients easily. It also passes to group extensions, but this is way less straightforward and will require lemma 2.26.

Lemma 2.25. *Let G be a locally compact group, and H be a closed normal subgroup of G . Then if G has property (T) , the quotient group G/H also does. Conversely, if G/H and H have property (T) , then G also does.*

Proof. We keep the notations of the statement of the lemma. Note that the fact that H is closed ensures that G/H is Hausdorff. This is not needed for the proof, but we added this conditions because we will only deal with Hausdorff groups in this text.

Let $\pi : G/H \rightarrow \mathcal{U}(\mathcal{H})$ a unitary representation which has almost invariant vectors. Consider $p : G \rightarrow G/H$ the canonical projection, which is a continuous morphism of groups. Then $\pi \circ p$ is a unitary representation of G , and it has almost invariant vectors : indeed, if K is a compact subset of G and $\varepsilon > 0$, then $p(K)$ is compact, thus since π has almost invariant vectors, there is $\xi \in \mathcal{H} \setminus \{0\}$ such that $\|\pi \circ p(g)(\xi) - \xi\| \leq \varepsilon \|\xi\|$. Since G has property (T) and $\pi \circ p$ has almost invariant vectors, there is a non-zero invariant vector under the action of G and thus under the action of G/H . This concludes the proof of the first part.

Suppose now that both G/H and H have property (T) , and consider a representation $\pi : G \rightarrow \mathcal{U}(\mathcal{H})$ which almost has invariant vectors. Then $\pi|_H$ also does, and therefore $W = \mathcal{H}^H$ is non-zero. Moreover, since H is normal in G this space is G -invariant. Thus upon factoring we get a representation $\tilde{\pi} : G/H \rightarrow \mathcal{U}(W)$. From the fact that π has almost invariant vectors combined with lemma 2.26 applied to H , we get that $\tilde{\pi}$ has almost invariant vectors and hence has invariant vectors since G/H has property (T) . Thus π has invariant vectors and we are done. $\square$

Lemma 2.26. *Let G be a locally compact group with property (T) . Then, there exists a compact subset $K \subseteq G$ such that the following is true.*

For every $\alpha > 0$, there exists $\delta > 0$ such that for every continuous unitary representation $\pi : G \rightarrow \mathcal{U}(\mathcal{H})$ and every $\xi \in \mathcal{H}$, one has the implication :

$$(\forall g \in K, \|\pi(g)(\xi) - \xi\| \leq \delta \|\xi\|) \Rightarrow \|P_{\mathcal{H}^G}(\xi) - \xi\| \leq \alpha \|\xi\|$$

Here, $P_{\mathcal{H}^G}$ denotes the orthogonal projection on the subspace $\mathcal{H}^G$ of fixed points under the action of G .

Proof. Since G has property (T), there exists $\varepsilon > 0$ and $K \subseteq G$ compact such that every unitary representation π of G having a unit vector ξ such that $\forall g \in K, \|\pi(g)(\xi) - \xi\| < \varepsilon$ actually has non-zero invariant vectors (this is lemma 2.24). Take now such a representation $\pi : G \rightarrow \mathcal{U}(\mathcal{H})$, and write P for the orthogonal projection on the fixed subspace $\mathcal{H}^G$. Now $\pi(G)$ acts on the orthogonal $(\mathcal{H}^G)^\perp$ and has no invariant vectors, thus for every $\xi \in (\mathcal{H}^G)^\perp$ there is some $g \in K$ such that $\|\pi(g)(\xi) - \xi\| \geq \varepsilon\|\xi\|$, and by linearity we deduce that for every $\xi \in \mathcal{H}$ one has :

$$\sup_{g \in K} \|\pi(g)(\xi) - \xi\| \geq \varepsilon \|x - P_{\mathcal{H}^G}(x)\|$$

and the desired result follows. $\square$

We can also note that, in some way, property (T) and amenability are opposite properties. The following gives a taste of this general idea.

Proposition 2.27. *Let G be a locally compact σ -compact group that is both amenable and has property (T). Then, G is compact.*

Proof. Consider the unitary representation π of G on $L^2(G)$. Let us prove that it has almost invariant vectors. Let $\varepsilon > 0$ and K be a compact subset of G . Since G is amenable, there is some measurable set $U \subseteq G$ such that $\mu_G(U) > 0$ and $\mu_G(U \Delta gU) < \varepsilon\mu_G(U)$ for every $g \in K$. Consider the indicator function 1_U of U , which is non-zero in $L^2(G)$ since $\mu_G(U) \neq 0$. One then has, for every $g \in K$:

$$\|\pi(g)(1_U) - 1_U\|_2 = \|1_{gU} - 1_U\|_2 = \mu_G(U \Delta gU) \leq \varepsilon\mu_G(U) = \varepsilon\|1_U\|$$

Therefore π has almost invariant vectors, and since G has property (T) there must be some non zero $f \in L^2(G)$ that is invariant under the action of the whole of G . Such a function must be essentially constant (this is not completely obvious, see if you can figure out why), therefore $1_G \in L^2(G)$ and it follows from the properties of the Haar measure that G must be compact. $\square$

This corollary will be the crown jewel in the proof of Margulis normal theorem.

Corollary 2.28. *Let G be a discrete countable group that is both amenable and has property (T). Then G is finite.*

Corollary 2.29. *A σ -compact locally compact group which has property (T) cannot have an infinite abelian quotient by a closed normal subgroup. In particular, the free group F_2 does not have property (T).*

On the contrary to amenability, property (T) does not pass to closed subgroups. However, it does when the subgroup is a lattice. This is the following result, due to Kazhdan. Note that the implication $G \Rightarrow \Gamma$ is rather simple to prove, while the other one requires quite a lot of effort.

Theorem 2.30. *Let G be a locally compact group, and Γ be a lattice in G . Then G has property (T) if and only if Γ does.*

Proof. Keep the above notations and suppose that G has property (T). Consider a continuous unitary representation $\pi : \Gamma \rightarrow \mathcal{H}$ which almost has invariant vectors. Then by the lemma 1.26, $\text{Ind}_\Gamma^G \pi$ also almost has invariant vectors, thus has invariant vectors since G has property (T). Now using lemma 1.26 once again, it follows that π has invariant vectors and we are done.

Suppose now that Γ has property (T). We will show that G also does. Consider a continuous unitary representation $\pi : G \rightarrow \mathcal{U}(H\mathcal{H})$ which almost has invariant vectors. Then clearly $\pi|_\Gamma$ also does, so by the fact that Γ has property (T) the space $\mathcal{H}^\Gamma$ of points fixed under the action of Γ is non-zero. Write P for the orthogonal projection on $\mathcal{H}^\Gamma$.

Now consider a sequence $(\xi_n)_{n \geq 0}$ of non-zero vectors of G such that for every compact set $K \subseteq G$ and every $\varepsilon > 0$, there is some $N \geq 0$ such that for every $n \geq N$ one has $\sup_{g \in K} \|\pi(g)(\xi_n) - \xi_n\| < \varepsilon \|\xi_n\|$: such a sequence can easily be constructed using the fact that π almost has invariant vectors. Write $\eta_n = P(\xi_n) \in \mathcal{H}^\Gamma$ for every $n \in \mathbb{N}$. Now due to lemma 2.26, the following is true :

For every $\varepsilon > 0$ and every compact set $K \subseteq G$, there is some $N \geq 0$ such that for every $n \geq N$ one has $\sup_{g \in K} \|\pi(g)(\eta_n) - \eta_n\| < \varepsilon \|\eta_n\|$. Upon normalizing, we may also suppose that all the η_n are unit vectors.

Now for every $n \in \mathbb{N}$ and every $x\Gamma \in G/\Gamma$, since $\eta_n \in \mathcal{H}^\Gamma$, we may define $\pi(x\Gamma)(\eta_n) \in \mathcal{H}$ without ambiguity as $\pi(x)(\eta_n)$. One can then check that the function $G/\Gamma \rightarrow \mathcal{H}$ defined by $x\Gamma \rightarrow \pi(x\Gamma)(\eta_n)$ is continuous and bounded in $\mathcal{H}$. Therefore, upon considering a left-invariant probability measure $\mu_{G/\Gamma}$ on G/Γ , we may define for every $n \in \mathbb{N}$:

$$z_n = \int_{G/\Gamma} \pi(x\Gamma)(\eta_n) d\mu(x\Gamma) \in \mathcal{H}$$

It is then clear by left-invariance of the measure $\mu_{G/\Gamma}$ that z_n is invariant under the action of G . To conclude, it therefore suffices to show that for n large enough one has $z_n \neq 0$. To do so, notice that for every $n \in \mathbb{N}$ one has :

$$\langle z_n, \eta_n \rangle = \int_{G/\Gamma} \langle \pi(x\Gamma)(\eta_n), \eta_n \rangle d\mu_{G/\Gamma} \quad (*)$$

Now from the property defining the η_n , we get that for every $x \in G$, one has $\|\pi(g)(\eta_n) - \eta_n\| \xrightarrow{n \rightarrow +\infty} 0$ and therefore $\langle \pi(g)(\eta_n), \eta_n \rangle \xrightarrow{n \rightarrow +\infty} \|\eta_n\|^2 = 1$. It follows that in (*) above, the integrand converges pointwise to 1, and it is uniformly bounded by 1. Thus by dominated convergence, the integral also goes to 1 as $n \rightarrow +\infty$. In particular for n large enough one has $z_n \neq 0$. Since z_n is invariant under the action of G we are done. $\square$

One easy though corollary of this remark is the following.

Corollary 2.31. *Let G, H be two real connected semi-simple Lie group which are locally isomorphic. Then G has property (T) if and only if H does.*

Proof. Just combine theorem 2.30 with corollary 1.42. $\square$

2.2.2 Higher rank lattices and property (T)

The following crucial theorem is due to Kazhdan.

Theorem 2.32. *Let G be a connected simple Lie group of real rank at least 2. Then G has property (T) .*

We will explain the main ideas involved in the proof of it in the two sections to come. In particular, we will fully prove it in the simpler case of $SL_3(\mathbb{R})$. In the meantime, let us note that using the fact that property (T) passes to lattices, the above result gives many examples of discrete groups having property (T) .

We now define the main object of interest of this text. Note that our definition requires the ambient Lie group to be simple (in the realm of Lie groups, that is has simple Lie algebra). This is more restrictive than what is commonly found in the literature : it is generally asked that the Lie group is semi-simple, and the lattice is irreducible. The irreducibility condition is of course vacuous in the case of a simple Lie group.

The reason for this choice is not only simplicity (pun intended) but also to avoid semi-simple Lie groups of rank greater than 2 which have a rank one factor, since they would not have property (T) which we want to use. Let us finally come to the definition.

Definition 2.33. A discrete group Γ is said to be a higher rank lattice if and only if it is isomorphic to a lattice in a connected real simple Lie group of real rank at least 2.

A Lie group is said to be of higher rank if and only if it is a connected real simple Lie group of rank at least 2.

Let us give a first example of what such lattices would look like.

Example 2.34. $SL_n(\mathbb{Z})$ for $n \geq 3$ and all of its finite index subgroups are higher rank lattices. It is also the case of $Sp_{2n}(\mathbb{Z})$ for $n \geq 2$.

We now state the main corollary of theorem 2.32, which we will be using extensively in what follows.

Corollary 2.35. *Higher rank lattices have property (T) .*

Proof. Just use theorem 2.30 combined with theorem 2.32. □

Let us now move to an attempt at an explanation of the proof of theorem 2.32. In the proof of it, we will crucially rely on the fact that both $SL_3(\mathbb{R})$ and $Sp_4(\mathbb{R})$ have property (T) . These facts we be addressed in the following section.

Sketch of the proof of theorem 2.32. First of all, note that the universal cover $\tilde{G}$ of G has the same rank as G , is also a connected simple Lie group, and that G is a finite quotient of $\tilde{G}$. Therefore, by lemma 2.25, G has property (T) if and only if $\tilde{G}$ has. So we may suppose without loss of generality that G is simply connected.

Now it is well known, and easily checkable from the classification of real simple Lie groups, that every real simply-connected simple Lie of rank at least 2 group contains a semi-simple

subgroup F that is locally isomorphic to either $\mathrm{SL}_3(\mathbb{R})$ or $\mathrm{Sp}_4(\mathbb{R})$. Now being locally isomorphic respects having property (T), due to lemma 2.25 and corollary 1.42. Hence F has property (T), since $\mathrm{SL}_3(\mathbb{R})$ and $\mathrm{Sp}_4(\mathbb{R})$ do (this fact is what will be addressed in the next section).

Now consider a continuous unitary representation $\pi : G \rightarrow \mathcal{U}(\mathcal{H})$ which almost has invariant vectors. Since F has property (T), we deduce that the space $\mathcal{H}^F = \{\xi \in \mathcal{H} \mid \forall g \in F, \pi(g)(\xi) = \xi\}$ is non-zero. Take some unit vector ξ in that space.

Consider now a maximal split torus S in F , $\mathfrak{h}_0$ its Lie algebra, and $\mathfrak{h}$ an Cartan algebra of $\mathfrak{g}$ containing $\mathfrak{h}_0$. We may then take some $s_0 \in \mathfrak{h}_0$ the action of which on $\mathfrak{f}$ comes with a negative eigenvalue : indeed, it suffices to fix a root of $\mathfrak{h}_0^*$ and take in $\mathfrak{h}_0$ an element whose image by this root is negative.

Define θ to be the set of simple roots of G such that $\alpha(s_0) < 0$, which is non-empty. Now $\mathrm{Int}(s)|_{V_\theta}$ is a contracting automorphism (see lemma 1.39), so $s^n y s^{-n} \xrightarrow{n \rightarrow +\infty} e$ for every $y \in V_\theta$, and since $\|\pi(s^n y s^{-n})(\xi) - \xi\| = \|\pi(y)(\xi) - \xi\|$ (here we use the fact that ξ is invariant under the action of s), it follows that $\pi(y)(\xi) = \xi$ for every $y \in V_\theta$. We prove similarly that V_θ^- leaves ξ fixed, hence since V_θ and V_θ^- generate G it follows that ξ is fixed under the whole of G , which is what we needed to prove. $\square$

2.2.3 $\mathrm{SL}_3(\mathbb{R})$ and $\mathrm{Sp}_4(\mathbb{R})$ have property (T)

In the proof of theorem 2.32, we crucially relied on the rank 2 case, and in particular on the fact that both $\mathrm{SL}_3(\mathbb{R})$ and $\mathrm{Sp}_4(\mathbb{R})$ enjoy property (T). In this section, we will complete the proof of this fact. We will only treat the case of $\mathrm{SL}_3(\mathbb{R})$, but the reader should be aware that $\mathrm{Sp}_4(\mathbb{R})$ can also be proven to have property (T) in a very similar fashion.

Theorem 2.36. *The group $\mathrm{SL}_3(\mathbb{R})$ has property (T).*

Proof. Consider the semi-direct product :

$$H = \mathrm{SL}_2(\mathbb{R}) \ltimes \mathbb{R}^2 = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & 1 \end{pmatrix} \subseteq \mathrm{SL}_3(\mathbb{R})$$

And consider a continuous unitary representation π of $\mathrm{SL}_3(\mathbb{R})$ on a Hilbert space $\mathcal{H}$ which almost has invariant vectors. Of course, one may restrict π to the subgroup H and it would still almost have invariant vectors. Write π_H for this restriction.

We will prove in the following lemma that π_H has a vector that is invariant under the subgroup $\mathbb{R}^2 \subseteq H$. Temporarily admitting that, it follows from the Mautners phenomenon (theorem 2.38) that the same vector is invariant under the action of $\mathrm{SL}_3(\mathbb{R})$. $\square$

To conclude the proof, it therefore remains to prove both lemma 2.37 and theorem 2.38.

Lemma 2.37. *Let $H = \mathrm{SL}_2(\mathbb{R}) \ltimes \mathbb{R}^2$ be the semi-direct product considered in the above proof. Let π be a continuous unitary representation of H into a Hilbert space $\mathcal{K}$ that almost has invariant vectors.*

Then, there is a non zero vector in $\mathcal{K}$ that is invariant under the action of $\mathbb{R}^2 \subseteq H$.

Proof. Let $(v_n)_{n \geq 0}$ be a sequence of unit vectors in $\mathcal{H}$ such that $\|\pi(h)(v_n) - v_n\| \xrightarrow{n \rightarrow +\infty} 0$.

Upon setting $\lambda_n(T) = \langle T v_n, v_n \rangle$ for every $T \in \mathcal{L}(\mathcal{H})$ and taking λ a weak-* accumulation point of the sequence $(\lambda_n)_{n \geq 0}$, we get a linear functional $\lambda : \mathcal{L}(\mathcal{H}) \rightarrow \mathbb{R}$ that is bi-invariant under the action of H and takes positive values on orthogonal projections.

Denote by R the copy of $\mathbb{R}^2$ in H . π_R is a continuous unitary representation of $\mathbb{R}^2$, thus by Fourier theory (see for example section D.3 in [1] for a survey of representations of locally compact abelian groups) there is a unique regular measure P on the Pontryagin dual $\hat{R}$ of R that takes values in the orthogonal projections of $\mathcal{K}$ such that for every $a \in R$, one has :

$$\pi(a) = \int_{\chi \in \hat{R}} \chi(a) dP(\chi)$$

This measure satisfies, for every $m \in \mathrm{SL}_2(\mathbb{R})$ and every Borel set $E \subseteq \hat{R} : \pi(m)P(E)\pi(m)^{-1} = P(m(E))$, where the action of $\mathrm{SL}_2(\mathbb{R})$ on $\hat{R}$ is given by the dual action of $\mathrm{SL}_2(\mathbb{R})$ on $R \simeq \mathbb{R}^2$. Now write for every Borel set E in $\hat{R} : \mu(E) = \lambda(P(E))$. This defines a probability measure on $\hat{R} = \mathbb{R}^2$, that is invariant under the action of $\mathrm{SL}_2(\mathbb{R})$. One claims that such a measure must be the Dirac mass supported on 0, and thus $P(\{0\})$ must be non-zero. But any vector in the image of $P(\{0\})$ is an invariant vector under the action of H , therefore π has a non zero invariant vector.

It remains to show that the only probability measure on $\mathbb{R}^2$ that is invariant under the action of $\mathrm{SL}_2(\mathbb{R})$ is the Dirac mass supported at zero. To that extent, let us consider such a probability measure μ . Write $h = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and consider $V = \{(x, y) \in \mathbb{R}^2 \mid y > |x|\}$. Then geometric considerations show that the $h^i V$ are disjoint for $i \in \mathbb{N}$, hence $\mu(V) = 0$. Now all of $\mathbb{R}^2 \setminus \{0\}$ can be covered by finitely many gV for $g \in \mathrm{SL}_2(\mathbb{R})$, hence the result. $\square$

2.2.4 Decay of matrix coefficients and Moore ergodicity theorem

Let us now move to the statement and proof of Mautners theorem.

Theorem 2.38 (Mautners phenomenon). *Let G be a simple connected real Lie group, π a continuous unitary representation of G on a Hilbert space $\mathcal{K}$. Let H a closed non-compact subgroup of G .*

Then every $\xi \in \mathcal{K}$ that is invariant under the action of H is also invariant under the action of G .

To prove it, we will need the following lemma on the decay of matrix coefficients.

Lemma 2.39. *Let G be a simple connected real Lie group, π a continuous unitary representation of G on a Hilbert space $\mathcal{K}$. Write $\mathcal{K}^G$ for the space of G -invariant vectors.*

Then for every $\xi, \eta \in (\mathcal{K}^G)^\perp$, and every $\varepsilon > 0$ there is some compact set $K \subseteq G$ such that for every $g \in G \setminus K$ one has $|\langle \pi(g)(\xi), \eta \rangle| < \varepsilon$.

The conclusion of this lemma is to be understood as : $\langle \pi(g)(\xi), \eta \rangle$ tends to 0 as g "tends to infinity" in whatever direction we like. Let us first deduce Mautners phenomenon from this decay lemma.

Proof of theorem 2.38 using lemma 2.39. Consider a vector $\xi \in \mathcal{K}$ that is invariant under the action of H and consider its orthogonal projection $\tilde{\xi}$ on $\mathcal{K}^G$. Write $\eta = \xi - \tilde{\xi}$. Now η is invariant under the action of H , hence for every $h \in H$ one has :

$$\langle \eta, \eta \rangle = \langle \pi(h)(\eta), \eta \rangle$$

and since H is non-compact we may let h go to infinity while staying in H in the above equation, and the decay lemma above shows that the right hand side goes to zero. It follows that $\langle \eta, \eta \rangle = 0$ and thus $\xi \in \mathcal{K}^G$. $\square$

It remains to prove the lemma about the decay of matrix coefficients, which is the technical heart of this section. We only prove it in the case of $\mathrm{SL}_2(\mathbb{R})$ for simplicity, but the reader aware of the theory of Weyl chambers should not have trouble generalizing this argument to general simple real Lie groups. Note that at some point you may need to use something of a flavour similar to that of lemma 1.39.

Proof of lemma 2.39. Let us only consider the easier case $G = \mathrm{SL}_2(\mathbb{R})$. Of course, upon restricting the representation π to a subspace we may and will suppose without loss of generality that $\mathcal{K}^G = \{0\}$. Let $\xi, \eta \in \mathcal{K}^G$.

Now moving to a sequence point of view, it suffices to show that for every sequence $(g_n)_{n \geq 0}$ of elements of G which goes to infinity (that is, eventually exits from every compact set), one has $\langle \pi(g_n)(\xi), \eta \rangle \xrightarrow{n \rightarrow +\infty} 0$. We thus suppose by way of contradiction that there is some $\xi, \eta \in \mathcal{K}$, some sequence $(g_n)_{n \geq 0}$ of elements of G which goes to infinity but such that $|\langle \pi(g_n)(\xi), \eta \rangle| > \varepsilon$ for every $n \in \mathbb{N}^*$, for some fixed $\varepsilon > 0$.

Write $g_n = s_n a_n t_n$ with $s_n, t_n \in \mathrm{SO}_2(\mathbb{R})$ and a_n diagonal (such a decomposition is given by the classical symmetric orthogonal decomposition). Upon extracting, we may suppose that $(s_n)_{n \geq 0}$ and $(t_n)_{n \geq 0}$ both converge to some matrices $s, t \in \mathrm{SO}_2(\mathbb{R})$. Then one has $\langle \pi(g_n)(\xi), \eta \rangle = \langle \pi(a_n)(\pi(s_n)(\xi)), \pi(t_n^{-1})(\eta) \rangle = \langle \pi(a_n)(\pi(s)(\xi)), \pi(t^{-1})(\eta) \rangle + o(1)$ so upon replacing ξ, η with $\pi(s)(\xi), \pi(t)(\eta)$ we may suppose that g_n is diagonal for every n . We do so in what follows.

Upon extracting, we may and will suppose that $\pi(g_n)$ converges weakly as n goes to $+\infty$ to some operator E (this is easily done using a diagonal argument on a countable Hilbert basis), that is $\langle \pi(g_n)(\varphi), \psi \rangle \xrightarrow{n \rightarrow +\infty} \langle E(\varphi), \psi \rangle$ for every $\xi, \eta \in \mathcal{K}$. We wish to prove that $E = 0$, as this will provide a contradiction with the definition of the sequence $(g_n)_{n \geq 0}$. Now write :

$$\begin{aligned} U^+ &= \{u \in G \mid g_n^{-1} u g_n \xrightarrow{n \rightarrow +\infty} e\} \\ U^- &= \{u \in G \mid g_n u g_n^{-1} \xrightarrow{n \rightarrow +\infty} e\} \end{aligned}$$

For every $u \in U^-$ and every $\xi, \eta \in \mathcal{K}$ one has :

$$\begin{aligned} \langle E\pi(u)(\varphi), \psi \rangle &= \lim_{n \rightarrow +\infty} \langle \pi(g_n u)(\varphi), \psi \rangle = \lim_{n \rightarrow +\infty} \langle \pi(g_n u g_n^{-1})(\pi(g_n)(\varphi)), \psi \rangle \\ &= \lim_{n \rightarrow +\infty} \langle \pi(g_n)(\varphi), \psi \rangle = \langle E(\varphi), \psi \rangle \end{aligned}$$

Therefore for every $u \in U^-$ one has $E\pi(u) = E$. Let us deduce that $(\mathcal{K}^{U^-})^\perp \subseteq \ker E$. To that extent, it suffices to show that $(\ker E)^\perp \mathcal{K}^{U^-}$. Consider some $\varphi \in (\ker E)^\perp$ and some $u \in U^-$. As $\pi(u^{-1})(\psi) - \psi \in \ker(E)$ one has $0 = \langle \varphi, \pi(u^{-1})(\psi) - \psi \rangle = \langle \pi(u)(\varphi) - \varphi, \psi \rangle$ for every $\psi \in \mathcal{K}$ hence $\pi(u)(\varphi) = \varphi$ thus $\varphi \in \mathcal{K}^{U^-}$. So finally one has $(\mathcal{K}^{U^-})^\perp \subseteq \ker E$. In exactly the same way, one shows that $E^*\pi(u) = E^*$ for every $u \in U^+$ and that $(\mathcal{K}^{U^+})^\perp \subseteq \ker E^*$. Moreover, taking to the limit the equation $\langle \pi(g_n)(\varphi), \pi(g_m)(\varphi) \rangle = \langle \pi(g_m^{-1})(\varphi), \pi(g_n^{-1})(\varphi) \rangle$ (here we use the fact that the g_i are diagonal and hence commute) we get $\|E(\varphi)\| = \|E^*(\varphi)\|$ for every $\varphi \in \mathcal{K}$, hence $\ker(E) = \ker(E^*)$.

It follows that :

$$\ker(E) = \ker(E) + \ker(E^*) \supseteq (\mathcal{K}^{U^-})^\perp + (\mathcal{K}^{U^+})^\perp \supseteq (\mathcal{K}^{(U^-, U^+)})^\perp$$

Now write $g_n = \begin{pmatrix} \lambda_n & 0 \\ 0 & \lambda_n^{-1} \end{pmatrix}$. Upon extracting we may either suppose that $\lambda_n \xrightarrow{n \rightarrow +\infty} +\infty$ or 0. Let us treat the case where it goes to $+\infty$, leaving the other identical case to the reader. In that case, it is easily seen that U^+ contains the upper triangular matrices and U^- the lower triangular matrices, hence $\langle U^-, U^+ \rangle = G$ and the above inclusions show that $E = 0$, as needed. $\square$

This theorem has the following implication in terms of ergodicity of certain actions.

Corollary 2.40. *Let G be a real connected simple Lie group, H a closed non-compact subgroup of G , and X a G -space of finite total mass on which G acts measure preserving. Suppose that G acts ergodically on X . Then H also does.*

Proof. Consider the unitary representation of G on $L^2(G)^0$, the space of L^2 functions on X of mean 0. Take a subset $Y \subseteq X$ that is invariant under the action of H . Then $\mu(X \setminus Y)1_Y - \mu(Y)1_{X \setminus Y} \in L^2(G)^0$ is an invariant vector under the action of H , hence also under the action of G due to theorem 2.38. Thus it must be constant, and this proves that $\mu(Y) \in \{0, 1\}$. $\square$

An example of this is the following result on the action of subgroups of G on the quotient by a lattice. It is a direct application of the above.

Corollary 2.41. *Let G be a real connected simple Lie group, Γ a lattice in G , and H a closed non-compact subgroup of G . Then, H acts ergodically on G/Γ .*

3 Actions of higher rank groups

3.1 Margulis factor theorem and normal subgroup theorem

3.1.1 Margulis normal subgroup theorem

The following theorem of Margulis makes strong usage of property (T) in order to prove that higher rank lattices are close to being simple groups.

Theorem 3.1 (Margulis). *Let Γ be a higher rank lattice, and N be a normal subgroup of Γ . Then N is either finite and central or has finite index in Γ .*

Throughout the proof of this theorem, we will suppose that N is not central and prove that Γ/N is finite. This will be done by proving that it is amenable and has property (T), which is enough due to corollary 2.28. The fact that it has property (T) is an easy consequence of corollary 2.35 combined with lemma 2.25.

Therefore, it will be sufficient to prove the following lemma, which is the highly technical part of the proof.

Lemma 3.2. *Let Γ be a higher rank lattice and N a normal subgroup of Γ that is not contained in the center of Γ . Then Γ/N is amenable.*

This lemma will be deduced from the Margulis factor theorem. Let us first state the said theorem, before explaining its importance and finally using it to prove the lemma 3.2.

Theorem 3.3 (Margulis factor theorem). *Let G be a semi-simple connected real Lie group of rank at least 2, and Γ an irreducible lattice in G . Let P be a minimal parabolic subgroup of G .*

Consider a Γ -space Z , and a Γ -equivariant measurable map that is measure class preserving $\Phi : G/P \rightarrow Z$.

Then for some parabolic subgroup Q , there is a Γ -equivariant measurable bijection $\psi : G/Q \rightarrow Z$ (up to null sets) such that the following diagram commutes almost everywhere :

$$\begin{array}{ccc} G/P & \xrightarrow{\Phi} & Z \\ & \searrow \pi & \uparrow \psi \\ & & G/Q \end{array}$$

Here, π is of course the canonical projection.

This theorem is a result of rigidity. In some way, it states that in the category of Γ -spaces, G/P has very few quotients.

Such a result should be very suprising. To grasp its importance, let us make a small detour through the actions of G itself. Of course, P being a minimal parabolic subgroup of G , G acts on the homogenous space G/P . Now if Q is any parabolic subgroup of G then as $P \subseteq Q$ one has a G -equivariant projection map $G/P \rightarrow G/Q$. In some way, G/Q is a quotient of G/P in the category of metrizable spaces acted on by G , the morphisms of this category

being the G -equivariant measurable maps.

It can be proven that these are essentially the only possible quotients of G/P : this is the content of the proposition 3.5, to be seen later on. The very surprising fact given by this theorem is that in the category of Γ -spaces, this result remains : any quotient of G/P is isomorphic to G/Q . This means in particular that every action of Γ on a quotient of G/P lifts to an action of G itself. In that sense, it can also be understood as a result of the same flavour as superrigidity theorems.

Let us now deduce the crucial lemma 3.2 from this classical theorem 3.3.

Proof of lemma 3.2. To prove that Γ/N is amenable, we use the characterization from theorem 2.21. Consider a compact Hausdorff metric space X endowed with an action of Γ by homeomorphisms that is trivial on N . To conclude, it suffices to find a Borel probability measure on X that is invariant under the action of Γ .

Upon letting the amenable group P (it is amenable since it is solvable, see corollary 2.14) act on the space $L^\infty(G, \text{Prob}(X))^\Gamma$, which is compact convex in a separable Banach space, we get a fixed point (see corollary 2.22) of this action, that is a Γ -equivariant map $G \rightarrow \text{Prob}(X)$ that is invariant under the action of P , and can thus be seen as a Γ -invariant measurable map $\varphi : G/P \rightarrow \text{Prob}(X)$. Then theorem 3.3 implies that the action of Γ on $\text{Prob}(X)$ is essentially isomorphic to that of Γ on G/Q for some parabolic subgroup Q of G . As N acts trivially on X , it must also act trivially on G/Q hence the kernel of the action of G on G/Q is infinite : as G is simple, this kernel must be G itself. In particular, Γ acts trivially on $\text{Prob}(X)$ so any measure in $\text{Prob}(X)$ is Γ -invariant and we are done. $\square$

What remains now is to prove the Margulis factor theorem 3.3. This will be done in what remains of this section. Let us recall the statement of the Margulis factor theorem.

3.1.2 The factor theorem and the subalgebra theorem

Theorem 3.3 (Margulis factor theorem). *Let G be a semi-simple real Lie group of rank at least 2, and Γ an irreducible lattice in G . Let P be a minimal parabolic subgroup of G .*

Consider a Γ -space Z , and a Γ -equivariant measurable map that is measure class preserving $\Phi : G/P \rightarrow Z$.

Then for some parabolic subgroup Q , there is a Γ -equivariant measurable bijection $\psi : G/Q \rightarrow Z$ (up to null sets) such that the following diagram commutes almost everywhere :

$$\begin{array}{ccc} G/P & \xrightarrow{\Phi} & Z \\ & \searrow \pi & \uparrow \psi \\ & & G/Q \end{array}$$

We will now reformulate this theorem in a more algebraic setting. To that extent, we will need to introduce some terminology.

Consider a locally compact σ -compact space (X, μ) with a positive regular measure. Define the algebra $\mathcal{M}(X)$ of all measurable sets of X taken up to equality almost everywhere. It

is endowed with the usual laws of countable union and intersection and complement, which make it what we call a σ -algebra. It is also endowed with a natural topology : a sequence (B_n) of elements of $\mathcal{M}(X)$ converge to some $B \in \mathcal{M}(X)$ if and only if for every compact $Y \subseteq X$ one has $\mu(Y \cap (B \Delta B_n)) \xrightarrow{n \rightarrow +\infty} 0$. It is clear that any sub- σ -algebra of $\mathcal{M}(X)$ is closed in this topology.

At this point, we should note that in order to define $\mathcal{M}(X)$, we did not use all of the features of the measure μ . Actually, any measure in the same measure class (that is, any measure μ' on X which has the same null sets as μ) would have given the same σ -algebra. Therefore, we only need a measure class and not necessarily a specific measure to define such an algebra. This fact will be used in the following.

Suppose now that X is a locally compact σ -compact group, endowed with its Haar measure μ_X . If F is a closed subgroup of X , there may not be a canonical measure on X/F that we can use to construct $\mathcal{M}(X/F)$. However, there exists a canonical measure class, hence the definition of $\mathcal{M}(X/F)$.

Now note that if H is a closed subgroup of X which contains F , one has a natural continuous surjection $X/F \rightarrow X/H$. Any null set in X/H pulls back to a null set in X/F , hence this surjection provides a map $\mathcal{M}(X/H) \rightarrow \mathcal{M}(X/F)$. This map is injective since $X/F \rightarrow X/H$ is surjective. Therefore, we have a copy of the σ -algebra $\mathcal{M}(X/H)$ that lies inside of $\mathcal{M}(X/F)$. This copy will be denoted by $\mathcal{M}(X/F, H)$ to avoid confusions.

It will turn out that theorem 3.3 can actually be deduced from this apparently weaker statement about invariant sub-algebras of $\mathcal{M}(G/P)$, where G is a semi-simple higher rank Lie group and P is a Borel subgroup of G .

Theorem 3.4 (Margulis subalgebra theorem). *Let G be a real semi-simple Lie higher rank Lie group, and P be a Borel subgroup of G . Let Γ be a lattice in G . Then all the Γ -invariant sub- σ -algebras of $\mathcal{M}(G/P)$ are the $\mathcal{M}(G/P, Q)$, for Q a parabolic subgroup of G containing P .*

We will soon explain how this result implies theorem 3.3. In the mean time, let us add a little more context to this statement.

Consider G a real semi-simple Lie group, and P a parabolic subgroup of G . It is well known that the only closed subgroups of G containing P are the parabolic subgroups referred to in the above statement. Now if Q is such a parabolic subgroup of G , then $\mathcal{M}(G/P, Q)$, which is a subalgebra of $\mathcal{M}(G/P)$, is clearly invariant under the action of G (say, on the left) since the map $G/P \rightarrow G/Q$ is G -equivariant.

It is in fact well known that these are the only examples of sub- σ -algebras of $\mathcal{M}(G/P)$ which are invariant under the action of G . See the following proposition 3.5 for more details about this statement.

Proposition 3.5. *Let X be a locally compact σ -compact group.*

If $\mathcal{B}$ is a sub- σ -algebra of $\mathcal{M}(X)$ that is invariant under the action of X , then one has $\mathcal{B} = \mathcal{M}(X, H)$ for H a closed subgroup of X .

More generally, if F is a closed subgroup of X and $\mathcal{B}$ is a sub- σ -algebra of $\mathcal{M}(X/F)$ that is

invariant under the action of X , then $\mathcal{B} = \mathcal{M}(X/F, H)$ for some closed subgroup H of X containing F .

Proof. See [8], IV.2.4. □

Thus we have a bijective correspondance between the sub- σ -algebras of $\mathcal{M}(G/P)$ which are invariant under the action of G and the parabolic subgroups of G containing P . Of course, the phrase "invariant under the action of G " is crucial here : there are many sub- σ -algebras of $\mathcal{M}(G/P)$ which have nothing to do with the $\mathcal{M}(G/P, Q)$.

The content of theorem 3.4 is to show that in fact, to reach the same conclusion, we do not need our subalgebra to be invariant under the whole of G : invariance under the action of a lattice suffices. Another way to state the conclusion of theorem 3.4 would therefore be the following : any sub- σ -algebra of $\mathcal{M}(G/P)$ that is invariant under the action of Γ is also invariant under the action of G . Let us now explain how the Margulis factor theorem can be deduced from the subalgebra theorem. It essentially comes down to categorical nonsense.

Deduction of theorem 3.3 from theorem 3.4. To do so, keep all of the assumptions of theorem 3.3 and assume theorem 3.4. Then using the Γ -equivariant $\Phi : G/P \rightarrow Z$, we can pull back $\mathcal{M}(Z)$ into a σ -algebra $\Phi^{-1}(\mathcal{M}(Z))$ which will be contained in $\mathcal{M}(G/P)$ and invariant under the action of Γ , since Φ is Γ -invariant. Now, we use theorem 3.4 and deduce that :

$$\mathcal{M}(Z) \simeq \Phi^{-1}(\mathcal{M}(Z)) = \mathcal{M}(G/P, Q) \simeq \mathcal{M}(G/Q)$$

for some parabolic subgroup Q , the isomorphisms being Γ -invariant and fitting into a commutative diagram with $\mathcal{M}(G/P)$.

The whole stake is now to bootstrap this isomorphism of σ -algebras into an isomorphism of Γ -spaces. To do so, note that $L^\infty(Z)$ is the linear completion of $\mathcal{M}(Z)$ and similarly for G/Q , hence this morphism extends to an isomorphism of von Neumann algebras :

$$L^\infty(Z) \simeq L^\infty(G/Q)$$

which is still Γ -invariant and fits into a commutative diagram with $L^\infty(G/P)$. Now it is well known that the category of von Neumann algebras is dual to the category of standard measured spaces (see for example the survey [3]), so this isomorphism gives rise to the desired isomorphism between Z and G/Q and we are done. □

Remark 3.6. *Note that we have essentially used a correspondence here between standard measured spaces, σ -algebras and abelian von Neumann algebras which preserves (or reverses, depending on the case) inclusions and maps. This correspondence will be used extensively throughout the rest of this text.*

The next two sections will be devoted to the proof of the subalgebra theorem, that is theorem 3.4.

3.1.3 Contracting automorphisms

This paragraph is mainly preliminaries for the proof of the subalgebra theorem.

Let $b > 0$ and X be a set. A non-negative symmetric function $\rho : X \times X \rightarrow \mathbb{R}_+$ is called a b -metric if and only if $\rho(x, y) = 0 \iff x = y$ and for every $x, y, z \in X$, $\rho(x, y) \leq b(\rho(x, z) + \rho(y, z))$.

If X is a topological space, we say that it is a b -metric topological space if its topology is generated by the $U(x, \varepsilon) = \{y \in X \mid \rho(x, y) \leq \varepsilon\}$ for $\varepsilon > 0$ and $x \in X$. If F is a subset of X , write $U(F, \varepsilon)$ for the union of the $U(x, \varepsilon)$ for $x \in F$.

If F is a subset of a b -metric space X , the diameter $\delta(F)$ of F is defined as usual by $\delta(F) = \sup_{x, y \in F} \rho(x, y)$.

We start by a highly technical lemma used in the proof of theorem 3.8 below.

Lemma 3.7. *Let X be a compact b -metric space and $B \subseteq X$. Suppose that a family $\mathcal{F}$ of closed subsets of X has the following property :*

For each $x \in X$ there exists $F \in \mathcal{F}$ with arbitrarily small strictly positive diameter $\delta(F)$ and containing x .

Then, either B is contained in a finite union of disjoint members of $\mathcal{F}$, or there is a countable family $(F_n)_{n \geq 1}$ of disjoint members of $\mathcal{F}$ such that :

$$B \subseteq F_1 \cup \dots \cup F_n \cup \bigcup_{k=n+1}^{+\infty} U(F_k, 3b\delta(F_k)) \quad (1)$$

Proof. We define the family $(F_n)_{n \geq 1}$ by induction, so that they are mutually disjoint at every stage of the construction. Let F_1 be an arbitrary member of $\mathcal{F}$. Suppose now that for some $k \geq 1$, the sets $F_1, \dots, F_k$ have already be constructed. If $B \subseteq F_1 \cup \dots \cup F_k$ then we are done. Otherwise, let $\varepsilon_k = \sup_{F \in \mathcal{F}_k} \delta(F)$ where $\mathcal{F}_k = \{F \in \mathcal{F} \mid \forall i \in \llbracket 1, k \rrbracket, F_i \cap U(F, \delta(F)) = \emptyset\}$. Since X is a b -metric space and $F_1 \cup \dots \cup F_k$ is closed in X , it follows from the assumption of the lemma that $\mathcal{F}_k$ is not empty and that $\varepsilon_k > 0$. We choose F_k to be an arbitrary element of $\mathcal{F}_k$ with $\delta(F_{k+1}) > \frac{2\varepsilon_k}{3}$.

Now suppose this process goes on to infinity, and let us prove that (1) is satisfied for every n . Suppose by way of contraction that there is some $n \geq 0$ such that (1) is not satisfied. Let p be an element of B which is not in the right-hand side of (1). Since the set $F_1 \cup \dots \cup F_n$ is closed, it follows from the assumption of the lemma that there is some $F \in \mathcal{F}_n$ with $p \in F$ and $\delta(F) > 0$.

Now suppose there is some $k \geq n$ such that $F \in \mathcal{F}_k$. Then $\delta(F) \leq \varepsilon_k$ hence $\delta(F_{k+1}) \geq \frac{2\delta(F)}{3}$. Since $p \in F$ this would imply that $U(F, \delta(F)) \subseteq U(p, 2nb\delta(F)) \subseteq U(p, 3b\delta(F_{k+1}))$. But as p cannot be in $U(F_{k+1}, 3b\delta(F_{k+1}))$ by definition, one must have $F_{k+1} \cap U(F, \delta(F)) = \emptyset$ so since $F \in \mathcal{F}_{k+1}$ we also have $F \in \mathcal{F}_k$. Therefore we have proven that for every $k \geq n$ one has $F \in \mathcal{F}_k \Rightarrow F \in \mathcal{F}_{k+1}$. From this and the fact that $F \in \mathcal{F}_n$, it follows by induction that F is in all of the $\mathcal{F}_k$ for $k \in \mathbb{N}^*$. Thus $\varepsilon_k \geq \delta(F) > 0$ for all $k \in \mathbb{N}^*$, hence $\delta(F_k) \geq \frac{2\delta(F)}{3}$ for all $k \in \mathbb{N}$.

Now consider an arbitrary sequence $(p_k)_{k \geq 1}$ in X with $p_k \in F_k$ for every $k \in \mathbb{N}^*$. If $i < j$, then $F_i \cap U(F_j, \delta(F_j)) = \emptyset$ by construction of the sequence $(F_n)_{n \geq 1}$, and so $\rho(p_i, p_j) \geq \delta(F_j) \geq \frac{2\delta(F)}{3}$. Thus the sequence $(p_k)_{k \geq 0}$ does not contain a convergent sub-sequence, which contradicts the compactness of X and concludes the proof. $\square$

Now let X be a b -metric topological space endowed with a positive measure μ . We say a set $B \subseteq X$ is covered in the sense of Vitali by a family $\mathcal{F}$ of closed subsets of X if and only if each $F \in \mathcal{F}$ has positive measure and there exists $\lambda > 0$ such that each point of B is contained in sets $F \in \mathcal{F}$ of arbitrarily small strictly positive diameter $\delta(F)$ such that :

$$\frac{\mu(U(F, 3b\delta(F)))}{\mu(F)} \leq \lambda$$

Theorem 3.8 (Vitali's covering theorem). *Let X be a locally compact σ -compact b -metric topological space and let μ be a positive regular Borel measure on X .*

Then if a family $\mathcal{F}$ of closed subsets of X cover some $C \subseteq X$ in the sense of Vitali, there exists a sequence $(F_n)_{n \geq 0}$ of disjoint sets in $\mathcal{F}$ such that $\mu(C \setminus \bigcup_{n \geq 0} F_n) = 0$.

Proof. We first consider the case where X is compact. In that case, the above lemma provides a sequence $(F_n)_{n \geq 0}$ of elements of $\mathcal{F}$ which are disjoint such that :

$$B \setminus (F_1 \cup \dots \cup F_n) \subseteq \bigcup_{k \geq n+1} U(F_k, 3b\delta(F_k))$$

for each n . Using the definition of a covering in the sense of Vitali, it follows that for every $n \in \mathbb{N}$ one has :

$$\mu(B \setminus (B_1 \cup \dots \cup B_n)) \leq \lambda \sum_{k \geq n+1} \mu(F_k)$$

and the right-hand side goes to zero as n goes to $+\infty$ since the sequence $(\mu(F_k))_{k \geq 0}$ has finite sum (here we use the fact that the F_k are disjoint and that X has finite measure). This concludes the proof in the case where X is compact.

Now for the general case, since X is σ -compact there is a sequence $(X_i)_{i \geq 1}$ of disjoint open relatively compact subsets of X the union of which is conull in X . The existence of such a sequence can be proven for example by the Urysohn lemma. Replacing X by the closure of X_i and B by $B_i = B \cap X_i$, as well as the family $\mathcal{F}$ by the sub-family $\mathcal{F}^i = \{F \in \mathcal{F} \mid F \subseteq X_i\}$, we reduce to the case where X is compact. The fact that B_i is covered by $\mathcal{F}^i$ in the sense of Vitali is a consequence the fact that X_i is open which ensures that elements of $\mathcal{F}$ with small enough diameter around some specified point of X_i actually is in $\mathcal{F}^i$. What remains is easy book-keeping. $\square$

Definition 3.9. Let X be a b -metric topological space endowed with a positive Borel measure μ . We say that the b -metric ρ is finite dimensional relative to μ if and only if $\mu(U(x, \varepsilon)) > 0$ for every $x \in X$, $\varepsilon > 0$ and :

$$\sup_{x \in X} \limsup_{\varepsilon \rightarrow 0} \frac{\mu(U(x, c\varepsilon))}{\mu(U(x, \varepsilon))} < +\infty$$

for every $c > 1$. Note that if this is satisfied for some $c > 1$ then it is of course satisfied for every $c > 1$. We say that a point $x \in X$ is a point of density of some measurable subset C if and only if one has :

$$\limsup_{\varepsilon \rightarrow 0} \frac{\mu(C \cap U(x, \varepsilon))}{\mu(U(x, \varepsilon))} = 1$$

Let us now state and prove the important result of this section, the purpose of which is essentially its corollary 3.11.

Theorem 3.10 (Density point theorem). *Let X be a locally compact σ -compact b -metric topological space and let μ be a positive regular Borel measure on X . Assume that the b -metric defined on X is finite dimensional relative to μ .*

Then for every measurable set $C \subseteq X$, almost every point of C is a point of density.

Proof. Let us fix a measurable set $C \subseteq X$ of X . Since the measure μ is regular, there is a sequence of open sets $(W_j)_{j \geq 1}$ containing C such that $\mu(W_j \setminus C) \xrightarrow{j \rightarrow +\infty} 0$. Now for every $i \in \mathbb{N}^*$, set :

$$B_i = \left\{ x \in C \mid \liminf_{\varepsilon \rightarrow 0} \frac{\mu(C \cap U(x, \varepsilon))}{\mu(U(x, \varepsilon))} < 1 - \frac{1}{i} \right\}$$

Since the sets W_j are open and the b -distance function on X is finite dimensional relative to μ , it follows that for every $i, j \in \mathbb{N}^*$ the family :

$$\mathcal{F}_{ij} = \left\{ U(x, \varepsilon) \mid x \in C, \varepsilon > 0, U(x, \varepsilon) \subseteq W_j, \frac{\mu(C \cap U(x, \varepsilon))}{\mu(U(x, \varepsilon))} < 1 - \frac{1}{i} \right\}$$

covers B_i in the sense of Vitali. We deduce from theorem 3.8 that there is a sequence $(F_{ij}^n)_{n \geq 1}$ of disjoint members of $\mathcal{F}_{ij}$ the union of which is conull in B_i . It then follows from the fact that all the F_{ij}^n are in $\mathcal{F}_{ij}$ and are disjoint that :

$$\mu(W_j \setminus C) \geq \sum_{n=1}^{+\infty} \mu(F_{ij}^n \setminus C) > \sum_{n=1}^{+\infty} \frac{1}{i} \mu(F_{ij}^n)$$

Now since $\mu(B_i \setminus \bigcup_{n=1}^{+\infty} F_{ij}^n) = 0$ we deduce that $\mu(B_i) = \sum_{n=1}^{+\infty} \mu(F_{ij}^n) < i \mu(W_j \setminus C)$ so upon letting $j \rightarrow \infty$ it follows that B_i has zero measure.

Now note that the set of elements of C which are not density points is exactly the union of the B_i 's for $i \in \mathbb{N}^*$. Thus it has zero-measure, which concludes the proof. $\square$

Corollary 3.11. *Let H be a locally compact σ -compact group, $\varphi : H \rightarrow H$ a contracting automorphism, and $B \subseteq H$ a μ_H -measurable subset. For $Y \subseteq H$, we set :*

$$\omega(Y) = \begin{cases} H & \text{if } e \in Y \\ \emptyset & \text{else.} \end{cases}$$

Then for almost every $h \in H$, the sequence $(\varphi^{-n}(hB))_{n \geq 0}$ converges in measure μ_H to $\omega(hB)$.

This is not surprising. Indeed, φ may be given the name of "expanding" automorphism as the inverse of a contracting automorphism. We therefore expect the iterated images of a Borel set B to fill in the whole of H if $e \in B$, and to escape to infinity if $e \notin B$. This corollary states that this is "most of the time" the case.

Proof. We prove that for every measurable set $C \subseteq H$ and every open relatively compact set $K \subseteq H$ containing the identity, one has :

$$\lim_{n \rightarrow +\infty} \frac{\mu_H(\varphi^{-n}(hC) \cap K)}{\mu_H(K)} = 1 \quad \text{for almost every } h \in C^{-1}.$$

It then suffices to apply this result to $C = B$ and $C = H \setminus B$ to conclude. Let C be a measurable subset of H .

Since φ is an automorphism of H , the uniqueness of the Haar measure ensures that the quotient $\frac{\mu_H(\varphi(A))}{\mu_H(\varphi(B))} = \frac{\mu_H(A)}{\mu_H(B)}$ for every measurable subsets A, B such that $\mu_H(B) > 0$. Hence it suffices to show that for every open relatively compact set $K \subseteq H$ containing the identity, one has :

$$\frac{\mu_H(C \cap h^{-1}\varphi^n(K))}{\mu_H(\varphi^n(K))} \xrightarrow{n \rightarrow +\infty} 1 \quad \text{for almost every } h \in C^{-1}. \quad (*)$$

Now, since φ is contracting and K is a neighborhood of e , one may take $N \in \mathbb{N}$ such that $\varphi^n((K \cup K^{-1})K) \subseteq K$ for every $n \geq N$. So upon replacing K by $\bigcup_{0 \leq n \leq N-1} \varphi^n(K \cup K^{-1})$, we may suppose that $K = K^{-1}$ and $\varphi(K) \subseteq K$.

Now for every $h_1, h_2 \in H$, set :

$$\begin{aligned} n_K(h_1, h_2) &= \max\{n \in \mathbb{Z} \mid h_1^{-1}h_2 \in \varphi^n(K)\} = -\min\{n \in \mathbb{Z} \mid \varphi^n(h_1^{-1}h_2) \in K\} \\ \rho_K(h_1, h_2) &= 2^{-n_K(h_1, h_2)/N} \end{aligned}$$

Now since φ is contracting, it is clear that $\rho_K(h_1, h_2)$ is positive and assumes the value zero only when $h_1 = h_2$. It is moreover symmetric since $K = K^{-1}$. Moreover, for every h_1, h_2, h_3 , if $h_1^{-1}h_2 \in \varphi^m(K)$ and $h_2^{-1}h_3 \in \varphi^n(K)$ then $h_1^{-1}h_3 = h_1^{-1}h_2h_2^{-1}h_3 \in \varphi^n(K)\varphi^m(K) \subseteq \varphi^{\max(n, m)}(K)\varphi^{\max(n, m)}(K) = \varphi^{\max(n, m)}(KK) \subseteq \varphi^{\max(n, m)-N}(K)$. Thus one has :

$$\rho_K(h_1, h_3) \leq 2^{\frac{-\max(n_K(h_1, h_2), n_K(h_2, h_3)) + N}{N}} = 2 \max(\rho_K(h_1, h_2), \rho_K(h_2, h_3))$$

and in particular ρ_K is a 2-metric. Write $U(x, \varepsilon) = \{y \in H \mid \rho_K(x, y) < \varepsilon\}$ for every $x \in H$ and every $\varepsilon > 0$. Since K is an open neighborhood of e and φ is contracting, it is rather easy to see that the $U(x, \varepsilon)$ form a basis of the topology of H , hence H is in fact a 2-metric topological space.

Now the fact that φ is μ_H -homogenous (that is, $\mu_H(\varphi(A))\mu_H(B) = \mu_H(\varphi(B))\mu_H(A)$ for every Borel sets A, B) implies that the distance function ρ_K is finite dimensional relative to μ_H . Hence the density point theorem shows that for almost every $h \in C$, one has :

$$\lim_{\varepsilon \rightarrow 0} \frac{\mu_H(C \cap U(h, \varepsilon))}{\mu_H(U(h, \varepsilon))} = 0$$

which is exactly equivalent to $(*)$, so we are done. $\square$

3.1.4 Proof of the subalgebra theorem

In all of this section, G is a real semi-simple Lie group of higher rank, Γ is a lattice in G and P is a minimal parabolic subgroup of G , that is a Borel subgroup. Let S be a maximal split $\mathbb{R}$ -torus contained in P . We refer to section 1.3 for details about these notations.

We start with a few lemmas, before moving to the proof of theorem 3.4.

Lemma 3.12. *Let $g \in G$ which acts ergodically on the left quotient $\Gamma \backslash G$. Then for almost all $g' \in G$, the sets $\bigcup_{n \in \mathbb{N}} \Gamma g' g^n$ and $\bigcup_{n \in \mathbb{N}} \Gamma g' g^{-n}$ are dense in G .*

Here, by " g acts ergodically on $\Gamma \backslash G$ " we mean that every Borel subset of $\Gamma \backslash G$ invariant under $x \mapsto xg$ has zero or full measure.

Proof. We claim that it suffices to show that for every open non-empty $B \subset G$, the sets $\bigcup_{n \in \mathbb{N}} \Gamma B g^n$ and $\bigcup_{n \in \mathbb{N}} \Gamma B g^{-n}$ are conull. Indeed, suppose that this is true. Then, take a family $(B_k)_{k \geq 0}$ of open sets which form a basis of the topology of G . Write W for the intersections of the $\bigcup_{n \in \mathbb{N}} \Gamma B_k g^n$ and $\bigcup_{n \in \mathbb{N}} \Gamma B_k g^{-n}$ for $k \in \mathbb{N}$. W is then conull, and for every $g' \in W$ one has the desired result. Let us therefore fix some open non-empty set B and prove that $X^+ = \bigcup_{n \in \mathbb{N}} \Gamma B g^n$ and $X^- = \bigcup_{n \in \mathbb{N}} \Gamma B g^{-n}$ are conull. To do so, note that if $\pi : G \rightarrow \Gamma \backslash G$ is the natural projection, then $\pi(X^+)$ is open non-empty and invariant under the action of G (since $\pi(X^+)g \subseteq \pi(X^+)$ and they have the same measure), thus conull. Its inverse image is conull as it is contained in a countable number of copies of $\pi(X^+)$ in G . Similarly, $\pi(X^-)$ satisfies $\pi(X^-)g^{-1} \subseteq \pi(X^-)$, that is $\pi(X^-) \subseteq \pi(X^-)g$ hence these two are almost equal since they have the same measure, and the same argument applies. $\square$

Now the next step is to decompose G as $V^- \cdot P$, and transform this result on almost every $g' \in G$ to a result about almost every $v \in V^-$. This is of course highly non trivial : the intersection of a conull subset of a product with one of the two factors need not be conull. Note that for this to work, we need to use some $s \in S$ in place of the $g \in G$ acting ergodically.

Lemma 3.13. *Let $s \in S$. Suppose $s \in R_\Delta$ and s acts ergodically on $\Gamma \backslash G$. Then for almost all $v \in V^-$ the set $\bigcup_{n \in \mathbb{N}} \Gamma v s^{-n}$ is dense in G .*

Proof. Using the above lemma, we get that the set $W = \{g \in G \mid \text{the set of elements } \gamma g s^{-n}, n \in \mathbb{N}, \gamma \in \Gamma \text{ is dense in } G\}$ is conull in G , and we want to prove that $V^- \cap W$ is conull in V^- . To that extent, we will use the decomposition $G = V^- \cdot P$ (which is true up to a null set), the map $(v, p) \mapsto vp^{-1}$ being a homeomorphism on its image that is measure class preserving. For every $v \in V^-$, write $C_v = \{p \in P \mid vp^{-1} \notin W\}$, and $Y = \{v \in V^- \mid \mu_P(C_v) = 0\}$. The fact that W is conull in G therefore implies that Y is conull in V^- , by Fubini's theorem :

$$\mu_{V^-}(V^- \setminus Y) = 0$$

We will prove that, in fact, Y is contained in $V^- \cap W$, which would conclude the proof. Let $v \in Y$, so that C_v is a null set. We wish to prove that $v \in W$. To do so, we will actually prove that $C_v = \emptyset$ and in particular $e \notin C_v$, which amounts to $v \in W$. To prove that $C_v = \emptyset$, since C_v has zero measure, it suffices to prove that C_v is open. This is what we will do next.

Take $p \in C_v$, write $g = vp^{-1} \in W$. We will prove that for q close enough to e , qp is actually in C_v . Write for every $q \in P$:

$$\Gamma v(qp)^{-1}s^{-n} = \Gamma gs^{-n}(s^n q^{-1}s^{-n}) \quad (1)$$

Now we know that the automorphism $x \mapsto sxs^{-1}$ of P is non-expanding (lemma 1.39, this is where we use the fact that $s \in R_\Delta$). Thus as q approaches e , all the $(s^n q^{-1}s^{-n})$ uniformly approach e . Since $p \in C_v$, there is some open non-empty subset U of G such that $\bigcup_{n \geq 0} \Gamma gs^{-n}$ does not intersect U . Now fix some relatively compact non-empty open subset U_1 of U and $x_1 \in U_1$. For q small enough one has $s^n qs^{-n} \subseteq U_1^{-1}x_1$ for every $n \in \mathbb{N}$, hence :

$$\bigcup_{n \in \mathbb{N}} \Gamma v(qp)^{-1}s^{-n} \subset (G \setminus U) \cdot U_1^{-1}x_1 \subseteq G \setminus \{x_1\}$$

Thus the left-hand side is contained in a closed set of G that is not G , hence is not dense in G , so $qp \in C_v$ and this is true for q small enough so C_v is open, so empty since it has zero measure and we are done. $\square$

Before stating the next lemma define for every subset B of G :

$$\Psi_\theta(B) = V_\theta^- \cdot (B \cap L_\theta^-)$$

Note that if $B \subseteq V_\theta^-$ then $\Psi_\theta(B) = V_\theta^-$ if $e \in B$ and $\emptyset$ else.

Lemma 3.14. *Let θ be a subset of the set of roots. Then the following two propositions are satisfied.*

- a) *If $s \in R_\theta$ and C is a measurable subset of V_θ^- , then for almost every $v \in V_\theta^-$, the sequence $(s^n v C s^{-n})_{n \geq 0}$ converges relative to the measure $\mu_{V_\theta^-}$ to $\Psi_\theta(vC)$.*
- b) *If $s \in D_\theta$ and C is a measurable subset of V^- then for almost all $v \in V^-$ the sequence $(s^n v C s^{-n})_{n \geq 0}$ converges relative to the measure μ_{V^-} to $\Psi_\theta(vC)$.*

Proof. a) is an application of corollary 3.11, using the fact that on the group V_θ^- , the automorphism $v \mapsto s^{-1}vs$ is contracting. The point b) is a consequence of a) and of the fact that $V^- = V_\theta^- \rtimes L_\theta^-$ and that D_θ acts trivially on L_θ^- . $\square$

These preliminary lemmas being proven, we may now move to stuff that explicitly involve sub- σ -algebras of $\mathcal{M}(G/P)$ invariant under the action of a lattice.

Before we do so, let us give an alternate description of $\mathcal{M}(G/P)$. Recall that since the multiplication map from $V^- \times P$ to G is a homeomorphism onto a dense open subset of G , we may identify G/P with V^- up to a null set. This provides a natural isomorphism between $\mathcal{M}(G/P)$ and $\mathcal{M}(V^-)$.

We may also like to understand to what sub- σ -algebra of $\mathcal{M}(V^-)$ does $\mathcal{M}(G/P, Q)$ correspond to, for Q a parabolic subgroup of G containing P . Write $Q = P_\theta$ for θ a subset of Δ . Then it turns out that through the isomorphism $\mathcal{M}(G/P) \simeq \mathcal{M}(V^-)$, the σ -algebra

$\mathcal{M}(G/P, P_\theta)$ corresponds to $\mathcal{M}(V^-, L_\theta^-)$ which is an isomorphic copy of $\mathcal{M}(V^-/L_\theta^-)$ lying in $\mathcal{M}(V^-)$. All these informations are summarized in the following diagram :

$$\begin{array}{ccc} \mathcal{M}(G/P) & \xrightarrow{\simeq} & \mathcal{M}(V^-) \\ \uparrow & & \uparrow \\ \mathcal{M}(G/P, P_\theta) & \xrightarrow{\simeq} & \mathcal{M}(V^-, L_\theta^-) \end{array}$$

Proposition 3.15. *Let $\mathcal{B}$ be a sub- σ -algebra of $\mathcal{M}(G/P) \simeq \mathcal{M}(V^-)$ that is invariant under the action of Γ on the left (that is, for every $B \in \mathcal{B}$ and every $\gamma \in \Gamma$ one has $\gamma B \in \mathcal{B}$).*

Let $B \in \mathcal{B}$ and θ a subset of positive roots. Suppose that some $s \in D_\theta$ acts ergodically on $\Gamma \backslash G$.

Then, for almost all $v \in V^-$ (relative to the measure μ_{V^-}) and for each $g \in G$ we have that $g\Psi_\theta(vB) \in \mathcal{B}$.

Proof. We set :

$$U_1 = \{v \in V^- \mid \bigcup_{n \in \mathbb{N}} \Gamma v^{-1} s^{-n} \text{ is dense in } G\}$$

$$U_2 = \{v \in V^- \mid s^n v B s^{-n} \text{ converges in measure relative to } \mu_{V^-} \text{ to } \Psi_\theta(vB)\}$$

and $U = U_1 \cap U_2$. The lemma 3.13 and lemma 3.14 ensure that U is conull in V^- . Now we aim to prove that for every $v \in U$ and every $g \in G$ we have $g\Psi_\theta(vB) \in \mathcal{B}$.

Let $v \in U$ and $g \in G$. Since $v \in U_1$, we get sequences $(g_n)_{n \geq 0} \in G^\mathbb{N}$ and $(\gamma_n)_{n \geq 0} \in \Gamma^\mathbb{N}$ such that $(g_n)_{n \geq 0}$ converges to $g \in G$ and $g_n = \gamma_n v^{-1} s^{-n}$, that is $\gamma_n = g_n s^n v$. But $v \in U_2$, hence the sequence $(\gamma_n B)_{n \geq 0} = (g_n s^n v B)_{n \geq 0} = (g_n s^n v B s^{-n})_{n \geq 0}$ converges to $g\Psi_\theta(vB)$ relative to the measure $\mu_{V_\theta^-}$ (here we used the fact that $B = B s^{-1}$ since $s \in P$, hence s acts trivially on the right on $\mathcal{M}(G/P)$).

Now for every $n \in \mathbb{N}$, $\gamma_n B \in \mathcal{B}$ since $\mathcal{B}$ is Γ -invariant. Every sub- σ -algebra of $\mathcal{M}(G/P)$ being closed for the convergence in measure (see the explanations following theorem 3.4), it follows that $g\Psi_\theta(vB) \in \mathcal{B}$ and we are done. $\square$

Here is the last lemma that we will need before going through the proof of theorem 3.4.

Lemma 3.16. *Let $\theta \subseteq \Delta$ and B an element of $\mathcal{M}(G/P)$. Assume that $B \notin \mathcal{M}(G/P, P_\theta)$. Then, up to identification between $\mathcal{M}(G/P)$ and $\mathcal{M}(V^-)$, there is some root $b \in \theta$ such that $BL_{\{b\}}^- \neq B$.*

Proof. Using the fact that the $P_{\{b\}}$ for $b \in \theta$ generate P_θ , one has :

$$\mathcal{M}(G/P, P_\theta) = \bigcap_{b \in \theta} \mathcal{M}(G/P, P_{\{b\}})$$

since for every parabolic subgroup Q one has :

$$\mathcal{M}(G/P, P_\theta) = \{B \in \mathcal{M}(G/P) \mid BP_\theta = B\}.$$

Therefore there is some $b \in \theta$ such that $B \notin \mathcal{M}(G/P, P_{\{b\}})$. But through the identification between G/P and V^- , $\mathcal{M}(G/P, P_{\{b\}})$ gets mapped to $\mathcal{M}(V^-, L_{\{b\}}^-)$, hence the desired result. $\square$

Note that, in the case where $\theta = \emptyset$, the σ -algebra $\mathcal{M}(G/P, P_\emptyset)$ is nothing but $\mathcal{M}(G/P)$ itself, hence the assumption that $B \notin \mathcal{M}(G/P, P_\theta)$ cannot be satisfied. We now have all of the ingredients needed to prove theorem 3.4.

Proof of theorem 3.4. Let $\mathcal{B}$ be a sub- σ -algebra of $\mathcal{M}(G/P)$ that is invariant under the action of Γ on the left. Let $Q = P_\theta$ be the smallest parabolic subgroup containing P such that $\mathcal{M}(G/P, P_\theta) \subseteq \mathcal{B} \subseteq \mathcal{M}(G/P)$, where $\theta \subseteq \Delta$. We aim to show that $\mathcal{B} = \mathcal{M}(G/P, P_\theta)$, so we suppose otherwise. We can therefore consider some Borel set $B \in \mathcal{B}$ that is not in $\mathcal{M}(G/P, P_\theta)$.

Using $\mathcal{B}$, we will construct a G -invariant sub- σ -algebra $\mathcal{B}_1$ of $\mathcal{B}$ which does not contain $\mathcal{M}(G/P, P_\theta)$, hence reaching a contradiction.

Now identify $\mathcal{M}(G/P)$ with $\mathcal{M}(V^-)$ as before. We have proven that there is some $b \in \theta$ such that $BL_{\{b\}}^- \neq B$ in $\mathcal{M}(V^-)$. Now set :

$$U = \{v \in V^- \mid \mu_{L_{\{b\}}^-}(L_{\{b\}}^- \cap vB) > 0 \text{ and } \mu_{L_{\{b\}}^-}(L_{\{b\}}^- \setminus vB) > 0\}$$

Suppose that $\mu_{V^-}(U) = 0$. Then for almost every $v \in V^-$ one has either $v^{-1}L_{\{b\}}^- \subseteq B$ or $v^{-1}L_{\{b\}}^- \subseteq V^- \setminus B$, up to a $\mu_{L_{\{b\}}^-}$ -null set. It follows that up to a μ_{V^-} -null set, B is a union of cosets of $L_{\{b\}}^-$ of the form $vL_{\{b\}}^-$, and in particular one has $BL_{\{b\}}^- = B$ which is excluded. This proves that $\mu_{V^-}(U) > 0$.

Now take some non-trivial element $s \in D_{\{b\}}$. Note that since $D_\Delta = \{e\}$, we need to use the fact that $\Delta \neq \{b\}$. This is where the higher rank assumption is crucial. Now s generates a non-compact group, hence by corollary 2.41 we get that s acts ergodically on the quotient $\Gamma \backslash G$. Thus by the preceding proposition 3.15, one has for almost every $v \in V^-$ and every $g \in G$: $g\Psi_\theta(vB) \in \mathcal{B}$. Since U has non zero measure, it follows that :

$$\exists v \in U, \forall g \in G, g\Psi_\theta(vB) \in \mathcal{B}$$

We fix such a $v \in U$ in what follows. Denote by $\mathcal{B}_1$ the sub- σ -algebra of $\mathcal{B}$ that is generated by the elements $g\Psi_\theta(vB)$ for $g \in G$. It is clearly invariant under the action of G , hence by proposition 3.5 there is some subset of positive roots θ' such that $\mathcal{B}_1 = \mathcal{M}(G/P, P_{\theta'})$. By minimality of θ the fact that $\mathcal{B}_1 \subseteq \mathcal{B}$ implies that $\theta \subseteq \theta'$. Hence $b \in \theta'$ and $\mathcal{B}_1 \subseteq \mathcal{M}(V^-, L_{\{b\}}^-)$.

In particular one has $\Psi_{\{b\}}(vB) \in \mathcal{M}(V^-, L_{\{b\}}^-)$ in $\mathcal{M}(V^-)$, hence $\Psi_{\{b\}}(vB)L_{\{b\}}^- = \Psi_{\{b\}}(vB)$. This leads to $(vB \cap L_{\{b\}}^-) \cdot L_{\{b\}}^- = vB \cap L_{\{b\}}^-$ which in turn implies that $vB \cap L_{\{b\}}^-$ is either empty or the whole of $L_{\{b\}}^-$. This contradicts the fact that $v \in U$ and we are done. $\square$

3.2 The Nevo-Zimmer theorem

3.2.1 Statement and motivation

Let us consider a higher rank semi-simple Lie group G , with a parabolic subgroup P of G . Now if Y is a G -space and we have a measure class preserving map that is G -invariant

$G/P \rightarrow Y$, we have seen before (it essentially is remark 3.6 combined with proposition 3.5) that Y must be isomorphic as a G -space to G/Q for some parabolic subgroup Q of G containing P .

In fact, this result which should not be very surprising has a very elegant generalization, the Nevo-Zimmer theorem.

Theorem 3.17 (Nevo-Zimmer). *Let G be a higher rank Lie group, P a Borel subgroup of G and X, Y two G -spaces, X being ergodic and probability measure preserving. Assume that there are measure-class preserving maps fitting in the diagram :*

$$G/P \times X \rightarrow Y \rightarrow X$$

Then, there exists a parabolic subgroup Q containing P and an isomorphism of G -spaces $Y \rightarrow G/Q \times X$ such that the following diagram commutes :

$$\begin{array}{ccccc} G/P \times X & \xrightarrow{\quad} & Y & \xrightarrow{\quad} & X \\ & \searrow & \downarrow & \nearrow & \\ & & G/Q \times X & & \end{array}$$

with the obvious arrows.

We will not go through the proof of this theorem, which was first stated and proven in [13], Theorem 4.1.

Of course, in the light of remark 3.6, we may state this theorem in a σ -algebras setting as follows.

Theorem 3.18 (Nevo-Zimmer, σ -algebras version). *Let G be a higher rank Lie group, P a Borel subgroup of G and X an ergodic probability measure preserving G -space. Every sub- σ -algebra of $\mathcal{M}(G/P \times X)$ containing $\mathcal{M}(X)$ is of the form $\mathcal{M}(G/Q \times X)$ for some parabolic subgroup Q of G containing P .*

Here, of course, the implied inclusions $\mathcal{M}(X) \subseteq \mathcal{M}(G/Q \times X) \subseteq \mathcal{M}(G/P \times X)$ are the ones coming from the natural arrows $G/P \times X \rightarrow G/Q \times X \rightarrow X$.

Note that quite surprisingly, the Margulis factor theorem which revolves around Γ -spaces and not G -spaces is actually a consequence of this.

Proof of theorem 3.3 using theorem 3.17. Let us consider a lattice Γ in G as well as a G/P factor Z , that is a measure class preserving Γ -equivariant map $G/P \rightarrow Z$ where Z is a Γ -space. Now consider the G -space $G \times_{\Gamma} Z$, which fits in the diagram :

$$G/\Gamma \times G/P \rightarrow G \times_{\Gamma} Z \rightarrow G/\Gamma$$

since $G \times_{\Gamma} G/P \simeq G/\Gamma \times G/P$. The Nevo-Zimmer theorem then gives a G -equivariant commutative diagram, the vertical arrow being an isomorphism :

$$\begin{array}{ccc} G/\Gamma \times G/Q & & \\ \downarrow & \searrow & \\ G \times_{\Gamma} Z & \xrightarrow{\quad} & G/\Gamma \end{array}$$

for some parabolic subgroup Q of G containing P . Now taking the fiber of $\{e\}$ above G/Γ , we get the desired isomorphism $G/Q \simeq Z$. $\square$

There is one subtlety in the last sentence of the above proof that was swept under the rug : since we only have equality almost everywhere, we cannot restrict functions to a specific fiber. We would instead have to choose a "generic fiber" on which to restrict, and the choice of that generic fiber would not matter.

3.2.2 The proof

Let us now move to the proof of the Nevo-Zimmer theorem. The proof has extremely close structure to that of the Margulis factor theorem, so we may be rather vague on some details already mentioned in a slightly less broad setting in section 3.1.4. The version of Nevo-Zimmer theorem we will prove is theorem 3.18, but theorem 3.17 can easily deduced from it using the ideas of remark 3.6.

Let G be a higher rank semi-simple Lie group, and X an ergodic G -space. Consider a G -space Y with measure class preserving maps $G/P \times X \rightarrow Y \rightarrow X$. We now have $\mathcal{M}(X) \subseteq \mathcal{M}(Y) \subseteq \mathcal{M}(G/P \times X)$, with all algebras being G -invariant. Write $B = \mathcal{M}(Y)$. It is a rather classical fact from abelian von Neumann algebras that we can write :

$$\mathcal{M}(Y) = \int_{x \in X}^{\oplus} \mathcal{M}(Y)_x d\mu(x)$$

For a family $(\mathcal{M}(Y)_x)_{x \in X}$ of sub-algebras of $\mathcal{M}(G/P)$ that is uniquely determined almost everywhere. G -invariance then implies that for almost every $x \in X$ and every $g \in G$ one has $g\mathcal{M}(Y)_x = \mathcal{M}(Y)_{gx}$.

Here is one proposition that will be crucial in the proof, and is the two variable analogue of proposition 3.15.

Proposition 3.19. *Let G be an higher rank semi-simple Lie group, and X be an ergodic G -space. Let P_θ be a parabolic subgroup containing the Borel subgroup P . Identify V^- with G/P . Let $C \in \mathcal{M}(Y) = B$ that we write $C = \int_{x \in X}^{\oplus} C_x$.*

Then for almost every $(x, u) \in X \times V^-$ one has $g\Psi_\theta(uC_x) \in B_x$ for all $g \in G$.

Here, of course, $\Psi_\theta(uC_x)$ keeps the same meaning as in the proof of the Margulis factor theorem, that is $\Psi_\theta(D) = V_\theta^- \cdot (C \cap L_\theta^-)$.

Proof. We begin by recalling that $\mathcal{M}(G/P)$ is embedded as a closed compact subset of the von Neumann algebra $L^\infty(G/P)$. We endow the set $\mathcal{C}$ of closed subsets of $\mathcal{M}(G/P)$ with its standard topology which makes it into a compact metric space, so that the map $X \mapsto \mathcal{C}$ sending x to B_x is measurable. Write ρ for the distance on $\mathcal{C}$.

Therefore, for every $N \geq 1$, one can write a finite union $X = \bigcup_{i \in I_N} X_i^N$ such that for every $x, y \in X_i^N$ one has $\rho(B_x, B_y) < \frac{1}{N}$.

Consider now $s \in D_\theta$. For each $N \geq 1$ and $i \in I_N$, write :

$$Z_i^N = \{(x, g) \in X_i^N \times G \mid \{\alpha g s^{-n} \mid n \geq 0, \alpha \in G \text{ with } \alpha \cdot x \in X_i^N\} \text{ is dense in } G\}$$

Now by Moore Ergodicity theorem (see especially corollary 2.41), s acts ergodically on X . From this, using the contracting automorphisms from lemma 1.39 we deduce that the action of $G \times \mathbb{Z}$ on $X \times G$ given by $(\alpha, n) \cdot (x, g) = (\alpha \cdot x, \alpha g s^{-n})$ is also ergodic. It follows that Z_i^N is conull in $X_i^N \times G$ and $Z^N = \bigcup_{i \in I_N} Z_i^N$ is conull in $X \times G$.

Now let $U^N = Z^N \cap (X \times V^-)$. We claim, and we will show in this paragraph, that U^N is conull in $X \times V^-$. Indeed, if $x \in X$ with $(x, g) \in Z^N$ for almost all g then for almost all $u \in V^-$ we have $(x, up) \in Z^N$ for almost all $p \in P$: this is just the fact that $G = V^- \cdot P$ up to a measure zero subset. Now the exact same argument as in lemma 3.13 show that for almost all x , almost all $v \in V^-$, $\{p \in P \mid (x, vp) \in Z^N\}$ is a closed set, so it must be equal to the whole of P and thus U^N is conull in $X \times V^-$. Now the intersection of all U^N , that we denote by U , is also conull in $X \times V^-$.

Upon having identified G/P with V^- , we now have an action of G on V^- that we denote by $(g, v) \mapsto g \circ v$. The identification between the algebras $\mathcal{M}(G/P)$ and $\mathcal{M}(V^-)$ allows us to see $\mathcal{M}(Y)_x$ as a sub-algebra of $\mathcal{M}(V^-)$ for almost every $x \in X$, and the G -invariance ensures that $g \circ \mathcal{M}(Y)_x = \mathcal{M}(Y)_{gx}$ for almost every $x \in X$ and every $g \in G$. Now, write $C = \int_{x \in X}^\oplus C_x$ and fix $g \in G$. Define :

$$W = \{(x, u) \in X \times V^- \mid s^n u C_x s^{-n} \text{ converges in measure } \mu_{V^-} \text{ to } \Psi_\theta(vC)\}$$

W is conull by lemma 3.14. Now if $(x, u) \in W \cap U$ we may choose a sequence $(\alpha_n)_{n \geq 0}$ of elements of G and a subsequence $(\alpha_{n_j})_{j \geq 0}$ such that :

- (i) $\alpha_{n_j} u^{-1} s^{-n_j} \rightarrow g$ (this is thanks to the fact that $(x, u) \in U$)
- (ii) $\rho(B_x, X_{\alpha_{n_j} x}) \rightarrow 0$.

Now, letting $g_n = \alpha_n u^{-1} s^{-n}$ so that $\alpha_n = g_n s^n u$, we have for almost every $x \in X$: $\alpha_n C_x \in \mathcal{M}(Y)_{\alpha_n x}$ by invariance under the action of G . But one has, using the fact that $C_x s^{-1} = C_x$ since $C_x \in \mathcal{M}(G/P)$:

$$\alpha_n \circ C_x = (g_n s^n u) \circ C_x = g_n (s^n u C_x s^{-n})$$

which converges to $g \Psi_\theta(vC_x)$ as stated in lemma 3.14. B_x being closed an invariant under the action of G , this forces the limit to be in B_x and we are done. $\square$

We now have enough background to prove theorem 3.18.

Proof of theorem 3.18. Let us consider some sub- σ -algebra $\mathcal{B}$ of $\mathcal{M}(X \times G/P)$ containing $\mathcal{M}(X)$ and invariant under the action of G . Consider $\theta \subseteq \Delta$ the smallest set of simple roots such that :

$$\mathcal{M}(X) \subseteq \mathcal{M}(X \times G/P_\theta) \subseteq \mathcal{B} \subseteq \mathcal{M}(X \times G/P)$$

Exactly as in the proof of theorem 3.4, suppose by way of contradiction that there is some $C \in \mathcal{B} \setminus \mathcal{M}(X \times G/P_\theta)$, and disintegrate it along X :

$$C = \int_{x \in X}^{\oplus} C_x \quad \text{where the } C_x \in \mathcal{M}(G/P).$$

Now for $x \in X$ in a set of positive measure one must have $C_x \notin \mathcal{M}(G/P_\theta)$ so arguing as in the proof of theorem 3.4 we actually get a root $b \in \theta$ such that $C_x \notin \mathcal{M}(G/P_{\{b\}})$ for x in a positive measure set.

Once again arguing as before, one has for every $x \in X$ with $C_x \notin \mathcal{M}(G/P_{\{b\}})$ that for a set of $u \in V^-$ of positive measure $L_{\{b\}}^- \cap uC_x$ is neither null nor conull in $L_{\{b\}}^-$. Using this remark as well as the above lemma, it follows that there exists some $u \in V^-$ with the following two properties :

- (i) For $x \in X$ in a set of positive measure, $L_{\{b\}}^- \cap uC_x$ is neither null nor conull in $L_{\{b\}}^-$
- (ii) For almost all $x \in X$, $g\Psi_{\{b\}}(uC_x) \in B_x$ for all $g \in G$.

Once again as in the proof of theorem 3.4, we deduce from (i) that for x in a set of positive measure, $\Psi_{\{b\}}(uC_x)$ cannot be in $\mathcal{M}(G/P, P_{\{b\}})$, so neither can it be in $\mathcal{M}(G/P, P_\theta)$.

Now let $B_1(x)$ the sub- σ -algebra of $\mathcal{M}(V^-) \simeq \mathcal{M}(G/P)$ generated by $\mathcal{M}(G/P, P_\theta)$ as well as the $g\Psi_{\{b\}}(uC_x)$ for $g \in G$. By (ii), one has for almost every $x \in X$:

$$\mathcal{M}(G/P, P_\theta) \subseteq B_1(x) \subseteq B_x \subseteq \mathcal{M}(G/P)$$

and moreover the first inclusion is strict for x in a set of positive measure. Since there is only a finite set of roots, we deduce the existence of θ' strictly contained in θ such that, for x in a set of positive measure :

$$\mathcal{M}(G/P, P_\theta) \subseteq B_1(x) = \mathcal{M}(G/P, P_{\theta'}) \subseteq B_x \subseteq \mathcal{M}(G/P)$$

However, using the fact that $gB_x = B_{gx}$ and ergodicity of the action of G on X , it follows that these inclusions are not only true on a set of positive measure but on the whole of G , thus one has $B_x \supseteq \mathcal{M}(G/P, P_{\theta'})$ for almost every $x \in X$ and therefore $\mathcal{B}$ contains $\mathcal{M}(G/P_{\theta'} \times X)$ hence the desired contradiction. $\square$

3.3 The Stuck-Zimmer theorem

3.3.1 Statement and motivation

The Stuck-Zimmer theorem says the following.

Theorem 3.20. *Let G be a real semi-simple connected Lie group whose simple factors are of higher rank.*

Then any faithful irreducible properly ergodic probability measure preserving G -space is essentially free.

For all the vocabulary relative to G -space, please refer to section 1.2.1.

The hypothesis about simple factors having higher rank is there to ensure G has property (T) : it is indeed a consequence of lemma 2.25 and theorem 2.32.

One also misses the definition of "irreducible", which is only relative to actions of real semi-simple Lie groups and not general locally compact σ -compact group. Here is said definition.

Definition 3.21. Let G be a real semi-simple Lie group.

An ergodic measure preserving action of G is said to be irreducible if and only if every non-central normal subgroup of G acts ergodically.

This theorem can be seen as a generalization of the normal subgroup theorem of Margulis. We will now explain how to deduce theorem 3.1 from theorem 3.20 in the case of simple Lie groups.

Proof of theorem 3.1 using theorem 3.20. Consider a simple Lie group G of real rank at least 2, and Γ a lattice in G . Let N be an infinite index normal subgroup of Γ . We aim to show that N is central in G . To do so, consider the group Γ/N . It is infinite and has property (T) (see corollary 2.35) hence it acts freely ergodically probability measure preserving on a probability space X (see [1], theorem 6.3.6). This action induces to G and gives a properly ergodic action of G on $G/\Gamma \times X$ (here we use the fact that the lattice Γ is irreducible in G), in which every stabilizer is conjugate to N . Then theorem 3.20 implies that N is in fact in the kernel of this action, so because G is simple N must be central and we are done. $\square$

To serve as motivation for this theorem, here is an other example of an application of it, which also leans towards rigidity of higher rank lattices, and in particular their actions on probability spaces.

Corollary 3.22. Let Γ be a higher rank lattice in a real connected simple Lie group.

Consider a measure preserving action on (X, μ) a probability space, that is faithful and properly ergodic. Then, the action is essentially free.

Proof. This action can be induced to an action of the ambient group G on $G/\Gamma \times X$. This action of G is still properly ergodic and faithful, and is of course irreducible since G is simple. Therefore the theorem 3.20 ensures it is essentially free. But the stabilizers of the G -action are just conjugates in G of stabilizers of the Γ -action, hence stabilizers of the Γ -action must also be trivial. $\square$

Let us now give an example. Consider $n \geq 3$, and the higher rank lattice $SL_n(\mathbb{Z})$ acting on the torus $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$. It obviously acts faithfully and also properly ergodically : one way to see that orbits have zero measure is to notice that they are countable. So the above corollary implies that the action of $SL_n(\mathbb{Z})$ on $\mathbb{T}^n$ is essentially free, that is almost every point in $\mathbb{T}^n$ has trivial stabilizer.

3.3.2 Actions, cocycles and amenability

Consider now two locally compact σ -compact groups G and H , and an ergodic G -space (X, μ) as well as an ergodic H -space (Y, ν) .

Definition 3.23. The two actions of G and H in X and Y are said to be orbit equivalent if, after possibly discarding null sets, there is a Borel isomorphism $X \rightarrow Y$ taking orbits to orbits.

That is to say, up to a null set we can find a measurable bijection which sends orbits to orbits. It is of course weaker than being equivalent as G -actions, as a G -equivariant bijection obviously preserves orbits.

Now if (X, μ) is a G -space, then we get a Borel equivalence relation R_G by setting $x \sim gy$ for every $x, y \in X$ and $g \in G$.

Amenable actions.

The aim of this paragraph is to define a notion of amenability for actions of locally compact groups.

Consider now H a topological group, which will often be taken to be the isometry group of a separable Banach space or even $\mathbb{R}$ itself, as well as a G -space (X, μ) .

Definition 3.24. Consider a Borel map $\alpha : G \times X \rightarrow H$.

- α is called a cocycle if and only if, for every $g \in G$, $h \in H$ one has $\alpha(gh, x) = \alpha(g, hx)\alpha(h, x)$ for almost every x .
- Two cocycles α, β are said to be equivalent if and only if there exists a Borel map $\varphi : X \rightarrow H$ such that for every $g \in G$ and for almost every $x \in X$ one has : $\beta(g, x) = \varphi(gx)^{-1}\alpha(g, x)\varphi(x)$.
- The cocycle α is said to be trivial if and only if it is equivalent to the cocycle that is constant equal to 1_H .
- The cocycle α is said to be orbital if for almost every $x \in X$ one has $\alpha(g, x) = e$ for every $g \in G_x = \{\gamma \in G \mid \gamma x = x\}$.

Now consider an action of H on a Borel space T , as well as a Borel function $f : X \rightarrow T$. One says that the function f is α -invariant (for a given cocycle $\alpha : G \times X \rightarrow H$) if and only if for each $g \in G$, one has $f(gx) = \alpha(g, x)f(x)$ for almost every $x \in X$.

Moreover, associated to the cocycle α there is a Borel action of G on $X \times T$ given by :

$$\forall g \in G, \forall x \in X, \forall t \in T, g \cdot (x, t) = (gx, \alpha(g, x)t)$$

We will write $X \times_\alpha T$ when we consider the space $X \times T$ endowed with this Borel action of G .

Consider now E a separable Banach space, as well as a cocycle $\alpha : G \times X \rightarrow \text{Iso}(E)$. Let $\alpha^* : (g, x) \mapsto (\alpha(g, x)^{-1})^*$ be the dual cocycle of α , taking values in $\text{Iso}(E^*)$. Let E_1^* be the closed unit ball in E^* endowed with its weak-* topology, which makes it a compact space on which $\text{Iso}(E^*)$ acts.

Definition 3.25. An affine G -space over X is a Borel subset $A \subseteq X \times_{\alpha^*} E_1^*$ such that the two following conditions are satisfied.

- (i) For each $x \in X$, the set $A_x = A \cap (\{x\} \times E_1^*)$ is a closed convex non-empty subset of E_1^* .
- (ii) For each $g \in G$ one has $\alpha^*(g, x)A_x = A_{gx}$ for almost every $x \in X$.

If the cocycle α is orbital then we say that A is an affine orbital G -space over X .

We can now finally define the notion of amenable action of G .

Definition 3.26. The action of G on X is said to be amenable (resp. weakly amenable) if and only if for every affine (resp. orbital affine) G -space A over X , there is an α^* -invariant Borel function $f : X \rightarrow E_1^*$ such that for each $x \in X$ one has $f(x) \in A_x$. Such a function is called a G -invariant section of A .

Let us now list a few properties of amenability of actions, the proof of which can be found in [12].

Proposition 3.27. *Let G be a locally compact second countable group.*

- (i) *If $X \rightarrow Y$ is a map of ergodic G -spaces and the action on Y is amenable, then so is the action on X .*
- (ii) *Let H be a closed subgroup of G . The action of G on G/H is amenable if and only if H is an amenable group.*

Proof. The proof of these can be found in [12]. See in particular Theorems 1.9 and 2.4. $\square$

Amenable relations.

Completely analogous to these definitions of amenability of actions, one can define amenability of a Borel equivalence relation. In all of this paragraph, a Borel equivalence relation over a topological space X will denote a Borel subset of $X \times X$ which is an equivalence relation. Of course, if X is a G -space then the orbits under the action of G form a Borel equivalence relation that we will call R_G .

Two Borel equivalence relations R, S on X and Y are said to be equivalent if there is a bijection (up to null sets) between X and Y which sends R to S and whose inverse sends S to R , still up to a null set. Upon unravelling the definition of orbit equivalence, the reader can now check that two actions are orbit equivalent if and only if the associated Borel equivalence relations are equivalent.

Let us now define a notion of amenability for equivalence relations that we will use in coordination with the group action analog.

Definition 3.28. Let X be a topological space, R a Borel equivalence relation on X . Let H be a topological group.

- A cocycle on the relation R is a Borel map $\alpha : R \rightarrow H$ such that $\alpha(x, y)\alpha(y, z) = \alpha(x, z)$ for every $x, y, z \in X$ in the same equivalence class.
- Two cocycles $\alpha, \beta : R \rightarrow H$ are equivalent if and only if there is a map $\varphi : X \rightarrow H$ such that, for every $(x, y) \in R$ one has $\beta(x, y) = \varphi(y)^{-1}\alpha(x, y)\varphi(x)$.
- A cocycle $\alpha : R \rightarrow H$ is said to be trivial if it is equivalent to the cocycle which sends every element of R to e .

Now consider the case where $H = \text{Iso}(E)$, where E is a separable Banach space. We get an adjoint cocycle m^* by setting $m^*(x, y) = (m(x, y)^*)^{-1}$ which lies in $\text{Iso}(E^*)$. We will denote by E_1^* the unit ball of E^* .

Definition 3.29. Let X be a topological space, R a Borel relation equivalence relation on X , E a separable Banach space and $\alpha : R \rightarrow \text{Iso}(E)$ a cocycle. An affine R -space over E is the data of a Borel subset $A \subseteq X \times E_1^*$ such that the following holds.

Denoting by A_x the projection of $A \cap (\{x\} \times E_1^*)$ on E_1^* , one has :

- (i) Almost every A_x is closed compact non empty.
- (ii) For almost every $(x, y) \in R$, one has $m^*(x, y)A_y = A_x$

Now define the notion of invariant section, similarly to what we did in the case of group actions.

Definition 3.30. Let R be a Borel equivalence relation on X , A an affine R -space over X with cocycle $\alpha : R \rightarrow \text{Iso}(E)$.

A α -invariant section of this affine R -space is a Borel function $f : X \rightarrow E_1^*$ such that $m^*(x, y)f(y) = f(x)$ for almost every $x, y \in X$, and for almost every $x \in X$ one has $f(x) \in A_x$.

Now we can state the definition of amenable equivalence relations.

Definition 3.31. Let R be a Borel equivalence relation on X . One says that R is amenable if and only if every affine R -space has an invariant section.

The following fact is rather easy to check from the definitions.

Proposition 3.32. *Let G be a locally compact σ -compact group and X a G -space. Then the action of G on X is weakly amenable if and only if the relation R_G on X is amenable.*

3.3.3 Weakly amenable actions of property (T) groups

The main goal of this section is to prove the following characterization of weakly amenable actions in the case where the group acting enjoys property (T). This will be the cornerstone of the proof of theorem 3.20.

Lemma 3.33. *Suppose G is a locally compact σ -compact group with property (T). Let X be an ergodic G -space with finite invariant measure.*

Then the action of G on X is weakly amenable if and only if it is essentially transitive.

By essentially transitive, we mean that there is a conull orbit in X . To prove lemma 3.33, we will need two preliminary lemmas that we state here.

Lemma 3.34. *Let G be a lcsc group acting ergodically on (X, μ) .*

If the action is weakly amenable then it is orbit equivalent to an action of either $\mathbb{R}$ or $\mathbb{Z}$.

Lemma 3.35. *Suppose that G acts properly ergodically and weakly amenably on (X, μ) .*

Then there is a non-trivial cocycle $\alpha : G \times X \rightarrow \mathbb{R}$.

Let us now deduce from these two the crucial lemma 3.33. The proof uses a few results on the interaction between cocycles and property (T) that are to be found in [14].

Proof of lemma 3.33 using lemma 3.34 and lemma 3.35. Suppose that the action is essentially transitive. Then one has some $x \in X$ such that Gx is conull in X . Then by [14], 4.2.12 and 4.2.14, every orbital cocycle on $G \times Gx$ is trivial, thus every orbital affine G -space A over Gx satisfies that $\varphi(y)A_y = \varphi(gy)A_{gy}$ for almost all y for some function $\varphi : X \rightarrow \text{Iso}(E_1^*)$. The orbit Gx being conull, it follows that $\varphi(y)A_y$ is almost constant over X , and in particular upon taking some $t \in A_x$ we get an invariant function on Gx by setting $f(gx) = \varphi(gx)^{-1}\varphi(x)t \in X_{gx}$ for all $g \in G$, which extends to an invariant function on X since Gx is conull. Hence the action is weakly amenable.

Conversely, suppose that the action is weakly amenable. Let us show that it is essentially transitive. Suppose it is not. Then it must be properly ergodic since it is ergodic : if it was not, then one would have an orbit of non zero measure which would be conull by ergodicity. Now, the preceding lemma 3.33 provides a non-trivial cocycle $\alpha : G \times X \rightarrow \mathbb{R}$. Now Kazhdan property (T) implies (see [14], Theorem 9.1.1) that α must be equivalent to a cocycle into a compact subgroup. This is a contradiction. $\square$

Let us now prove the two preliminary lemmas. The first one is a classification result most of which is well-known, that will only be used to prove the second one.

Proof of lemma 3.34. Let us consider G a locally compact σ -compact group acting ergodically and weakly amenably on (X, μ) . We want to see that it is orbit equivalent to an action of $\mathbb{R}$ or $\mathbb{Z}$. Upon quotienting G by the kernel of the action (this does not change orbit equivalence), we may and will suppose without loss of generality that the action is faithful. Let us first consider the case where G is countable. Then the equivalence relation R_G is discrete and ergodic. It is thus amenable by [4], Proposition 7 and by [15], Theorem 3.6. The main result of [4] then ensures that there is an ergodic, measure class preserving action of $\mathbb{Z}$ on (X, μ) which generates the relation R_G , and is thus equivalent to that of G . This proves the lemma in the countable case.

Let us now move to the uncountable case. Let I be the unit interval endowed with its Lebesgue measure λ , and let R_I be the total relation on I (all elements are in relation with all elements). Then by [6], Theorems 2.8 and 5.6, the following are true (up to a null set) : There is a measurable subset $T \subseteq X$, a measure ν on T , and a Borel isomorphism $\varphi : (X, \mu) \rightarrow (I \times T, \lambda \times \nu)$ such that :

- (1) GT is conull in X .
- (2) For each $x \in X$, $T \cap Gx$ is countable.
- (3) $\varphi|_{\{0\} \times T}^{-1}$ is the inclusion of T in X .
- (4) φ induces an isomorphism $\hat{\varphi} : R_G \rightarrow R_I \times R_{G|T}$.

We now claim that the equivalence relation $R_{G|T}$ is amenable. To prove this, let us consider a cocycle $\alpha : R_{G|T} \rightarrow \text{Iso}(E)$, and $A \subseteq T \times E_1^*$ an affine $R_{G|T}$ -space over T . Define $\tilde{\alpha} : R_G \rightarrow \text{Iso}(E)$ by setting $\tilde{\alpha}(x, y) = \alpha(\varphi_2(x), \varphi_2(y))$, where φ_2 denotes the composition of φ with the projection $I \times T \rightarrow T$. Write :

$$\tilde{A} = (\varphi^{-1} \times \text{Id})(I \times A) \subseteq X \times E_1^*$$

It can then be checked that $\tilde{\alpha}$ is a cocycle and $\tilde{A}$ is an affine R_G -space over X , hence by amenability of R_G there is and $\tilde{\alpha}^*$ -invariant section $\tilde{f} : X \rightarrow \tilde{A}$. The cocycle $\tilde{\alpha}$ is constant

on $\varphi^{-1}(I \times \{t\})$ for each $t \in T$. The set A_x also is constant on $\varphi^{-1}(I \times \{t\})$, since for every $s \in I$ one has :

$$\tilde{A}_{\varphi^{-1}(s,t)} = \tilde{A} \cap (\{s\} \times \{t\} \times E_1^*) = A_t$$

By $\tilde{\alpha}^*$ -invariance, it follows that for almost every $t \in T$, the map $\tilde{f}|_{\varphi^{-1}(I \times T)}$ is an essentially constant map to A_t , hence $\tilde{f}$ must factor through φ_2 : there is (up to a null set) a map $f : T \rightarrow A$ such that $\tilde{f} = f \circ \varphi_2$. Then f is an α^* -invariant section of A so $R_{G|T}$ is amenable. Now by the countable case, $R_{G|T}$ is generated by an ergodic action of $\mathbb{Z}$. Now the equivalence relation $R_I \times R_{G|T}$ is generated by the suspension (see below) of this action, hence R_G is generated by an action of $\mathbb{R}$. $\square$

The above proof uses the notion of suspension of an action of $\mathbb{Z}$. Let us detail here what it means. Consider an action of $\mathbb{Z}$ on a measured space X , and $t : X \rightarrow X$ the map given by the action of 1 on X . Write Y for the quotient of $X \times \mathbb{R}$ by the relation generated by $(x, s) \simeq (t^n(x), s + n)$ for every $x \in X$, $n \in \mathbb{Z}$, $s \in \mathbb{R}$. $\mathbb{R}$ clearly acts on $X \times \mathbb{R}$ and this action respects the relation, thus $\mathbb{R}$ acts on the quotient Y . This action is what we call the suspension of the action of $\mathbb{Z}$.

The proof of lemma 3.35 essentially boils down to seeing that thanks to lemma 3.34, it suffices to treat the cases of $\mathbb{R}$ and $\mathbb{Z}$, which follow from [11].

Proof of lemma 3.35. By lemma 3.34, the G -action is orbit equivalent to an ergodic action of M , where M is $\mathbb{Z}$ or $\mathbb{R}$, and proper ergodicity is preserved by this construction. Upon examining the proof and using the fact that any properly ergodic action of $\mathbb{Z}$ is essentially free, we get that the action of M is also essentially free. Now by [11], Corollary 3.4, there exists a non trivial cocycle $\alpha : M \times X \rightarrow \mathbb{R}$. It must be orbital since the action of M is essentially free. Using equivalence between the actions of G and M , we deduce a non trivial cocycle $G \times X \rightarrow \mathbb{R}$ that is orbital since α is. $\square$

3.3.4 The proof

We can now move to the proof of theorem 3.20. It uses a generalization of the Margulis factor theorem.

Proof of theorem 3.20. Let us consider an ergodic G -space (X, μ) with probability measure μ that is invariant under the action of G . Suppose that the action is faithful, irreducible and properly ergodic. Let $Z(G)$ be the center of G . It can be checked that $Z(G)$ acts essentially freely on X so we may replace G by $G/Z(G)$ and X by $X/Z(G)$ and suppose that G has finite center : this will not break the assumptions, and essential freeness of the action of $Z(G)$ allows us to bootstrap to conclusion of essential freeness of the action of $G/Z(G)$ to that of G . Therefore, in the following, we assume that G has trivial center.

Since G has Property (T), the lemma 3.33 ensures that the action of G on X is not weakly amenable: indeed, it cannot be essentially transitive since it is properly ergodic. By definition of weak amenability, it follows that there is some separable Banach space E , an orbital cocycle $\alpha : G \times X \rightarrow \text{Iso}(E)$ and a G -invariant subset $A \subseteq X \rtimes_{\alpha} E_1^*$ that is an affine orbital G -space

and has no α^* -invariant section. Now consider P a Borel subgroup of G , that is a minimal parabolic subgroup. Define a cocycle :

$$\tilde{\alpha} : G \times (G/P \times X) \rightarrow \text{Iso}(E)$$

by setting $\tilde{\alpha}(g, hP, x) = \alpha(g, x)$, let $\tilde{A} = G/P \times A$. Then $\tilde{A}$ is an affine orbital G -space over $G/P \times X$ relative to the cocycle $\tilde{\alpha}^*$. Now it is well known that the action of G on $G/P \times X$ is amenable (use proposition 3.27 (i) and (ii) and the fact that P is solvable hence amenable). Therefore we get an invariant section $\tilde{f} : G/P \times X \rightarrow E_1^*$. Now define :

$$f : G/P \times X \rightarrow X \rtimes_{\alpha} E_1^*$$

by $f(hP, x) = (x, \tilde{f}(hP, x))$. Then f takes values in A , and it is easily seen to be G -equivariant.

We now have the measure class preserving G -equivariant maps :

$$G/P \times X \xrightarrow{f} A \xrightarrow{p} X$$

where p is the projection deduced from the inclusion $A \subseteq X \rtimes_{\alpha} E_1^*$. Now using theorem 3.17, it follows that A is (isomorphic to) $G/Q \times X$ for some parabolic subgroup Q of G containing P . If $Q = G$ then the said isomorphism is a map $X \rightarrow A$, which sends every point $x \in X$ to A_x : it is thus a α^* -invariant section of A , and we have supposed that no such section exists. Hence we may suppose that Q is strict.

Now all of the following arguments will be made "up to a null set". For details, we refer to the paper of Stuck and Zimmer [10]. The fact that α is orbital ensures that for every $x \in X$, one has $\alpha^*(G_x, x) = e$, thus G_x acts trivially on $p^{-1}(x)$. Identifying A with $G/Q \times X$, this ensures that G_x acts trivially on $A_x \simeq G/Q$, hence G_x is contained in the kernel of the action of G on $A_x \simeq G/Q$. The action of G on G/Q has proper kernel in G , hence G_x is contained in a proper normal subgroup of G .

Now we have proven that for almost every $x \in X$, G_x is contained in a proper normal subgroup of G . Thus if G is simple we are done. Let us now treat the case where G is not simple. Consider H a proper non trivial normal subgroup of G and $M = Z_G(H)$ its centralizer in G . Since G acts properly ergodically, H acts ergodically on X . Then the map $x \mapsto M_x$ is H -invariant since M and H commute, thus since H acts ergodically it must be constant and faithfulness of the action ensures that it is constant equal to $\{e\}$. Hence we have proven that M_x is trivial for almost every $x \in X$, and so is H_x . Finally, every proper normal subgroup acts essentially freely on X . Since all the stabilizers are contained in such a proper normal subgroup, the action is free and we are done. $\square$

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