Dualities between genus zero wild nonabelian Hodge spaces

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General context: isomorphisms of wild character varieties

Wild character varieties = moduli spaces of generalized monodromy data (Stokes data) of meromorphic connections with irregular singularities.

A WCV depends on a choice of "wild Riemann surface": a curve Σ together with singularity data.

WCVs coming from different wild Riemann surfaces can be isomorphic.
$\Rightarrow$ several "representations" of the same abstract moduli space.

A manifestation of this is the existence of several Lax pairs for Painlevé-type equations.

Big question: can one classify wild character varieties?

Today: In genus zero, describe a combinatorial way to obtain several (expected to be) representations of the same moduli space.

This generalizes some known dualities to the case of arbitrary irregular types.

1. Stokes data and wild character varieties

2. Dualities between wild character varieties

Meromorphic connections

- Let Σ be a smooth complex algebraic curve. Here $\Sigma = \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$.
- Consider (E, ∇) vector bundle with algebraic connection on $\Sigma^o = \Sigma \setminus \{a_1, \ldots, a_m\}$.
- In a local trivialization and with a choice of coordinate z:

$$\nabla = d - A(z)dz,$$

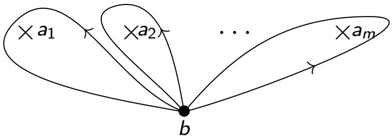
  with $A(z)$ $n \times n$ matrix having poles at singular points.
- This corresponds to the system of linear differential equations

$$\frac{dY}{dz} = A(z)Y.$$

Moduli spaces of monodromy data: character varieties

To ∇ we associate its monodromy representation $\rho : \pi_1(\Sigma^o, b) \to GL_n(\mathbb{C})$.

Choose some paths $\gamma_1, \ldots, \gamma_m$ around a_i generating $\pi_1(\Sigma^o, b)$



Let $M_i = \rho(\gamma_i) \in G = GL_n(\mathbb{C})$.

The moduli space of monodromy data is the *character variety*

$$\mathcal{M}_B(\Sigma, \mathbf{a}) = \{M_1, \ldots, M_m \,|\, M_1 \ldots M_m = 1\}/G.$$

It is a Poisson manifold (Atiyah-Bott, Goldman).

Symplectic leaves $\mathcal{M}_B(\Sigma, \mathbf{a}, \boldsymbol{\mathcal{C}})$ obtained by fixing conjugacy classes $M_i \in \mathcal{C}_i$.

Regular Riemann-Hilbert correspondence

Case of regular singularities (i.e basically simple poles): monodromy data characterize entirely connections

de Rham moduli space:

$$\mathcal{M}_{dR}(\Sigma, \mathbf{a}) = \{\text{connections with regular singularities on } \Sigma \setminus \mathbf{a}\} / \sim$$

Here $\sim$ corresponds to *gauge transformations* i.e. changes of trivialization $g : \Sigma^o \to GL_n(\mathbb{C})$, doing

$$A \mapsto gAg^{-1} - dg\, g^{-1}.$$

For the system $Y' = AY$, it corresponds to change of variable $Z = g(z)Y$.

Riemann-Hilbert correspondence:

$$\boxed{\mathcal{M}_{dR}(\Sigma, \mathbf{a}) \simeq \mathcal{M}_B(\Sigma, \mathbf{a})}$$

Regular vs irregular singularities

Irregular singularities: higher order poles

$$\nabla = d - A(z)dz, \qquad A(z) = \frac{A_s}{z^s} + \cdots + \frac{A_1}{z} + \dots$$

Now monodromy is not enough to reconstruct the connection.

Example:

- Regular $\nabla = d - \frac{\lambda}{z}dz$, monodromy $e^{2i\pi\lambda}$.
- Irregular $\nabla = d - dq - \frac{\lambda}{z}dz$, with $q \in z^{-1}\mathbb{C}[z^{-1}]$ has monodromy $e^{2i\pi\lambda}$ for any q.

$\Rightarrow$ Need generalized monodromy data for a topological description of irregular connections.

Formal data

Turritin-Levelt theorem: it is possible to "diagonalise" ∇ using formal gauge transformations to a normal form:

$$\nabla^0 = d - dQ - \frac{\Lambda}{z}dz, \quad Q = \begin{pmatrix} q_1 & & \\ & \ddots & \\ & & q_n \end{pmatrix}, \quad q_i \in z^{-1/r}\mathbb{C}[z^{-1/r}],$$

where

- q_i: exponential factors of ∇,
- Q: irregular type of ∇, r ramification order, Q is untwisted if $r = 1$.
- Regular singularity if $Q = 0$.
- Λ: exponent of formal monodromy (constant and block diagonal with blocks corresponding to the distinct q_i).

$\Rightarrow$ Local formal data: $(\Theta, \mathcal{C})$ with

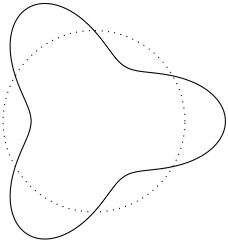
- Irregular class $\Theta = n_1\langle q_1\rangle + \dots n_k\langle q_k\rangle$.
- $\mathcal{C} = (\mathcal{C}_1, \dots, \mathcal{C}_k)$ conjugacy classes of blocks of Λ.

Stokes phenomenon

A fundamental solution of ∇^0 is $e^Q z^\Lambda$.

True solutions asymptotic to this can only be found in sectors around the singularity

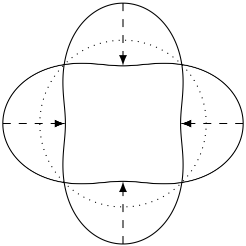
This is because growth rates of e^{q_i} change when going around the singularity



Stokes diagram: draw $e^{\mathrm{Re}(q(\epsilon i\theta))}$, here $q = z^3$, $\epsilon \ll 1$.

Stokes matrices

Change of dominance between the exponential factors e^{q_i} depending on the direction.



Singular directions: $d \in \mathbb{A}$ where $q_i - q_j \in \mathbb{R}_{<0}$ for some (i, j).
Stokes arrow $\langle q_i \rangle \to \langle q_j \rangle$

Associated block in Stokes matrix $S_d = \begin{pmatrix} 1 & 0 & * \\ 0 & \ddots & 0 \\ 0 & 0 & 1 \end{pmatrix}$ in position (i, j).

Wild character varieties

Global formal data $(\boldsymbol{\Theta}, \boldsymbol{\mathcal{C}}) = (\Theta_i, \mathcal{C}_i)_i$.

Wild character variety:

$$\mathcal{M}_B(\boldsymbol{\Theta}) = \left\{ (C^{(i)}, h^{(i)}, S_k^{(i)}) \;\middle|\; \prod_i (C^{(i)^{-1}} h^{(i)} S_{k_i}^{(i)} \dots S_1^{(i)} C^{(i)}) = 1 \right\} / G$$

with $C^{(i)} \in G = \mathrm{GL}_n(\mathbb{C})$ connection matrices, $h^{(i)} \in H_i \subset G$ formal monodromies, $S^{(i)}$ Stokes matrices.

It has a quasi-Hamiltonian structure (Boalch).

Get symplectic variety $\mathcal{M}_B(\boldsymbol{\Theta}, \boldsymbol{\mathcal{C}}) = \mathcal{M}_B(\boldsymbol{\Theta}) /\!\!/_{\boldsymbol{\mathcal{C}}} \mathbf{H}$, where $\mathbf{H} = \prod_i H_i$, if we fix conjugacy classes of $h^{(i)}$.

Riemann-Hilbert-Birkhoff correspondence:

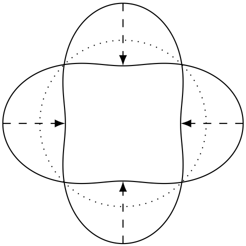
$$\boxed{\mathcal{M}_{dR}(\boldsymbol{\Theta}) \cong \mathcal{M}_B(\boldsymbol{\Theta}).}$$

Example (pure gaussian case)

- Singularity at infinity, two exponential factors $q_1 = z^2$, $q_2 = -z^2$, 4 singular directions, 4 Stokes matrices.
- Moduli space

$$\mathcal{M}_B = \{hS_4S_3S_2S_1 = 1\}$$

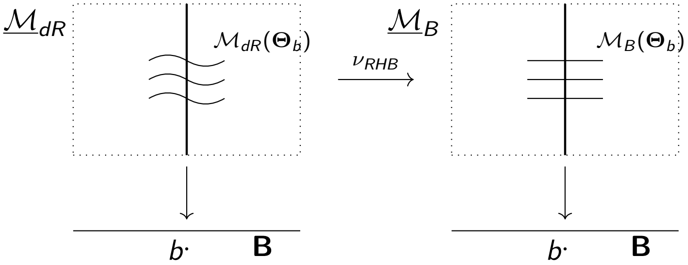
with $S_{2i+1} = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$, $S_{2i} = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}$, $h = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$.



Isomonodromic deformations

Let's move the positions of singularities and irregular classes (regular and irregular "times"): how should ∇ change for the Stokes data to remain constant?

This can be viewed as a flat (Ehresmann) connection on an admissible family of wild character varieties $(\mathcal{M}_{dR}(\mathbf{\Theta}_b))_{b\in\mathbf{B}}$.



$\mathbf{B}$: space of "times"

All Painlevé equations can be obtained in that way

Regular case

Isomonodromic "times" : positions of singularities a_i.

Only constraint: must have $a_i \neq a_j$ for $i \neq j$.

$\Rightarrow$ Space of times $\mathbf{B} = \mathrm{Conf}_n = \{a_1, \ldots, a_m \in \mathbb{C} \,|\, a_i \neq a_j \text{ for } i \neq j\}$.

Example: rank 2 connections on $\mathbb{P}^1$ with 4 simple poles

$$A = \frac{A_1}{z - a_1} + \frac{A_2}{z - a_2} + \frac{A_3}{z - a_3} + \frac{A_4}{z - a_4}.$$

Isomonodromy leads to Painlevé VI:

$$\frac{d^2 y}{dt^2} = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)\frac{dy}{dt}$$
$$+ \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{y^2} + \gamma\frac{t-1}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}\right)$$

Irregular case: admissible deformations

Local case: we fix a singularity, only vary the irregular type

$$Q = \begin{pmatrix} q_1 & & \\ & \ddots & \\ & & q_n \end{pmatrix}, \quad q_i \in z^{-1}\mathbb{C}[z^{-1}].$$

Set $Q = \frac{A_s}{z^s} + \cdots + \frac{A_1}{z}$, with $A_i = \mathrm{diag}(a_{1i}, \ldots, a_{ni})$ i.e. $q_i = \sum_j a_{ij} z^{-j}$.

Admissibility constraint: we need to have

$$\boxed{d_{ij} = \deg(q_i - q_j) = constant.}$$

What are equivalence classes of admissible deformations?

Generic case: if A_s has distinct eigenvalues, $d_{ij} = n$ for all $i \neq j$

Only constraint when deforming is to keep $a_{si} \neq a_{sj}$

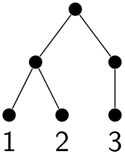
$\Rightarrow$ The space of times **B** is homotopy equivalent to Conf_n.

Fission trees

Otherwise, look at how eigenspaces of A_s split as eigenspaces of A_{s-1} etc

Example: $Q = \frac{A_2}{z^2} + \frac{A_1}{z}$, $\quad A_2 = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 2 \end{pmatrix}$, $\quad A_1 = \begin{pmatrix} -1 & & \\ & 1 & \\ & & 0 \end{pmatrix}$

Fission tree $\mathcal{T}(Q)$:



Theorem (Boalch-D.-Rembado-Tamiozzo)

Fission trees $\Leftrightarrow$ eq. classes of admissble defs. of (local) irreg. classes

The space of irregular times $\mathbf{B}$ can be explicitly determined from the tree.

Globally: Fission forest $\mathbf{F} \Leftrightarrow$ class of admissible defs. of irreg class $\boldsymbol{\Theta}$

Question: find all classes of representations $(\mathbf{F}, \mathcal{C})$ of a genus zero wild nonabelian Hodge space $\mathcal{M}$.

Harnad duality

Relates connections of the form

$$d + (A + P(Y - z)^{-1}Q)dz, \qquad \text{and} \qquad d + (Y + Q(A - z)^{-1}P)d\lambda,$$

with A, P, Q, Y linear maps between V, W given by:

$$\begin{tikzcd} V \arrow[loop left, "A"] \arrow[r, bend right, "Q"'] & W \arrow[l, bend right, "P"'] \arrow[loop right, "Y"] \end{tikzcd}$$

Both sides are connections on $\mathbb{P}^1$ with

- One irregular sing. at ∞ of order 2
- Regular singularities at finite distance

This induces an isomorphism between the corresponding moduli spaces.

The isomonodromic deformation equations on both sides are equivalent.

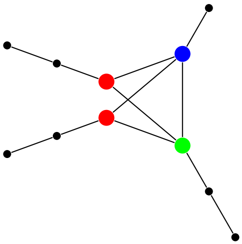
The simply-laced case (Boalch 08,12,15)

Connections on $\mathbb{P}^1$ with:

- Irreg. sing. at ∞ of order ≤ 3,
- Regular singularities at finite distance.

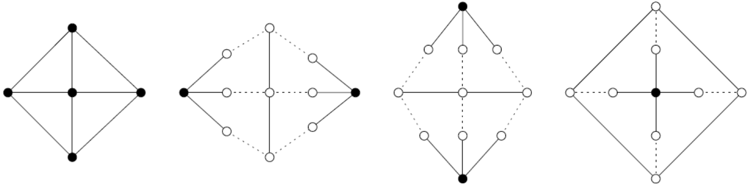
Each such connection defines a supernova quiver (+extra data)

- Core diagram: complete k-partite graph, basically one vertex for each exponential factor.
- Glue "legs", encoding conjugacy classes of formal monodromies.



Reading the diagram

Given such a diagram $k+1$ ways to "read" it, i.e. get $k+1$ moduli spaces of connections, all isomorphic:



For each reading, open dense part $\mathcal{M}^* \subset \mathcal{M}$ of the moduli space is isomorphic to a Nakajima quiver variety defined by the quiver

The isomonodromy systems for each reading are equivalent

$\mathrm{SL}_2(\mathbb{C})$ symmetry

Weyl algebra $A_1 = \mathbb{C}[z, \partial_z]$ of diff. operators on $\mathbb{C} = \mathbb{P}^1 \setminus \infty$.

Any $A \in \mathrm{SL}_2(\mathbb{C})$ defines an automorphism of A_1

$$\begin{aligned} z &\mapsto az + b\partial_z \\ \partial_z &\mapsto cz + d\partial_z \end{aligned}$$

The ODE $P(z, \partial_z)y = 0$ becomes $P(az + b\partial_z, cz + d\partial_z)y = 0$.

This induces a transformation on (irreducible) connections on Zariski open susbets of $\mathbb{P}^1$.

Being of simply-laced type is preserved by this.

The $k+1$ readings of the quivers correspond to the different types of formal data that appear in a $\mathrm{SL}_2(\mathbb{C})$ orbit.

Arbitrary formal data: diagrams

One can define a diagram for arbitrary formal data on $\mathbb{P}^1$ (D. 21).

Given (E, ∇) with formal data $(\boldsymbol{\Theta}, \mathcal{C})$, diagram $\Gamma(E, \nabla) = \Gamma(\boldsymbol{\Theta}, \mathcal{C})$.

Same structure as before: core + legs.

The core vertices (almost) correspond to the Stokes circles of $\boldsymbol{\Theta}$.

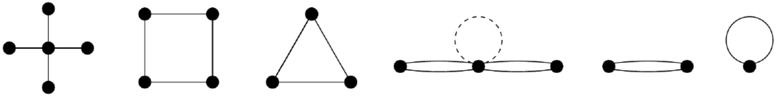
Now the core can have edges/loops with negative multiplicity.

Quiver variety not defined but formula for the dimension remains true.

Theorem (D. 21)

The diagram is invariant under $\mathrm{SL}_2(\mathbb{C})$.

Examples: diagrams for Painlevé moduli spaces



Generic form

Problem: in general the diagram is not enough to reconstruct the formal data of elements of an $\mathrm{SL}_2(\mathbb{C})$ orbit.

$\Rightarrow$ Need some extra data! Fission trees will help us.

Say that an exponential factor is of generic form if it is at infinity and of slope ≤ 2, i.e.

$$q(z) = -\frac{a}{2}z^2 + q_{<2}(z).$$

(E, ∇) on $\mathbb{P}^1$ is of generic form it all its exponential factors are of generic form.

Then (E, ∇) has just one singularity at ∞.

Lemma

$A \cdot (E, \nabla)$ is of generic form for A in an open dense subset of $\mathrm{SL}_2(\mathbb{C})$.

Remark: the simply-laced case is exactly when this generic form has unramified irregular class

The nearby representations

Consider

- $\mathbf{T}$: fission tree of generic form (all vertices have height ≤ 2)
- $\mathcal{C}$: datum of a conjugacy class for each leaf of $\mathbf{T}$

Let $k = \sharp\,\{\text{principal subtrees of } \mathbf{T}\} = \sharp\,\{\text{vertices of height } 2\}$

Theorem (D. 24)

We construct explicitly k classes of admissible defs. $(\mathbf{F}_i, \mathcal{C}_i)$, $1 \leq i \leq k$. $(\{\mathbf{T}\}, \mathcal{C})$ and $(\mathbf{F}_i, \mathcal{C}_i)$ are the classes of admissible defs. of formal data of elements of the orbit $\mathrm{SL}_2(\mathbb{C}) \cdot (E, \nabla)$, for any (E, ∇) of generic form with formal data of class $(\{\mathbf{T}\}, \mathcal{C})$.

$\Rightarrow$ Expect them to be $k+1$ classes of representations of the same moduli space.

We call

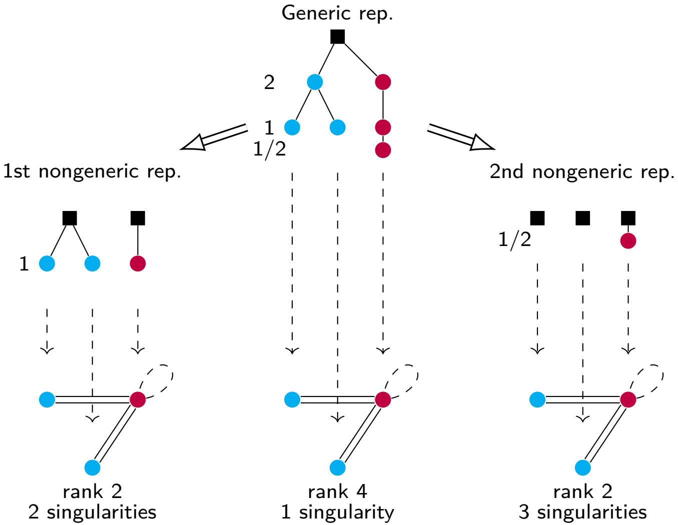
- $(\{\mathbf{T}\}, \mathcal{C})$ the generic class of reps,
- $(\mathbf{F}_i, \mathcal{C}_i)$ the i-th nongeneric class of reps.

Properties of representations

- Link with diagrams:
  - The diagram of any such (E, ∇) is determined by $(\mathbf{T}, \mathcal{C})$.
  - The k principal subtrees define a canonical partition of the vertices of the core diagram: $N = N_1 \sqcup \cdots \sqcup N_k$
- Define N_i^+= leaves in N_i with an ancestor of height $1 < h < 2$ and $N_i^- = N_i \setminus N_i^+$.

  In the i-th nongeneric representation:
  - Elements of N_i^+ correspond to Stokes circles of slope > 1 at ∞.
  - Elements of N_i^- correspond to Stokes circles at finite distance
  - Elements of N_j for $j \neq i$ are of generic form
- If $A \cdot (E, \nabla)$ is in the i-th nongeneric rep.
  $\sharp\, \{\text{singularities} \neq \infty\} = \sharp\, \{\text{vertices of height 1 above leaves in } N_i^-\}$.
- Have an explicit formula for the rank of each rep.

Example: Painlevé III



Fourier sphere

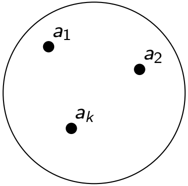
For a Stokes circle $\langle q\rangle$, define its Fourier sphere coeff. $\lambda(\langle q\rangle)$ by

- If $q = -\frac{a}{2}z^2 + q_{<2}(z)$ at ∞ of generic form, let $\lambda(\langle q\rangle) := a$;
- Otherwise $\lambda(\langle q\rangle) := \infty$.

Lemma

$\mathrm{SL}_2(\mathbb{C})$ acts by homographies on the Fourier sphere:
$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acts by $\lambda \mapsto \frac{a\lambda+b}{c\lambda+d}$

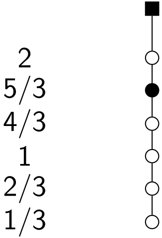
$\mathbb{P}^1$



$k = \sharp$ {distinct Fourier sphere coeffs. of Stokes circles of (E, ∇)}.

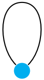
- Generic reading: $a_j \neq \infty$ for all j.
- i-th nongeneric reading: $a_i = \infty$.

Painlevé I



	Rank	Number of singularities
Generic rep.	3	1
Nongeneric rep.	2	1

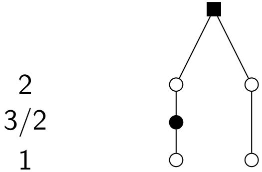
Two readings of the diagram:



$\langle \alpha z^{5/2} \rangle_\infty$

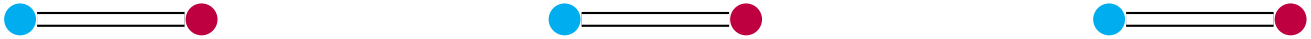
$\langle \lambda z^2 + \beta z^{5/3} \rangle_\infty$

Painlevé II



	Rank	Number of singularities
Generic rep.	3	1
1st nongeneric rep.	2	1
2nd nongeneric rep.	2	2

Three readings of the diagram:



$\langle \alpha z^3 \rangle_\infty \ \langle 0 \rangle_\infty \quad \langle \lambda z^2 + \alpha z^{3/2} \rangle_\infty \ \langle \mu z^2 \rangle_\infty \quad \langle \alpha z^{3/2} \rangle_\infty \quad \langle 0 \rangle_0$

Further questions

- Can we prove that $\mathcal{M}_B(\{\mathbf{T}\},\boldsymbol{\mathcal{C}})$, $\mathcal{M}_B(\mathbf{F}_i,\boldsymbol{\mathcal{C}}_i)$ are isomorphic? (cf. Andreas' talk)
- Do the isomorphisms preserve the symplectic structure?
- Deligne-Simpson problem: when is $\mathcal{M}_B$ nonempty?
- Can we also relate the spaces of isomonodromic deformations, and the full isomonodromy systems?
- Here we view Lax representations of Painlevé-type equations in an abstract sense. Can one write down more explicit parametrizations of the Lax representations?