Isomonodromic deformations and generalised braid groups

Jean Douçot

GFM, Universidade de Lisboa

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Motivation

Isomonodromic deformations:

- Give rise to many integrable systems.
- Give rise to new special functions, e.g. Painlevé transcendents.
- Links with geometry: Frobenius manifolds, wall-crossing...

Main goals today:

- Geometric point of view on isomonodromic deformations.
- Tell some things about the wild case, i.e. irregular singularities.
- Describe the spaces of "times" for isomonodromic deformations and their topology.

Outline

I would like to explain:

- Isomonodromic deformations of linear systems can be seen as a (nonlinear) connection on a fibration whose fibres are moduli spaces of linear systems, or equivalently spaces of monodromy data, aka character varieties.
- Considering the monodromy of these fibrations when we move the positions of singularities, we get actions of braid groups on character varieties.
- We can generalise this picture to connections with irregular singularities: we will get *wild* mapping class group actions on *wild* character varieties.

The last part is based on recent and ongoing work with: G. Rembado, M. Tamiozzo, P. Boalch.

Linear differential systems

We are interested in systems of linear differential equations on the Riemann sphere $\Sigma = \mathbb{P}^1$.

$$\frac{dY}{dz} = A(z)Y,$$

where

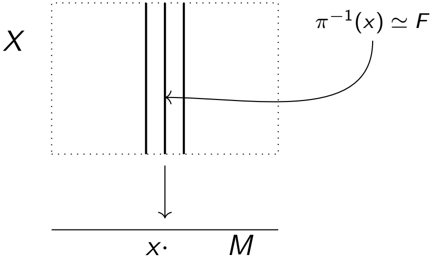
- $A(z)$ is a $n \times n$ matrix whose coefficients are holomorphic functions of z, with poles at points $a_1, \dots, a_m \in \mathbb{P}^1$.
- the unknown Y is a vector whose entries are holomorphic functions of z on $\Sigma^o := \mathbb{P}^1 \setminus \{a_1, \dots, a_m\}$.

We will first conisder the case of regular singularities, i.e. simple poles.

Link with connections

Fiber bundle X on a manifold M: a manifold X with a map $\pi : X \to M$ such that locally on $U \subset M$ the situation is diffeomorphic to a projection

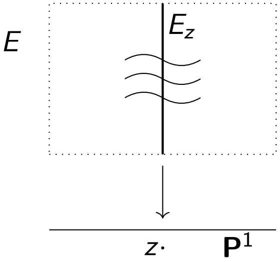
$$F \times U \mapsto U.$$



No canonical way to identify nearby fibres!

Link with connections

Connection on a vector bundle $E \to \mathbb{P}^1$: way to move "horizontally" between fibres:



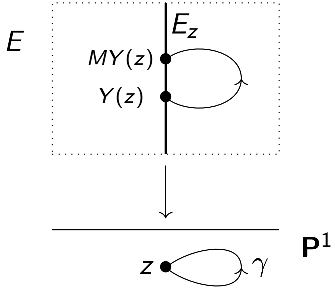
In our setting, consider

$$\nabla = d - A(z)dz.$$

A horizontal section of ∇ is Y such that $\nabla Y = 0$, i.e. $Y' = AY$.

(Linear) monodromy

Consider a solution Y of the equation. If we go around one singularity: $Y(z) \mapsto MY(z)$, with $M \in GL_n(\mathbb{C})$.

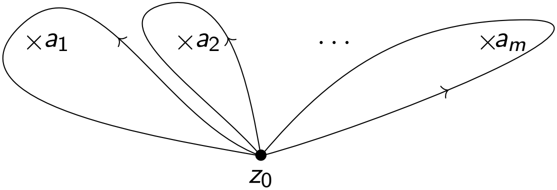


Example: if $\nabla = d - \frac{\lambda}{z} dz$, solution $y(z) = z^{\lambda}$, monodromy $e^{2i\pi\lambda}$.

If ∇ is flat this only depends on the homotopy class of γ. To ∇ we associate its monodromy representation $\rho : \pi_1(\Sigma) \to GL_n(\mathbb{C})$.

Moduli spaces of monodromy data: character varieties

Choose some paths $\gamma_1, \ldots, \gamma_m$ around a_i generating $\pi_1(\Sigma^o, z_0)$



Let $M_i = \rho(\gamma_i) \in G = GL_n(\mathbb{C})$.

The moduli space of monodromy data is the *character variety*

$$\mathcal{M}_B(\Sigma, \mathbf{a}) = \{M_1, \ldots, M_m \mid M_1 \ldots M_m = 1\}/G.$$

It is a Poisson manifold (from Atiyah-Bott, Goldman).

Isonodromic deformations

Now we want to move the positions a_i of the singularities.

We wish to deform $\nabla = d - Adz$ so that the M_i remain constant

$\Rightarrow$ consider $A = A(z; a_i)$, isomonodromy gives a nonlinear PDEs satisfied by the coefficients of A.

Schlesinger equations: for $A = \sum_i \frac{A_i}{z-a_i}$, we get

$$\frac{\partial A_i}{\partial a_j} = \frac{[A_i, A_j]}{a_i - a_j}, \qquad j \neq i, \tag{0.1}$$

$$\frac{\partial A_i}{\partial a_i} = -\sum_{j \neq i} \frac{[A_i, A_j]}{a_i - a_j}. \tag{0.2}$$

The a_i are the "times" for the isomonodromic deformations.

Moduli spaces of connections with regular singularities

We consider the de Rham moduli space

$$\mathcal{M}_{dR}(\Sigma, \mathbf{a}) = \{\text{connections with regular singularities on } \Sigma \setminus \mathbf{a}\} \,/ \sim$$

where $\sim$ corresponds to *gauge transformations*.

This means a change of trivialisation $g : \Sigma^o \to GL_n(\mathbb{C})$, doing

$$A \mapsto gAg^{-1} - dg\, g^{-1}.$$

For the system $Y' = AY$, it corresponds to change of variable $Z = g(z)Y$.

A connection ∇ defines a point in $\mathcal{M}_{dR}(\Sigma, \mathbf{a})$.

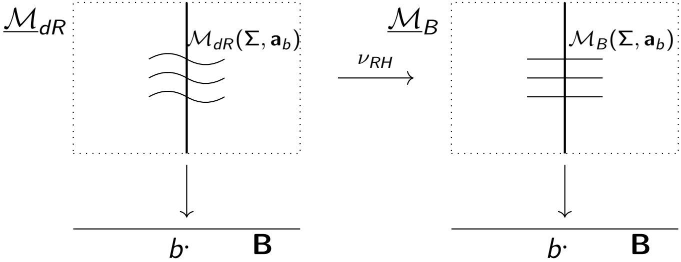
Riemann-Hilbert correspondence:

$$\boxed{\mathcal{M}_{dR}(\Sigma, \mathbf{a}) \simeq \mathcal{M}_{B}(\Sigma, \mathbf{a})}$$

Both sides are Poisson manifolds, and the RH map preserves the Poisson structure.

Geometric point of view on isomonodromy

Isomonodromy as an (Ehresmann) connection on a family of moduli spaces $(\mathcal{M}_{dR}(\Sigma, \mathbf{a}_b))_{b\in\mathbf{B}}$.



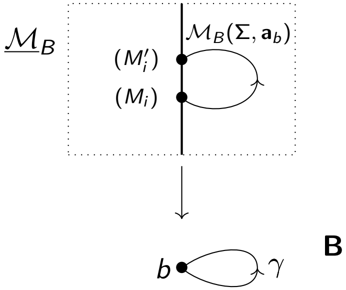
$\mathbf{B}$: space of deformation parameters

On the RHS: connection given by locally constant monodromy.

Using Riemann-Hilbert, get a connection on the LHS $\Rightarrow$ isomonodromic deformations.

The monodromy of isomonodromy

But now we can consider the (nonlinear) monodromy of this connection!
A loop $\gamma \in \pi_1(\mathbf{B}, b)$ in the base $\mathbf{B}$ induces an automorphism of the character variety $\mathcal{M}_B(\Sigma, \mathbf{a}_b)$.



A loop $\gamma \in \pi_1(\mathbf{B}, b)$ in the base $\mathbf{B}$ induces an automorphism of the character variety $\mathcal{M}_B(\Sigma, \mathbf{a}_b)$.

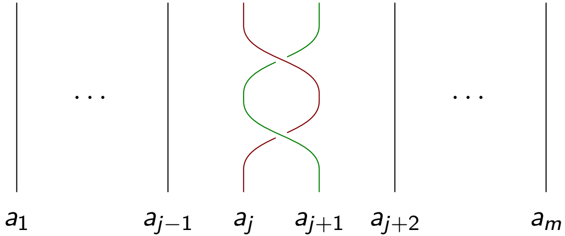
Braid group actions on character varieties

The natural universal space of deformation parameters is

$$\mathbf{B} = \mathrm{Conf}_n = \{a_1, \dots, a_m \in \mathbb{C} \,|\, a_i \neq a_j \text{ for } i \neq j\}\,.$$

$\pi_1(\mathbf{B})$ is the braid group on n strands B_n, which is also the mapping class group of the disc with n marked points.

It is generated by s_i, $i = 1, \dots, n-1$.



$\in PB_m\,.$

Thus B_n acts on the character variety: s_j acts as $M_k \mapsto M_k$ for $k \neq j, j+1$ and

$$(M_j, M_{j+1}) \mapsto (M_{j+1}^{-1} M_j M_{j+1}, M_{j+1}^{-1} M_j^{-1} M_{j+1} M_j M_{j+1}).$$

An example: Painlevé VI

$$\frac{d^2y}{dt^2} = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)\frac{dy}{dt}$$
$$+ \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{y^2} + \gamma\frac{t-1}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}\right)$$

It comes from isomonodromic deformations of rank 2 connections on $\mathbb{P}^1$ with 4 simple poles, i.e.

$$A = \frac{A_1}{z-a_1} + \frac{A_2}{z-a_2} + \frac{A_3}{z-a_3} + \frac{A_4}{z-a_4}.$$

We can use automorphisms of $\mathbf{P}^1$ to send $(a_1, a_2, a_3, a_4) \mapsto (0, 1, \infty, t)$.
Only one "time" variable t

t going around $0, 1, \infty \longrightarrow$ monodromy of solutions of PVI.

$(M_i)_i$ with finite orbits give algebraic solutions of PVI (cf. Dubrovin-Mazzocco, Boalch, Lisovyy).

Generalisation to the irregular case: outline

Motivation:

- Include cases related to other Painlevé equations.
- Links (e.g Fourier transform) between regular and irregular cases.

Main differences with regular case:

- Need generalised monodromy data, a.k.a Stokes data to get analogue of Riemann-Hilbert correspondence.
- The moduli spaces, the wild character varieties $\mathcal{M}_B(\Sigma, \mathbf{a}, \mathbf{Q})$, now also depend on *irregular types*.
- The irregular types give new deformation parameters, and new groups acting on wild character varieties.

Irregular singularities

Irregular singularities: higher order poles

$$\nabla = d - A(z)dz, \qquad A(z) = \frac{A_s}{z^s} + \cdots + \frac{A_1}{z} + \dots$$

Monodromy is not enough to reconstruct the connection.

Example:

- Regular $\nabla = d - \frac{\lambda}{z}dz$, monodromy $e^{2i\pi\lambda}$.
- Irregular $\nabla = d - dq - \frac{\lambda}{z}dz$, with $q \in z^{-1}\mathbb{C}[z^{-1}]$ has monodromy $e^{2i\pi\lambda}$ for any q.

Irregular types

Turritin-Levelt theorem: it is possible to "diagonalise" ∇ using formal gauge transformations to a normal form

$$\nabla_0 = d - dQ - \frac{\Lambda}{z}dz, \quad Q = \begin{pmatrix} q_1 & & \\ & \ddots & \\ & & q_n \end{pmatrix}, \quad q_i \in z^{-1/r}\mathbb{C}[z^{-1/r}],$$

where

- q_i: exponential factors of ∇,
- Q: irregular type of ∇, r ramification order, Q is untwisted if $r = 1$.
- Λ: exponent of formal monodromy.

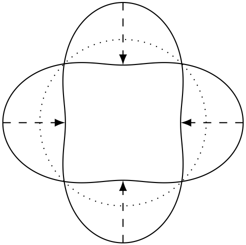
A solution of ∇_0 is $e^Q z^\Lambda$.

Main idea: asymptotic behaviour of e^{q_i} when $z \to 0$ changes depending on the direction.

The Stokes phenomenon

Consider the growth rates of q_i, q_j for $z \to 0$:

- Changes of dominance between e^{q_i} and e^{q_j} at Stokes directions.
- Anti-Stokes/singular directions where the difference of the growth rates is largest.



Example: $q_1 = z^2, q_2 = -z^2$.

Each pair (i, j) gives $d_{ij} = \deg(q_i - q_j)$ (anti)Stokes directions.

Wild character varieties

Generalised monodromy:

- One Stokes matrix S_d for each singular direction d.
- Each pair (i, j) gives d_{ij} nontrivial Stokes matrix entries.

Consider ∇ on $\Sigma \setminus \mathbf{a}$, with irregular type Q_i at a_i.

To pass to the wild case: replace M_i by product

$$C^{(i)^{-1}} h^{(i)} S^{(i)}_{k_i} \dots S^{(i)}_1 C^{(i)}$$

with $h^{(i)}$ encoding the formal monodromy, $S^{(i)}_j$ Stokes matrices.

Space of generalised mondromy data = wild character variety:

$$\mathcal{M}_B(\Sigma, \mathbf{a}, \mathbf{Q}) = \left\{ C^{(i)}, h^{(i)}, S^{(i)}_j \;\middle|\; \prod_i (C^{(i)^{-1}} h^{(i)} S^{(i)}_{k_i} \dots S^{(i)}_1 C^{(i)}) = 1 \right\} / G$$

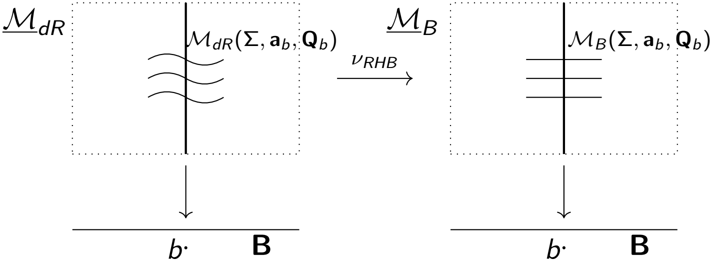
It is a Poisson manifold.

Wild isomonodromic deformations

Riemann-Hilbert-Birkhoff correspondence:

$$\boxed{\mathcal{M}_{dR}(\Sigma, \mathbf{a}, \mathbf{Q}) \simeq \mathcal{M}_B(\Sigma, \mathbf{a}, \mathbf{Q})}$$

Similar picture as before: (Ehresmann) connection on an admissible family of wild character varieties $(\mathcal{M}_{dR}(\Sigma, \mathbf{a}_b, \mathbf{Q}_b))_{b\in\mathbf{B}}$.



The irregular types give new deformation parameters: "irregular times". One now varies a "wild Riemann surface" $(\Sigma, \mathbf{a}, \mathbf{Q})$.

Admissible deformations

Local case: we fix a singularity, only vary the irregular type

$$Q = \begin{pmatrix} q_1 & & \\ & \ddots & \\ & & q_n \end{pmatrix}, \quad q_i \in z^{-1}\mathbb{C}[z^{-1}].$$

Set $Q = \frac{A_s}{z^s} + \cdots + \frac{A_1}{z}$, with $A_i = \mathrm{diag}(a_{1i}, \dots, a_{ni})$ i.e. $q_i = \sum_j a_{ij} z^{-j}$.

Admissibility constraint: we need to have

$$\boxed{d_{ij} = \deg(q_i - q_j) = \textit{constant}.}$$

Question: what is the analogue of Conf_n, the universal base $\mathbf{B}$?

If A_s has distinct eigenvalues, $d_{ij} = n$ for all $i \neq j$, the only constraint is that $a_{si} \neq a_{sj}$, and again $\pi_1(\mathbf{B}) = PB_n$.

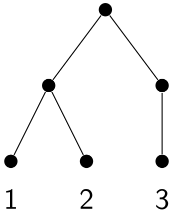
Fission trees

Otherwise, we have to look at how eigenspaces of A_s split as eigenspaces of A_{s-1} and so on

Example:

$$Q = \frac{A_2}{z^2} + \frac{A_1}{z}, \quad A_2 = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 2 \end{pmatrix}, \quad A_1 = \begin{pmatrix} -1 & & \\ & 1 & \\ & & 0 \end{pmatrix},$$

Corresponding fission tree $\mathcal{T}(Q)$:



Space of admissible deformations

Admissible deformations of Q:

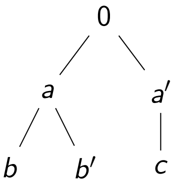
$$Q' = \frac{A_2'}{z^2} + \frac{A_1'}{z}, \quad \text{with } A_2' = \begin{pmatrix} a & & \\ & a & \\ & & a' \end{pmatrix}, \quad A_1' = \begin{pmatrix} b & & \\ & b' & \\ & & c \end{pmatrix},$$

with $a, a', b, b', c \in \mathbb{C}$ such that $a \neq a'$ and $b \neq b'$.

The universal space of admissble deformations is

$$\mathbf{B}(Q) = \{a, a', b, b', c \mid a \neq a', b \neq b'\} \simeq \mathrm{Conf}_2 \times \mathrm{Conf}_2 \times \mathbb{C}.$$

Link with the fission tree:



Wild mapping class group actions

Here the pure WMCG is $\Gamma(Q) = \pi_1(\mathbf{B}(Q)) \simeq PB_2 \times PB_2 \simeq \mathbb{Z}^2$.
Two generators:

- σ_2: a and a' go around each other.
- σ_1: b and b' go around each other.

Wild character variety:

$$\mathcal{M}_B(Q) = \left\{ (h, B_1^1, B_3^1, B_1^2, B_2^2, B_3^2, B_4^2) \;\middle|\; h(B_3^1 B_1^1)(B_4^2 B_3^2 B_2^2 B_1^2) = 1 \right\},$$

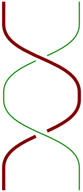
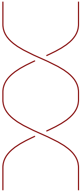
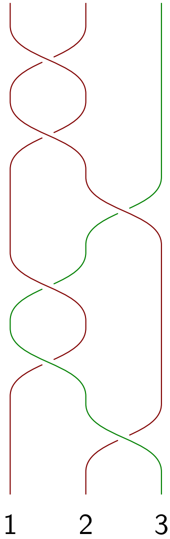
with B_i^j Stokes factors.

Action of the WMCG: $\sigma_1 = s_1^2$, $\sigma_1 = s_1^2$, with

$$s_1(h, B_1^1, B_3^1, B_i^2) = (h, B_3^1, h^{-1} B_1^1 h, B_1^1 B_i^2 {B_1^1}^{-1}),$$
$$s_2(h, B_1^1, B_3^1, B_1^2, B_2^2, B_3^2, B_4^2) = (h, B_1^1, B_3^1, B_3^2, B_4^2, h_1^{-1} B_1^2 h_1, h_1^{-1} B_2^2 h_1).$$

Cabling of braids

$\Gamma(Q)$ is a subgroup of PB_3 via cabling of braids

$\sigma_2 \leftrightarrow$  $\sigma_1 \leftrightarrow$  $\sigma_2 \circ \sigma_1 \leftrightarrow$ 

This formalises an intuition of Ramis

Extension to the nonpure case

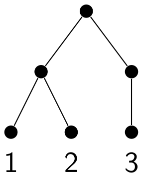
In the tame case:

- Full braid group B_n: exchanges the singularities.
- Pure braid group $PB_n \subset B_n$: fixes their order.

Wild analogue:

- Pure local wild mapping class group: fix the order of the q_i.
- Full local wild mapping class group: allow to exchange the q_i.

Not any two q_i can be exchanged, we have to look at automorphisms of the tree.



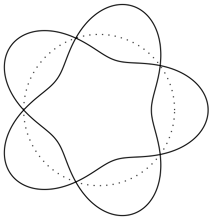
Here only q_1 and q_2 can be exchanged.

Twisted irregular types

Twisted case: $q_i \in \mathbb{C}[z^{-1/r}]$, with $r > 1$ ramification order.

In this case $q_i(z)$ is multivalued, comes with Galois conjugates $q_i(e^{2ik\pi/r}z)$.

Example: Painlevé I, related to $q = z^{5/2}$.



Get new admissibility constraints of the form

$$a \neq e^{2ik\pi/r} a'.$$

This gives new types of pieces for $\pi_1(\mathbf{B})$ (related to complex hyperplane arrangements).

Extension to principal bundles

Similar (untwisted) story for principal G-bundles, for a complex reductive Lie group G.

Main differences:

- Irregular types $Q = \sum_{i=1}^{s} A_i z^i$ with $A_i \in \mathfrak{t}$ Cartan subalgebra of $\mathfrak{g} = \mathrm{Lie}(G)$.
- Instead of differences $q_i - q_j$, get $\alpha(Q)$ with α root of $\mathfrak{g}$.
- Generic case: instead of A_s with distinct eigenvalues, get $A_s \in \mathfrak{t}_{reg}$, i.e. $\alpha(A_s) \neq 0$ for any root.
  The wild mapping class group is $\pi_1(\mathfrak{t}_{reg})$, the Artin braid group of $\mathfrak{g}$.
- For classical Lie simple algebras: we can still define fission trees, but need coloured fission trees.
- This is related to breaking the Dynkin diagram in several pieces.

Relating different isomonodromy systems

Sometimes we can have $\mathcal{M}_B(\mathbf{P}^1, \mathbf{a}, \mathbf{Q}) \simeq \mathcal{M}_B(\mathbf{P}^1, \mathbf{a}', \mathbf{Q}')$ for wild Riemann surfaces with different ranks, numbers of singularities...

Example: Fourier transform/Harnad dual of the standard Painlevé VI Lax pair, relating the cases:

- $\mathbf{a} = (a_1, a_2, a_3, \infty)$, $\mathbf{Q}$: rank 2, simple poles.
- $\mathbf{a}' = (0, \infty)$, $\mathbf{Q}'$: rank 3, simple pole at 0, second order pole at ∞,

$$Q_\infty = \begin{pmatrix} -a_1 z & & \\ & -a_2 z & \\ & & -a_3 z \end{pmatrix}.$$

The regular and irregular times are exchanged! The braidings match on both sides.

Some further questions

- Are the spaces of times and wild mapping class groups on both sides of the Fourier transform related in general?
- The global version: move the irregular types and the positions of singularities, include automorphisms of Σ.
- Write down explicitly the wild mapping class group actions.
- The twisted case outside type A?
- Many people work on the dynamics of mapping class groups actions on character varieties ("representations of surface groups"). Can we say some things about the wild dynamics?
- Quantization of this picture?