On the Fourier transform of Stokes data of irregular connections

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General context

Wild character varieties = moduli spaces of generalised monodromy data (Stokes data) of meromorphic connections with irregular singularities.

They have a rich structure: they are symplectic, even hyperkähler

Why are they interesting in mathematical physics?

- Give rise via isomonodromic deformations to many integrable systems, eg. Painlevé equations
- Appear as phase spaces of some theories (e.g. Coulomb branches of some d=4 N=2 supersymmetric QFTs)

Motivation: dualities of isomonodromy systems

A WCV depends on a choice of "wild Riemann surface" : a curve Σ together with singularity data $\mathbf{\Theta}$.

WCVs coming from different wild Riemann surfaces can be isomorphic.

A manifestation of this is the existence of different Lax pairs for the same Painlevé equation

Can we understand better these "dualities" between different isomonodromy systems?

More specifically: there is a notion of Fourier transform for connections which induces such isomorphisms.

Question: how does the Fourier transform act on generalised monodromy data?

Outline

1. Wild character varieties and isomonodromic deformations

2. Fourier transform of irregular connections

Meromorphic connections

- Let Σ be a smooth complex algebraic curve. Here $\Sigma = \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$.
- Consider (E, ∇) vector bundle with algebraic connection on $\Sigma \setminus \{a_1, \ldots, a_m\}$.
- In a local trivialization and with a choice of coordinate:

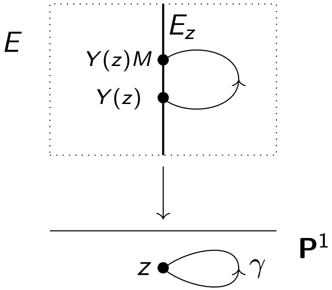
$$\nabla = d - A(z)dz,$$

  with A having poles at singular points.
- This corresponds to the system of linear differential equations

$$\frac{dY}{dz} = A(z)Y$$

Monodromy

Consider a solution Y of the equation. If we go around one singularity: $Y(z) \mapsto Y(z)M$, with $M \in GL_n(\mathbb{C})$.

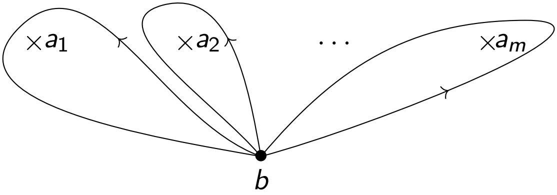


Example: if $\nabla = d - \frac{\lambda}{z}dz$, solution $y(z) = z^{\lambda}$, monodromy $e^{2i\pi\lambda}$.

Here ∇ is flat so this only depends on the homotopy class of γ. To ∇ we associate its monodromy representation $\rho : \pi_1(\Sigma) \to GL_n(\mathbb{C})$.

Moduli spaces of monodromy data: character varieties

Choose some paths $\gamma_1, \ldots, \gamma_m$ around a_i generating $\pi_1(\Sigma^o, b)$



Let $M_i = \rho(\gamma_i) \in G = GL_n(\mathbb{C})$.

The moduli space of monodromy data is the *character variety*

$$\mathcal{M}_B(\Sigma, \mathbf{a}) = \{M_1, \ldots, M_m \,|\, M_1 \ldots M_m = 1\}/G.$$

It is a Poisson manifold (Atiyah-Bott, Goldman).

Regular Riemann-Hilbert correspondence

Case of regular singularities (i.e basically simple poles)

de Rham moduli space:

$$\mathcal{M}_{dR}(\Sigma, \mathbf{a}) = \{\text{connections with regular singularities on } \Sigma \setminus \mathbf{a}\} \,/ \sim$$

Here $\sim$ corresponds to *gauge transformations* i.e. changes of trivialisation $g : \Sigma^o \to GL_n(\mathbb{C})$, doing

$$A \mapsto gAg^{-1} - dg\, g^{-1}.$$

For the system $Y' = AY$, it corresponds to change of variable $Z = g(z)Y$.

Riemann-Hilbert correspondence (Deligne):

$$\boxed{\mathcal{M}_{dR}(\Sigma, \mathbf{a}) \simeq \mathcal{M}_{B}(\Sigma, \mathbf{a})}$$

Regular vs irregular singularities

Irregular singularities: higher order poles

$$\nabla = d - A(z)dz, \qquad A(z) = \frac{A_s}{z^s} + \cdots + \frac{A_1}{z} + \dots$$

Monodromy is not enough to reconstruct the connection.

Example:

- Regular $\nabla = d - \frac{\lambda}{z}dz$, monodromy $e^{2i\pi\lambda}$.
- Irregular $\nabla = d - dq - \frac{\lambda}{z}dz$, with $q \in z^{-1}\mathbb{C}[z^{-1}]$ has monodromy $e^{2i\pi\lambda}$ for any q.

$\Rightarrow$ need generalised monodromy data for a topological description of irregular connections

Formal data

Turritin-Levelt theorem: it is possible to "diagonalise" ∇ using formal gauge transformations to a normal form

$$\nabla^0 = d - dQ - \frac{\Lambda}{z}dz, \quad Q = \begin{pmatrix} q_1 & & \\ & \ddots & \\ & & q_n \end{pmatrix}, \quad q_i \in z^{-1/r}\mathbb{C}[z^{-1/r}],$$

where

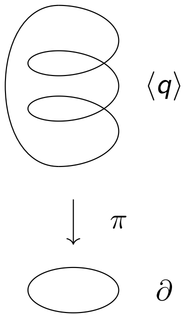
- q_i: exponential factors of ∇,
- Q: irregular type of ∇, r ramification order, Q is untwisted if $r = 1$.
- Regular singularity if $Q = 0$.
- Λ: exponent of formal monodromy.

A fundamental solution of ∇^0 is $e^Q z^\Lambda$.

Stokes circles

∂ : circle of directions around singularity $z = 0$

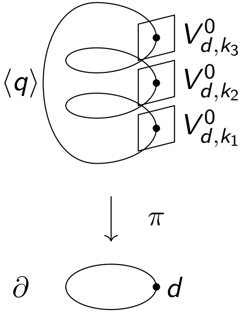
Exponential factors q as germs of functions on ∂, sections of the exponential local system $\pi : \mathcal{I} \to \partial$.



Connected components: Stokes circles $\langle q \rangle$

The map $\langle q \rangle \to \partial$ is $r : 1$ with r=ramification index of q (=3 on the picture)

Geometric description of formal data

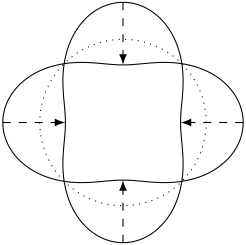


- Irregular class $\Theta = \sum_i n_i \langle q_i \rangle$, $n \geq 1$.
- Local system of formal solutions $V^0 \to \partial$, with a grading $V^0_d = \bigoplus_{\pi(i)=d} V^0_{d,i}$, and $\dim(V^0_{d,k}) = n_i$ if $k \in \langle q_i \rangle$.
- The formal monodromy correponds to the monodromy of V^0.

Stokes phenomenon

Regular singularities: formal solutions are actually convergent

Irregular singularities: when resumming formal solutions, we get analytic solutions which jump at singular (or anti-Stokes) directions.



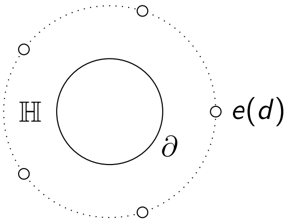
Stokes diagram: draw growth rate $\mathrm{Re}(q_i(z))$ for $|z| \to 0$ as a function of the direction (here $q_1 = z^{-2}$, $q_2 = -z^{-2}$)

Stokes arrow $i \leftarrow j$ at d if $e^{q_i - q_j}$ has maximal decay when $z \to 0$ along d.

Gluing formal and analytic solutions

Modified surface $\tilde{\Sigma}(\Theta)$:

- Take the real blow up at $z = 0$ (i.e. replace the singularity by ∂)
- Add tangential puncture $e(d)$ for each singular direction d



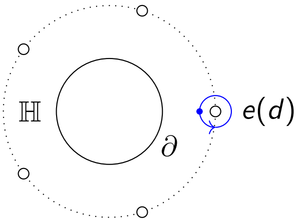
Consider:

- On the halo $\mathbb{H}$: Formal local system V^0
- Outside: Local system of analytic solutions V

$\Rightarrow$ Canonical way to glue them except at tangential punctures (Martinet-Ramis, Loday-Richaud).

Stokes local systems

One gets a "Stokes local system" (Boalch) $\mathbb{V}$ on $\tilde{\Sigma}(\Theta)$.



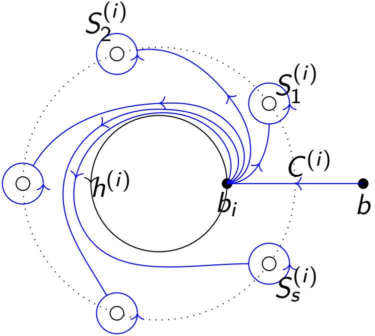
Properties: if ρ is the parallel transport in $\mathbb{V}$,

- For γ_d loop around $e(d)$, $\rho(\gamma_d)$ belongs to the Stokes group $\mathrm{Sto_d} \subset \mathrm{GL}(V_d^0)$
  - Identity blocks on the diagonal
  - Other nontrivial blocks $V_{d,j}^0 \to V_{d,i}^0$ for each Stokes arrow $i \leftarrow_d j$.
- Formal monodromy $\rho(\partial)$ compatible with grading of V^0.

Explicit description

Doing this for each singularity a_i, get global modified surface $\tilde{\Sigma}(\mathbf{\Theta})$

Choosing a basepoint b, get wild monodromy representation $\rho : \pi_1(\tilde{\Sigma}(\mathbf{\Theta}), b) \to G$.



The monodromy around a_i is the product $M_i = {C^{(i)}}^{-1} h^{(i)} S_{k_i}^{(i)} \dots S_1^{(i)} C^{(i)}$

Wild character varieties

Get the wild character variety

$$\mathcal{M}_B(\mathbf{a},\boldsymbol{\Theta}) = \left\{ (C^{(i)}, h^{(i)}, S_k^{(i)}) \,\middle|\, \prod_i (C^{(i)^{-1}} h^{(i)} S_{k_i}^{(i)} \dots S_1^{(i)} C^{(i)}) = 1 \right\} / G \times \mathbf{H}$$

where $\mathbf{H} = \prod_i H_i$ and $H_i \subset G$ corresponding to changes of graded framings of $\mathbb{V}_{b_i}$.

It has a quasi-Hamiltonian structure (Boalch)

Riemann-Hilbert-Birkhoff correspondence (Deligne-Malgrange):

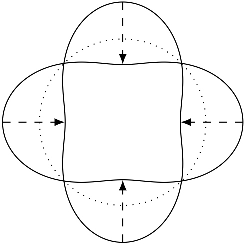
$$\boxed{\mathcal{M}_{dR}(\mathbf{a},\boldsymbol{\Theta}) \cong \mathcal{M}_B(\mathbf{a},\boldsymbol{\Theta}).}$$

Untwisted example (Pure gaussian case)

- Singularity at infinity, two exponential factors $q_1 = z^2$, $q_2 = -z^2$, 4 singular directions, 4 Stokes matrices.
- Moduli space

$$\mathcal{M}_B = \{hS_4S_3S_2S_1 = 1\}/H$$

with $S_{2i+1} = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$, $S_{2i} = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}$, $h = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$

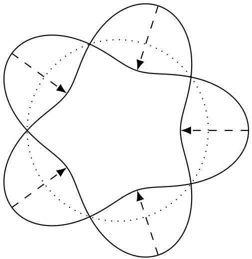


Twisted example (Painlevé I)

- Singularity at infinity, one exponential factor $z^{5/2}$, 5 singular directions, 5 Stokes matrices.
- Moduli space (of dimension 2)

$$\mathcal{M}_B = \{hS_5S_4S_3S_2S_1 = 1\}/H$$

with $S_{2i+1} = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$, $S_{2i} = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}$, $h = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$



Isomonodromic deformations: basic case

Regular case: we want to move the positions a_i of the singularities.

We wish to deform $\nabla = d - Adz$ so that the monodromies M_i remain constant

$\Rightarrow$ consider $A = A(z; a_i)$, isomonodromy gives a nonlinear PDEs satisfied by the coefficients of A.

Schlesinger equations: for $A = \sum_i \frac{A_i}{z-a_i}$, we get

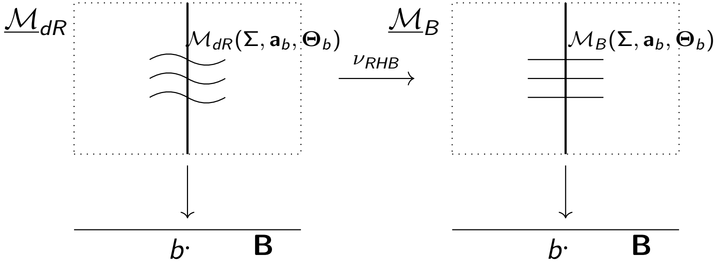
$$\frac{\partial A_i}{\partial a_j} = \frac{[A_i, A_j]}{a_i - a_j}, \qquad j \neq i,$$
$$\frac{\partial A_i}{\partial a_i} = -\sum_{j \neq i} \frac{[A_i, A_j]}{a_i - a_j}.$$

The a_i are the "times" for the isomonodromic deformations.

For rank 2, 4 singularities, we get Painlevé VI

Geometric POV on (irregular) isomonodromic deformations

Isomonodromy as an (Ehresmann) connection on an admissible family of wild character varieties $(\mathcal{M}_{dR}(\Sigma, \mathbf{a}_b, \mathbf{\Theta}_b))_{b \in \mathbf{B}}$.



$\mathbf{B}$: space of deformation parameters

The irregular class $\mathbf{\Theta}$ gives extra deformation parameters: "irregular times" t_i.

"Integrability" here: the connection is flat, the flows ∂_{t_i} commute

All Painlevé equations can be obtained in that way

1. Wild character varieties and isomonodromic deformations

2. Fourier transform of irregular connections

The Fourier transform

- Connections on the Riemann sphere closely related to modules on the Weyl algebra $A_1 = \mathbb{C}[z]\langle\partial_z\rangle$, with $[\partial_z, z] = 1$.
- Fourier transform: automorphism of the Weyl algebra:

$$\begin{cases} z & \mapsto -\partial_z \\ \partial_z & \mapsto z \end{cases}$$

- If M module over the Weyl algebra, Fourier transform $\Rightarrow \mathcal{F}M$
- More generally: we can act with any matrix $A \in SL_2(\mathbb{C})$

The stationary phase formula [Malgrange 91, Fang 09, Sabbah 08]

It relates the irregular class of a connection and its Fourier transform.

- Solutions are linear combinations of terms of the form

$$f(z) = e^{q(z)} g(z),$$

- The Fourier transform is an integral $\widehat{f}(\xi) = \int_\gamma e^{q(z)-\xi z} g(z)$
- The behaviour of the integral when $\xi \to \infty$ is determined by the critical point of the exponential factor, i.e. z_0 such that

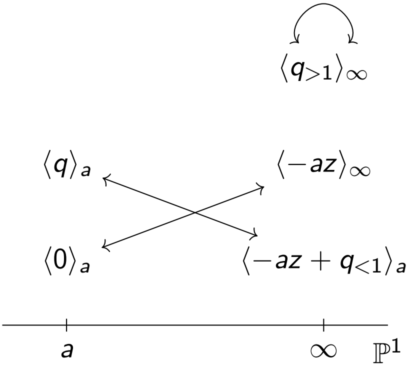
$$\frac{\partial q}{\partial z}(z_0) = \xi.$$

- New exponential factor $\widetilde{q}(\xi) = q(z_0(\xi)) - \xi z_0(\xi)$
  $\Rightarrow$ Legendre transform of q.

The stationary phase formula

Different types of circles:

1. The *pure circles* at infinity, of the form $\langle \alpha z \rangle_\infty$, with $\alpha \in \mathbb{C}$.
2. Other circles of slope ≤ 1 at infinity, of the form $\langle \alpha z + q \rangle_\infty$, with $\alpha \in \mathbb{C}$, and $q \neq 0$ of slope < 1,
3. Circles $\langle q \rangle_\infty$ of slope > 1 at infinity,
4. Irregular circles at finite distance $\langle q \rangle_a$, with $q \neq 0$, $a \in \mathbb{C}$.
5. The tame circles $\langle 0 \rangle_a$, $a \in \mathbb{C}$ at finite distance.



Diagrams

The Fourier transform (and $\mathrm{SL}_2(\mathbb{C})$ action) should induces isomorphisms between moduli spaces.

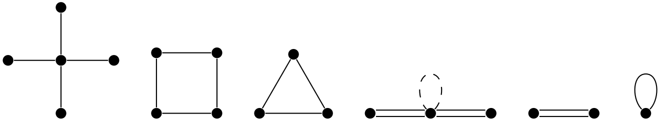
In many cases, an open dense subset $\mathcal{M}^* \subset \mathcal{M}_B$ is a quiver variety

(Crawley-Boevey, Boalch, Hiroe-Yamakawa)

Different readings of the quiver correspond to isomorphisms between moduli spaces with different $(\mathbf{a}, \boldsymbol{\Theta})$

More generally it is possible to define a diagram which is invariant under $\mathrm{SL}_2(\mathbb{C})$ for an arbitrary connection on $\mathbb{P}^1$. (D.)

For moduli spaces corresponding to Painlevé equations, the diagrams are related to the Okamoto symmetries of the equations



Fourier transform of Stokes data: some history

Well-known case (Balser-Jurkat-Lutz, Malgrange, Boalch, d'Agnolo-Hien-Morando-Sabbah)

- One singularity of order 2 at ∞,
- Regular singularities at finite distance.

In the "simply-laced case" (one pole of order less than 3 at infinity + regular singularities at finite distance), some symplectic isomorphisms obtained (Boalch), but unclear if there are the ones induced by Fourier.

In general, not many explicit examples.

General approaches (Malgrange 1991, T. Mochizuki 2010, 2018): general results but not very explicit

The setting

Joint work with A. Hohl: we use results of Mochizuki to obtain explicit isomorphisms in a large class of cases.

In brief:

- Translate a class of cases of T. Mochizuki's "Stokes shells and Fourier transform" (2018) into the language of Stokes local systems
- Get explicit formulas for the transformation of Stokes matrices

Assumption:

- Only Stokes circles of slope >1 at ∞
- Circles of pure level $r/s > 1$ with s, r coprime $q_i = a_i z^{s/r}$.
- Extra hypothesis $|a_i| = 1$.

Stronger version of Legendre transform

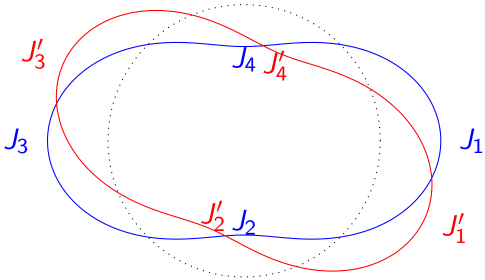
Main idea: the Legendre transform as an homeomorphism between circles $\ell : \langle q \rangle \cong \langle \hat{q} \rangle$.

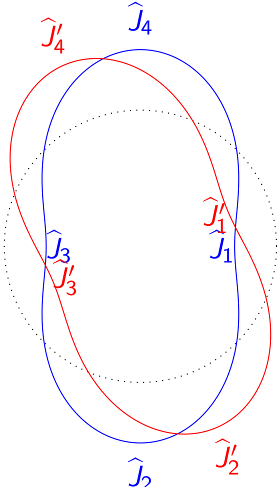
One can use ℓ to transport the nontrivial entries of Stokes data (up to signs)

Distinguished intervals

On each Stokes circle, intervals J where q is either increasing or decreasing when $|z| \to 0$.

Increasing intervals are sent by ℓ to decreasing ones and vice versa

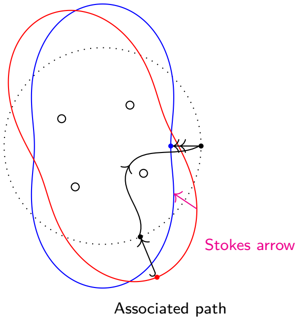




Stokes paths

Nontrivial entry of Stokes matrix $\leftrightarrow$ entry of parallel transport in Stokes local system along a path $\gamma_{i \to j}$

If the Stokes arrow goes from sector I to J, $i, j \in \partial$ are the midpoints of I, J.



The Stokes local system can be reconstructed from these entries of the parallel transport along these Stokes paths

The algorithm

Start with connection (E, ∇) on $\mathbb{C}$ with irregular class Θ, formal local system V^0, Stokes local system $\mathbb{V}$.

The corresponding objets $\widehat{\Theta}$, $\widehat{V}^0$, $\widehat{\mathbb{V}}$ for the Fourier transform are determined as follows:

- Formal part: $\widehat{V}^0$ obtained from $\ell^* V^0$ by adding some signs when passing from one distinguished sector to the next
- Stokes data: for any Stokes path $\gamma_{i \to j}$, the parallel transport is

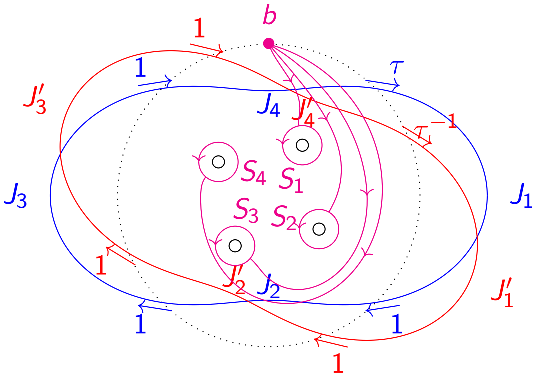
$$\widehat{\rho}(\gamma_{i \to j}) = \pm \rho(\gamma_{\ell^{-1}(i) \to \ell^{-1}(j)})$$

  with an explicitly determined sign.

The nontrivial entries of Stokes matrices are exactly the "deformation data" considered by Mochizuki

Example : pure gaussian case

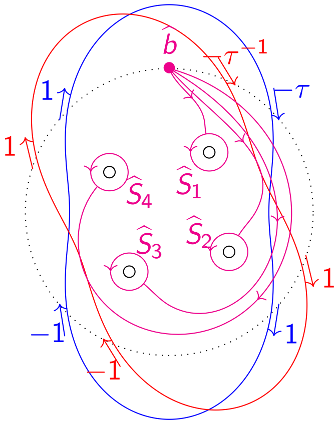
Initial irregular class $\Theta = \langle z^2 \rangle + \langle \frac{1+i}{\sqrt{2}} z^2 \rangle$.



$$S_1 = \begin{pmatrix} 1 & 0 \\ s_1 & 1 \end{pmatrix} \quad S_2 = \begin{pmatrix} 1 & s_2 \\ 0 & 1 \end{pmatrix} \quad S_3 = \begin{pmatrix} 1 & 0 \\ s_3 & 1 \end{pmatrix} \quad S_4 = \begin{pmatrix} 1 & s_4 \\ 0 & 1 \end{pmatrix} \quad h = \begin{pmatrix} \tau & 0 \\ 0 & \tau' \end{pmatrix}$$

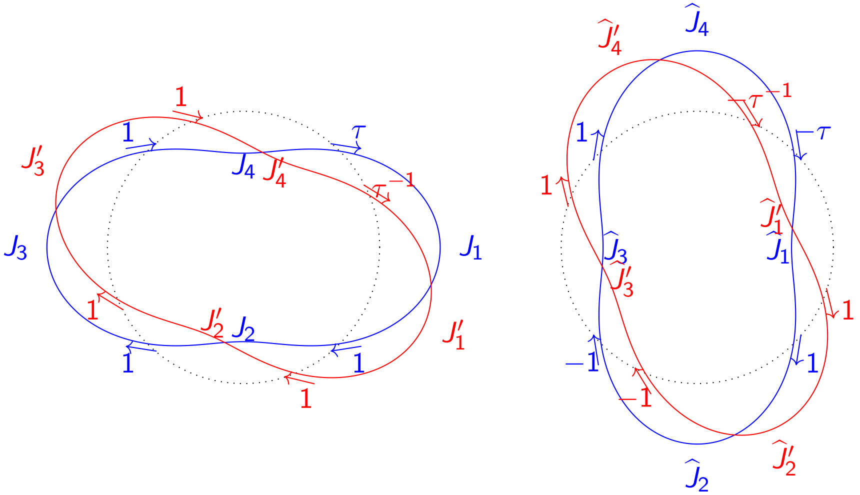
$$\mathcal{M}_B(\Theta) = \{h, S_1, S_2, S_3, S_4 \mid hS_4S_3S_2S_1 = 1\}/H.$$

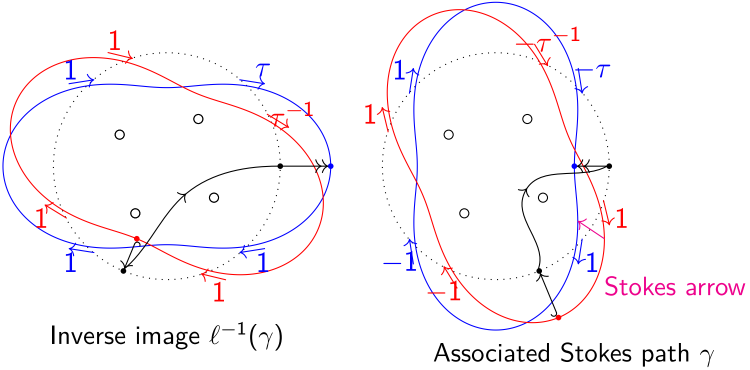
New irregular class $\widehat{\Theta} = \langle -z^2 \rangle + \langle \frac{-1+i}{\sqrt{2}} z^2 \rangle$.



$$\mathcal{M}_B(\widehat{\Theta}) = \{\widehat{h}, \widehat{S}_1, \widehat{S}_2, \widehat{S}_3, \widehat{S}_4 \mid hS_4S_3S_2S_1 = 1\}/H.$$

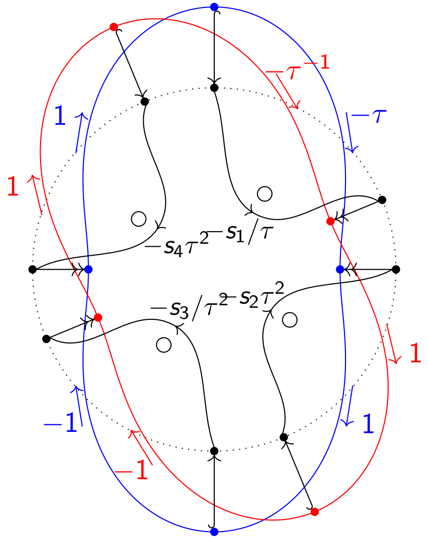
Correspondence between distinguished intervals and transformation of the formal data



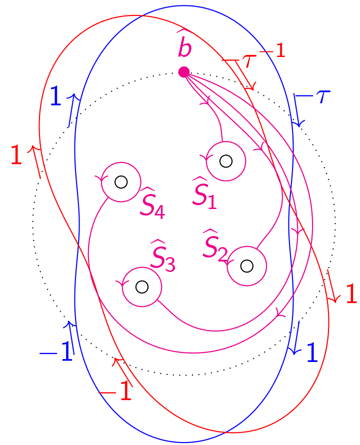


With the Legendre transform, transport γ to the initial Stokes diagram

One obtains the entries of the parallel transport along the Stokes paths



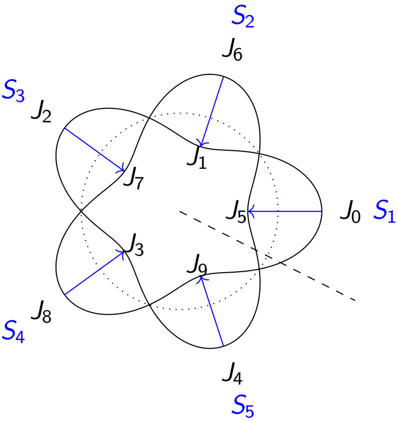
Finally we get the new Stokes matrices



$$\boxed{\widehat{S}_i = S_i, \qquad \widehat{h} = h.}$$

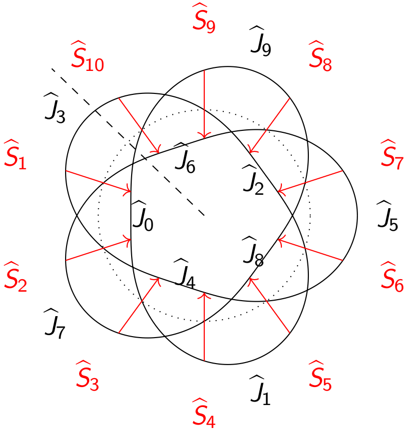
Example: Painlevé I case

$\Theta = \langle -z^{5/2} \rangle$



$\mathcal{M}_B = \{hS_5S_4S_3S_2S_1 = 1\}$ with $h = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, S_1 = \begin{pmatrix} 1 & s_1 \\ 0 & 1 \end{pmatrix}, S_2 = \begin{pmatrix} 1 & 0 \\ s_2 & 1 \end{pmatrix}, \ldots$

Fourier transform $\widehat{\Theta} = \langle z^{5/3} \rangle$.



$\mathcal{M}'_B = \{kT_{10} \dots T_1 = 1\}$ with $k = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, $T_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ t_1 & 0 & 1 \end{pmatrix}$, $T_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t_2 & 1 \end{pmatrix}$

$\mathcal{M}_B \simeq \mathcal{M}'_B$ via $\Phi : (s_1, s_2, s_3, s_4, s_5) \mapsto (s_3, -s_5, -s_2, s_4, s_1, -s_3, s_5, s_2, -s_4, -s_1)$

Computation of the isomorphism

coefficient	Stokes arrow	Stokes matrix entry	extra sign
t_1	$3 \to 0$	$-s_5$	$+$
t_2	$7 \to 0$	$-s_2$	$+$
t_3	$7 \to 4$	s_4	$+$
t_4	$1 \to 4$	$-s_1$	$-$
t_5	$1 \to 8$	s_3	$-$
t_6	$5 \to 8$	$-s_5$	$-$
t_7	$5 \to 2$	s_2	$+$
t_8	$9 \to 2$	$-s_4$	$+$
t_9	$9 \to 6$	$-s_1$	$+$
t_{10}	$3 \to 6$	$-s_3$	$-$

The isomorphism is symplectic!

Questions

Conjecture: the isomorphisms induced by the Fourier transform preserve the symplectic structure.

Further questions:

- Can we show this?
- Obtain explicit isomorphisms for more general situations (several irregular singularities, etc...)?
- How these isomorphisms behave in families: can we relate the corresponding spaces of times and isomonodromy systems?