Diagrams and the classification of wild character varieties

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(irregular) nonabelian Hodge spaces

$$
\begin{array}{ccll}
 & \nearrow & \mathcal{M}_{Dol} & \text{mero.Higgs bundles} \\
\begin{array}{c}\mathfrak{M} \\ \text{nonabelian Hodge space}\end{array} & \longrightarrow & \mathcal{M}_{dR} & \text{mero. connections} \\
 & \searrow & \mathcal{M}_{B} & \text{(Stokes) local systems}
\end{array}
$$

- $\mathcal{M}_{Dol} \simeq \mathcal{M}_{dR}$: (wild) nonabelian Hodge correspondence
- $\mathcal{M}_{dR} \simeq \mathcal{M}_{B}$: (irregular) Riemann-Hilbert correspondence
- $\mathcal{M}_{B}$ is a (wild) character variety.

Hyperkähler manifold, 3 different algebraic structures.

Outline

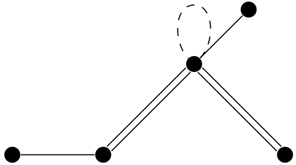
The moduli space depends on the choice of a "wild Riemann surface", i.e. a Riemann surface Σ together with singularity data $(\Theta, \mathcal{C})$.

Several different singularity data can give rise to isomorphic moduli spaces: different "representations" of the same space

Question: can we classify wild character varieties? Is there a systematic way to find all wild Riemann surfaces giving rise to the same moduli space?

Idea: associate a diagram to singularity data

$(\Theta, \mathcal{C}) \longrightarrow$



such that, at least in some cases, wild Riemann surfaces corresponding to isomorphic moduli spaces have the same diagram.

1. Wild character varieties
2. Construction of diagrams
3. Applications to Painlevé equations
4. Further questions

Meromorphic connections

- Let Σ be a smooth complex algebraic curve. Here $\Sigma = \mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\}$.
- Consider (E, ∇) vector bundle with algebraic connection on $\Sigma \smallsetminus \{a_1, \ldots, a_m\}$.
- In a trivialization and with a choice of coordinate:

$$\nabla = d - A(z)dz,$$

with A having poles at singular points.
$\rightarrow$ corresponds to a system of linear differential equations.

Formal normal form

Turritin-Levelt theorem

After passing to a finite cover $t = z^{1/r}$, any connection on the formal punctured disk is formally isomorphic (via formal gauge transformations) to a connection of the form

$$d - (dQ + \Lambda \frac{dz}{z})$$

where $Q = \mathrm{diag}(q_1, \ldots, q_k)$ diagonal matrix with coefficients in $z^{-1/r}\mathbb{C}[z^{-1/r}]$, Λ constant block diagonal.

The q_i are the *exponential factors* of the connection: indeed formal solutions $e^Q z^\Lambda$ are linear combinations of terms with $e^{q_i(z)}$.
Λ: formal monodromy

Untwisted case: when no ramification, i.e. $r = 1$.

The exponential local system

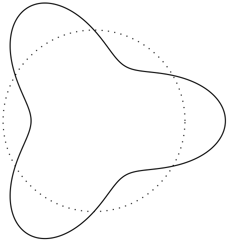
- Let ∂ be the circle of directions around $z = 0$.
- Exponential local system $\mathcal{I} \to \partial$: sections on sectors are germs of functions of the form

$$q(z) = \sum_{i=1}^{s} b_i z^{-i/r},$$

- Connected components are *circles* $\langle q \rangle$. $\langle q \rangle \to \partial$ cover of ∂ of order r. $\mathcal{I}$ is a collection of circles
- $r = \mathrm{ram}(q)$ ramification index,
  $s = \mathrm{Irr}(q)$ irregularity.
  s/r is the slope of q.
- (local) singularity data: choice of active circles $\langle q_i \rangle$, with multiplicities n_i, and conjugacy classes $\mathcal{C}_i \subset GL_{n_i}(\mathbb{C})$
  - Irregular class $\Theta = n_1\langle q_1 \rangle + \ldots n_k \langle q_k \rangle$.
  - $\mathcal{C} = (\mathcal{C}_1, \ldots, \mathcal{C}_k)$ conjugacy classes of blocks of Λ.

Stokes phenomenon

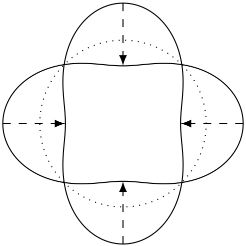
Main idea: growth rates of e^{q_i} change when going around the singularity



Stokes diagram: draw $e^{\mathrm{Re}(q(\epsilon i\theta))}$, $\epsilon \ll 1$, here for $q = z^3$.

Stokes matrices

Change of dominance between the exponential factors e^{q_i} depending on the direction.



Singular directions: $d \in \mathbb{A}$ where $q_i - q_j \in \mathbb{R}_{<0}$ for some (i, j).
Stokes arrow $\langle q_i \rangle \to \langle q_j \rangle$

Associated block in Stokes matrix $S_d = \begin{pmatrix} 1 & 0 & * \\ 0 & \ddots & 0 \\ 0 & 0 & 1 \end{pmatrix}$ in position (i, j).

Explicit presentation of wild character varieties

$$\mathcal{M}_B \cong \{(C_1^{-1} h_1 S_{k_1}^{(1)} \dots S_1^{(1)} C_1) \dots (C_m^{-1} h_m S_{k_m}^{(m)} \dots S_1^{(m)} C_m) = 1\}/G \times \boldsymbol{H},$$

This corresponds to a quasi-Hamiltonian description

- Fission spaces: $\mathcal{A}(V_{a_i}^0) \ni (C_i, h_i, S_1^{(i)}, \dots, S_{k_i}^{(i)})$
- For several singularities, fusion of fission spaces (here $G = GL_n(\mathbb{C})$)

$$\mathrm{Hom}_{\mathbb{S}}(\Theta) \simeq \mathcal{A}(V_{a_1}^0) \circledast \cdots \circledast \mathcal{A}(V_{a_m}^0) /\!\!/ G.$$

- $\mathcal{M}_B$ is now obtained by taking the multiplicative symplectic reduction at the conjugacy classes of formal monodromies.

$$\mathcal{M}_B(\Theta, \mathcal{C}) = \mathrm{Hom}_{\mathbb{S}}(\Theta) /\!\!/_{\mathcal{C}} \mathbf{H}.$$

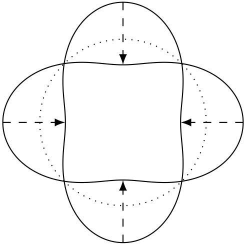
where $\mathbf{H} = H_1 \times \cdots \times H_m$, H_i is a subgroup of G of block diagonal matrices

Untwisted example

- Singularity at infinity, two exponential factors $q_1 = z^2$, $q_2 = -z^2$, 4 singular directions, 4 Stokes matrices.
- Moduli space

$$\mathcal{M}_B = \{hS_4 S_3 S_2 S_1 = 1\} /\!\!/ H$$

with $S_{2i+1} = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$, $S_{2i} = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}$, $h = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$

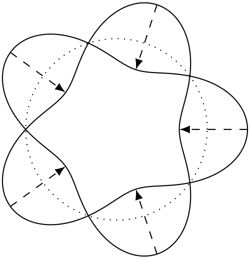


Twisted example

- Singularity at infinity, one exponential factor $z^{5/2}$, 5 singular directions, 5 Stokes matrices.
- Moduli space (of dimension 2)

$$\mathcal{M}_B = \{hS_5S_4S_3S_2S_1 = 1\} /\!\!/ H$$

with $S_{2i+1} = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$, $S_{2i} = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}$, $h = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$



1 Wild character varieties

2 Construction of diagrams

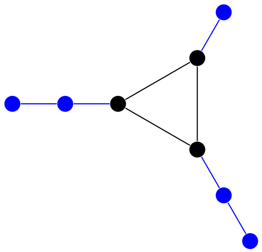
3 Applications to Painlevé equations

4 Further questions

Diagram associated to a connection

General structure:

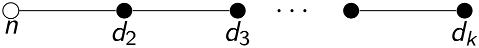
- Core diagram encoding the irregular class Θ
- "Legs" (linear quivers) encoding the conjugacy classes of formal monodromies



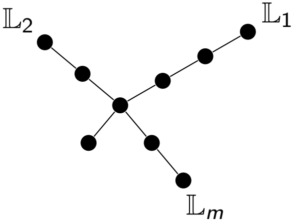
This relies on the fact that a conjugacy class $\mathcal{C} \subset GL(N)$ can be characterized by a a linear quiver

Regular connections

- Singularity data: conjugacy classes $\mathcal{C}$ of formal monodromies.
- Quivers from conjugacy classes [Kraft-Procesi 82, Crawley-Boevey 01-06]. Let $\mathcal{C} \subset GL(N)$ conjugacy class. Choose *marking* i.e. polynomial $P = (X - \xi_1) \dots (X - \xi_k)$ s.t $P(A) = 0$ for any $A \in \mathcal{C}$. Define



  with $d_i = \mathrm{rank}(A - \xi_1) \dots (A - \xi_{i-1})$.
- Glue those legs $\rightarrow$ star-shaped quiver associated to the connection.



One irregular singularity: core diagram [Boalch-Yamakawa 20]

Setting: only one singularity at infinity, exponential factors q_i with ramification orders r_i.

The vertices correspond to the exponential factors q_i.

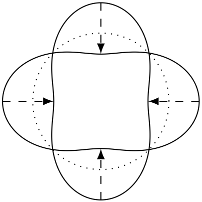
To find the number of edges/loops between q_i and q_j (one edge $=$ one arrow in each direction)

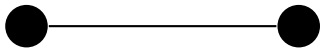
- Count the number of Stokes arrows between the corresponding circles in the Stokes diagrams
- substract $r_i r_j$ if $i \neq j$, or $r_i^2 - 1$ if $i = j$ (corresponds to relations in the quasi-Hamiltonian presentation)

Notice there can be edges/loops with negative multiplicities

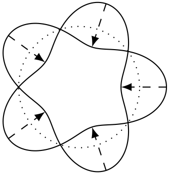
Simple examples

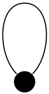
Untwisted example: $2 - 1 = 1$ edges between $\langle q_1 \rangle$ and $\langle q_2 \rangle$





Twisted example: $5 - 2^2 + 1 = 2$ oriented loops at $\langle q \rangle$





Properties of the quivers

Dimension of the wild character variety: $\dim \mathcal{M}_B = 2 - (\mathbf{d}, \mathbf{d})$,
with $\mathbf{d}$ dimension vector, $(\cdot, \cdot)$ bilinear form defined by the Cartan matrix of the diagram.

In the case of regular singularities, or in the "simply-laced" case (one untwisted irregular singularity of order ≤ 3 + simple poles), we have more:

- Quiver modularity theorem: The additive moduli space $\mathcal{M}^* \subset \mathcal{M}_B$ is isomorphic to the Nakajima quiver variety of Γ

$$\mathcal{M}^* \simeq N(\Gamma, \mathbf{d}, \lambda).$$

- $\mathcal{M}_B$ can be seen as a multiplicative quiver variety. The several readings correspond to isomorphisms between the wild character varieties.
- Weyl group action: basically corresponds to exchanging eigenvalues of formal monodromies.

Extension to several irregular singularities

Idea: use transformations to reduce to the case with one singularity at infinity.

Action of $SL_2(\mathbb{C})$ on singularity data, including Fourier-Laplace transform.

Theorem (D.)

There is a well-defined way to associate to any connection (E, ∇) on a Zariski open subset of the affine line, with modified formal data $(\breve{\boldsymbol{\Theta}}, \breve{\mathcal{C}})$ a core diagram $\Gamma(\breve{\boldsymbol{\Theta}}, \breve{\mathcal{C}})$ such that

1. *If $\breve{\boldsymbol{\Theta}}$ has support at infinity, then $\Gamma(\breve{\boldsymbol{\Theta}}, \breve{\mathcal{C}})$ is the diagram of [BY] .*
2. $\Gamma(\breve{\boldsymbol{\Theta}}, \breve{\mathcal{C}}) = \Gamma(A \cdot (\breve{\boldsymbol{\Theta}}, \breve{\mathcal{C}}))$ *for any $A \in SL_2(\mathbb{C})$.*

The diagram comes together with a dimension vector $\mathbf{d} \in \mathbb{Z}^I$, and a vector of labels $\boldsymbol{q} \in (\mathbb{C}^*)^I$ (where I is the set of vertices).

Explicit description

- As before, the diagram consists of a core to which we attach legs
- The vertices of the core correspond to the active circles
- The multiplicity of edges/loops in the core is given by

Definition

Suppose i, j are active circles at a_i, a_j, with $i = \langle q_i \rangle$, $j = \langle q_j \rangle$ are circles at a_i, a_j respectively. Let $\alpha_i = \mathrm{Irr}(q_i), \beta_i = \mathrm{ram}(q_i)$ and similarly for j. If $a_i = a_j$ let us denote by $B^{\infty}_{i,j}$ the number of edges between i and j in the diagram of [BY].

1. If $a_i = a_j = \infty$ then $B_{ij} = B^{\infty}_{ij}$.
2. If $a_i = \infty$ and $a_j \neq \infty$ then $B_{ij} = B_{ji} = \beta_i(\alpha_j + \beta_j)$.
3. If $a_i \neq \infty, a_j \neq \infty$ and $a_i \neq a_j$ then $B_{ij} = B_{ji} = 0$.
4. If $a_i = a_j \neq \infty$ then $B_{ij} = B_{ji} = B^{\infty}_{ij} - \alpha_i\beta_j - \alpha_j\beta_i$.

Fourier-Laplace transformation

- Connections on the Riemann sphere closely related to modules on the Weyl algebra $A_1 = \mathbb{C}[z]\langle\partial_z\rangle$, with $[\partial_z, z] = 1$.
- Fourier transform: automorphism of the Weyl algebra:

$$\begin{cases} z & \mapsto -\partial_z \\ \partial_z & \mapsto z \end{cases}$$

- If M module over the Weyl algebra, Fourier transform $\Rightarrow \mathcal{F}M$
- More generally: we can act with any matrix $A \in SL_2(\mathbb{C})$.
- Such symplectic transformations are generated by 3 types of elementary transformations.
  - Fourier-Laplace
  - Scalings $z \mapsto z/\lambda$
  - Twists, corresponding to $\nabla \mapsto \nabla + \lambda z dz$.

The stationary phase formula [Malgrange 91, Fang 09, Sabbah 08]

It relates the irregular class of a connection and its Fourier transform.

- Solutions are linear combinations of terms of the form

$$f(z) = e^{q(z)} g(z),$$

- The Fourier transform is an integral $\widehat{f}(\xi) = \int_\gamma e^{q(z) - \xi z} g(z)$
- The behaviour of the integral when $\xi \to \infty$ is determined by the critical point of the exponential factor, i.e. z_0 such that

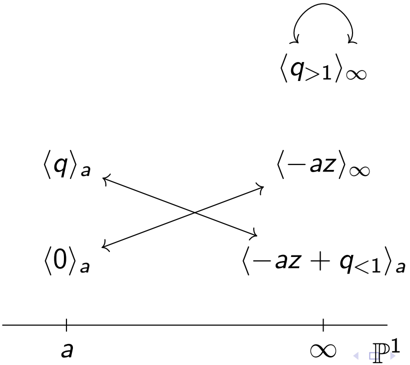
$$\frac{\partial q}{\partial z}(z_0) = \xi.$$

- New exponential factor $\widetilde{q}(\xi) = q(z_0(\xi)) - \xi z_0(\xi)$
  $\Rightarrow$ Legendre transform of q.

The stationary phase formula

Different types of circles:

1. The *pure circles* at infinity, of the form $\langle \alpha z\rangle_\infty$, with $\alpha \in \mathbb{C}$.
2. Other circles of slope ≤ 1 at infinity, of the form $\langle \alpha z + q\rangle_\infty$, with $\alpha \in \mathbb{C}$, and $q \neq 0$ of slope < 1,
3. Circles $\langle q\rangle_\infty$ of slope > 1 at infinity,
4. Irregular circles at finite distance $\langle q\rangle_a$, with $q \neq 0$, $a \in \mathbb{C}$.
5. The tame circles $\langle 0\rangle_a$, $a \in \mathbb{C}$ at finite distance.



Main idea

Putting everything at infinity

- Apply twist at infinity $q \mapsto q + \lambda z^2$.
  Now all active circles at ∞ have slope > 1.
- Now apply Fourier: only one singularity at infinity.

Crucial fact: this does not depend on the choices. Indeed

Theorem

If Θ is an irregular class at infinity and $A \in SL_2(\mathbb{C})$ such that $A \cdot \Theta$ is also at infinity, then

$$\Gamma_c(A \cdot \Theta) = \Gamma_c(\Theta).$$

Main ingredient of proof: the diagram is invariant under Fourier transform.

Properties of the diagram

Dimension of the wild character variety: we still have

$$\dim \mathcal{M}_B(E, \nabla) = 2 - (\mathbf{d}, \mathbf{d}).$$

Idea of proof: use the quasi-Hamiltonian description of the wild character variety

$$\mathcal{M}_B(V) = \mathrm{Hom}_{\mathbb{S}}(E, \nabla) /\!\!/_{\mathcal{C}} \mathbf{H}.$$

From the invariance under $SL_2(\mathbb{C})$, get several readings of the diagram

It is still possible to interpret some Weyl reflections on the diagram as coming from operations on connections

1 Wild character varieties

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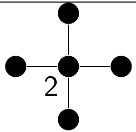
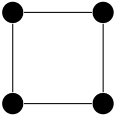
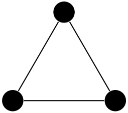
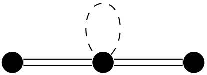
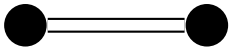
Diagrams for Painlevé equations

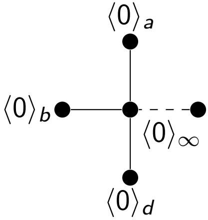
- Painlevé equations can be obtained from isomonodromic deformation equations of some meromorphic connections. Such a connection is a Lax pair for the Painlevé equation.
- They correspond to two-dimensional moduli spaces.
- Standard (rank 2) Lax pairs and diagrams:

VI	V	IV	III	II
1+1+1+1	2+1+1	3+1	2+2	4
				

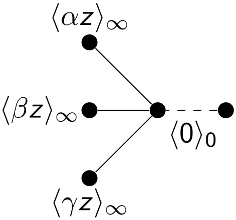
- Get affine Dynkin diagrams corresponding to the Okamoto symmetries of Painlevé equations

Painlevé VI: several Lax pairs

Standard Lax pair: rank 2, 4 simple poles (including one at infinity)

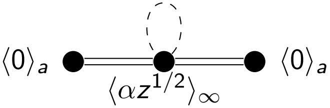


Harnad dual: rank 3, one irreg sing. at infinity, and one simple pole at finite distance

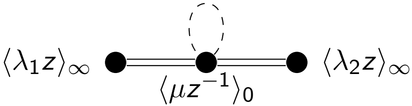


Painlevé III

From alternative Lax pair [Boalch-Yamakawa 20]: one pole of order 2, twisted, 2 simple poles, rank 2



From the usual Lax pair: two irregular poles of order 2



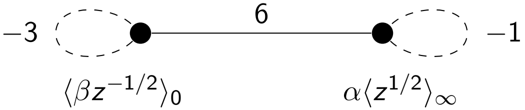
Degenerate Painlevé equations

Degenerate PIII: two second order poles, one of them twisted (slope $1/2$).

$$\breve{\Theta} = \langle \alpha z^{1/2} \rangle_\infty + \langle \beta z^{-1} \rangle_0$$

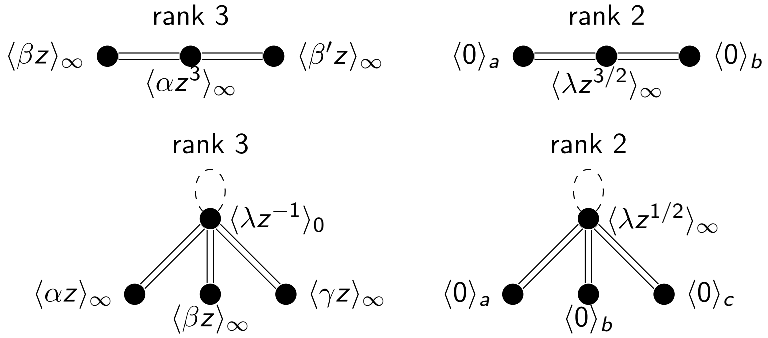
Doubly degenerate PIII: two twisted second order poles (slopes $1/2$)

$$\breve{\Theta} = \langle \alpha z^{1/2} \rangle_\infty + \langle \beta z^{-1/2} \rangle_0$$



Higher dimensional cases

Among 4-dimensional Painlevé-type equations, two examples of different known Lax pairs corresponding to the same diagram:



Proposition

All Lax pairs for a same given 4-dimensional Painlevé-type equation listed by [Kawakami et al. 18] correspond to different readings of the same diagram.

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Orbits under basic operations

Basic operations:

- $SL_2(\mathbb{C})$
- Twists
- Möbius transformations (change the point at infinity)

The diagram is invariant under $SL_2(\mathbb{C})$ but not under other basic operations.

Question: orbits under repeated application of basic operations? Is there a minimal diagram?
In the rigid case:

Theorem (Katz '96, Arinkin '10)

Any rigid irreducible connection on $\mathbb{P}^1$ can be brought to the trivial rank one connection by repeated application of basic operations.

Diagrams with one vertex and few loops

Consider an exponential factor $\langle q \rangle$ at infinity, let $k = B_{\langle q \rangle, \langle q \rangle}/2$ the corresponding number of loops.

Theorem

If $k = 0, 1, 2$, $\langle q \rangle$ can be brought by repeated application of basic operations to an exponential factor which is an admissible deformation of

k	simple exponential factor
0	0
1	$z^{5/3}$
2	$z^{7/5}$

Towards explicit isomorphisms?

- One expects the different readings of the diagram correspond to isomorphisms of the moduli spaces, compatible with the symplectic structure
- Such isomorphisms should be obtained by the transformation of Stokes data under Fourier transform
- This has been studied by many people [Malgrange, Mochizuki, Sabbah, Hien, D'Agnolo...], but difficult to get explicit formulas beyond the simply laced case.
- A simple twisted case : connections with only 1 exponential factor $\langle z^{\alpha/\beta}\rangle_\infty$ and $\langle z^{\alpha/\alpha-\beta}\rangle_\infty$ with $\alpha > \beta$.

Isomorphism for slopes $5/2 \leftrightarrow 5/3$

- $\mathcal{M}_B = \{s_1, \ldots, s_5,\ hS_5S_4S_3S_2S_1 = 1\}$ with
  $h = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, S_1 = \begin{pmatrix} 1 & s_1 \\ 0 & 1 \end{pmatrix}, S_2 = \begin{pmatrix} 1 & 0 \\ s_2 & 1 \end{pmatrix}, \ldots$
- $\mathcal{M}'_B = \{t_1, \ldots, t_{10},\ kT_{10} \ldots T_1 = 1\}$ with
  $k = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, T_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ t_1 & 0 & 1 \end{pmatrix}, T_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t_2 & 1 \end{pmatrix}, \ldots$
- $\mathcal{M}_B \simeq \mathcal{M}'_B$ via
  $\Phi : (s_1, s_2, s_3, s_4, s_5) \mapsto (s_3, -s_5, -s_2, s_4, s_1, -s_3, s_5, s_2, -s_4, -s_1)$
- The isomorphism is compatible with the symplectic structures